Quantisation of the Laplacian and a Curved Version of Geometric Quantisation

Julien Meyer

Département de Mathématique
Université libre de Bruxelles

This dissertation is submitted for the degree of
Docteur en Sciences

Supervisor : Prof. Joel Fine

April 2016

Acknowledgements

This thesis would not have been possible without the friendship, help and motivation of a lot of people.

Foremost I want to thank my supervisor Joel Fine who guided me through this work. Your way of thinking about mathematics was always as inspiring as motivating for me. I am also very grateful for the freedom you let me on how I chose to organise my work.

Furthermore I want to thank my co-authors Julien Keller and Reza Seyyedali with whom I wrote the paper *Quantization of the Laplacian operator on vector bundles, I*. Collaborating with you taught me a lot and showed me how common interests bring people together coming from every corner of the world.

I have had a lot of conversations with different people over the years who influenced my understanding of geometry and quantisation and whom I want to thank. These include Michel Cahen, Ruadhai Dervan, Simone Gutt, Laurent La Fuente-Gravy, Julien Keller, Mehdi Lejmi, Julian Pook, Rosa Sena-Dias, Reza Seyyedali and Patrick Weber. I also want to thank all the members of the mathematics department at the ULB, especially those who taught me mathematics when I was an undergraduate.

I am very grateful to my friends in Brussels who gave me shelter when I needed it, namely Cédric, Mattieu W., Kim, Tamy, Anissa, Patrick, Julie, Matthieu S., Thierry and Corinne.

Special thanks go to

- Cédric, Olivier and Patrick. Your friendship is invaluable!
- my parents Roland and Bruna, for your love and for always supporting me.
- my brother Daniel and his wife Raphaelle, for all the laughs and jokes and drinks and smokes…

Finally I want to thank Julie who I love and whose presence brightens my life every day.

I also warmly want to thank all the members of my jury, namely Mélanie Bertelson, Joel Fine, Simone Gutt, Georges Marinescu and Julius Ross.

This thesis was funded by an AFR PhD grant from the Fonds National de la Recherche du Luxembourg.

Table of contents

1 Introduction **1**

2 Quantisation of the Laplacian **7**

2.1 Introduction 7
2.2 The moment map picture 9
2.2.1 The infinite dimensional picture 10
2.2.2 The finite dimensional picture 12
2.3 Some preliminaries 14
2.3.1 A quick review of the Fubini–Study geometry of Grassmannians . . 14
2.3.2 Asymptotic expansions of Bergman kernels and Toeplitz operators . 15
2.3.3 The Hilb_k and the FS_k maps 19
2.3.4 The Lichnerowicz operator 20
2.4 Asymptotics of $P_k^* P_k$ 20
2.5 Eigenvalues and Eigenspaces 27
2.5.1 Initial step of the induction 29
2.5.2 Upper bound on the eigenvalues 29
2.5.3 Some estimates 31
2.5.4 Lower bound for the eigenvalues 33
2.5.5 Completing the proof of the induction, steps 2 and 3 43
2.6 Applications 46
2.6.1 Quantisation of the Laplacian for balanced metrics 46
2.6.2 Spectral measures 49
2.7 Explicit calculations for $\mathbb{C}P^1$ 50

3 Quantising Solutions to the Heat Equation **57**

3.1 Introduction 57
3.2 Relation with other work 59
3.3 Proof of the results 60

3.3.1 Proof of Proposition 3.3.3 61

4 Geometric Quantisation and the Derivative of Hilb$_k$ 75

4.1 Introduction 75

4.2 The geometric structures on $\mathscr{H}$ and $\mathscr{B}_k$ 77

4.3 The derivatives of Hilb$_k$ and FS$_k$ 78

4.4 Relation between geometric quantisation and the Hilb$_k$-map. 80

4.5 Relation between Berezin quantisation and the FS$_k$-map 82

4.6 The Hessian of Hilb$_k$ 84

4.6.1 Application 85

4.6.2 Proof of theorem 4.6.2 87

References 93

Chapter 1

Introduction

The goal of this thesis is to investigate some aspects of the quantisation of a polarised Kähler manifold $L \to X$.

Let (E,h) be a Hermitian holomorphic vector bundle of rank r over X. To ease notation we put $E(k) = E \otimes L^k$. Denote by $H^0(X, E(k))$ the space of holomorphic sections from X to $E(k)$ and by $\mathbb{G}_k$ the Grassmannian of r-dimensional subspaces of $H^0(X, E(k))^*$. For k big enough, evaluation at a point defines a map $\mathrm{ev}_k : X \to \mathbb{G}_k$. The philosophy of quantisation is to associate to each *classical* object defined on $E \to X$ a sequence of *quantised* objects defined on the dual of the tautological bundle over $\mathbb{G}_k$. These quantised objects should be defined purely in terms of the projective geometry of $\mathbb{G}_k$ and the maps ev_k and should converge to the classical object as $k \to \infty$ in an appropriate sense.

In the second chapter of this thesis we quantise the Laplacian operator acting on $C^\infty(X, \mathrm{End}_h(E))$, the space of smooth sections of the bundle of h-Hermitian endomorphisms of E. If E has rank 1, $\mathrm{End}_h(E) \cong \underline{\mathbb{R}}$ and one recovers the Laplacian acting on smooth functions. The main results of this chapter have been published in an article called *Quantization of the Laplacian operator on vector bundles I* and are joint work with Julien Keller and Reza Seyyedali, [14]. To describe them, fix a Hermitian metric on L whose curvature defines a Kähler form ω on X. This endows $H^0(X, E(k))$ with an L^2-inner product and hence the Grassmannian $\mathbb{G}_k$ with a Fubini–Study metric. Write V_k for the space of Hermitian endomorphisms of $H^0(X, E(k))$. One defines a self-adjoint operator on V_k as follows. Any element in V_k induces a holomorphic vector field on $\mathbb{G}_k$. Restricting this vector field to $\mathrm{ev}_k(X)$ defines a map $P_k : V_k \to C^\infty\left(X, \mathrm{ev}_k^*(T\mathbb{G}_k)\right)$. The spaces V_k and $C^\infty\left(X, \mathrm{ev}_k^*(T\mathbb{G}_k)\right)$ are naturally endowed with inner products given by the trace and the L^2-inner product respectively. Here the L^2-inner product is defined using the Fubini–Study metric on the fibers and the (fixed) volume form $\omega^n/n!$ coming from the metric on L. Write P_k^* for the adjoint of P_k with respect to these inner products. The composition $P_k^*P_k$ then defines a self-adjoint operator on V_k for each k. In the first chapter we show that these operators quantise the Laplacian. Note that the operators $P_k^*P_k$ are not produced by applying some general quantisation scheme to the Laplacian but are defined purely in terms of the projective geometry of $\mathbb{G}_k$ and the maps ev_k. They make sense for an arbitrary complex submanifold of a Grassmannian.

The definition of the operators $P_k^*P_k$ is motivated by the following moment map picture. Denote by $\mathscr{A}^{1,1}$ the set of unitary connections on E whose curvature is of type $(1,1)$. Since the work of Atiyah and Bott [1], it is well known that the map

$$\mu_\infty : A \mapsto \Lambda_\omega F_A$$

can be seen as a moment map for the action of the unitary gauge group $\mathscr{G}_h$ of E. Building on earlier work of Donaldson [7], Wang explains in [24] that the moment map μ_∞ can be seen as the limit of moment maps—denoted by $\bar{\mu}_k$—on the space of holomorphic embeddings of X into the Grassmannians $\mathbb{G}_k$. Noticing that the group actions extend to the complexified groups one can push Wang's picture a step further and integrate the moment maps to get the so-called Kempf–Ness functions which we denote by F_∞ and F_k respectively. From this point of view, Wang's results can be rephrased by saying that the first order derivatives of the Kempf–Ness functions converge. It is then natural to ask what happens for their Hessians. It turns out that on one hand the Hessian of F_∞ is nothing else than the Laplacian and on the other hand the Hessians of F_k are given by the sequence of operators $P_k^*P_k$ described above. Consequently they are natural candidates to consider as an intrinsic quantisation of the Laplacian.

A priori, the Laplacian and the operators $P_k^*P_k$ are completely different and in order to compare them we first need to explain the link between the spaces they act on. On one hand, a smooth section of $\mathrm{End}_h(E)$ can be viewed as an infinitesimal change of the metric on E and hence of the L^2-inner product on $H^0(X,E(k))$. In other words we get an element of V_k. This defines a map $C^\infty(\mathrm{End}_h(E)) \to V_k : \phi \mapsto Q_{k,\phi}$. On the other hand, any element in V_k gives an infinitesimal change of the inner product on $H^0(X,E(k))$ and hence induces an infinitesimal change in the metric on $E(k)$. This in turn corresponds to a Hermitian section of $\mathrm{End}(E(k))$ and hence also of $\mathrm{End}(E)$. In this way we get a map : $V_k \to C^\infty(\mathrm{End}_h(E)) : A \mapsto H_A$.

Using this notation, we are now able to state the results of the first chapter.

Theorem 1.0.1. *For each section ϕ of $\mathrm{End}_h(E)$ there is an asymptotic expansion*

$$\mathrm{Tr}\left(Q_{k,\phi}P_k^*P_kQ_{k,\phi}\right) = \frac{1}{4\pi k}\int_X \mathrm{Tr}\left(\phi\Delta^E\phi\right)\frac{\omega^n}{n!} + O\left(k^{-2}\right).$$

These estimates are uniform in the endomorphism ϕ if ϕ varies in a subset of $C^\infty(\mathrm{End}_h(E))$ which is compact for the C^∞-topology. Moreover the estimate is uniform when the metric on E varies in a set of uniformly equivalent metrics lying in a compact set for the C^∞-topology.

When E is the trivial line bundle with the flat metric we also compute the next order term of this expansion.

In the case when the bundle E is simple, meaning that its only holomorphic automorphisms are multiplication by a constant, we show that the eigenvalues of $P_k^*P_k$ converge to those of the Laplacian. Furthermore under the maps $Q_{k,(\cdot)}$ and $H_{(\cdot)}$ described above, the eigenspaces of $P_k^*P_k$ and Δ^E asymptotically line up isometrically. More precisely if we denote by $\nu_{k,j}$ and λ_j the j-th eigenvalue of $P_k^*P_k$ and Δ^E respectively we have the following.

Theorem 1.0.2. *Assume that E is simple. For each $j \geq 0$, we have*

$$\nu_{k,j} = \frac{\lambda_j}{4\pi k^{n+1}} + O\left(k^{-n-2}\right)$$

when $k \to +\infty$.

Define $F_{k,r}$ to be the space generated by the $\nu_{k,j}$-eigenspaces of $P_k^*P_k$ for $0 \leq j \leq r$ and write $F_{k,p,q}$ for the span of the $\nu_{k,j}$-eigenspaces with $p \leq j \leq q$.

Theorem 1.0.3. *Assume that E is simple. For each integer $r > 0$ there is a constant C such that for all $A, B \in F_{k,r}$,*

$$\left|\mathrm{Tr}(AB) - k^n\langle H_A, H_B\rangle_{L^2}\right| \leq Ck^{-1}\mathrm{Tr}(A^2)^{1/2}\mathrm{Tr}(B^2)^{1/2}.$$

Moreover, let us fix integers $0 < p < q$ such that $\lambda_{p-1} < \lambda_p = \lambda_{p+1} = \ldots = \lambda_q < \lambda_{q+1}$. Given an eigenvector $\phi \in \mathrm{Ker}(\Delta^E - \lambda_p \mathrm{Id})$, let $A_{\phi,k}$ denote the point in $F_{k,p,q}$ with $H_{A_{\phi,k}}$ nearest to ϕ as measured in L^2. Then

$$\|H_{A_{\phi,k}} - \phi\|_{L^2}^2 = O\left(k^{-1}\right),$$

and this estimate is uniform in ϕ if we require that $\|\phi\|_{L^2} = 1$.

Using the fact that we have some flexibility in our results (they are uniform if we vary the metric or endomorphism in compact sets with respect to the smooth topology) we derive as corollaries some quantisation results for sequences of balanced metrics when E is assumed to be Gieseker stable, see theorem 2.6.5.

As another application of the convergence results of the eigenvalues we quantise some spectral measures associated to the Laplacian, see section 2.6.2 and speculate about further implications involving the heat kernel. To conclude the chapter we illustrate our results via some direct computations in the special case when the manifold is the one dimensional complex projective space polarized by the dual of the tautological line bundle, see section 2.7.

In the third and fourth chapter of the thesis we restrict to the case when the vector bundle E is the trivial line bundle with the flat metric.

The third chapter is devoted to study the asymptotics of solutions to the “heat equation" associated to the (rescaled) operators $P_k^*P_k$. Heuristically speaking we show that the heat operator associated to $P_k^*P_k$ converges to the genuine heat operator on X as $k\to\infty$. More precisely let $f\in C^\infty(X,\mathbb{R})$ and write $f(x,t)$ for the solution to the equation

$$\begin{cases}\partial_t f(x,t)+\Delta f(x,t)=0\\ f(x,0)=f(x).\end{cases} \tag{1.0.1}$$

Furthermore given an Hermitian endomorphisms A of $H^0\left(X,L^k\right)$ we write $A(t)$ for the solution to the system

$$\begin{cases}\partial_t A(t)+4\pi k^{n+1}P_k^*P_k\left(A(t)\right)=0\\ A(0)=A.\end{cases} \tag{1.0.2}$$

Our main result of this chapter is the following.

Theorem 1.0.4. *There is a constant C such that for all $t\in[0,T]$ we have*

$$\|Q_{k,f(t,x)}-Q_{k,f(x)}\left(t\right)\|_k^2\leq\frac{C}{k}$$

where the norm $\|\cdot\|_k$ is defined by $\|A\|_k^2=\frac{1}{k^n}\mathrm{Tr}(A^2)$.

Corollary 1.0.5. *There is a constant C such that for all $t\in[0,T]$ we have*

$$\|f(x,t)-H_{Q_{k,f(x)}(t)}\|_{L^2}\leq\frac{C}{k}.$$

The fourth chapter of this thesis is dedicated to an intriguing link between geometric quantisation and a program initiated by Donaldson to study the geometry of the space of Kähler metrics in a fixed cohomology class using finite dimensional approximations. In order to ease notation, write $\mathscr{H}$ for the space of all Hermitian metrics on L for which the curvature is positive. Hence any element h in $\mathscr{H}$ induces Kähler metric ω_h on X. Furthermore let $\mathscr{B}_k$ be the *Bergman space of level k*, i.e. the space of Hermitian inner products on $H^0\left(X, L^k\right)$. The space $\mathscr{H}$ carries the structure of an infinite dimensional manifold and its tangent space at a point h is naturally identified with the space of smooth functions on X. On the other hand the tangent space to $\mathscr{B}_k$ is identified with the space V_k of Hermitian endomorphisms of $H^0\left(X, L^k\right)$.

Choosing a metric on L induces an L^2-inner product on $H^0\left(X, L^k\right)$ and hence a map $\mathrm{Hilb}_k : \mathscr{H} \to \mathscr{B}_k$. Explicitly,

$$\mathrm{Hilb}_k(h)(s,t) = \int_X h^k(s(x), t(x)) \frac{\omega_h^n}{n!}.$$

There is also a map in the other direction called the Fubini–Study map which can be described as follows. First recall that by Kodaira's embedding theorem, evaluation at a point defines an embedding $X \to \mathbb{P}\left(H^0\left(X, L^k\right)^*\right)$. A point $b \in \mathscr{B}_k$ endowes the hyperplane bundle of $\mathbb{P}\left(H^0\left(X, L^k\right)^*\right)$ with a Fubini–Study metric which can be pulled back to give a metric on L^k. Taking its k-th rooth defines a genuine metric on L which we call $\mathrm{FS}_k(b)$. Hence we have a map $\mathrm{FS}_k : \mathscr{B}_k \to \mathscr{H}$.

A lot of research has been devoted to the geometric quantisation of Kostant [16] and Souriau [21]. Roughly speaking it explains how to naturally associate to every classical observable (i.e. a smooth function on X), a quantum observable (i.e. a Hermitian operator on a Hilbert space). Here the manifold should be thought off as the phase space of a classical mecanical system. In our setup the Hilbert space in question is nothing else than $H^0\left(X, L^k\right)$ together with the L^2-inner product and the quantum observables are defined as follows. One first associates to every function $f \in C^\infty(X, \mathbb{R})$ the pre-quantum operator $\tilde{\sigma}_{k,f}$ acting on the space $C^\infty\left(X, L^k\right)$ of smooth sections from X to L^k by

$$\tilde{\sigma}_{k,f} = 2\pi k f + i\nabla^{(k)}_{X_f}$$

Here X_f denotes the Hamiltonian vector field associated to f and $\nabla^{(k)}$ is the Chern connection on L with respect to the metric h^k. To define the genuine quantum operators $\sigma_{k,f}$ one simply takes the holomorphic part of a pre-quantum operator by composing $\tilde{\sigma}_{k,f}$ with the orthogonal projecting onto the sub-space of holomorphic sections. In this way we obtain a map

$$f \in C^{\infty}(X,\mathbb{R}) \mapsto \sigma_{k,f} \in V_k.$$

The starting observation is that *this map is nothing else than the derivative of* $Hilb_k$ (see section 4.4). In this sense one can think of the Hilb_k-map as a *curved* version of geometric quantisation. It is then natural to ask of what use the higher derivatives of the Hilb_k-map could be. Furthermore one might think whether the differential of the FS_k-map also has an interpretation in terms of some dequantisation. And indeed we show in section 4.5 that its derivative is nothing else than *Berezin's covariant symbol*. Using expansions of Toeplitz operators one easily sees that the composition $d\mathrm{FS}_k \circ d\mathrm{Hilb}_k$ tends to the identity as k goes to infinity and hence, at least asymptotically, Berezin's covariant symbol can be interpreted as the inverse of geometric quantisation. This sheds new light on some of the results obtained by Cahen, Gutt and Rawnsley in [20].

Motivated by the fact that the linearisation of Hilb_k gives geometric quantisation we compute its next order approximation, namely its Hessian. To state the result, define $\mathscr{D}$: $C^{\infty}(X,\mathbb{R}) \to \Omega^{0,1}(TX)$ to be the operator given by

$$\mathscr{D}(f) = \bar{\partial}\left(X_f\right).$$

$\mathscr{D}(f)$ measures the failure of the Hamiltonian vector field X_f of being holomorphic and is know as the *Lichnerowicz operator*.

Theorem 1.0.6. *The Hessian of* $Hilb_k : \mathscr{H} \to \mathscr{B}_k$ *admits an asymptotic expansion in which the leading order term in given by the leading order of the Toeplitz operator associated to the function* $(\mathscr{D}f, \mathscr{D}g)$*. More precisely, as* $k \to \infty$*, one has*

$$(\nabla dHilb_k)_{\phi}(f,g) = T_{(\mathscr{D}f,\mathscr{D}g)} + O\left(k^{n-1}\right).$$

As a corollary we reprove a result by Fine (theorem 2 in [11]) saying that the Hessian of balancing energy converges to the Hessian of Mabuchi energy (see theorem 4.6.4).

Chapter 2

Quantisation of the Laplacian

2.1 Introduction

Let L be an ample line bundle over a compact complex manifold X of complex dimension n. Fix a Hermitian metric σ on L whose curvature F_σ gives a Kähler form $\omega = \frac{i}{2\pi}F_\sigma$ on X. Let E be a holomorphic vector bundle of rank r over X with Hermitian metric h. This data allows us to define the induced Bochner Laplacian acting on smooth endomorphisms of E,

$$\Delta^E : C^\infty(X, \mathrm{End}(E)) \to C^\infty(X, \mathrm{End}(E)).$$

Explicitely one puts $\Delta^E = -\mathrm{tr}_g \nabla^2$, where g is the Riemannian metric on X associated to ω and the complex structure and $\nabla^2 : C^\infty(\mathrm{End}(E)) \to C^\infty(T^*X \otimes T^*X \otimes \mathrm{End}(E))$.

Note that in the case of a line bundle or a Hermitian-Einstein metric, this reduces to the Kodaira Laplacian $\Delta_\partial = \partial^{*_h}_{\mathrm{End}(E)} \partial_{\mathrm{End}(E)}$ up to a constant factor. In general, both Laplacians are related by a Weitzenböck type formula, namely

$$\Delta_\partial = \frac{1}{2}\left(\Delta^E - i\Lambda_\omega F_{\mathrm{End}(E)} \cdot\right) = \frac{1}{2}\left(\Delta^E - [i\Lambda_\omega F_h, \cdot]\right). \tag{2.1.1}$$

Here $F_h \in \Omega^{1,1}(X, \mathrm{End}(E))$ denotes the curvature endomorphism of the metric h. Moreover the Bochner–Kodaira–Nakano formula gives the link between Δ_∂ and $\Delta_{\bar\partial}$,

$$\Delta_{\bar\partial} = \Delta_\partial - [i\Lambda_\omega F_h, \cdot]. \tag{2.1.2}$$

See [4] or [18] as a reference.

There is a decomposition $\mathrm{End}(E) = \mathrm{End}_{sh}(E) \oplus \mathrm{End}_h(E)$ where $\mathrm{End}_{sh}(E)$ and $\mathrm{End}_h(E)$ denote the bundles of skew-Hermitian and Hermitian endomorphisms of (E,h) respectively. It is easy to check that the Laplacian Δ^E actually preserves this decomposition. Throughout this chapter, we restrict the action of the Bochner Laplacian to the space of sections of the bundle of Hermitian endomorphisms of E, which is the central object of our study. Note that when E is a line bundle, $\mathrm{End}(E) = E \otimes E^* \cong \underline{\mathbb{C}}$ and so $C^\infty(X, \mathrm{End}(E)) \cong C^\infty(X, \mathbb{C})$. Sections of *Hermitian* endomorphisms are then given by *real* valued functions on X.

Denote by $\mathscr{A}^{1,1}$ the set of unitary connections on E whose curvature is of type $(1,1)$. Since the work of Atiyah and Bott [1] it is well known that the map

$$\mu_\infty : A \mapsto \Lambda_\omega F_A$$

can be seen as a moment map for the action of the unitary gauge group $\mathscr{G}_h$ of E. Building on earlier work of Donaldson [7], Wang explains in [24] that the moment map μ_∞ can be seen as the limit of moment maps—later denoted by $\bar{\mu}_k$—on the space of holomorphic embeddings of X into some Grassmannians denoted by $\mathbb{G}_k$. In section 2.2 we briefly recall this moment map picture and show that the group actions extend to the complexified groups. General Kempf–Ness theory tells us how to integrate the moment maps to get so-called Kempf–Ness functions F_∞ and F_k respectively. From this point of view, Wang's results can be rephrased by saying that the first order derivatives of F_k converge to those of F_∞. The goal of this chapter is then to show that their Hessians converge too. It turns out that on one hand the Hessian of F_∞ is nothing else than the Laplacian and on the other hand, the Hessians of F_k are given by a sequence of operator $P_k^* P_k$ acting on some finite dimensional vector spaces whose dimension grows in k. Consequently they are natural candidates to consider as an intrinsic quantisation of the Laplacian.

As a first result we shall see the following. Any $\phi \in C^\infty(X, \mathrm{End}_h(E))$ can be thought of as an infinitesimal change of the metric on E and hence of the L^2-inner product induced by h and the (fixed) volume form $\omega^n/n!$ on the space of holomorphic sections of $E(k) := E \otimes L^k$. Thus we naturally obtain an Hermitian endomorphism $Q_{k,\phi}$ of $H^0(X, E(k))$. Given, $\phi, \psi \in C^\infty(X, \mathrm{End}_h(E))$, we prove that the map

$$(\phi, \psi) \mapsto \mathrm{Tr}\left(Q_{k,\phi} P_k^* P_k \left(Q_{k,\psi}\right)\right)$$

admits an asymptotic expansion where the leading order term is given by the trace of the map

$$(\phi, \psi) \mapsto \int_M \phi \Delta_\partial \psi \frac{\omega^n}{n!}$$

after renormalization, see theorem 2.4.1. This first theorem relies deeply on the asymptotic expansions of Bergman kernels and Toeplitz operators due to Ma and Marinescu [19]. We briefly recall their results in section 2.3.2.

In a second step we analyse the spectrum of the operators $P_k^* P_k$. If E is simple, we prove that the eigenvalues of $P_k^* P_k$ converge (after renormalisation) towards the eigenvalues of the Laplacian Δ^E, see theorem 2.5.2. Finally, theorem 2.5.3 says that the eigenspaces of $P_k^* P_k$ converge isometrically to the eigenspaces of Δ^E in an appropriate sense. The proof of these two theorems follows a strategy used by Fine in his quantisation of the Hessian of Mabuchi energy [11].

Note that since the operators $P_k^* P_k$ act on finite dimensional spaces these results make it in principle possible to approximate the eigenvalues of the Laplacian algorithmically using a computer program.

We remark that there is some flexibility in the results is the sense that they are still valid if we vary the data (metric or endomorphism) in compact sets with respect to the smooth topology. Using this fact, we derive as an application some quantisation results for sequences of balanced metrics when E is assumed to be Gieseker stable (see theorem 2.6.5).

2.2 The moment map picture

In this section we recall the moment map picture mentioned above. To fix notation let us start with some general theory.

Let G be a Lie group acting on a symplectic manifold (X, ω) by symplectomorphisms. Differentiating the action provides a Lie algebra map

$$\mathfrak{g} \to \mathfrak{X}(X) : \phi \mapsto \xi^\phi$$

where $\mathfrak{X}(X)$ denotes the space of vector fields on X.

Definition 2.2.1. A moment map is a G-equivariant map $\mu : X \to \mathfrak{g}^*$ such that for any $\phi \in \mathfrak{g}$ the function $\mu(\phi) : X \to \mathbb{R}$ satisfies

$$d\mu(\phi) = \omega\left(\xi^\phi, \cdot\right).$$

Among many others, one important application of moment maps comes from Kempf–Ness theory. Suppose that the manifold in question is not only symplectic but Kähler and that the group G acts holomorphically and isometrically on X. Furthermore suppose that the action of G extends to a holomorphic (but no longer isometric) action of the complexification $G^{\mathbb{C}}$ of G. In this context one defines the so called *Kempf–Ness* function

$$F : G^{\mathbb{C}}/G \to \mathbb{R}$$

by integrating up the moment map. The space $G^{\mathbb{C}}/G$ can be endowed with a symmetric metric g turning it into a non-positively curved symmetric space. The function F enjoys the following two important properties:

1. $\frac{d}{dt} F\left(e^{ti\phi}h\right)\big|_{t=0} = \mu_{h\cdot p}(\phi)$;
2. $\frac{d^2}{dt^2} F\left(e^{ti\phi}h\right)\big|_{t=0} = g\left(\xi^\phi_{h\cdot p}, \xi^\phi_{h\cdot p}\right)$.

where $h \in G^{\mathbb{C}}$, $\phi \in \mathfrak{g}$ and $p \in X$. The first property implies that critical points of F correspond to zeros of the moment map in the complex orbits and the second property tells us that F is convex along geodesics in $G^{\mathbb{C}}/G$. In finite dimensions this implies for instance that, up to the action of G, zeros of the moment map are unique within a complex orbit (assuming that ξ^ϕ never vanishes). We refer to [22] for further details.

2.2.1 The infinite dimensional picture

Let (E,h) be a Hermitian vector bundle over a compact Kähler manifold (X,ω). Write $\mathscr{A}^{1,1}$ for the space of unitary connections on E whose curvature is of type $(1,1)$ and denote by $\mathscr{G}_h$ the unitary gauge group, i.e. the group of unitary automorphisms of (E,h). $\mathscr{G}_h$ acts on $\mathscr{A}^{1,1}$ by conjugation. The space of *all* unitary connections is an affine space modelled on $\Omega^1(X, \mathrm{End}_{sh}(E))$, the space of 1-forms with values in the bundle of skew-Hermitian endomorphisms of (E,h). The infinitesimal change of such a connection d_A in the direction a induces an infinitesimal change in the curvature given by $d_A a$. To make sure a is a tangent vector to $\mathscr{A}^{1,1}$, the $(0,2)$-part of $d_A a$ (and hence its $(2,0)$-part) must vanish. In other words,

$$T_{d_A}\mathscr{A}^{1,1} = \{a \in \Omega^1(X, \mathrm{End}_{sh}(E)) \mid \bar{\partial}_A a^{0,1} = 0\}.$$

There is a natural symplectic form on $\mathscr{A}^{1,1}$ given by

$$\Omega_A(a,b) := -\int_X \Lambda_\omega \mathrm{Tr}(a \wedge b)\frac{\omega^n}{n!}$$

where $a, b \in T_{d_A}\mathscr{A}^{1,1}$. The compex strucure on the space of 1-forms on X induces a complex structure J on $\Omega^1(X, \mathrm{End}_{sh}(E))$ and turns $\mathscr{A}^{1,1}$ into an infinite dimensional Kähler manifold with Kähler metric

$$(a,b)_A := \Omega_A(a, Jb).$$

The action of the unitary gauge group $\mathscr{G}_h$ on $\mathscr{A}^{1,1}$ preserves the symplectic form and thus it makes sense to ask whether it admits a moment map or not. And indeed,

Theorem 2.2.2 (Atiyah–Bott, [1]). *Under the identification of the Lie algebra $\Omega^0(X, \mathrm{End}_{sh}(E))$ of $\mathscr{G}_h$ with its dual using the L^2-inner product defined by the Killing form on the unitary group and the volume form $\omega^n/n!$, the map $\mu_\infty : \mathscr{A}^{1,1} \to \Omega^0(X, \mathrm{End}_{sh}(E))$ defined by*

$$\mu_\infty(A) := \Lambda_\omega F_A$$

is a moment map.

The action of $\mathscr{G}_h$ not only preserves the symplectic form but also the complex structure. Moreover $\mathscr{G}_h$ admits a complexification given by the *complex gauge group* $\mathscr{G}_E$, the group of complex linear automorphisms of E and the action of $\mathscr{G}_h$ on $\mathscr{A}^{1,1}$ extends to an action of $\mathscr{G}_E$ by

$$g \cdot \bar{\partial}_A = g \circ \bar{\partial}_A \circ g^{-1}$$
$$g \cdot \partial_A = (g^*)^{-1} \circ \partial_A \circ g^*$$

where $d_A \in \mathscr{A}^{1,1}$ and $g \in \mathscr{G}_E$. Thus we are in the context of Kempf–Ness theory. As mentioned above we can integrate the moment map to get a function

$$F_\infty : \mathscr{G}_E/\mathscr{G}_h \to \mathbb{R}.$$

The space $\mathscr{G}_E/\mathscr{G}_h$ actually has a nice interpretation: $\mathscr{G}_E$ acts transitively on $\mathscr{H}$, the space of all Hermitian metrics on E, and the stabilizer of the point $h \in \mathscr{H}$ is $\mathscr{G}_h$. Hence we get an identification

$$\mathscr{H} \cong \mathscr{G}_E/\mathscr{G}_h$$

and we can think of F_∞ as a function on $\mathscr{H}$. Moreover from general theory we see that the Hessian of F_∞ is given by

$$\mathrm{Hess}(F_\infty)_h(\phi,\phi) = \int_X (d_A\phi, d_A\phi)\frac{\omega^n}{n!} = \int_X \mathrm{Tr}(\phi\Delta_A\phi)\frac{\omega^n}{n!}.$$

2.2.2 The finite dimensional picture

The infinite dimensional moment map picture described in the previous section has a finite dimensional analogue as described in [24]. First recall that the unitary group $U(N)$ acts isometrically and holomorphically on the standard Grassmannian $\mathbb{G}(r,N)$ of r-dimensional subspaces in $\mathbb{C}^N$ endowed with the standard Fubini–Study metric. The action admits a moment map

$$\mu : \mathbb{G}(r,N) \to i\mathfrak{u}(N)$$

where we implicitly identified the dual of the Lie algebra $\mathfrak{u}(N)$ with $\mathfrak{u}(N)$ using the Killing form and multiplied the result by i. Thinking of the Grassmannian as

$$\mathbb{G}(r,N) = \frac{\mathscr{M}^0_{r\times N}}{\sim},$$

where $\mathscr{M}^0_{r\times N}$ is the space of rank r $r\times N$-matrices and where $z\sim w$ if and only if there exists $P \in GL(r,\mathbb{C})$ such that $z = Pw$, the explicit formula for μ is given by

$$\mu([z]) = z^*(zz^*)^{-1}z.$$

Let (X,ω) be a compact Kähler manifold of complex dimension n and let (L,σ) be an ample Hermitian line bundle over X who's curvature F_σ satisfies $\frac{i}{2\pi}F_\sigma = \omega$. Let E be a holomorphic vector bundle of rank r over X. Since L is ample we can use holomorphic sections of $E(k) = E\otimes L^k$ to embed X into $\mathbb{G}(r,N_k)$ for $k \gg 0$. Indeed, for any $x\in X$, we have the evaluation map $H^0(X,E(k)) \to E(k)_x$, which sends s to $s(x)$. Since $E(k)$ is globally generated, this map is a surjection. So its dual is an inclusion of $E(k)^*_x \hookrightarrow H^0(X,E(k))^*$, which determines an r-dimensional subspace of $H^0(X,E(k))^*$. Therefore we get a map $\iota : X \to \mathbb{G}(r,H^0(X,E(k))^*)$. Since L is ample, ι is an embedding for $k\gg 0$. Clearly we have $\iota^*U_r^* = E(k)^*$, where U_r^* is the tautological vector bundle on $\mathbb{G}(r,H^0(X,E(k))^*)$, i.e. at any r-plane in $\mathbb{G}(r,H^0(X,E(k))^*)$, the fibre of U_r^* is exactly that r-plane. Any choice of basis $\underline{s} = s_1,\dots,s_{N_k}$ for $H^0(X,E(k))$ gives an isomorphism between $\mathbb{G}(r,H^0(X,E(k))^*)$ and the standard Grassmannian $\mathbb{G}(r,N_k)$ and hence such a choice defines an embedding

$$\iota_{\underline{s}} : X \to \mathbb{G}(r,N_k).$$

Denote by $\tilde{\mathscr{B}}_k$ the space of all basis of $H^0(X,E(k))$ or equivalently the space of embeddings from X to $\mathbb{G}(r,N_k)$ which are projectively equivalent to a given one. There is a natural symplectic structure on $\tilde{\mathscr{B}}_k$ given by

$$\varpi(a,b) = \int_X \langle a,b\rangle_{FS} \frac{\omega^n}{n!},$$

for $a,b \in T_{\underline{s}}\tilde{\mathscr{B}}$ and $\langle\cdot,\cdot\rangle_{FS}$ the Fubini–Study inner product. The action of the unitary group admits a moment map $\bar{\mu} : \tilde{\mathscr{B}}_k \to i\mathfrak{u}(N_k)$. Explicitly,

$$\bar{\mu}(\underline{s}) = \int_X \mu \circ \iota_{\underline{s}} \frac{\omega^n}{n!}.$$

Since the complexification of the unitary group $U(N_k)$ is given by the general linear group $\mathrm{GL}(N_k,\mathbb{C})$, this gives rise to a function

$$F_k : \mathscr{B}_k = \mathrm{GL}(N_k,\mathbb{C})/U(N_k) \to \mathbb{R}$$

by integrating up the moment map. The homogeneous space $\mathscr{B}_k$ is called *Bergman space* and can be thought of as the space of Hermitian inner products on $H^0(X,E(k))$.

Fix $b \in \mathscr{B}_k$ and let $\underline{s}$ be any b-orthonormal basis of $H^0(X,E(k))$. Define the operator

$$P : i\mathfrak{u}(N_k) \to C^\infty(X, T\mathbb{G}(r,N_k)|_{\iota_{\underline{s}}(X)}) \tag{2.2.1}$$

by $P(A) = \xi^A|_{\iota_{\underline{s}}(X)}$. Using the Fubini-Study metric on $T\mathbb{G}(r,N_k)|_{\iota_{\underline{s}}(X)}$ and the volume form $\frac{\omega^n}{n!}$ on X, one obtains a L^2 inner product on $C^\infty(T\mathbb{G}(r,N_k)|_{\iota_{\underline{s}}(X)})$. Together with the trace on $i\mathfrak{u}(N_k)$ this allows us to define the adjoint map

$$P^* : C^\infty(T\mathbb{G}(r,N_k)_{|\iota_{\underline{s}}(X)}) \to i\mathfrak{u}(N_k)$$

and hence we get a self-adjoint operator

$$P^*P : i\mathfrak{u}(N_k) \to i\mathfrak{u}(N_k).$$

Note that the operator P and its adjoint P^* depend on the inner product b.

By general Kempf–Ness theory, the Hessian of F_k, thought of as a bilinear form on $i\mathfrak{u}(N_k)$, is given by

$$(\operatorname{Hess} F_k)_b(A,B) = \int_{\iota_{\underline{s}}(X)} \langle P(A), P(B)\rangle_{FS} \frac{\omega^n}{n!} = \operatorname{Tr}(AP^*PB).$$

2.3 Some preliminaries

Before we start with the proofs of our results let us first recall some general theory which will be needed.

2.3.1 A quick review of the Fubini–Study geometry of Grassmannians

Denote the space of all matrices $z \in \mathscr{M}_{r\times N}(\mathbb{C})$ with rank r by $\mathscr{M}^0_{r\times N}$. By definition,

$$\mathbb{G}(r,N) = \frac{\mathscr{M}^0_{r\times N}}{\sim},$$

where $z \sim w$ if and only if there exists $P \in GL(r,\mathbb{C})$ such that $z = Pw$. Note that $\mathbb{G}(r,N)$ can be identified with the space of all r-dimensional subspaces of $\mathbb{C}^N$. The tangent bundle of $\mathbb{G}(r,N)$ is given by

$$\frac{\left\{(z,X)\,|\,z \in \mathscr{M}^0_{r\times N},\, X \in \mathscr{M}_{r\times N}\right\}}{\sim'},$$

where $(z,X) \sim' (w,Y)$ if and only if there exists $P \in GL(r,\mathbb{C})$ and $Q \in \mathscr{M}_{r\times r}(\mathbb{C})$ such that

$$z = Pw,\ X = PY + Qw.$$

Under the equivalence relation $\sim'$, an element of $T\mathbb{G}(r,N)$ is denoted $[z,X]$. The Fubini-Study metric on $T\mathbb{G}(r,N)$ is given by

$$\langle [(z,X)], [(z,Y)]\rangle_{FS} = \operatorname{Tr}(Y^*(zz^*)^{-1}X) - \operatorname{Tr}((zz^*)^{-1}zY^*(zz^*)^{-1}Xz^*).$$

Let $U_r \to \mathbb{G}(r,N)$ be the dual of the tautological bundle

$$U_r^* = \{(P,v) \in \mathbb{G}(r,N)\times\mathbb{C}^N\,|\,v \in P\}.$$

The standard Hermitian inner product on $\mathbb{C}^N$ induces a Fubini–Study metric on U_r and U_r^* by restriction. There is a 1-1 correspondence between the space of linear forms on $\mathbb{C}^N$ and the holomorphic sections of U_r. Explicitely,

$$f \in (\mathbb{C}^N)^* \mapsto s_f \in H^0(\mathbb{G}(r,N), U_r)$$

where

$$s_f[z](v) = f(v),$$

for any $[z] \in \mathbb{G}(r,N)$ and $v \in (U_r^*)_{[z]}$. Let $e_1, \dots e_N$ be the standard basis of $\mathbb{C}^N$ and define $s_i = s_{e_i^*}$. Then $s_1, \dots s_N$ is a basis for $H^0(\mathbb{G}(r,N), U_r)$ satisfying

$$\sum_{i=1}^{N} s_i \otimes s_i^{*_{h_{FS}}} = \mathrm{Id}_{U_r}$$

as an endomorphism over $\mathbb{G}(r,N)$.

Definition 2.3.1. Any $A \in i\mathfrak{u}(N)$ defines a smooth Hermitian endomorphism (with respect to the Fubini-Study metric h_{FS}) H_A of U_r by

$$H_A = \sum_{i,j=1}^{N} A_{ji} s_i \otimes s_j^{*_{h_{FS}}}.$$

Definition 2.3.2. Any $A \in i\mathfrak{u}(N)$ induces a holomorphic vector field ξ^A on $\mathbb{G}(r,N)$ given by

$$\xi^A(z) := [z, zA]. \tag{2.3.1}$$

2.3.2 Asymptotic expansions of Bergman kernels and Toeplitz operators

Part of the main technical tools in this thesis are the asymptotic expansions of Bergman kernels and Toeplitz operators.

Let (X, ω) be a compact Kähler manifold of complex dimension n and let (L, σ) be a Hermitian line bundle over X such that the curvature F_σ of the Chern connection satisfies $F_\sigma = -2\pi i \omega$. Let (E, h) be a Hermitian holomorphic vector bundle on X of rank r and put $E(k) = E \otimes L^k$. Write $L^2(X, E(k))$ for the completion of $C^\infty(X, E(k))$ with respect to the L^2-inner product induced by the metric $h \otimes \sigma^k$ on $E(k)$ and the volume form $\omega^n/n!$. Denote by $\Pi_k : L^2(X, E(k)) \to H^0(X, E(k))$ the orthogonal projection onto the finite dimensional subspace of holomorphic sections. Its integral kernel $B_k(x,y)$, also called the *Bergman kernel*, is a smooth section of the pull-back bundle $E(k) \boxtimes E(k)^* \to X \times X$. Explicitly at a point

$(x,y) \in X \times X$ it is given by

$$B_k(x,y) = \sum_{i=1}^{N_k} s_i(x) \otimes s_i^*(y)$$

where $\underline{s} = s_1, \dots, s_{N_k}$ is an L^2-orthonormal basis of $H^0(X, E(k))$. Furthermore the restriction of B_k to the diagonal is naturally identified with a section of $\mathrm{End}(E)$.

Remark 2.3.3. We say that a sequence of $\Theta_k \in C^\infty(X, \mathrm{End}(E))$ has an asymptotic expansion of the form

$$\Theta_k(x) = \sum_{j=0}^{\infty} A_j(x) k^{n-j}$$

where $A_j \in C^\infty(X, \mathrm{End}(E))$, if for any $r, M > 0$ there exists a constant $C_{r,M}$ such that

$$\left| \Theta_k - \sum_{j=0}^{M} A_j k^{n-j} \right|_{C^r(X)} \leq C_{r,M} k^{n-M-1}$$

where $|\cdot|_{C^r(X)}$ denotes the C^r-norm.

The following result has been proved in various degrees of generality by Zelditch [25], Catlin [3], Lu [17] and Bouche [2] in the case where E is a line bundle. The general case was studied later by Wang [24] and in a more general setting by Ma and Marinescu, see [18] and [19].

Theorem 2.3.4 (Theorem 4.1.2. in [18])**.** *For any Hermitian metric h on E and Kähler form $\omega \in c_1(L)$, there exists smooth endomorphisms $b_i(h, \omega) \in C^\infty(X, \mathrm{End}(E))$ such that the restriction to the diagonal of the Bergman kernel, denoted by $B_k(h, \omega)$, admits an asymptotic expansion as $k \to \infty$,*

$$B_k(h, \omega) = k^n + b_1(h, \omega) k^{n-1} + \dots$$

In particular

$$b_1(h, \omega) = \frac{i}{2\pi} \Lambda_\omega F_{(E,h)} + \frac{1}{8\pi} S(\omega) \mathrm{Id}_E.$$

Moreover the expansion is uniform in h and ω if they vary in compact subsets for the C^∞-topology.

Definition 2.3.5. Let $f \in C^\infty(X, \mathrm{End}(E))$. We define the *Toeplitz operator* $T_{k,f} : L^2(X, E(k)) \to L^2(X, E(k))$ by

$$T_{k,f} = \Pi_k \circ f \circ \Pi_k.$$

One checks that in terms of an L^2-orthonormal basis $\underline{s} = s_1, \ldots, s_{N_k}$ of $H^0(X, E(k))$ the integral kernel of $T_{k,f}$ can explicitly be written as

$$K_{k,f}(x,y) = \sum_{i,j=1}^{N_k} \int_X \langle f s_i, s_j \rangle_{h\otimes\sigma^k}(z)\, s_j(x) \otimes s_i^*(y)\, \frac{\omega_z^n}{n!}.$$

Theorem 2.3.6 (Theorem 0.1 in [19]). *Let $f \in C^\infty(X, \mathrm{End}(E))$. The restriction to the diagonal of the Toeplitz kernel admits the following asymptotic expansions as $k \to +\infty$,*

$$K_{k,f} = b_{0,f}k^n + b_{1,f}k^{n-1} + b_{2,f}k^{n-2} + O(k^{n-3})$$

where

$$\begin{aligned} b_{0,f} &= f, \\ b_{1,f} &= \frac{S(\omega)}{8\pi} f + \frac{i}{4\pi}(\Lambda_\omega F_{h_E} f + f\Lambda_\omega F_{h_E}) - \frac{1}{4\pi}\Delta^E f. \end{aligned}$$

In the case when E is the trivial line bundle with the flat metric one has

$$b_{2,f} = b_2 f + \frac{1}{32\pi^2}\Delta^2 f - \frac{1}{32\pi^2} S(\omega)\Delta f + \frac{1}{8\pi^2}(Ric, i\bar{\partial}\partial f).$$

This expansion is uniform in the endomorphism f if f varies in a subset of $C^\infty(X, \mathrm{End}(E))$ which is compact for the C^∞-topology. Eventually the expansions are uniform when the metric h on E varies in a set of uniformly equivalent metrics lying in a compact set for the C^∞-topology.

We come now to the composition of Toeplitz operators. For $f, g \in C^\infty(X, \mathrm{End}(E))$ consider $T_{k,f,g} = T_{k,f} \circ T_{k,g}$ and denote by $K_{k,f,g}$ its integral kernel which can be written explicitly in terms of $K_{k,f}$ and $K_{k,g}$ as

$$K_{k,f,g}(x,y) = \int_X K_{k,f}(z,y) \circ K_{k,g}(x,z)\, \frac{\omega_z^n}{n!}.$$

Theorem 2.3.7 (Theorem 0.2 in [19]). *Let $f, g \in C^\infty(X, \mathrm{End}(E))$. As $k \to \infty$ the restriction to the diagonal of the kernel $K_{k,f,g}$ admits an asymptotic expansion*

$$K_{k,f,g} = b_{0,f,g}k^n + b_{1,f,g}k^{n-1} + O(k^{n-2}).$$

In particular,

$$\begin{aligned} b_{0,f,g} &= fg, \\ b_{1,f,g} &= \frac{1}{8\pi}S(\omega)fg + \frac{i}{4\pi}\left(\Lambda_\omega F_{h_E} fg + fg\Lambda_\omega F_{h_E}\right) \\ &\qquad - \frac{1}{4\pi}\left(f\Delta^E g + \left(\Delta^E f\right)g\right) + \frac{1}{2\pi}\langle \bar{\partial}^E f, \nabla^{1,0} g\rangle. \end{aligned}$$

Moreover this expansion is uniform in the endomorphisms f,g if f and g vary in a subset of $C^\infty(X, End(E))$ which is compact for the C^∞-topology. Eventually the expansions are uniform when the metric h and ω vary in sets of uniformly equivalent metrics lying in a compact set for the C^∞-topology.

Finally the last result we need is the following expansion of the composition of two Toeplitz operators.

Theorem 2.3.8 (Theorem 0.3 in [19])**.** *Let $f,g \in C^\infty(X, \mathrm{End}(E))$. The composition of the Toeplitz operators $T_{k,f}$ and $T_{k,g}$ is again a Toeplitz operator and admits the asymptotic expansion*

$$T_{k,f} \circ T_{k,g} = T_{k,C_0(f,g)} + T_{k,C_1(f,g)} k^{-1} + O\left(k^{-2}\right).$$

where C_r are bidifferential operators, in the sense that for any $r \geq 0$, there exists constants $c_r > 0$ with

$$||T_{k,f} \circ T_{k,g} - \sum_{j=0}^{r} k^{-j} T_{k,C_j(f,g)}|| \leq c_r k^{-r-1}$$

where $\|\cdot\|$ denotes the operator norm. Moreover the first order terms are given by

$$\begin{aligned} C_{0,f,g} &= fg, \\ C_{1,f,g} &= -\frac{1}{2\pi}\langle \nabla^{1,0} f, \bar{\partial}^E g\rangle, \\ C_{2,f,g} &= b_{2,f,g} - b_{2,fg} - b_{1,C_1(f,g)}. \end{aligned}$$

If $f,g \in C^\infty(X,\mathbb{R})$ then

$$C_{2,f,g} = \frac{1}{8\pi^2}\langle D^{1,0}\partial f, D^{0,1}\bar{\partial} g\rangle + \frac{i}{4\pi^2}\langle Ric, \partial f \wedge \bar{\partial} g\rangle - \frac{1}{4\pi^2}\langle \partial f \wedge \bar{\partial} g, F_h\rangle.$$

Here $D^{1,0}$ and $D^{0,1}$ are the $(1,0)$ and $(0,1)$ components of the connection

$$D^{T^*X} : C^\infty(X, T^*X) \to C^\infty(X, T^*X \otimes T^*X)$$

induced by the Levi-Civita connection on X.

2.3.3 The Hilb_k and the FS_k maps

In order to show that the operators Δ and $P_k^* P_k$ are linked, we will need to understand the relation between the spaces they act on. As before denote by $\mathscr{H}$ the space of Hermitian metrics on E and by $\mathscr{B}_k$ the space of Hermitian inner products on $H^0(X, E(k))$. Following Donaldson [9] we define maps from $\mathscr{H}$ to $\mathscr{B}_k$ and back:

- Define $\mathrm{Hilb}_k : \mathscr{H} \to \mathscr{B}_k$ by
$$\mathrm{Hilb}_k(h)(s,t) = \int_X \langle s(x), t(x) \rangle_{h \otimes \sigma^k} \frac{\omega^n}{n!}$$
for any $s, t \in H^0(X, E(k))$.
- To define the map in the other direction, let $b \in \mathscr{B}_k$ and pick any b-orthonormal basis $\underline{s} = s_1, \ldots, s_{N_k}$ of $H^0(X, E(k))$. As explained in section 2.2.2 this defines an embedding of X into $\mathbb{G}(r, N_k)$. Pulling-back the Fubini–Study metric from the dual of the tautological bundle defines an Hermitian metric on $E(k)$ and hence on E which we call $\mathrm{FS}_k(b)$. Hence we have a map $\mathrm{FS}_k : \mathscr{B}_k \to \mathscr{H}$. Equivalently $\mathrm{FS}_k(b)$ is the unique metric on E such that
$$\sum s_i \otimes s_i^{*_{\mathrm{FS}_k(b) \otimes \sigma^k}} = \mathrm{Id}_E.$$

Furthermore for $h \in \mathscr{H}$, consider the map

$$d(\mathrm{Hilb}_k)_h : T_h \mathscr{H} \cong C^\infty(X, \mathrm{End}_h(E)) \to T_{\mathrm{Hilb}_k(h)} \mathscr{B}_k \cong i\mathfrak{u}(N_k).$$

In order to simplify notations we put

$$Q_{k,\phi} := d(\mathrm{Hilb}_k)_h(\phi).$$

In terms of an $\mathrm{Hilb}_k(h)$-orthonormal basis $s_1, \ldots, s_N$ of $H^0(X, E(k))$ we have

$$\left(Q_{k,\phi}\right)_{ij} = \int_X \langle s_i, \phi s_j \rangle_{h \otimes \sigma^k} \frac{\omega^n}{n!}.$$

To see this, it suffices to differentiate Hilb_k along the path of Hermitian metrics given by

$$\langle r, s \rangle_t = \langle r, (\mathrm{Id} + t\phi) s \rangle_{h \otimes \sigma^k}$$

where $r, s \in H^0(X, E(k))$ and where t is sufficiently small.

2.3.4 The Lichnerowicz operator

The so called *Lichnerowicz operator* plays an important role in the last chapter of this thesis. Furthermore, it appears in the second order term in the asymptotic expansion of $P_k^* P_k$ in the case when E is the trivial line bundle, see theorem 2.4.1.

Let $L \to X$ be an ample holomorphic line bundle over a compact Kähler manifold. Fix a Hermitian metric σ on L such that its curvature F_σ gives a Kähler form $\omega = \frac{i}{2\pi} F_\sigma$.

Definition 2.3.9. The Lichnerowicz operator $\mathscr{D} : C^\infty(X, \mathbb{R}) \to \Omega^{0,1}(TX)$ is defined to be the operator given by

$$\mathscr{D}(f) = \bar{\partial}(v_f)$$

where v_f is the Hamiltonian vector field corresponding to f via ω. $\mathscr{D}(f)$ measures the failure of the Hamiltonian vector field v_f of being holomorphic.

Write $\mathscr{D}^*$ for its L^2-adjoint. The composition $\mathscr{D}^*\mathscr{D}$ is linked to the linearisation of the scalar curvature D as follows. If σ_t is the path of Hermitian metrics on L given by $\sigma_t = e^{4\pi f t}$, we put

$$D(f) = \frac{\partial S(\sigma_t)}{\partial t}.$$

Lemma 2.3.10.

$$D(f) = \Delta^2 f - 2(Ric, 2i\bar{\partial}\partial f)$$

and

$$\mathscr{D}^*\mathscr{D}(f) = D(f) + (dS, df)$$

See for example [5] or [11] as a reference.

2.4 Asymptotics of $P_k^* P_k$

Fix an Hermitian metric h on E. Applying the Hilb_k-map we get an Hermitian inner product on $H^0(X, E(k))$ for each k. Choosing any Hilb_k-orthonormal basis of $H^0(X, E(k))$ identifies $\mathbb{G}\left(r, H^0(X, E(k))^*\right)$ with the standard Grassmannian $\mathbb{G}(r, N_k)$. As explained in section 2.2.2 this gives us a sequence of embeddings $\iota_k : X \mapsto X_k \subset \mathbb{G}(r, N_k)$. Let us recall the construction of the operator $P_k^* P_k$ in this setting.

Any $A \in i\mathfrak{u}(N_k)$ defines a holomorphic vector field ξ^A on $\mathbb{G}(r, N_k)$. Following the discussion from section 2.2.2, we define

$$P_k : i\mathfrak{u}(N_k) \to C^\infty(X, T\mathbb{G}(r, N_k)|_{X_k})$$

by $P_k(A) = \xi_A|_{X_k}$. Using the trace on $i\mathfrak{u}(N_k)$ and the L^2 inner product on $C^\infty(X, T\mathbb{G}(r, N_k)|_{X_k})$ induced by the Fubini–Study metric on the tangent space we, get an adjoint map

$$P_k^* : C^\infty(X, T\mathbb{G}(r, N_k)|_{X_k}) \to i\mathfrak{u}(N_k).$$

Thus, we have a sequence of self-adjoint operators $P_k^* P_k : i\mathfrak{u}(N_k) \to i\mathfrak{u}(N_k)$.

Theorem 2.4.1. *Let $\phi \in C^\infty(X, End_h(E))$. The pullback of $P_k^* P_k$ under the map $Hilb_k : \mathscr{H} \to \mathscr{B}_k$ admits an asymptotic expansion as $k \to \infty$. More precisely,*

$$\mathrm{Tr}\left(Q_{k,\phi} P_k^* P_k(Q_{k,\phi})\right) = a_1 k^{-1} + a_2 k^{-2} + O(k^{-3})$$

where the leading order term a_1 is given by

$$\begin{aligned} a_1 &= \frac{1}{4\pi} \int_X \mathrm{Tr}\left(\phi \Delta^E \phi\right) \frac{\omega^n}{n!}, \\ &= \frac{1}{2\pi} \int_X \mathrm{Tr}\left(\phi \Delta_\partial \phi\right) \frac{\omega^n}{n!}, \\ &= \frac{1}{2\pi} \int_X \mathrm{Tr}\left(\phi \Delta_{\bar\partial} \phi\right) \frac{\omega^n}{n!}. \end{aligned}$$

Moreover, in the case where E is the trivial flat line bundle, the second order coefficient is given for all $\phi \in C^\infty(X, \mathbb{R})$ by

$$a_2 = \frac{1}{32\pi^2} \int_X \left(\phi \mathscr{D}^* \mathscr{D} \phi - 4\phi \Delta^2 \phi\right) \frac{\omega^n}{n!}.$$

These estimates are uniform in the endomorphism ϕ if ϕ varies in a subset of $C^\infty(X, End_h(E))$ which is compact for the C^∞-topology. The estimate is uniform when the metric h on E varies in a set of uniformly equivalent metrics lying in a compact set for the C^∞-topology.

The proof of this theorem relies on the following lemma which is a generalisation of lemma 18 in [10]. The notation is that of sections 2.2.2 and 2.3.1.

Lemma 2.4.2. *For any $A, B \in i\mathfrak{u}(N)$, the following identitiy holds pointwise in $\mathbb{G}(r, N)$,*

$$\mathrm{Tr}(H_A H_B) + \langle \xi^A, \xi^B \rangle_{FS} = \mathrm{Tr}(AB\mu). \tag{2.4.1}$$

Proof. By $U(N)$ equivariance it is sufficient to prove the formula at one particular point in $\mathbb{G}(r,N)$, let's say the r-dimensional subspace in $\mathbb{C}^N$ generated by the r first basis-vectors of the standard basis $e_1,\dots,e_N$. Denote this point by $[z]$ where z is the $r\times N$ matrix given by

$$z = \begin{pmatrix} \mathrm{Id}_{r\times r} & 0_{r\times N-r} \end{pmatrix}.$$

In order to fix notation, note that any $A \in i\mathfrak{u}(N)$ can be decomposed as

$$A = \begin{pmatrix} A_{r\times r} & A_{r\times N-r} \\ A_{N-r\times r} & A_{N-r\times N-r} \end{pmatrix}.$$

Let's start computing the first term of the left-hand side of equation (2.4.1). For any $A \in i\mathfrak{u}(N)$ the matrix of H_A at the point $[z]$ is given by

$$\begin{pmatrix} A_{r\times r} & 0_{r\times N-r} \\ 0_{N-r\times r} & 0_{N-r\times N-r} \end{pmatrix}$$

and hence

$$\mathrm{Tr}\left(H_A\left([z]\right)H_B\left([z]\right)\right) = \mathrm{Tr}\left(A_{r\times r}B_{r\times r}\right).$$

In order to compute the second term of the left-hand side in (2.4.1), recall that by definition, the vector field ξ^A induced by $A \in i\mathfrak{u}(N)$ at $[z]$ is given by

$$\begin{aligned} \xi^A([z]) &= [z, zA] \\ &= \left[\begin{pmatrix} \mathrm{Id}_{r\times r} & 0_{r\times N-r} \end{pmatrix}, \begin{pmatrix} A_{r\times r} & A_{r\times N-r} \end{pmatrix}\right]. \end{aligned}$$

Moreover from section 2.3.1 we know that the general formula for the Fubini–Study metric on $\mathbb{G}(r,N)$ is given by

$$\langle [(z,X)],[(z,Y)]\rangle_{FS} = \mathrm{Tr}(Y^*(zz^*)^{-1}X) - \mathrm{Tr}((zz^*)^{-1}zY^*(zz^*)^{-1}Xz^*).$$

In our situation, $zz^* = \mathrm{Id}_{r\times r}$ and a straightforward computation shows that

$$\langle [(z,zA)],[(z,zB)]\rangle_{FS} = \mathrm{Tr}\left((B_{r\times N-r})^* A_{r\times N-r}\right).$$

In order to compute the right-hand side of (2.4.1), first note that

$$\mu\left([z]\right) = z^*\left(zz^*\right)z = \begin{pmatrix} \mathrm{Id}_{r\times r} & 0_{r\times N-r} \\ 0_{N-r\times r} & 0_{N-r\times N-r} \end{pmatrix}.$$

After a short calculation, one gets

$$\operatorname{Tr}(AB\mu([z])) = \operatorname{Tr}(A_{r\times r}B_{r\times r}) + \operatorname{Tr}(A_{r\times N-r}B_{N-r\times r}).$$

The fact that B is Hermitian implies that $B_{N-r\times r} = B^*_{r\times N-r}$. Putting everything together yields the result.

$\square$

Proof of theorem 2.4.1

In order to simplify notation, we denote the volume form $\frac{\omega^n}{n!}$ by Ω through the proof.

Integrating the formula from lemma 2.4.2 over X, one has for all $A, B \in i\mathfrak{u}(N_k)$,

$$\int_X \langle \xi^A, \xi^B \rangle_{FS}\, \Omega = \operatorname{Tr}(AB\bar{\mu}_k) - \langle H_A, H_B \rangle_{L^2}$$

Moreover, by definition of the operator P_k,

$$\operatorname{Tr}(AP_k^* P_k(B)) = \int_X \langle \xi^A, \xi^B \rangle_{FS}\, \Omega.$$

Putting these together, we get that for any $\phi \in C^\infty(X, End_h(E))$,

$$\operatorname{Tr}\left(Q_{k,\phi} P_k^* P_k(Q_{k,\phi})\right) = \operatorname{Tr}\left(Q_{k,\phi}^2 \bar{\mu}_k\right) - \int_X \operatorname{Tr}\left(H^2_{Q_{k,\phi}}\right)\Omega. \tag{2.4.2}$$

Let $\underline{s} = s_1, \ldots, s_{N_k}$ be a $\mathrm{Hilb}_k(h)$-orthonormal basis of $H^0(X, E(k))$. We start computing the first term of the right-hand side of this formula.

$$\begin{aligned}
\operatorname{Tr}\left(Q_{k,\phi}^2 \bar{\mu}_k\right) &= \sum_{i,j,\ell} \int_X \langle s_i, \phi s_j \rangle\, \Omega \int_X \langle s_j, \phi s_\ell \rangle\, \Omega \int_X \langle s_\ell, B_k^{-1} s_i \rangle\, \Omega \\
&= \operatorname{Tr} \int_{X\times X\times X} \sum_{i,j,\ell} \langle s_i, \phi s_j \rangle(y) \langle s_j, \phi s_\ell \rangle(z) s_\ell \otimes s_i^*(x) \circ B_k^{-1}(x)\, \Omega \wedge \Omega \wedge \Omega \\
&= \operatorname{Tr} \int_X K_{\phi,\phi,k}(x)(B_k^{-1}(x))\, \Omega.
\end{aligned}$$

Using theorem 2.3.4 and proposition 2.3.6, one can compute the first two terms in the asymptotic expansion of this expression. After a short computation, one gets

$$\int_X \mathrm{Tr}(\phi^2)\,\Omega + k^{-1}\mathrm{Tr}\left(\int_X \frac{1}{2\pi}\langle\bar\partial\phi, \nabla^{1,0}\phi\rangle\,\Omega - \int_X \frac{1}{2\pi}\phi\Delta^E\phi\right)\Omega + O\left(k^{-2}\right)$$
$$= \int_X \mathrm{Tr}(\phi^2)\,\Omega + \frac{k^{-1}}{2\pi}\int_X \left(\left|\bar\partial\phi\right|^2 - \mathrm{Tr}\left(\phi\Delta^E\phi\right)\right)\Omega + O(k^{-2}).$$

On the other hand,

$$\begin{aligned} H_{Q_{k,\phi}}(x) &= \sum_{i,j}\int_X \langle s_i, \phi s_j\rangle(y) s_j \otimes s_i^{*\mathrm{FS}_k(h)}(x)\,\Omega \\ &= \sum_{i,j}\int_X \langle s_i, \phi s_j\rangle(y) s_j \otimes s_i^{*_{B_k^{-1}h}}(x)\,\Omega \\ &= \sum_{i,j}\int_X \langle s_i, \phi s_j\rangle(y) s_j \otimes s_i^{*h}\circ B_k^{-1}(x)\,\Omega \\ &= K_{k,\phi}\circ B_k^{-1}(x). \end{aligned}$$

Using the asymptotic expansions for B_k and $K_{k,\phi}$, we have

$$\begin{aligned} H_{Q_{k,\phi}} = &\left(\phi + k^{-1}\Big(\frac{S(\omega)}{8\pi}\phi + \frac{i}{4\pi}(\Lambda F_h\phi + \phi\Lambda F_h) - \frac{1}{4\pi}\Delta^E\phi\Big) + \dots\right) \\ &\times\left(\mathrm{Id} - k^{-1}\Big(\frac{S(\omega)}{8\pi} + \frac{i}{2\pi}\Lambda F_h\Big) + \dots\right) \\ =&\phi + k^{-1}\left(\frac{i}{4\pi}(\Lambda F_h\phi + \phi\Lambda F_h - 2\phi\Lambda F_h) - \frac{1}{4\pi}\Delta^E\phi\right) + O(k^{-2}). \end{aligned}$$

Therefore,

$$\mathrm{Tr}(H^2_{Q_{k,\phi}}) = \mathrm{Tr}(\phi^2) - \frac{k^{-1}}{2\pi}\mathrm{Tr}(\phi\Delta^E\phi) + O(k^{-2}).$$

Thus, putting everything together yields

$$\begin{aligned} \mathrm{Tr}\left(Q_{k,\phi}P_k^*P_kQ_{k,\phi}\right) &= \int_X \mathrm{Tr}(Q_{k,\phi}\mu)\,\Omega - \int_X \mathrm{Tr}(H^2_{Q_{k,\phi}})\,\Omega \\ &= \frac{k^{-1}}{2\pi}\int_X |\bar\partial\phi|^2\Omega + O\left(k^{-2}\right). \end{aligned}$$

Moreover, by the Weitzenböck formula (2.1.1) and the Bochner–Kodaira–Nakano identity (2.1.2), we have

$$\begin{aligned}\frac{1}{2\pi}\int_X |\bar{\partial}\phi|^2\,\Omega &= \frac{1}{2\pi}\int_X \mathrm{Tr}(\phi\Delta_{\bar{\partial}}\phi)\,\Omega\\ &= \frac{1}{4\pi}\int_X \mathrm{Tr}(\phi\Delta^E\phi)\,\Omega + \frac{i}{2\pi}\int_X \mathrm{Tr}(\phi[\Lambda F_h,\phi])\,\Omega\\ &= \frac{1}{4\pi}\int_X \mathrm{Tr}(\phi\Delta^E\phi)\,\Omega.\end{aligned}$$

We now explain how to compute the second order coefficient in the case E is the trivial line bundle with the flat metric. The only thing we have to do is to go one term further is the asymptotic expansion of the two terms in equation (2.4.2).

Let us start with the first term. We already know that

$$\int_X \mathrm{Tr}(Q_{k,\phi}^2\mu)\,\Omega = \int_X K_{\phi,\phi,k}(x)(B_k^{-1}(x))\,\Omega$$

Using theorem 2.3.4 one gets

$$B_k^{-1} = k^{-n}\left(1-\frac{S}{8\pi}k^{-1}+\left(\frac{S^2}{64\pi^2}-b_2\right)k^{-2}+O\left(k^{-3}\right)\right).$$

Moreover, the asymptotic expansion for $K_{\phi,\phi,k}$ given in theorem 2.3.6 implies

$$\begin{aligned}K_{\phi,\phi,k}B_k^{-1} &= \left(1-\frac{S}{8\pi}k^{-1}+\left(\frac{S^2}{64\pi^2}-b_2\right)k^{-2}+O(k^{-3})\right)\\ &\quad\times\left\{\phi^2+\left(\frac{S}{8\pi}\phi^2-\frac{1}{2\pi}\phi\Delta\phi+\frac{1}{4\pi}|df|^2\right)k^{-1}+b_{2,\phi,\phi}k^{-2}+O\left(k^{-3}\right)\right\}.\end{aligned}$$

Since we already computed the k^0 and k^{-1}-terms above, we only focus on the k^{-2}-term. An straightforward computation shows that the coefficient of the k^{-2} term is given by

$$b_{2,\phi,\phi}+\frac{S}{16\pi^2}\phi\Delta\phi-\frac{S}{32\pi^2}|d\phi|^2-b_2\phi^2,$$

and hence the k^{-2}-coefficient of $\int_X \mathrm{Tr}(Q_{k,\phi}^2\mu)\Omega$ is

$$\int_X\left(b_{2,\phi,\phi}+\frac{S}{16\pi^2}\phi\Delta\phi-\frac{S}{32\pi^2}|d\phi|^2-b_2\phi^2\right)\Omega. \tag{2.4.3}$$

Now consider the second term of the right-hand side of equation (2.4.2). We need the asymptotic expansion of $H_{Q_{k,\phi}}$ up to the k^{-2} term. We have

$$\begin{aligned}
H_{Q_{k,\phi}} &= K_{k,\phi}B_k^{-1} \\
&= \left(1-\frac{S}{8\pi}k^{-1}+\left(\frac{S^2}{64\pi^2}-b_2\right)k^{-2}+O(k^{-3})\right) \\
&\qquad \times\left(\phi+\left(\frac{S}{8\pi}\phi-\frac{\Delta\phi}{4\pi}\right)k^{-1}+b_{2,\phi}k^{-2}+O(k^{-3})\right).
\end{aligned}$$

Expanding this expression, we get

$$\phi-\frac{\Delta\phi}{4\pi}k^{-1}+\left(b_{2,\phi}+\frac{S\Delta\phi}{32\pi^2}-b_2\phi\right)k^{-2}+O(k^{-3}).$$

An easy calculation shows then that the k^{-2}-coefficient of $H^2_{Q_{k,\phi}}$ is given by

$$2\phi b_{2,\phi}+\frac{S\phi\Delta\phi}{16\pi^2}-2b_2\phi^2+\frac{(\Delta\phi)^2}{16\pi^2}.$$

Hence the k^{-2}-coefficient of $\int_X H^2_{Q_{k,\phi}}\Omega$ is

$$\int_X\left(2\phi b_{2,\phi}+\frac{S\phi\Delta\phi}{16\pi^2}-2b_2\phi^2+\frac{(\Delta\phi)^2}{16\pi^2}\right)\Omega. \tag{2.4.4}$$

Putting equations 2.4.3 and 2.4.4 together we get the k^{-2}-coefficient of $\mathrm{Tr}(Q_{k,\phi}P_k^*P_kQ_k\phi)$:

$$\int_X\left(b_{2,\phi,\phi}-\frac{S|d\phi|^2}{32\pi^2}-2\phi b_{2,\phi}+b_2\phi^2-\frac{\phi\Delta^2\phi}{16\pi^2}\right)\Omega.$$

Now we use the fact that $\int_X K_{\phi,\psi,k}=\int_X \phi K_{\psi,k}$ which implies that $\int_X b_{\phi,\psi,k}=\int_X \phi b_{\psi,k}$. Using the formula for $b_{2,\phi}$ given in proposition 2.3.6 we get

$$\int_X\left(-\frac{3\phi\Delta^2\phi}{32\pi^2}+\frac{S}{32\pi^2}\left(\phi\Delta\phi-|d\phi|^2\right)-\frac{\phi}{8\pi^2}\left(Ric,i\bar{\partial}\partial\phi\right)\right)\Omega.$$

We will simplify this by using the following identities which can be proven using Leibniz's rule and integration by parts:

$$\int_X\phi(dS,d\phi)\,\Omega=\frac{1}{2}\int_X\phi^2\Delta S\,\Omega=\int_X S(\phi\Delta\phi-|d\phi|^2)\,\Omega.$$

We get

$$\frac{1}{32\pi^2}\int_X \phi\left((dS,d\phi)-3\Delta^2\phi-2(Ric,2i\bar{\partial}\partial\phi)\right)\Omega.$$

Using the formulas from lemma 2.3.10 for the Lichnerowicz operator, this can be written as

$$\frac{1}{32\pi^2}\int_X \left(\phi\mathscr{D}^*\mathscr{D}\phi-4\phi\Delta^2\phi\right)\Omega,$$

which concludes the proof. □

By symmetry, we get the following corollary.

Corollary 2.4.3. *Let h be a Hermitian metric on E and $\phi,\psi\in C^\infty(X,\mathrm{End}_h(E))$. Then we have the asymptotics*

$$\mathrm{Tr}\left(Q_{k,\phi}P_k^*P_kQ_{k,\psi}\right)=\frac{1}{8\pi k}\int_X \mathrm{Tr}\left(\phi\Delta^E\psi+\psi\Delta^E\phi\right)\Omega+O(k^{-2}).$$

Moreover, in the case where E is the trivial flat line bundle, the second order coefficient is given for all $\phi,\psi\in C^\infty(X,\mathbb{R})$ by

$$\frac{1}{32\pi^2}\int_X \left(\phi\mathscr{D}^*\mathscr{D}\psi-4\phi\Delta^2\psi\right)\frac{\omega^n}{n!}.$$

This estimate is uniform in the endomorphisms ϕ,ψ if ϕ,ψ vary in a subset which is compact for the C^∞-topology. The estimate is uniform when the metric h on E varies in a set of uniformly equivalent metrics lying in a compact set for the C^∞-topology.

2.5 Eigenvalues and Eigenspaces

For $j\geq 0$, let λ_j be the eigenvalues of the Bochner Laplacian $\Delta=\Delta^E$ acting on the space of smooth sections of the bundle $\mathrm{End}_h(E)$ of h-Hermitian endomorphisms of E. We use the convention that $0\leq\lambda_j\leq\lambda_{j+1}$. If we set E_r to be the space generated by the eigenspaces

$$\{v\in C^\infty(\mathrm{End}_h(E))\mid\left(\Delta^E-\lambda_j\mathrm{Id}\right)v=0\}$$

for $0\leq j\leq r$, then

$$\lambda_{r+1}=\min_{\phi\in E_r^\perp}\frac{\|\nabla\phi\|_{L^2}^2}{\|\phi\|_{L^2}^2}.$$

Note that $\dim E_r\geq r+1$ and equality holds if and only if $\lambda_{r+1}>\lambda_r$. Let $\nu_{0,k}\leq\ldots\leq\nu_{M_k,k}$ the eigenvalues of the operator $P_k^*P_k$, where $M_k+1=\dim\mathfrak{u}(N_k)=N_k^2$. Define $F_{r,k}$ to be the space generated by the eigenspaces

$$\{A \in i\mathfrak{u}(N_k) \mid \left(P_k^* P_k - \nu_{j,k}\mathrm{Id}\right) A = 0\}$$

for $0 \leq j \leq r$. Then

$$\nu_{r+1,k} = \min_{B \in F_{r,k}^{\perp}} \frac{\|P_k B\|^2}{\|B\|^2}.$$

Note that $\dim F_{r,k} \geq r+1$ and equality holds if $\nu_{r+1,k} > \nu_{r,k}$. We write $F_{p,q,k} \subset i\mathfrak{u}(N_k)$ for the span of $\nu_{j,k}$-eigenspaces of $P_k^* P_k$ with $p \leq j \leq q$.

Definition 2.5.1. A holomorphic vector bundle is *simple* if its only holomorphic automorphisms are multiplication by a constant.

Theorem 2.5.2. *Suppose that E is a simple vector bundle. For each $j \geq 0$, one has*

$$\nu_{j,k} = \frac{\lambda_j}{4\pi k^{n+1}} + O(k^{-n-2}),$$

as $k \to +\infty$.

Theorem 2.5.3. *Under the setting as above assume that E is a simple vector bundle. Fix an integer $r > 0$. There is a constant $C > 0$ such that for all $A, B \in F_{r,k}$,*

$$\left|\mathrm{Tr}(AB) - k^n \langle H_A, H_B \rangle_{L^2}\right| \leq C k^{-1} \mathrm{Tr}(A^2)^{1/2} \mathrm{Tr}(B^2)^{1/2}.$$

Moreover, let us fix integers $0 < p < q$ such that

$$\lambda_{p-1} < \lambda_p = \lambda_{p+1} = \ldots = \lambda_q < \lambda_{q+1}.$$

Given an eigenvector $\phi \in \mathrm{Ker}(\Delta^E - \lambda_p \mathrm{Id})$, let $A_{\phi,k}$ denote the point in $F_{p,q,k}$ with $H_{A_{\phi,k}}$ nearest to ϕ as measured in L^2. Then

$$\|H_{A_{\phi,k}} - \phi\|_{L^2}^2 = O(k^{-1}),$$

and this estimate is uniform in ϕ if we require that $\|\phi\|_{L^2} = 1$.

In both theorems, the estimates are uniform when the metric varies in a family of uniformly equivalent metrics which is compact for the C^∞-topology.

We will prove theorems 2.5.2 and 2.5.3 simultaneously by induction. Let r be a non-negative integer. We call the following statement the r^{th} inductive hypotheses.

Induction hypotheses

1. For each $j = 0, \dots, r$,
$$\nu_{j,k} = \frac{\lambda_j}{4\pi k^{n+1}} + O(k^{-n-2}).$$
2. There exists a constant $C > 0$ such that for all $A, B \in F_{r,k}$,
$$\left| \mathrm{Tr}(AB) - k^n \langle H_A, H_B \rangle_{L^2} \right| \leq C k^{-1} \mathrm{Tr}(A^2)^{1/2} \mathrm{Tr}(B^2)^{1/2}.$$
3. Fix integers $0 < p < q \leq r$, such that $\lambda_{p-1} < \lambda_p = \lambda_{p+1} = \ldots = \lambda_q < \lambda_{q+1}$. Given $\phi \in \mathrm{Ker}(\Delta^E - \lambda_p \mathrm{Id})$, let $A_{\phi,k}$ denote the point in $F_{p,q,k}$ with $H_{A_{\phi,k}}$ nearest to ϕ as measured in L^2. Then
$$\|H_{A_{\phi,k}} - \phi\|^2_{L^2} = O(k^{-1})$$
and this estimate is uniform in ϕ if we require that $\|\phi\|_{L^2} = 1$.

2.5.1 Initial step of the induction

The first eigenvalue of Δ^E is $\lambda_0 = 0$ and $\mathrm{Ker}(\Delta) = \mathbb{R} \cdot \mathrm{Id}_E$ since E is simple. Therefore, $\lambda_1 > \lambda_0$. Since the first eigenvalue $\nu_{0,k}$ of $P_k^* P_k$ is also 0, step 1 of the induction obviously holds. Furthermore, note that Id spans the $\nu_{0,k}$-eigenspace. With $\mathrm{Tr}(\mathrm{Id}^2) = N_k = rVk^n + O(k^{n-1})$ and $H_{\mathrm{Id}} = \mathrm{Id}_E$, one gets easily Step 2 since $\int_X \langle H_{\mathrm{Id}}, H_{\mathrm{Id}} \rangle \Omega = rV$. Finally, H sends the $\nu_{0,k}$-eigenspace isomorphically to the λ_0-eigenspace. Hence, the induction process is valid at the base level.

Let us assume that the eigenvalues satisfy $\lambda_r < \lambda_{r+1} = \ldots = \lambda_s < \lambda_{s+1}$. To carry out the induction we will prove that the r^{th} inductive hypotheses imply the s^{th} inductive hypotheses.

2.5.2 Upper bound on the eigenvalues

We start giving an asymptotic upper bound of the eigenvalues of the operator $P_k^* P_k$. Define
$$\pi_k : C^\infty(X, \mathrm{End}_h(E)) \to F_{r,k},$$
to be the orthogonal projection of $Q_{k,\phi}$ onto $F_{r,k}$.

Lemma 2.5.4. *Assume that $\lambda_r < \lambda_{r+1} = ... = \lambda_s < \lambda_{s+1}$ and that the inductive hypothesis holds at level r. Then for all $j = r+1, ..., s$ one has*

$$\nu_{j,k} \leq \frac{\lambda_j}{4\pi k^{n+1}} + O(k^{-n-2}).$$

Proof. Let us start with the case of λ_{r+1}. Note that $\dim E_{r+1} \geq r+2$. Define $I = \{k \mid \dim F_{r,k} = r+1\}$. Hence, $k \in I$ if and only if $\nu_{r,k} < \nu_{r+1,k}$. On one hand, the induction hypotheses implies that for $k \notin I$,

$$\nu_{r+1,k} = \nu_{r,k} \leq \frac{\lambda_r}{4\pi k^{n+1}} + O(k^{-n-1}) \leq \frac{\lambda_{r+1}}{4\pi k^{n+1}} + O(k^{-n-1}).$$

On the other hand, for $k \in I$, we have $\mathrm{Ker}(\pi_k|_{E_{r+1}}) \neq \{0\}$ since $\dim F_{r,k} < \dim E_{r+1}$. Let $\phi_k \in \mathrm{Ker}(\pi_k|_{E_{r+1}})$ such that $\|\phi_k\|_{L^2} = 1$. Then

$$\nu_{r+1,k} \leq \frac{\|P_k Q_{k,\phi_k}\|^2}{\mathrm{Tr}(Q_{k,\phi_k}^2)}$$

since $Q_{k,\phi_k} \perp F_{r,k}$. Moreover by theorem 2.4.1, we get for all ℓ and all $k > 0$,

$$\|P_\ell Q_{\ell,\phi_k}\|^2 \leq \frac{1}{4\pi\ell}\lambda_{r+1} + C\ell^{-2}.$$

The constant C in this estimate is actually uniform in k. In fact, all the ϕ_k are lying in the unit sphere of E_{r+1} which is compact in the C^∞-topology. Putting $\ell = k$ yields

$$\|P_k Q_{k,\phi_k}\|^2 \leq \frac{1}{4\pi k}\lambda_{r+1} + Ck^{-2}.$$

Similarly,

$$\mathrm{Tr}(Q_{\ell,\phi_k}^2) = \ell^n \int \mathrm{Tr}(\phi_k^2) + O(\ell^{n-1}) = \ell^n + O(\ell^{n-1}).$$

Again, this estimate is uniform in k and putting $k = \ell$ implies

$$\mathrm{Tr}(Q_{k,\phi_k}^2) = k^n + O(\ell^{n-1}).$$

Hence for any $k \in I$, we have

$$\nu_{r+1,k} \leq \frac{\lambda_{r+1}}{4\pi k^{n+1}} + O(k^{-n-2}).$$

This settles the case when $j = r+1$. To get the bound for $j = r+2$, one repeats the same argument with $I = \{k \,|\, \dim F_{r+1,k} = r+2\}$. This time the easy case $k \notin I$ is given by the bound of $\nu_{r+1,k}$ we just got above. Carrying on this way until $j = s$ concludes the proof.

□

2.5.3 Some estimates

We will now prove some estimates which will be useful in the proof of the lower bound afterwards. They also appear (in a slightly less general setup) in [11].

Lemma 2.5.5. *As $k \to +\infty$, we have*

$$\|\bar{\mu}_k - k^{-n}\mathrm{Id}_k\|_{op} = O\left(k^{-n-1}\right).$$

where Id_k is the identity matrix in $i\mathfrak{u}(N_k)$.

Proof. Let $\{s_1, \ldots, s_{N_k}\}$ be a $\mathrm{Hilb}_k(h)$-orthonormal basis of $H^0(X, E(k))$, i.e

$$\int_X \langle s_i, s_j \rangle_{h\otimes\sigma^k} \frac{\omega^n}{n!} = \delta_{ij}.$$

We have

$$\begin{aligned}
(\bar{\mu}_k)_{ij} &= \int_X \langle s_i, B_k^{-1}(h) s_j \rangle_{h\otimes\sigma^k} \frac{\omega^n}{n!} \\
&= k^{-n} \int_X \langle s_i, (\mathrm{Id}_E + \varepsilon_k) s_j \rangle_{h\otimes\sigma^k} \frac{\omega^n}{n!} \\
&= k^{-n}\delta_{ij} + k^{-n} \int_X \langle s_i, \varepsilon_k s_j \rangle_{h\otimes\sigma^k} \frac{\omega^n}{n!},
\end{aligned}$$

where $\varepsilon_k = O\left(k^{-1}\right)$. Following Donaldson [7] and Fine [10], we put for any $\phi \in L^2\left(X, \mathrm{End}\left(E\right)\right)$,

$$(A_\phi)_{ij} = \int_X \langle s_i, \phi s_j \rangle_{h\otimes\sigma^k} \frac{\omega^n}{n!}.$$

A_ϕ defines a linear map from $H^0\left(X, E\left(k\right)\right)$ into itself. Moreover, $A_\phi = \pi \circ M_\phi \circ j$, where $j : H^0(E(k)) \to L^2(X, E(k))$ is the inclusion, $\pi : L^2(X, E(k)) \to H^0(E(k))$ is the orthogonal projection and $M_\phi : L^2(X, E(k)) \to L^2(X, E(k))$ is defined by $M_\phi s = \phi s$. Hence,

$$\|A_\phi\|_{op} = \|\pi \circ M_\phi \circ j\|_{op} \leq \|M_\phi\|_{op} \leq \|\phi\|_{C^0}.$$

Applying this to ε_k, we have

$$\begin{aligned}\|\bar{\mu}_k - k^{-n}\mathrm{Id}_k\|_{op} &= k^{-n}\Big\| \int_X \langle s_i, \varepsilon_k s_j\rangle_{h\otimes\sigma^k} \frac{\omega^n}{n!}\Big\|_{op} \\ &= k^{-n}\|A_{\varepsilon_k}\|_{op} \\ &\leq k^{-n}\|\varepsilon_k\|_{C^0} \\ &= O\left(k^{-n-1}\right).\end{aligned}$$

□

A consequence of lemma 2.5.5 is the following.

Lemma 2.5.6. *There is a constant $C > 0$ such that for any $A, B \in i\mathfrak{u}(N_k)$, one has*

$$|\,\mathrm{Tr}(AB\bar{\mu}) - \frac{1}{k^n}\mathrm{Tr}(AB)\,| \leq Ck^{-n-1}\mathrm{Tr}(A^2)^{1/2}\mathrm{Tr}(B^2)^{1/2}.$$

Proof. Let $M := \bar{\mu}_k - k^{-n}\mathrm{Id}_k$. We have

$$\begin{aligned}|\mathrm{Tr}(AB\bar{\mu}) - \frac{1}{k^n}\mathrm{Tr}(AB)| &= |\mathrm{Tr}(ABM)| \\ &\leq \|M\|_{op}|\mathrm{Tr}(AB)| \\ &\leq Ck^{-n-1}\|A\|\|B\|.\end{aligned}$$

□

Lemma 2.5.7. *There is a constant $C > 0$ such that for any $A \in i\mathfrak{u}(N_k)$, one has*

$$\|H_A\|^2_{L^2} \leq \frac{1}{k^n}\left(1 + Ck^{-1}\right)\mathrm{Tr}(A^2)$$

Proof. By lemma 2.4.1, we have

$$\int_X \mathrm{Tr}(H_A^2)\,\Omega + \|\xi_A\|^2_{L^2} = \mathrm{Tr}(A^2\bar{\mu}).$$

Thus,

$$\|H_A\|^2_{L^2} \leq \mathrm{Tr}(A^2\bar{\mu}) \leq \frac{1}{k^n}\left(1 + Ck^{-1}\right)\mathrm{Tr}(A^2).$$

□

The next lemma shows that the map H asymptotically preserves orthogonality along eigenspaces.

Lemma 2.5.8. *Let $\psi \in L^2$ and let $M_k \in i\mathfrak{u}(N_k)$ be a sequence of $P_k^* P_k$-eigenvectors satisfying the following conditions*

1. $\mathrm{Tr}(M_k^2) = k^n + O(k^{n-1})$
2. $\|H_{M_k} - \psi\|_{L^2}^2 = O(k^{-1})$,

then there is a constant $C > 0$ such that for all $B \in i\mathfrak{u}(N_k)$ with $\mathrm{Tr}(BM_k) = 0$, we have

$$|\langle H_B, \psi \rangle_{L^2}|^2 \leq Ck^{-n-1}\mathrm{Tr}(B^2).$$

Proof. Integrating the formula in lemma 2.4.2 implies that

$$\langle H_B, H_{M_k} \rangle_{L^2} = -\langle \xi_B, \xi_{M_k} \rangle_{L^2} + \mathrm{Tr}(BM_k\bar{\mu}_k).$$

By definition,

$$\langle \xi_B, \xi_{M_k} \rangle_{L^2} = \mathrm{Tr}\left(BP_k^* P_k\left(M_k\right)\right) = \lambda \mathrm{Tr}\left(B, M_k\right) = 0.$$

Moreover lemma 2.5.6 implies then that

$$|\mathrm{Tr}(BM_k\bar{\mu}_k)| \leq Ck^{-n-1}\|B\|\|M_k\|.$$

Using these estimates and lemma 2.5.7 we get,

$$\begin{aligned}
|\langle H_B, \psi \rangle_{L^2}| &\leq |\langle H_B, H_{M_k} \rangle_{L^2}| + |\langle H_B, H_{M_k} - \psi \rangle_{L^2}| \\
&\leq |\langle H_B, H_{M_k} \rangle_{L^2}| + \|H_B\|_{L^2}\|H_{M_k} - \psi\|_{L^2} \\
&= |\mathrm{Tr}(BM_k\bar{\mu}_k)| + \|H_B\|_{L^2}\|H_{M_k} - \psi\|_{L^2} \\
&\leq Ck^{-\frac{n+1}{2}}\mathrm{Tr}(B^2)^{1/2}.
\end{aligned}$$

□

2.5.4 Lower bound for the eigenvalues

The goal of this section is to prove the following lower bound for the eigenvalues which turns out to be much harder than the upper bound.

Proposition 2.5.9. *Assume that $\lambda_r < \lambda_{r+1}$ and that the inductive hypothesis holds at level r. Then one has the following bound*

$$\nu_{r+1,k} \geq \frac{\lambda_{r+1}}{4\pi k^{n+1}} + O\left(k^{-n-2}\right).$$

Remark 2.5.10. Note that the if we have for example $\lambda_{r+2} = \lambda_{r+1}$ then this proposition immediately implies that $\nu_{r+2,k} \geq \frac{\lambda_{r+2}}{4\pi k^{n+1}} + O\left(k^{-n-2}\right)$ too, since by definition $\nu_{r+2,k} \geq \nu_{r+1,k}$.

The crucial step in the proof of proposition 2.5.9 is the following key-estimate

Proposition 2.5.11. *For any $A \in i\mathfrak{u}(N_k)$, we have*

$$\|\nabla H_A\|_{L^2}^2 \leq (4\pi k + O(1)) \|P_k(A)\|^2.$$

The proof of this result makes use of the second fundamental form of a couple of holomorphic sub-bundles. In order to set things clear and for the sake of completeness, we begin by recalling some general theory. The way we present it here is close to Fine's treatment in [11]. Let $V \to X$ be a holomorphic Hermitian vector bundle over a complex manifold. Suppose S is a holomorphic sub-bundle of V with quotient Q. In other words, we have a short exact sequence

$$0 \to S \to V \to Q \to 0. \tag{2.5.1}$$

Denote by ∇^V the Chern connection on V. By restriction we also get a Hermitian metric on S. Moreover, the Hermitian metric allows us to identify the quotient bundle Q with $S^\perp$ as smooth vector bundles, so that we have a smooth splitting

$$V = S \oplus Q.$$

Hence we also obtain a Hermitian metric on Q which allows us to define Chern connections ∇^S and ∇^Q on S and Q respectively. One can check that ∇^S is the composition of ∇^V followed by the projection to S.

There are two ways to look at the second fundamental form of a short exact sequence as in (2.5.1). Either you measure the failure of S to be a parallel sub-bundle of V, or you look at $S^\perp$ and measure its failure of being a holomorphic sub-bundle. The first point of view can be described as follows. Denote by F the composition of ∇^V with the projection to Q. This defines an operator

$$F : C^\infty(S) \xrightarrow{\nabla^V} \Omega^1(V) \to \Omega^1(Q)$$

called the *second fundamental form* of (2.5.1). Note that since S is a holomorphic sub-bundle, the $(0,1)$-part of ∇^V leaves S invariant and thus F is a section of the bundle $\Lambda^{1,0} \otimes \mathrm{Hom}(S,Q)$.

On the other hand, observe that if $S^\perp$ was a holomorphic sub-bundle, it would be invariant under $\bar{\partial}^V$. The failure of $S^\perp$ of being a holomorphic sub-bundle can then be measured by the composition of $\bar{\partial}^V$ with the projection to S. This defines a map

$$\tilde{F} : C^\infty(S^\perp) \xrightarrow{\bar{\partial}^V} \Omega^{0,1}(V) \to \Omega^{0,1}(S). \tag{2.5.2}$$

Hence we can think of $\tilde{F}$ as a section of $\Lambda^{0,1} \otimes \mathrm{Hom}(S^\perp, S)$. One can check that under the identification $Q \simeq S^\perp$ the map $\tilde{F}$ is nothing else than F^*, the dual of F obtained by using conjugation in the $(1,0)$-form factor and taking the usual adjoint in the $\mathrm{Hom}(S,Q)$ factor.

On one hand, write

$$F \wedge F^* \in \Lambda^{1,1} \otimes \mathrm{End}(Q)$$

where we take the genuine wedge product on the form part and composition on the homomorphism part. On the other hand, we consider

$$F^* \wedge F \in \Lambda^{1,1} \otimes \mathrm{End}(S).$$

Denote by $R(S)$, $R(Q)$ and $R(V)$ the curvatures of the Chern connections of S, Q and V respectively. By the splitting of $V = S \oplus Q$ as smooth vector bundles, we get an induced splitting

$$\mathrm{End}(V) = \mathrm{End}(S) \oplus \mathrm{Hom}(S,Q) \oplus \mathrm{Hom}(Q,S) \oplus \mathrm{End}(Q).$$

If we write now $R(V)|_S$ and $R(V)|_Q$ for the components of $R(V)$ in $\mathrm{End}(S)$ and $\mathrm{End}(Q)$ respectively, we have the following standard lemma. See for instance page 78 of [12] for a proof.

Lemma 2.5.12.

$$F^* \wedge F = R(S) - R(V)|_S$$
$$F \wedge F^* = R(Q) - R(V)|_Q.$$

Assuming that the complex manifold X carries a Hermitian metric, we can identify

$$\Lambda^{1,0} \simeq (\Lambda^{0,1})^*.$$

Using this, F can be interpreted as a homomorphism

$$F : \Lambda^{0,1} \otimes S \to Q$$

and similarly, F^* can be thought of as a homomorphism

$$F^* : Q \to \Lambda^{0,1} \otimes S.$$

These two maps are adjoint with respect to the fibrewise Hermitian metrics on $\Lambda^{0,1} \otimes S$ and Q. Furthermore we will be interested in the compositions FF^* and F^*F of these maps. Namely

$$\Lambda^{0,1} \otimes S \xrightarrow{F} Q \xrightarrow{F^*} \Lambda^{0,1} \otimes S$$

and

$$Q \xrightarrow{F^*} \Lambda^{0,1} \otimes S \xrightarrow{F} Q.$$

One can then check that under these identifications, F^*F is identified with $-F^* \wedge F$ whereas FF^* is identified with $\mathrm{Tr}_X(F \wedge F^*)$. Here the trace is taken over the $\Lambda^{1,1}$-component of $F \wedge F^*$ using the Hermitian metric on X (see Fine [11] page 28).

We will now use this general theory in the following situation. Still suppose that we have a short exact sequence of holomorphic vector bundles

$$0 \to S \to V \to Q \to 0.$$

Taking duals, we get another short exact sequence

$$0 \to Q^* \to V^* \to S^* \to 0$$

and taking the tensor product with the bundle V yields

$$0 \to \mathrm{Hom}(Q,V) \to \mathrm{End}(V) \to \mathrm{Hom}(S,V) \to 0. \tag{2.5.3}$$

The Hermitian metric on V induces metrics on all of these bundles. Let $A \in C^\infty(\mathrm{End}(V))$ be Hermitian and covariant constant with respect to the Chern connection on $\mathrm{End}(V)$, i.e.

$$\nabla^{\mathrm{End}(V)} A = 0.$$

If we use the metric on $\mathrm{End}(V)$ to split

$$\mathrm{End}(V) = \mathrm{Hom}(Q,V) \oplus \mathrm{Hom}(S,V)$$

as smooth vector bundles, we can write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where $A_1 \in C^\infty(\mathrm{Hom}(Q,V))$ and $A_2 \in C^\infty(\mathrm{Hom}(S,V))$. Furthermore we have

$$\bar{\partial}^{\mathrm{End}(V)} = \begin{pmatrix} \bar{\partial}^{\mathrm{Hom}(Q,V)} & \eta^* \\ 0 & \bar{\partial}^{\mathrm{Hom}(S,V)} \end{pmatrix}$$

where η^* is the dual of the second fundamental form of the short exact sequence given in (2.5.3), defined as in (2.5.2). Applying it to our covariant constant section A yields

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \bar{\partial}^{\mathrm{End}(V)} A = \begin{pmatrix} \bar{\partial}^{\mathrm{Hom}(Q,V)} A_1 + \eta^* A_2 \\ \bar{\partial}^{\mathrm{Hom}(S,V)} A_2 \end{pmatrix}.$$

In particular,

$$\bar{\partial}^{\mathrm{Hom}(S,V)} A_2 = 0 \tag{2.5.4}$$

meaning that $A_2 \in C^\infty(\mathrm{Hom}(S,V))$ is a holomorphic section.

Now $\mathrm{End}(S)$ is a holomorphic sub-bundle of $\mathrm{Hom}(S,V)$ with quotient $\mathrm{Hom}(S,Q)$. In other words we have another short exact sequence

$$0 \to \mathrm{End}(S) \to \mathrm{Hom}(S,V) \to \mathrm{Hom}(S,Q) \to 0. \tag{2.5.5}$$

Again we use the Hermitian metric to split this sequence and write

$$A_2 = \begin{pmatrix} \tilde{H}_A \\ P_A \end{pmatrix}$$

where $\tilde{H}_A \in C^\infty(\mathrm{End}(S))$ and $P_A \in C^\infty(\mathrm{Hom}(S,Q))$. Writing

$$\bar{\partial}^{\mathrm{Hom}(S,V)} = \begin{pmatrix} \bar{\partial}^{\mathrm{End}(S)} & F^* \\ 0 & \bar{\partial}^{\mathrm{Hom}(S,Q)} \end{pmatrix}$$

and applying it to the holomorphic section A_2 gives in particular

$$\bar{\partial}^{\mathrm{End}(S)} \tilde{H}_A = -F^* P_A. \tag{2.5.6}$$

This formula is crucial for what follows. In fact it gives the geometric relation between the derivative of $\tilde{H}_A$ in terms of P_A.

In order to prove proposition 2.5.11 we will now apply the above discussion to our picture. Recall that we used higher and higher powers of the line bundle L tensored with E to get a sequence of embeddings of X into the Grassmannians $\mathbb{G}(r, N_k)$ which can be summarized by the following diagram,

$$\begin{array}{ccc} E(k) = \iota_k^* U_r & \longrightarrow & U_r \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota_k} & G(r, N_k) \end{array}$$

We have the following short exact sequence of holomorphic vector bundles

$$0 \to U_r^* \to \underline{\mathbb{C}}^{N_k} \to Q \to 0$$

where $\underline{\mathbb{C}}^{N_k}$ denotes the trivial bundle over $\iota_k(X)$ inside the Grassmannian and U_r^* and Q are restricted to $\iota_k(X)$ as well. As explained, we can use the metric to identify the quotient Q with $(U_r^*)^\perp$ as smooth vector bundles.

Let $A \in i\mathfrak{u}(N_k)$. We may think of A as a constant section of $\mathrm{End}(\underline{\mathbb{C}}^{N_k})$ so that we can apply the discussion from above with $V = \underline{\mathbb{C}}^{N_k}$. It is then just a matter of unwinding the definitions to see that the dual of the Hermitian endomorphism H_A of U_r defined in (2.3.1) coincides with $\tilde{H}_A$ described in the discussion above. Furthermore the holomorphic tangent bundle on the Grassmannian can be identified with $\mathrm{End}\left(U_r^*, (U_r^*)^\perp\right)$. Under this identification, the section $P_k(A)$ of $T\mathbb{G}(r,N)_{|\iota_k(X)}$ defined in equation (2.2.1) corresponds to the restriction to $\iota_k(X)$ of what we called P_A just above. Formula (2.5.6) gives then the link between the derivative of H_A^* and $P_k(A)$ by

$$\bar{\partial}^{\mathrm{End}(U_r^*)} H_A^* = -F_k^* P_k(A). \tag{2.5.7}$$

The next step in our discussion will be to control the asymptotics of the operator $F_k F_k^*$. However, it turns out to be easier to consider the operator $F_k^* F_k$ first and then pass to $F_k^* F_k$.

Lemma 2.5.13. *We have that* $\|F_k^* F_k - 2\pi k \mathrm{Id}\|_{C^0(\mathrm{op})} = O(1)$. *Here* Id *denotes the identity in* $\mathrm{End}\left(\Lambda^{0,1} \otimes \mathrm{End}\left(U_r^*\right)\right)$ *and* $C^0(\mathrm{op})$ *is the* C^0*-norm on sections of* $\mathrm{End}\left(\Lambda^{0,1} \otimes \mathrm{End}\left(U_r^*\right)\right)$ *associated to the fibrewise operator norm.*

Proof. Recall that under the identification of $\Lambda^{1,0}$ with $(\Lambda^{0,1})^*$, $F_k^* F_k$ is identified with $-F_k^* \wedge F_k$. Moreover by lemma 2.5.12 we know that

$$-F_k^* \wedge F_k = R\left(\mathrm{Hom}(U_r^*, \underline{\mathbb{C}}^{N_k})\right)|_{\mathrm{End}(U_r^*)} - R\left(\mathrm{End}(U_r^*)\right)$$

Let's start computing the first term of the right-hand side. Since $\underline{\mathbb{C}}^{N_k}$ is flat, we get

$$R\left(\mathrm{Hom}(U_r^*, \underline{\mathbb{C}}^{N_k})\right) = R\left(U_r\right) \otimes \mathrm{Id}_{\underline{\mathbb{C}}^{N_k}}. \tag{2.5.8}$$

So we see that it boils down to calculate the curvature of U_r, the dual of the tautological bundle restricted to $\iota_k(X)$, or in other words, the curvature of $E(k) = E \otimes L^k$, computed with respect to the metric $\mathrm{FS}_k(h) \otimes \sigma^k$. Since on one hand $R(L^k) = -2\pi k i\omega$ and on the other hand $R(E)$ isn't growing in k, we get

$$R(U_r) = O(1) + \mathrm{Id}_E \otimes R(L^k) = -2\pi k i\omega \otimes \mathrm{Id}_{U_r} + O(1).$$

Putting these together, we see that

$$R\left(\mathrm{Hom}(U_r^*, \underline{\mathbb{C}}^{N_k})\right)|_{\mathrm{End}(U_r^*)} = -2\pi k i\omega \otimes \mathrm{Id}_{\mathrm{End}(U_r^*)} + O(1).$$

Furthermore, since $\mathrm{End}\left(U_r^*\right) = U_r \otimes U_r^* = E(k) \otimes E(k)^* = E \otimes E^*$, the curvature of $\mathrm{End}(U_r^*)$ isn't growing in k. Putting these into equation (3.3.8) we get

$$-F_k^* \wedge F_k = -2\pi k i\omega \otimes \mathrm{Id}_{\mathrm{End}(U_r^*)} + O(1).$$

Raising indices to pass form $-F_k^* \wedge F_k$ to $F_k^* F_k$ proves the Lemma. □

We will now explain how to pass from $F_k^* F_k$ to $F_k F_k^*$. Denote by $T_k \in \mathrm{End}(\mathrm{Hom}(E(k), Q))$ the orthogonal projection onto the image of $F_k : \Lambda^{0,1} \otimes \mathrm{End}(E(k)) \to \mathrm{Hom}(E(k), Q)$.

Lemma 2.5.14. $\|F_k F_k^* - 2\pi k T_k\|_{C^0(\mathrm{op})} = O(1)$, *where we use the* C^0*-norm on sections of* $\mathrm{End}(\mathrm{Hom}(U_r^*, Q))$ *associated to the fibrewise operator norm.*

Proof. The argument is essentially the same as the proof of lemma 33 in [11], adapted to our situation. Accordingly, we give nearly word-by-word the same proof. Clearly we have that $\ker F_k F_k^* = \ker T_k$ and since $F_k F_k^*$ is self-adjoint, it is enough to prove that all the non-zero eigenvalues are given by $2\pi k + O(1)$. But the non-zero eigenvalues of $F_k F_k^*$ and $F_k^* F_k$ are the same since the eigenvectors are matched up by F_k^*. The result then follows from the previous lemma. □

Having gathered all of these pre-requisites, we are finally in position to prove proposition 2.5.11.

Proof of proposition 2.5.11. Let $A \in i\mathfrak{u}(N_k)$, we have

$$\begin{aligned}\|\nabla^{\mathrm{End}(E)} H_A\|_{L^2}^2 &= \int_X \mathrm{Tr}\left(H_A \Delta^{\mathrm{End}(E)} H_A\right)\Omega,\\ &= \int_X \mathrm{Tr}\left(H_A\left(2\Delta_{\bar\partial} H_A - i[\Lambda F, H_A]\right)\right)\Omega,\\ &= 2\int_X \mathrm{Tr}\left(H_A \Delta_{\bar\partial} H_A\right)\Omega,\\ &= 2\int_X |\bar\partial H_A|^2\,\Omega.\end{aligned}$$

Using the relation $\bar\partial H_A^* = -F_k^* P_k(A)$ given in (2.5.7) and lemma 2.5.14 we further get that

$$\begin{aligned}\int_X |\bar\partial H_A|^2\,\Omega &= \langle P_k(A), F_k F_k^* P_k(A)\rangle,\\ &= \langle P_k(A), (2\pi k T_k + O(1))\, P_k(A)\rangle,\\ &\le (2\pi k + O(1))\|P_k(A)\|^2.\end{aligned}$$

This concludes the proof of proposition 2.5.11. □

Assume that the induction hypothesis holds at level r and let $\lambda_r < \lambda_{r+1}$. We have the following.

Lemma 2.5.15. *Let $\phi_0, \dots, \phi_r$ be an L^2-orthonormal basis for E_r such that $\Delta^E \phi_i = \lambda_i \phi_i$. For integers $0 < p < q \le r$, satisfying $\lambda_{p-1} < \lambda_p = \lambda_{p+1} = \dots = \lambda_q < \lambda_{q+1}$ and $p \le j \le q$, let $A_{j,k} \in F_{p,q,k}$ be given by the induction hypotheses. Let $W_k \subset F_{r,k}$ be the span of the vectors $A_{j,k}$ $(0 \le j \le r)$. Then*

$$\nu_{r+1,k} \ge \min_{B \in W_k^\perp} \frac{\|P_k B\|^2}{\mathrm{Tr}(B^2)}.$$

Proof. By hypothesis (2) of the induction (I), there is a constant C such that

$$|\mathrm{Tr}(A_{i,k}A_{j,k}) - k^n \langle H_{A_{i,k}} H_{A_{j,k}}\rangle_{L^2_\Omega}| \le C k^{-1}\mathrm{Tr}\left(A_{i,k}^2\right)^{1/2}\mathrm{Tr}\left(A_{j,k}^2\right)^{1/2}.$$

Using the estimate

$$\mathrm{Tr}(A_{i,k}^2) = k^n + O(k^{n-1})$$

we get

$$\mathrm{Tr}(A_{i,k}A_{j,k}) = O(k^{n-1/2}) \ \text{ if } \ i \ne j,$$

since $H_{A_{i,k}} = \phi + O(k^{-1})$ uniformly by the induction hypotheses. Hence the vectors $A_{i,k}$ are linearly independent (otherwise their inner product would be of a similar order than their norms). Thus $\dim(W_k) = r+1$. The minimal eigenvalue of $P_k^* P_k$ on $W_k^\perp$ is at least the $(r+2)^{th}$ eigenvalue $\nu_{r+1,k}$. Using the variational characterization of eigenvalues, one gets the required inequality. □

Proposition 2.5.16. *There exists a constant C such that*

$$\|P_k B\|^2 \geq \Big(\frac{\lambda_{r+1}}{4\pi k^{n+1}} + \frac{C}{k^{n+2}}\Big)\mathrm{Tr}(B^2),$$

for all $B \in W_k^\perp$.

Proof. **Step I:** Integrating lemma 2.4.2 we get

$$\|H_B\|_{L^2}^2 + \|P_k B\|_{L^2}^2 = \mathrm{Tr}(B^2 \bar{\mu})$$

which together with lemma 2.5.6, imply that

$$\|H_B\|_{L^2}^2 + \|P_k B\|_{L^2}^2 \geq \frac{1}{k^n}\left(1 + O\left(k^{-1}\right)\right)\mathrm{Tr}\left(B^2\right). \tag{2.5.9}$$

Step II: Let $\phi_0, \dots, \phi_r$ be an L^2-orthonormal basis for E_r such that $\Delta^E \phi_i = \lambda_i \phi_i$ and let

$$H_B = \sum_{j=0}^{r} \langle H_B, \phi_j \rangle_{L^2} \phi_j + \widetilde{H},$$

where $\widetilde{H}$ is orthogonal to E_r. Applying lemma 2.5.8 to $A_{j,k}$, shows that there exists a constant C such that

$$\left|\langle H_B, \phi_j \rangle_{L^2}\right|^2 \leq C k^{-n-1}\mathrm{Tr}(B^2),$$

for all $B \in W_k^\perp$ and $0 \leq j \leq r$. Therefore,

$$\|H_B\|_{L^2}^2 = \sum_{j=0}^{r} \left|\langle H_B, \phi_j \rangle_{L^2}\right|^2 + \|\widetilde{H}\|_{L^2}^2 \leq C k^{-n-1}\mathrm{Tr}(B^2) + \|\widetilde{H}\|_{L^2}^2, \tag{2.5.10}$$

for all $B \in W_k^\perp$. We will now estimate $\|\widetilde{H}\|_{L^2}^2$. By definition, we have that

$$\lambda_{r+1} = \min_{\phi \in E_r^\perp} \frac{\|\nabla \phi\|_{L^2}^2}{\|\phi\|_{L^2}^2} \leq \frac{\|\nabla \widetilde{H}\|_{L^2}^2}{\|\widetilde{H}\|_{L^2}^2}$$

which implies

$$\|\widetilde{H}\|_{L^2}^2 \leq \frac{1}{\lambda_{r+1}}\|\nabla\widetilde{H}\|_{L^2}^2.$$

On the other hand,

$$\begin{aligned}\|\nabla H_B\|_{L^2}^2 =&\|\nabla\widetilde{H}\|_{L^2}^2 + \|\nabla(H_B-\widetilde{H})\|_{L^2}^2 \\ &+ 2Re\langle\nabla\widetilde{H},\nabla(H_B-\widetilde{H})\rangle_{L^2} \\ =&\|\nabla\widetilde{H}\|_{L^2}^2 + \|\nabla(H_B-\widetilde{H})\|_{L^2}^2.\end{aligned}$$

The second equality follows from

$$\begin{aligned}\langle\nabla\widetilde{H},\nabla(H_B-\widetilde{H})\rangle_{L^2} &= \langle\widetilde{H},\Delta^E(H_B-\widetilde{H})\rangle_{L^2} \\ &= \sum_{j=0}^{r}\lambda_j\overline{\langle H_B,\phi_j\rangle_{L^2}}\langle\widetilde{H},\phi_j\rangle_{L^2} \\ &= 0.\end{aligned}$$

Hence,

$$\|\widetilde{H}\|_{L^2}^2 \leq \frac{1}{\lambda_{r+1}}\|\nabla H_B\|_{L^2}^2.$$

Putting this into equation (2.5.10) implies

$$\|H_B\|_{L^2}^2 \leq \frac{1}{\lambda_{r+1}}\|\nabla H_B\|_{L^2}^2 + Ck^{-n-1}\mathrm{Tr}(B^2)$$

Step III: From step II and proposition 2.5.11 we get

$$\begin{aligned}\|H_B\|_{L^2}^2 &\leq \frac{1}{\lambda_{r+1}}\|\nabla H_B\|_{L^2}^2 + Ck^{-n-1}\mathrm{Tr}(B^2) \\ &\leq \frac{4\pi k}{\lambda_{r+1}}\|P_kB\|^2 + O(1)\|P_kB\|^2 + Ck^{-n-1}\mathrm{Tr}(B^2).\end{aligned}$$

This together with equation (2.5.9) from step I conclude the proof. □

Corollary 2.5.17. *Assume that $\lambda_r < \lambda_{r+1}$ and that the inductive hypothesis at level r holds. Then one has the lower bound,*

$$\nu_{r+1,k} \geq \frac{\lambda_{r+1}}{4\pi k^{n+1}} + O(k^{-n-2}).$$

Proof. We have by lemma 2.5.15 and from the previous proposition

$$\nu_{r+1,k} \geq \min_{B \in W_k^{\perp}} \frac{\|P_k B\|^2}{\operatorname{Tr}(B^2)} \geq \frac{\lambda_{r+1}}{4\pi k^{n+1}} + O\left(k^{-n-2}\right).$$

□

2.5.5 Completing the proof of the induction, steps 2 and 3

In this subsection, we fix positive integers r and s such that $\lambda_r < \lambda_{r+1} = .. = \lambda_s < \lambda_{s+1}$. We start proving step 2 of the induction.

Proposition 2.5.18. *If the r^{th} inductive hypotheses hold, then there is a constant C such that for all $A, B \in F_{s,k}$,*

$$|\operatorname{Tr}(AB) - k^n \langle H_A, H_B \rangle_{L^2}| \leq Ck^{-1} \operatorname{Tr}(A^2)^{1/2} \operatorname{Tr}(B^2)^{1/2}.$$

Proof. From lemma 2.5.6, we know that there is a uniform constant $C > 0$ such that

$$|\operatorname{Tr}(AB\bar{\mu}) - \frac{1}{k^n}\operatorname{Tr}(AB)| \leq Ck^{-n-1} \operatorname{Tr}\left(A^2\right)^{1/2} \operatorname{Tr}\left(B^2\right)^{1/2}.$$

Lemma 2.4.2 implies that $\operatorname{Tr}(AB\bar{\mu}) = \operatorname{Tr}(AP_k^* P_k B) + \langle H_A, H_B \rangle_{L^2}$. Moreover, using the facts that A and B lie in $F_{s,k}$ and $\nu_{s,k} = O(k^{-n-1})$ we see that

$$|\operatorname{Tr}(AP_k^* P_k B)| \leq \frac{C}{k^{n+1}} \operatorname{Tr}\left(A^2\right)^{1/2} \operatorname{Tr}\left(B^2\right)^{1/2}.$$

Putting these estimates together concludes the proof. □

Next, we prove that the step 3 of the induction holds. For any $A \in i\mathfrak{u}(N)$, we write

$$H_A = {H_A}^{<} + {H_A}^{r+1} + {H_A}^{>} \tag{2.5.11}$$

where ${H_A}^{<}$ is the component of H_A lying in E_r, ${H_A}^{>}$ lies in the span of the eigenspaces associated to eigenvalues strictly greater than λ_{r+1} and ${H_A}^{r+1}$ is the component of H_A in the span of the eigenspaces having eigenvalue λ_{r+1}.

Lemma 2.5.19. *Assume that the r^{th} inductive hypotheses hold. There exists a constant C such that for any $A \in F_{r+1,s,k}$, we have*

$$\begin{aligned}
\|{H_A}^{<}\|_{L^2}^2 &\le Ck^{-n-1}\mathrm{Tr}(A^2),\\
\|{H_A}^{>}\|_{L^2}^2 &\le Ck^{-n-1}\mathrm{Tr}(A^2),\\
\left|k^n\|{H_A}^{r+1}\|_{L^2}^2 - \mathrm{Tr}(A^2)\right| &\le Ck^{-1}\mathrm{Tr}(A^2).
\end{aligned}$$

Proof. Without loss of generality, we may assume that $A \in F_{r+1,s,k}$ is a $\nu_{j,k}$ eigenvector of $P_k^*P_k$ with $r+1 \le j \le s$. Let $\phi_0, \dots, \phi_r$ be an orthonormal basis for E_r such that $\Delta^E \phi_j = \lambda_j \phi_j$. By the induction hypotheses, there are eigenvectors $A_{j,k}$ with eigenvalues $\nu_{j,k} \le \nu_{r,k}$ of $P_k^*P_k$ satisfying

$$\mathrm{Tr}(A_{j,k}^2) = k^n + O(k^{n-1})$$

and

$$\|H_{A_{j,k}} - \phi_j\|_{L^2} = O(k^{-1/2}).$$

Since $A \perp A_{j,k}, 0 \le j \le r$, lemma 2.5.8 implies that

$$|\langle H_A, \phi_j\rangle_{L^2}|^2 \le Ck^{-n-1}\mathrm{Tr}(A^2).$$

Thus we get the first inequality $\|{H_A}^{<}\|_{L^2}^2 \le Ck^{-n-1}\mathrm{Tr}(A^2)$.

Moreover, proposition 2.5.18 implies that

$$\|{H_A}^{<}\|_{L^2}^2 + \|{H_A}^{r+1}\|_{L^2}^2 + \|{H_A}^{>}\|_{L^2}^2 = \frac{1}{k^n}\left(1 + O\left(k^{-1}\right)\right)\mathrm{Tr}\left(A^2\right)$$

and hence

$$\|{H_A}^{r+1}\|_{L^2}^2 + \|{H_A}^{>}\|_{L^2}^2 = \frac{1}{k^n}\left(1 + O\left(k^{-1}\right)\right)\mathrm{Tr}(A^2). \tag{2.5.12}$$

On the other hand, for any $A \in F_{r+1,s,k}$, an eigenvector associated to the eigenvalue $\nu_{j,k}$ $(r+1 \le j \le s)$, we have by proposition 2.5.11 that

$$\begin{aligned}
\|\nabla H_A\|_{L^2}^2 &\le 4\pi(k + O(1))\|P_k A\|^2\\
&= 4\pi(k + O(1))\nu_{j,k}\mathrm{Tr}(A^2)\\
&= \frac{\lambda_j + O(k^{-1})}{k^n}\mathrm{Tr}(A^2),
\end{aligned}$$

since $\nu_{j,k} = \frac{\lambda_j}{4\pi k^{n+1}} + O(k^{-n-2})$. Using the splitting (2.5.11) and the fact that ${H_A}^{r+1}$ lies in the λ_{r+1} eigenspace, we obtain

$$\|\nabla {H_A}^{<}\|_{L^2}^2 + \lambda_{r+1}\|{H_A}^{r+1}\|_{L^2}^2 + \|\nabla {H_A}^{>}\|_{L^2}^2 \leq \frac{\lambda_{r+1} + O(k^{-1})}{k^n}\mathrm{Tr}(A^2). \tag{2.5.13}$$

The variational property for eigenvalues of Δ^E implies that $\lambda_{s+1} = \min_{\phi \in E_s^{\perp}} \frac{\|\nabla\phi\|_{L^2}^2}{\|\phi\|_{L^2}^2}$. Therefore,

$$\lambda_{s+1} \leq \frac{\|\nabla {H_A}^{>}\|_{L^2}^2}{\|{H_A}^{>}\|_{L^2}^2},$$

since ${H_A}^{>} \in E_s^{\perp}$. Thus, using the fact that $\|\nabla {H_A}^{<}\|_{L^2}^2 \leq Ck^{-n-1}\mathrm{Tr}(A^2)$, we obtain thanks to (2.5.13),

$$\lambda_{r+1}\|{H_A}^{r+1}\|_{L^2}^2 + \lambda_{s+1}\|{H_A}^{>}\|_{L^2}^2 \leq \frac{1}{k^n}\left(\lambda_{r+1} + O\left(k^{-1}\right)\right)\mathrm{Tr}(A^2). \tag{2.5.14}$$

Since $\lambda_{s+1} > \lambda_{r+1}$, the system formed by the equations (2.5.12), (2.5.14) ensures the existence of a constant $C > 0$ such that

$$\|{H_A}^{>}\|_{L^2}^2 \leq Ck^{-n-1}\mathrm{Tr}(A^2),$$
$$\left|k^n\|{H_A}^{r+1}\|_{L^2}^2 - \mathrm{Tr}(A^2)\right| \leq Ck^{-1}\mathrm{Tr}(A^2).$$

This concludes the proof of the lemma. □

With this last proposition below, we obtain the induction at step $r+1$.

Proposition 2.5.20. *Assume that the r^{th} inductive hypotheses hold. Given $\phi \in \mathrm{Ker}(\Delta^E - \lambda_{r+1}\mathrm{Id})$, let $A_{\phi,k}$ be the point of $F_{r+1,s}$ for which $H_{A_{\phi,k}}$ is nearest to ϕ as measured in L^2. Then,*

$$\|H_{A_{\phi,k}} - \phi\|_{L^2}^2 = O(k^{-1})$$

and this estimate is uniform in ϕ if in addition we require $\|\phi\|_{L^2} = 1$.

Proof. First we show that the linear map

$$A \in F_{r+1,s,k} \to {H_A}^{r+1} \in V_{r+1}$$

is an isomorphism for $k \gg 0$, where V_{r+1} is the eigenspace of Δ^E associated to the eigenvalue λ_{r+1}. Suppose that $A \in F_{r+1,s,k}$ and ${H_A}^{r+1} = 0$. Then applying lemma 2.5.19, we have

$$\left|\mathrm{Tr}(A^2)\right| \leq Ck^{-1}\mathrm{Tr}(A^2).$$

This implies that $A = 0$ if $k \gg 0$. Note that $\dim F_{r+1,s,k} \geq s - r = \dim V_{r+1}$. Therefore, the linear map is an isomorphism. This implies that for any $\phi \in V_{r+1}$, there exists a unique $A_{\phi,k}$ such that ${H_{A_{\phi,k}}}^{r+1} = \phi$. Applying lemma 2.5.19, we have

$$\|H_{A_{\phi,k}} - \phi\|_{L^2}^2 = \|{H_{A_{\phi,k}}}^{<} + {H_{A_{\phi,k}}}^{>}\|_{L^2}^2 = O(k^{-1}).$$

□

2.6 Applications

2.6.1 Quantisation of the Laplacian for balanced metrics

In the former sections we quantised the Laplacian operator associated to a fixed Hermitian metric h on the vector bundle E. This was done using operators $P_k^* P_k$ defined with respect to the specific sequence $\mathrm{Hilb}_k(h) \in \mathscr{B}_k$. We will now extend our results to a different, *canonical* sequence which makes sense a priori, without specifying a Hermitian metric on E.

Let us recall Wang's results from [23] and [24]. See also section 5.2.3 in [18].

Definition 2.6.1. We say that a holomorphic vector bundle E over a polarized complex manifold $L \to X$ is *Gieseker stable* if for all proper coherent subsheaves $F \subset E$, one has for k sufficiently big,

$$\frac{\chi\left(X, F \otimes L^k\right)}{\mathrm{rk}(F)} < \frac{\chi\left(X, E \otimes L^k\right)}{\mathrm{rk}(E)}.$$

Let us remark that a Gieseker stable vector bundle is simple, see [15].

Definition 2.6.2. A pair (b, h) for $b \in \mathscr{B}_k$ and $h \in \mathscr{H}$ is said to be *balanced* if

$$b = \frac{N_k}{\mathrm{Vol}(X)\mathrm{rk}(E)}\mathrm{Hilb}_k(h), \quad \text{and} \quad h = \mathrm{FS}_k(b).$$

In this situation, we call $h \in \mathscr{H}$ and $b \in \mathscr{B}_k$ balanced.

Theorem 2.6.3 (Wang, [23]). *E is Gieseker stable if and only if for each k sufficiently big, there is a balanced metric h_k on E.*

Theorem 2.6.4 (Wang, [24]). *Suppose E is Gieseker stable. The sequence h_k from theorem 2.6.3 converges to some metric h_∞ on E in C^∞ if and only if h_∞ solves the following weakly Hermitian–Einstein equation*

$$\frac{i}{2\pi}\Lambda_\omega(R_\infty) = \left(\frac{\deg(E)}{\mathrm{Vol}(X)\,\mathrm{rk}(E)\,(n-1)!} - \frac{1}{8\pi}\left(S(\omega)-\bar{S}\right)\right)\mathrm{Id}_E, \tag{2.6.1}$$

where R_∞ is the curvature associated to h_∞ on E.

We are now going to extend our results to sequences of balanced metrics.

Theorem 2.6.5. *Assume E is Gieseker stable. For k big, write b_k for the balanced point in $\mathscr{B}_k$. In the following, all objects ($Q_{k,\phi}$, $P_k^* P_k, \dots$) are computed with respect to b_k. Let h_∞ be the almost Hermitian-Einstein metric on E satisfying equation* (2.6.1).

1. *For any $\phi \in C^\infty(X, End(E))$, Hermitian with respect to h_∞, one has*

$$\mathrm{Tr}(Q_{k,\phi} P_k^* P_k Q_{k,\phi}) \to \frac{1}{4\pi k}\int_X \mathrm{Tr}(\phi \Delta^{E,h_\infty}\phi)\,\Omega \tag{2.6.2}$$

where the Laplacian Δ^{E,h_∞} is computed with respect to h_∞. The result still holds if ϕ varies in a compact set of Hermitian endomorphisms in the C^∞-topology.

2. *One has convergence of the eigenvalues $\nu_{j,k}$ of the operator $P_k^* P_k$ towards the eigenvalues of Δ^{E,h_∞} after renormalization, i.e*

$$4\pi k^{n+1}\nu_{j,k} \to \lambda_j.$$

3. *Fix an integer $r > 0$. There is a constant $C > 0$ such that for all $A, B \in F_{r,k}$,*

$$\left|\mathrm{Tr}(AB) - k^n \langle H_A, H_B\rangle_{L^2_\Omega}\right| \leq Ck^{-1}\mathrm{Tr}(A^2)^{1/2}\mathrm{Tr}(B^2)^{1/2}.$$

4. *Let us fix integers $0 < p < q$ such that $\lambda_{p-1} < \lambda_p = \lambda_{p+1} = \dots = \lambda_q < \lambda_{q+1}$. Given $\phi \in \mathrm{Ker}\left(\Delta^{E,h_\infty} - \lambda_p \mathrm{Id}\right)$, let $A_{\phi,k}$ denote the point in $F_{p,q,k}$ with $H_{A_{\phi,k}}$ nearest to ϕ as measured in the L^2_Ω-norm. Then $H_{A_{\phi,k}}$ converges to ϕ in L^2_Ω and this convergence is uniform in ϕ if we require that $\|\phi\|_{L^2_\Omega} = 1$.*

Proof. We adapt the proof of theorem 7 in [11] to our situation. Denote by $h_k \in \mathscr{H}$ the balanced metric at level k. If we consider the sequence

$$b_{k,\ell} = \frac{N_\ell}{\mathrm{Vol}\,(X)\,\mathrm{rk}\,(E)} \mathrm{Hilb}_\ell(h_k) \in \mathscr{B}_\ell,$$

then by definition, the diagonal sequence $b_{k,k}$ is formed by balanced metrics. Let's apply theorem 2.4.1 to the metrics h_k. For ℓ large enough, denote by $Q_{\ell,\phi,h_k} \in \mathscr{B}_\ell$, the operator $Q_{\ell,\phi}$ computed with respect to the metric h_k. Of course, $Q_{k,\phi,h_k} = Q_{k,\phi}$. Similarly we introduce the operators P_{ℓ,h_k} that specify to P_k when $\ell = k$. By construction of the balanced metric (see [24]), ϕ is also Hermitian with respect to all the h_k. Hence we can apply our previous results, so that for each k large enough, there is a constant C such that

$$\left| \mathrm{Tr}\left(Q_{\ell,\phi,h_k} P_k^* P_k\left(Q_{\ell,\phi,h_k}\right)\right) - \frac{1}{4\pi\ell} \int_X \mathrm{Tr}\left(\phi \Delta^{E,h_k}(\phi)\right) \Omega \right| \leq C\ell^{-2}.$$

Since h_k converges to h_∞, the h_k form a family which is compact in the C^∞ topology and hence we can choose the constant in the estimate independently of k. Putting $k = \ell$, we get

$$\left| \mathrm{Tr}\left(Q_{k,\phi} P_k^* P_k\left(Q_{k,\phi}\right)\right) - \frac{1}{4\pi k} \int_X \mathrm{Tr}\left(\phi \Delta^{E,h_k}(\phi)\right) \Omega \right| \leq Ck^{-2}.$$

From [24], we know that $h_k = h_\infty + O\left(k^{-1}\right)$ in C^∞ and hence

$$\int_X \mathrm{Tr}\left(\phi \Delta^{E,h_k}(\phi)\right) \Omega = \int_X \mathrm{Tr}\left(\phi \Delta^{E,h_\infty}(\phi)\right) \Omega + O\left(k^{-1}\right).$$

This proves the first assertion.

The proof of the second and third assertion follows the same argument. Just note that to get convergence of the eigenvalues, we use the uniformity from theorem 2.5.2 and the fact that the eigenvalues depend continuously on the metric.

Let's give a few more details for point 4. Given $\phi \in \mathrm{Ker}\left(\Delta^{E,h_\infty} - \lambda_p \mathrm{Id}\right)$, recall that $A_{\phi,k} \in F_{p,q,k}$ is defined to be the $v_{j,k}$-eigenvector of $P_k^* P_k$ with $p \leq j \leq q$ and with $H_{A_{\phi,k}}$ nearest to ϕ as measured in L^2_Ω. Since h_k converges to h_∞ in C^∞, there is a sequence ϕ_k of Δ^{E,h_k} eigenvectors with eigenvalue $\lambda_j(h_k)$ converging to ϕ in L^2_Ω. Applying theorem 2.5.3 on each of these ϕ_k's gives for each k a sequence $A_{k,\ell}$ such that $H_{A_{k,\ell,\phi}}$ is nearest to ϕ_k. Restricting to the diagonal $k = \ell$ produces a single sequence $A_{k,k,\phi}$ such that

$$\|H_{A_{k,k,\phi}} - \phi_k\|_{L^2_\Omega} < Ck^{-1}$$

for some constant C. Moreover, since ϕ_k converges to ϕ in L^2, we get that $H_{A_{k,k,\phi}}$ converges to ϕ in L^2. Note that $A_{k,k,\phi}$ is not necessarily the same as $A_{k,\phi}$ but by definition $H_{A_{k,\phi}}$ is closer to ϕ than $H_{A_{k,k,\phi}}$ and hence $H_{A_{k,\phi}}$ also converges to ϕ in L^2_Ω. This concludes the proof. □

2.6.2 Spectral measures

In this section we use the convergence results of the eigenvalues to get quantisations of some spectral measures. For the sake of completeness, recall that we denote the eigenvalues of the Laplacian by $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ and the ones of $P_k^* P_k$ by $\nu_{k,0} \leq \nu_{k,1} \leq \dots \leq \nu_{k,N_k^2}$, repeated according to their multiplicities. In the case when the vector bundle E is simple, we showed in theorem 2.5.2 that for each $j = 0, 1, 2, \dots$

$$\nu_{k,j} = \frac{\lambda_j}{4\pi k^{n+1}} + O\left(k^{-n-2}\right).$$

To any compactly supported smooth function $\rho : \mathbb{R} \to \mathbb{R}$ we associate the following spectral measures

$$m(\rho) = \sum_{j=0}^{\infty} \rho(\lambda_j)$$

and

$$m_k(\rho) = \sum_{j=0}^{N_k^2} \rho(4\pi k^{n+1} \nu_{k,j}).$$

Theorem 2.6.6. *In the case E is simple, we have that*

$$m_k(\rho) = m(\rho) + O\left(k^{-1}\right).$$

Proof. Since the function ρ has compact support, the number of terms in the sum defining $m(\rho)$ is finite. Using the asymptotics from theorem 2.5.2 and Taylor expansions of ρ yields the result. □

A similar statement obviously holds in the case when E is Gieseker stable and the spectral measures m_k and m are defined with respect to the balanced point $b_k \in \mathscr{B}_k$ and the almost Hermitian–Einstein metric respectively. Then for any compactly supported smooth function $\rho : \mathbb{R} \to \mathbb{R}$, $m_k(\rho)$ converges to $m(\rho)$ as k tends to infinity.

It is noteworthy that we can only prove this theorem for functions ρ which are compactly supported. However it would be very interesting to know if the result still holds for a bigger class of functions. Here the motivation comes from the fact that for $\rho(x) = e^{-tx}$, $m(\rho)$ is nothing else than the trace of the heat operator

$$\operatorname{Tr}\left(e^{-t\Delta}\right) = \sum_{j=0}^{\infty} e^{-t\lambda_j}.$$

Being able to quantise this object would probably lead to further interesting results, one of them being the quantisation of the zeta function of the Laplacian. Let us briefly sketch how this could work. For $s \in \mathbb{C}$ one defines the zeta function of Δ^E by

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s}.$$

There is a trick one can use to rewrite this in terms of the trace of the heat operator and the Γ-function. A simple change of variable in

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

yields

$$\lambda_j^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-t\lambda_j} dt.$$

Then one can rewrite the zeta function (at least wherever it converges) as

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_j e^{-t\lambda_j} - \dim\ker\Delta^E \right) dt \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \operatorname{Tr}\left(e^{-t\Delta} - P\right) dt,
\end{aligned}$$

where P denotes the orthogonal projection onto $\ker\Delta^E$. Hence the quantisation of the trace of the heat operator implies the quantisation of the zeta function.

2.7 Explicit calculations for $\mathbb{C}P^1$

We will now illustrate our results via a direct computation in the special case when our manifold is $\mathbb{C}P^1$, polarized by the dual of the tautological line bundle.

Denote by $[Z : W]$ homogeneous coordinates on $\mathbb{C}P^1$. We start recalling the spectral theorem for $\mathbb{C}P^1$.

Theorem 2.7.1 ([13]). *If Δ denotes the Laplacian on $\mathbb{C}P^1$ with respect to the Fubini-Study metric, one has the following:*

1. *The eigenvalues of Δ are given by $\lambda_\ell = 4\pi\ell(\ell+1)$ where $\ell \in \mathbb{N}$.*
2. *Denote by $\mathscr{W}_\ell = \{f \in C^\infty(\mathbb{C}P^1) \,|\, \Delta f = \lambda_\ell f\}$ the ℓ-th eigenspace of Δ. Then*
$$L^2(\mathbb{C}P^1) = \bigoplus_{\ell=0}^{\infty} \mathscr{W}_\ell.$$
3. *$U(2)$ acts on $\mathbb{C}P^1$ and induces an action on $\mathscr{W}_\ell$. Moreover $\mathscr{W}_\ell$ is an irreducible representation of $SU(2)$.*
4. *$\mathscr{W}_\ell$ consists of all functions of the form*
$$\frac{\sum_{i,j=0}^{\ell} \sqrt{\binom{\ell}{i}\binom{\ell}{j}} a_{ij} Z^i W^{\ell-i} \bar{Z}^j \bar{W}^{\ell-j}}{(|Z|^2+|W|^2)^\ell}$$
where $\sum_{i,j=0}^{\ell} \sqrt{\binom{\ell}{i}\binom{\ell}{j}} a_{ij} Z^i W^{\ell-i} \bar{Z}^j \bar{W}^{\ell-j}$ is a harmonic homogeneous polynomial of degree (ℓ,ℓ) on $\mathbb{C}^2$ and $a_{ij} \in i\mathfrak{u}(\ell+1)$. Moreover, the dimension of $\mathscr{W}_\ell$ is $2\ell+1$.

The homogeneous polynomials
$$\sqrt{k+1}\sqrt{\binom{k}{j}} Z^j W^{k-j}$$
for $j = 0, \ldots, k$ define a basis of $H^0(\mathbb{C}P^1, O(k))$ and one can check that the embedding $\iota_k : \mathbb{C}P^1 \to \mathbb{C}P^k$ they define is balanced. By lemma 2.4.2 we know that for all $A, B \in i\mathfrak{u}(k+1)$ one has
$$\operatorname{Tr}(AP_k^* P_k(B)) = \operatorname{Tr}(AB\bar{\mu}_k) - \int_{\mathbb{C}P^1} H_A H_B \,\omega_{FS} \tag{2.7.1}$$
and since the embeddings ι_k are balanced, $(\bar{\mu}_k)_{ij} = \frac{1}{k+1}\delta_{ij}$ so that the first term of the right-hand side of (2.7.1) reduces to $\frac{1}{k+1}\operatorname{Tr}(AB)$.

Fix $k \gg 0$ and define
$$U_\ell = \{A \in i\mathfrak{u}(k+1) \,|\, H_A \in \mathscr{W}_\ell\}.$$

Our goal is to show that the leading order of $P_k^* P_k$ restricted to U_ℓ is a multiple of the identity and that this multiple is precisely the ℓ-th eigenvalue of Δ. This illustrates our general results from theorems 2.5.2 and 2.5.3 that the eigenvalues of $P_k^* P_k$ converge to those of Δ and that eigenvectors converge isometrically under $H : i\mathfrak{u}(k+1) \to C^\infty(X, \mathbb{R})$.

First observe that the map H is $U(k+1)$-equivariant. In fact, for $U \in U(k+1)$, $A \in i\mathfrak{u}(k+1)$ and $z \in \mathbb{C}P^k$ we have

$$\begin{aligned} H_{U \cdot A}(z) &= H_{UAU^{-1}}(z) \\ &= \mathrm{Tr}(AU^{-1}\mu_k(z)U) \\ &= \mathrm{Tr}(A\mu_k(zU^{-1})) = H_A(zU^{-1}). \end{aligned}$$

Moreover, by the spectral theorem,

$$H : i\mathfrak{u}(k+1) \to \bigoplus_{\ell=0}^{k} \mathscr{W}_\ell.$$

To show that this is actually an isomorphism (for each k), it suffices to compare dimensions. On one hand, $\dim(i\mathfrak{u}(k+1)) = (k+1)^2$ and on the other hand,

$$\dim\Big(\bigoplus_{\ell=0}^{k} \mathscr{W}_\ell\Big) = \sum_{\ell=0}^{k} (2\ell+1) = (k+1)^2.$$

Defining $U_\ell = H^{-1}\big(\mathscr{W}_\ell\big)$, the map

$$H : U_\ell \to \mathscr{W}_\ell$$

is still an isomorphism.

Since $\mathscr{W}_\ell$ is an irreducible real representation of $U(k+1)$ so is U_ℓ. Furthermore it is easy to see that

$$\langle A, B\rangle_1 = \mathrm{Tr}(AB)$$

and

$$\langle A, B\rangle_2 = \int_{\mathbb{C}P^1} H_A H_B \, \omega_{FS}$$

define both $U(k+1)$-invariant inner products on U_ℓ. This is trivial for the first one since the action of the unitary group on $i\mathfrak{u}(k+1)$ is given by conjugation. For the second one, observe that

$$\begin{aligned}\langle U\cdot A, U\cdot B\rangle_2 &= \int_{\mathbb{C}P^1} H_A(zU^{-1})H_B(zU^{-1})\,\omega_{FS}\\ &= \int_{\mathbb{C}P^1} H_A(z)H_B(z)\,\omega_{FS}.\end{aligned}$$

Here the last equality follows from a change of variables and the fact that our embedding of $\mathbb{C}P^1$ into $\mathbb{C}P^k$ is balanced. This implies that the volume form ω_{FS} is invariant. It follows then from a real version of Schur's lemma that both inner products only differ by a multiplicative constant. For the sake of completeness, let us briefly recall how this works. By the $U(k+1)$-invariance, there is an equivariant symmetric map $\varphi : U_\ell \to U_\ell$ such that

$$\langle A,B\rangle_2 = \langle \varphi(A), B\rangle_1.$$

On the other hand, since φ is symmetric, φ has a real eigenvalue and its associated eigenspace is invariant under $U(k+1)$. The irreducibility of the representation then implies that the eigenspace is U_ℓ. Therefore, φ is a scalar matrix and thus the inner products differ by a real multiplicative constant. Hence there exists $C_{\ell,k}\in\mathbb{R}$ such that for any $A,B\in U_\ell$ we have

$$\langle A,B\rangle_2 = C_{\ell,k}\langle A,B\rangle_1.$$

Furthermore in the balanced case, equation (2.7.1) implies that

$$\langle A, P_k^*P_k(B)\rangle_1 = \left(\frac{1}{k+1} - C_{\ell,k}\right)\langle A,B\rangle_1 \tag{2.7.2}$$

which shows that $P_k^*P_k$ restricted to U_ℓ is a multiple of the identity. As a corollary, we get that H sends eigenspaces of $P_k^*P_k$ isometrically to eigenspaces of Δ, at least up to a constant.

The end of this section is devoted to compute the constants $C_{\ell,k}$. Clearly it is sufficient to find a particular $A\in U_\ell$ for which we can calculate both $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ explicitely. Their quotient gives then the required constant.

Consider the eigenfunction

$$H_A = \frac{Z^\ell\bar{W}^\ell + \bar{Z}^\ell W^\ell}{(|Z|^2+|W|^2)^\ell} \in \mathscr{W}_\ell$$

for some appropriate Hermitian matrix A. After multiplying the numerator and the denominator by $\left(|Z|^2+|W|^2\right)^{k-\ell}$ and developing that term we can write

$$
\begin{aligned}
H_A &= \frac{\sum_{j=0}^{k-\ell}\binom{k-\ell}{j}|Z|^{2j}|W|^{2(k-\ell-j)}\left(Z^\ell\bar{W}^\ell+\bar{Z}^\ell W^\ell\right)}{(|Z|^2+|W|^2)^k}\\
&= 2\frac{\sum_{j=0}^{k-\ell}\sqrt{\binom{k}{\ell+j}\binom{k}{j}}^{-1}\binom{k-\ell}{j}\mathrm{Re}\left(\sqrt{\binom{k}{\ell+j}}Z^{\ell+j}W^{k-(\ell+j)}\sqrt{\binom{k}{j}}\bar{Z}^j\bar{W}^{k-j}\right)}{(|Z|^2+|W|^2)^k}
\end{aligned}
$$

From here we can read of the corresponding matrix A and get

$$
\begin{aligned}
\mathrm{Tr}(A^2) &= 2\sum_{j=0}^{k-\ell}\frac{\binom{k-\ell}{j}^2}{\binom{k}{\ell+j}\binom{k}{j}},\\
&= 2\frac{\sum_{j=0}^{k-\ell}(k-j)\cdots(k-j-\ell+1)(j+\ell)\cdots(j+1)}{\left(k(k-1)\cdots(k-\ell+1)\right)^2}.
\end{aligned}
$$

A tedious but straightforward computation implies that on one hand

$$
\begin{aligned}
&\sum_{j=0}^{k-\ell}(k-j)\cdots(k-j-\ell+1)(j+\ell)\cdots(j+1)\\
&\qquad = \frac{1}{(2\ell+1)\binom{2\ell}{\ell}}k^{2\ell+1}+\frac{1}{\binom{2\ell}{\ell}}k^{2\ell}+O(k^{2\ell-1}).
\end{aligned}
$$

On the other hand we get

$$
\begin{aligned}
\left(k(k-1)\cdots(k-\ell+1)\right)^2 &= \left(k^\ell-\frac{\ell(\ell-1)}{2}k^{\ell-1}+O(k^{\ell-2})\right)^2\\
&= k^{2\ell}-\ell(\ell-1)k^{2\ell-1}+O(k^{2\ell-2}).
\end{aligned}
$$

Putting these together, we get the following formula

$$
\begin{aligned}
\|A\|_1^2 = \mathrm{Tr}(A^2) &= \alpha\, k\,\frac{1+(2\ell+1)k^{-1}+O(k^{-2})}{1-\ell(\ell-1)k^{-1}+O(k^{-2})},\\
&= \alpha k\left(1+(\ell^2+\ell+1)k^{-1}+O(k^{-2})\right),
\end{aligned}
$$

where

$$
\alpha = \frac{2}{(2\ell+1)\binom{2\ell}{\ell}}.
$$

To be able to deduce the constants $C_{\ell,k}$ we now only have to compute $\|A\|_2^2$.

$$\begin{aligned}\langle A,A\rangle_2 &= \int_{\mathbb{C}P^1} H_A^2\,\omega_{FS},\\ &= \int_{\mathbb{C}P^1} \frac{\left(Z^\ell \bar{W}^\ell + \bar{Z}^\ell W^\ell\right)^2}{\left(|Z|^2+|W|^2\right)^{2\ell}}\,\omega_{FS},\\ &= 2\mathrm{Re}\int_{\mathbb{C}P^1} \frac{Z^{2\ell}\bar{W}^{2\ell}}{\left(|Z|^2+|W|^2\right)^{2\ell}}\,\omega_{FS} + 2\int_{\mathbb{C}P^1} \frac{|Z|^{2\ell}|W|^{2\ell}}{\left(|Z|^2+|W|^2\right)^{2\ell}}\,\omega_{FS}.\end{aligned}$$

The first of these integrals vanishes and the second one can be evaluated explicitly using the local coordinate $z = W/Z$. In fact we get

$$\begin{aligned}2\int_{\mathbb{C}P^1} \frac{|Z|^{2\ell}|W|^{2\ell}}{\left(|Z|^2+|W|^2\right)^{2\ell}}\,\omega_{FS} &= \frac{i}{\pi}\int_{\mathbb{C}} \frac{|z|^{2\ell}}{(1+|z|^2)^{2\ell+2}}\,dz\wedge d\bar{z}\\ &= \frac{2\pi}{\pi(2\ell+1)\binom{2\ell}{\ell}} = \alpha.\end{aligned}$$

Hence we proved the following Lemma.

Lemma 2.7.2. *For any $A,B \in U_\ell \subseteq i\mathfrak{u}(Sym^k\mathbb{C}^2)$ one has*

$$\int_{\mathbb{C}P^1} H_A H_B\,\omega_{FS} = \frac{1}{k}\left(1-(\ell^2+\ell+1)k^{-1}+O(k^{-2})\right)\mathrm{Tr}(AB).$$

From here it is now easy to get the leading order term of the eigenvalues of $P_k^*P_k$ as expected from theorem 2.5.2.

Proposition 2.7.3. *For any $A,B \in U_\ell \subseteq i\mathfrak{u}(Sym^k\mathbb{C}^2)$ one has*

$$\mathrm{Tr}\left(AP_k^*P_k(B)\right) = \left(\frac{4\pi\ell(\ell+1)}{4\pi k^2} + O(k^{-3})\right)\mathrm{Tr}(AB).$$

Proof. We have

$$\begin{aligned}\mathrm{Tr}\left(AP_k^*P_k(B)\right) &= \frac{1}{k+1}\mathrm{Tr}(AB) - \int_{\mathbb{C}P^1} H_A H_B,\\ &= \left(\frac{1}{k+1} - \frac{1}{k}\left(1-(\ell^2+\ell+1)k^{-1}+O(k^{-2})\right)\right)\mathrm{Tr}(AB),\\ &= \left(\frac{1}{k} - \frac{1}{k^2} - \frac{1}{k} + \frac{\ell^2+\ell+1}{k^2} + O(k^{-3})\right)\mathrm{Tr}(AB),\\ &= \left(\frac{\ell(\ell+1)}{k^2} + O(k^{-3})\right)\mathrm{Tr}(AB).\end{aligned}$$

Chapter 3

Quantising Solutions to the Heat Equation

3.1 Introduction

Let $L \to X$ be an ample line bundle over a compact Kähler manifold of complex dimension n. Fix some positively curved Hermitian metric σ on L such that the curvature F_σ defines a Kähler form $\omega = \frac{i}{2\pi} F_\sigma$. Furthermore denote by Δ the Bochner Laplacian on X. Recall that in this context, there is a sequence of non-negative real numbers $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ and an L^2-orthonormal basis $\phi_0, \phi_1, \phi_2, \dots$ of real-valued, smooth functions on X satisfying $\Delta \phi_j = \lambda_j \phi_j$, for each $j = 0, 1, 2, \dots$. The λ_j's are called the eigenvalues and the ϕ_j's the eigenfunctions of the Laplacian.

In the previous chapter we provided a quantisation of the eigenvalues and eigenfunctions of the Bochner Laplacian acting on smooth sections of the bundle $\mathrm{End}_h(E)$ where (E, h) is a Hermitian, holomorphic vector bundle. This was done by constructing a sequence of operators $P_k^* P_k$ which acted on the finite dimensional vector spaces of Hermitian endomorphisms of $H^0\left(X, E \otimes L^k\right)$. To recover the Laplacian on the manifold from above, we restrict now to the case where the vector bundle E is the trivial, flat line bundle $\underline{\mathbb{C}} \to X$.

In this chapter we are interested in solutions to the heat equation

$$\begin{cases} \partial_t f_t(x) + \Delta f_t(x) = 0 \\ f_0(x) = f(x) \end{cases} \tag{3.1.1}$$

where f is a smooth, real-valued function on X. One can prove that for any initial condition $f \in C^\infty(X,\mathbb{R})$ this Cauchy problem admits a unique solution $f(x,t)$ which can be written explicitely in terms of the *heat kernel* $p(t,x,y) \in C^\infty\left(\mathbb{R}^0_+ \times X \times X\right)$ as

$$f_t(x) = \int_X p(t,x,y) f(y) \frac{\omega_y^n}{n!}.$$

Using the eigenvalues and eigenfunctions of Δ, the heat kernel may be written as

$$p(t,x,y) = \sum_{j=0}^{\infty} e^{-t\lambda_j} \phi_j(x)\phi_j(y).$$

Moreover for each Hermitian endomorphism A of $H^0\left(X,L^k\right)$ we consider the equation

$$\begin{cases} \partial_t A(t) + 4\pi k^{n+1} P_k^* P_k\left(A(t)\right) = 0 \\ A(0) = A. \end{cases} \tag{3.1.2}$$

This equation is even easier to solve than the genuine heat equation since it is defined on a finite dimensional vector space. If in the case of the heat equation, the existence of the heat kernel is a highly non-trivial theorem to prove, one immediately checks that a fundamental solution to (3.1.2) is given by

$$p_k(t) = \sum_{j=1}^{d_k} e^{-t4\pi k^{n+1}\nu_{k,j}} \phi_{k,j}\phi_{k,j}.$$

Remark 3.1.1. From this point of view one might expect that the quantisation of the heat kernel and hence the quantisation of solutions to the heat equation readily follows from our results in the previous chapter. We proved for instance that if $\nu_{k,j}$ denotes the j-th eigenvalue of $P_k^* P_k$ and λ_j the j-th eigenvalue of the Laplacian, then

$$\nu_{k,j} = \frac{\lambda_j}{4\pi k^{n+1}} + O(k^{-n-2}).$$

However we have no bound on the error term which is uniform for *all* the eigenvalues and hence such an attempt must necessarily fail.

As in the previous chapter, N_k denotes the dimension of the space $H^0\left(X,L^k\right)$. The maps $Q_{k,f}$ and H_A introduced in section 2.3.3 and 2.3.1 can be written in terms of an L^2-orthonormal basis $\underline{s}$ of $H^0\left(X,L^k\right)$ as

$$Q_{k,f} = \int_X f(x) \langle s_i(x), s_j(x)\rangle_{\sigma^k} \frac{\omega^n}{n!}$$

for any $f \in C(X,\mathbb{R})$, and
$$H_A = \mathrm{Tr}(A\mu_k).$$
for any $A \in i\mathfrak{u}(N_k)$.

Our main result of this chapter states that solutions to the equation (3.1.2) quantise solutions to (3.1.1).

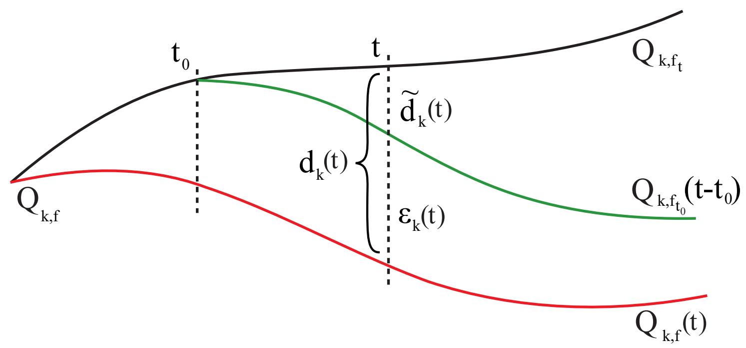
Theorem 3.1.2. *Let $f \in C^\infty(X,\mathbb{R})$ and denote by f_t a solution to the heat equation* (3.1.1) *starting at f. Moreover, denote by $Q_{k,f}(t)$ a solution to the equation* (3.1.2) *starting at $Q_{k,f}$. There is a constant C such that for all $t \in [0,T]$ we have*
$$\left\| Q_{k,f_t} - Q_{k,f}(t) \right\|_k^2 \leq \frac{C}{k}$$
where the norm $\|\cdot\|_k$ is defined by $\|A\|_k^2 = k^{-n}\mathrm{Tr}(A^2)$.

Corollary 3.1.3. *Under the same assumptions, there is a constant C such that for all $t \in [0,T]$ we have*
$$\left\| f_t(x) - H_{Q_{k,f(x)}(t)} \right\|_{L^2} \leq \frac{C}{k}.$$

The corollary readily follows from lemma 2.5.7 and the fact that $H_{Q_{k,f}}$ can be written as $B_k^{-1}K_{k,f}$ where B_k is the Bergman function and $K_{k,f}$ is the restriction to the diagonal of the Toeplitz kernel, see section 2.3.2.

3.2 Relation with other work

Similar to our quantisation of the Laplacian, Fine quantises in [11] another elliptic operator $\mathscr{D}^*\mathscr{D} : C^\infty(X,\mathbb{R}) \to C^\infty(X,\mathbb{R})$. This operator is nothing else than the Hessian of Mabuchi energy, see section 2.3.4. Similarly to what we do for solutions to the genuine heat equation, one can try to quantise solutions to the “heat equation" associated to $\mathscr{D}^*\mathscr{D}$. And indeed, all the steps in the proof go through except that we are not able to rewrite the proof of proposition 3.3.3 in this context. The problem is that as in our result, one must prove a complicated relationship between various coefficients in the asymptotic expansion of Bergman kernels and Toeplitz operators. Unfortunately the depth to which one must go in these expansions implies this time that we are not able to do the computations anymore. However we conjecture that the result still holds.



3.3 Proof of the results

Similarly to the technique used by Fine in [10], we first show that it is enough to prove an infinitesimal version of what we actually want to prove. In order to simplify notations, we define

$$d_k(t) = \left\| Q_{k,f_t} - Q_{k,f}(t) \right\|_k.$$

Now fix some $t_0 > 0$. For $t > t_0$ we put

$$\tilde{d}_k(t) = \left\| Q_{k,f_t} - Q_{k,f_{t_0}}(t - t_0) \right\|_k$$

and

$$\varepsilon_k(t) = \left\| Q_{k,f_{t_0}}(t - t_0) - Q_{k,f}(t) \right\|_k$$

Lemma 3.3.1. *$\varepsilon_k(t)$ is decreasing.*

Proof. Putting

$$A_k = Q_{f_{t_0}} - Q_{k,f}(t_0)$$

we can rewrite ε_k in a more condensed way as

$$\varepsilon_k(t) = \left\| A_k(t - t_0) \right\|_k.$$

Now let $\phi_{k,j}$ be an orthonormal basis of eigenvectors of $4\pi k^{n+1} P_k^* P_k$ with associated eigenvalues $\tilde{v}_{k,j}$. Then

$$\varepsilon_k(t)^2 = \sum_j e^{-2(t-t_0)\tilde{v}_{k,j}} \langle A_k, \phi_{k,j} \rangle^2$$

and hence

$$\frac{d}{dt} \varepsilon_k(t)^2 \Big|_{t_0} = -2 \sum_j \tilde{v}_{k,j} \langle A_k, \phi_{k,j} \rangle^2 \leq 0.$$

□

Lemma 3.3.2.
$$\frac{d}{dt}d_k(t)\Big|_{t_0} \leq \frac{d}{dt}\tilde{d}_k(t)\Big|_{t_0}$$

Proof. By the triangular inequality, we have that

$$d_k(t_0+\Delta t) \leq \tilde{d}_k(t_0+\Delta t) + \varepsilon_k(t_0+\Delta t).$$

Hence we can write

$$\frac{d}{dt}d_k(t)\Big|_{t_0} \leq \lim_{\Delta t\to 0}\frac{\tilde{d}_k(t_0+\Delta t)+\varepsilon_k(t_0+\Delta t)-d_k(t_0)}{\Delta t}.$$

Moreover, since $\varepsilon_k(t)$ is decreasing, and since $\varepsilon_k(t_0) = d_k(t_0)$ we are done. □

Clearly these arguments hold for any time $t_0 \geq 0$. The last ingredient in the proof is be the following proposition. Its proof is not necessarily complicated but since the calculations involved are very lenghty we postpone them to the next section.

Proposition 3.3.3. *The following estimate holds uniformly for* $t_0 \geq 0$*,*

$$\frac{d}{dt}\tilde{d}_k(t)\Big|_{t_0} = O\left(k^{-1/2}\right).$$

We are now in position to complete the proof of theorem 3.1.2. Integrating the inequality in lemma 3.3.2 for t_0 going from 0 to T, we get

$$d_k(T) \leq \int_0^T \frac{d}{dt}\tilde{d}_k(t)\Big|_{t_0} dt_0.$$

Since by proposition 3.3.3, $\frac{d}{dt}\tilde{d}_k(t)\big|_{t_0} = O\left(k^{-1/2}\right)$ uniformly in t_0, there is a constant C such that

$$d_k(T) \leq \frac{CT}{k^{1/2}}.$$

which proves the result.

3.3.1 Proof of Proposition 3.3.3

We saw in the last section that the remaining step in proving Theorem 3.1.2 was to show that $\frac{d}{dt}\tilde{d}_k(t)\big|_{t_0} = O\left(k^{-1/2}\right)$ uniformly in t_0.

Lemma 3.3.4. *Let $(X, \|\cdot\|)$ be a normed space. Let $t \mapsto \gamma(t)$ and $t \mapsto \eta(t)$ be two smooth paths in X such that $\gamma(0) = \eta(0)$. Then*

$$\left| \frac{d}{dt} \big\|\gamma(t) - \eta(t)\big\| \Big|_{t=0} \right| = \big\|\dot{\gamma}(0) - \dot{\eta}(0)\big\|.$$

Proof. We compute

$$\begin{aligned}
\left| \frac{d}{dt} \big\|\gamma(t) - \eta(t)\big\| \Big|_{t=0} \right| &= \left| \lim_{t\to 0} \frac{\|\gamma(t) - \eta(t)\|}{t} \right| \\
&= \left\| \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t} - \lim_{t\to 0} \frac{\eta(t) - \eta(0)}{t} \right\| \\
&= \big\|\dot{\gamma}(0) - \dot{\eta}(0)\big\|.
\end{aligned}$$

□

Applying the lemma to the quantity

$$\frac{d}{dt} \tilde{d}_k(t)\,\big|_{t_0} = \frac{d}{dt} \big\|Q_{k,f_t} - Q_{k,f_{t_0}}(t - t_0)\,\big\|_k \Big|_{t_0}$$

and using the fact that f_t and $Q_{k,f_{t_0}}(t - t_0)$ solve equations (3.1.1) and (3.1.2) respectively, one sees that in order to prove proposition 3.3.3 it is sufficient to show that

$$\big\|Q_{k,\Delta f} - 4\pi k^{n+1} P_k^* P_k(Q_{k,f})\big\|_k^2 = O(k^{-1}),$$

uniformly on compact subsets of $C^\infty(X, \mathbb{R})$ in the C^∞-topology.

First observe that

$$\begin{aligned}
&\big\|Q_{k,\Delta f} - 4\pi k^{n+1} P_k^* P_k(Q_{k,f})\big\|_k^2 \\
&= \big\|Q_{k,\Delta f}\big\|_k^2 + \big\|4\pi k^{n+1} P_k^* P_k(Q_{k,f})\big\|_k^2 - 2\,\langle Q_{k,\Delta f}, 4\pi k^{n+1} P_k^* P_k(Q_{k,f})\rangle_k \\
&= \frac{1}{k^n} \operatorname{Tr}\left(Q_{k,\Delta f}^2\right) + 16\pi^2 k^{n+2} \operatorname{Tr}\left(\left(P_k^* P_k(Q_{k,f})\right)^2\right) - 8\pi k \operatorname{Tr}\left(Q_{k,\Delta f} P_k^* P_k\left(Q_{k,f}\right)\right)
\end{aligned}$$

Theorem 2.4.1 from chapter 2, gives us the asymptotics of the last term of this expression,

$$\operatorname{Tr}\left(Q_{k,\Delta f} P_k^* P_k\left(Q_{k,f}\right)\right) = \frac{1}{4\pi k} \int_X (\Delta f)^2 \frac{\omega^n}{n!} + O(k^{-2}). \tag{3.3.1}$$

In the next lemma, we compute the nessessary asymptotics for the first term.

Lemma 3.3.5. *There is an asymptotic expansion*

$$\operatorname{Tr}\left(Q_{k,f}^2\right) = k^n \int_X f^2 \frac{\omega^n}{n!} + O(k^{n-1}).$$

Proof. Recall that by definition,

$$\left(Q_{k,f}\right)_{\alpha\beta} = \int_X f \langle s_\alpha, s_\beta \rangle \frac{\omega^n}{n!}.$$

Hence, we can write

$$\begin{aligned}\operatorname{Tr}\left(Q_{k,f}^2\right) &= \int_X \left(f(x) \sum_{\alpha\beta} \int_X f(y) \left(s_\alpha, s_\beta\right)(x) \left(s_\beta, s_\alpha\right)(y) \frac{\omega_y^n}{n!}\right) \frac{\omega_x^n}{n!} \\ &= \int_X f(x) K_{f,k}(x) \frac{\omega^n}{n!}.\end{aligned}$$

Using the asymptotic expansion for $K_{k,f}$ from theorem 2.3.6 yields the result.

□

From equation 3.3.1 and lemma 3.3.5 it follows that

$$\|Q_{k,\Delta f} - 16\pi^2 k P_k^* P_k(Q_{k,f})\|_k^2 = 16\pi^2 k^{n+2} \operatorname{Tr}\left(\left(P_k^* P_k(Q_{k,f})\right)^2\right) - \int_X f \Delta^2 f \frac{\omega^n}{n!} + O(k^{-1}).$$

Hence it is sufficient to show the following asymtotic result:

Proposition 3.3.6. *Let $f \in C^\infty(X, \mathbb{R})$. There is an asymptotic expansion*

$$\operatorname{Tr}\left(\left(P_k^* P_k(Q_{k,f})\right)^2\right) = \frac{1}{16\pi^2 k^{n+2}} \int_X f \Delta^2 f \frac{\omega^n}{n!} + O(k^{-n-3}).$$

The proof of this proposition is *in principle* not more difficult than the proof of the various asymptotic expansions we already got. However the computations become much more involved. Before we start with the actual proof, let us first recall the following general fact.

Lemma 3.3.7. *Any matrix A can be decomposed into the sum of a Hermitian and a skew-Hermitian matrix,*

$$A = A_H + A_{SH}$$

where

$$A_H = \frac{1}{2}\left(A + A^\dagger\right)$$
$$A_{SH} = \frac{1}{2}\left(A - A^\dagger\right).$$

Moreover if A is Hermitian and B skew-Hermitian, then

$$\mathrm{Tr}\,(AB) = -\overline{\mathrm{Tr}\,(AB)}$$

which implies that

$$\mathrm{Re}\,(\mathrm{Tr}\,(AB)) = 0.$$

The starting point in the proof of proposition 3.3.6 is the following lemma.

Lemma 3.3.8. *For any $A \in i\mathfrak{u}\,(N_k)$, we have*

$$\mathrm{Tr}\left(\left(P_k^* P_k(A)\right)^2\right) = \mathrm{Re}\,\mathrm{Tr}\left(\left(A\bar{\mu}_k\right)^2\right) - 2\,\mathrm{Re}\int_X \mathrm{Tr}\,(A\bar{\mu}_k \mu_k(x))\, H_A(x)\frac{\omega^n}{n!} + \int_{X\times X} H_A(x) H_A(y) H_{\mu_k(x)}(y)\frac{\omega^n \wedge \omega^n}{(n!)^2}.$$

Proof. Integrating lemma 2.4.2 over X we see that for all $A, B \in i\mathfrak{u}\,(N_k)$,

$$\mathrm{Tr}\,(AP_k^* P_k B) = \mathrm{Re}\,\mathrm{Tr}\,(AB\bar{\mu}_k) - \int_X H_A H_B \frac{\omega^n}{n!}. \tag{3.3.2}$$

Hence we can write

$$\mathrm{Tr}\left(\left(P_k^* P_k(A)\right)^2\right) = \mathrm{Re}\,\mathrm{Tr}\,(P_k^* P_k(A) A\bar{\mu}_k) - \int_X H_{P_k^* P_k(A)} H_A \frac{\omega^n}{n!}. \tag{3.3.3}$$

Applying formula (3.3.2) a second time to the first term of the right-hand side of (3.3.3) yields

$$\begin{aligned} \mathrm{Re}\,\mathrm{Tr}\,(P_k^* P_k(A) A\bar{\mu}_k) &= \mathrm{Re}\,\mathrm{Tr}\,((A\bar{\mu}_k)_H P_k^* P_k(A)) \\ &= \mathrm{Re}\,\mathrm{Tr}\left((A\bar{\mu}_k)^2\right) - \int_X H_{(A\bar{\mu}_k)_H}(x) H_A(x)\frac{\omega^n}{n!} \end{aligned}$$

Here we had to pay attention since the matrix $A\bar{\mu}_k$ is not necessarily Hermitian and we need to restrict $A\bar{\mu}_k$ to its Hermitian part, see lemma 3.3.7. Moreover, we used the fact that the trace of a skew-Hermitian matrix is imaginary to get the first term of the right-hand side.

Equation (3.3.3) can now be rewritten as

$$\operatorname{Tr}\left(\left(P_k^* P_k(A)\right)^2\right) = \operatorname{Re}\operatorname{Tr}\left(\left(A\bar{\mu}_k\right)^2\right) - \int_X \left(H_{(A\bar{\mu}_k)_H}(x) + H_{P_k^* P_k(A)}(x)\right) H_A(x) \frac{\omega^n}{n!}. \quad (3.3.4)$$

Furthermore, by definition

$$\begin{aligned} H_{(A\bar{\mu}_k)_H}(x) + H_{P_k^* P_k(A)}(x) &= \operatorname{Tr}\left(\left(A\bar{\mu}_k\right)_H \mu_k(x)\right) + \operatorname{Tr}\left(P_k^* P_k(A) \mu_k(x)\right) \\ &= \operatorname{Re}\operatorname{Tr}\left(A\bar{\mu}_k \mu_k(x)\right) + \operatorname{Re}\operatorname{Tr}\left(\mu_k(x) A\bar{\mu}_k\right) - \int_X H_A(y) H_{\mu_k(x)}(y) \frac{\omega_y^n}{n!} \\ &= 2\operatorname{Re}\operatorname{Tr}\left(A\bar{\mu}_k \mu_k(x)\right) - \int_X H_A(y) H_{\mu_k(x)}(y) \frac{\omega_y^n}{n!}. \end{aligned}$$

Putting this into (3.3.4) yields the result.

□

To prove proposition 3.3.6 we apply lemma 3.3.8 with $A = Q_{k,f}$. We start computing the three terms separately up to the 3rd order. We use the same notation as in section 2.3.2.

We start with the asymptotic expansion of the inverse of the Bergman function since it shows up ubiquitously in the calculations.

Lemma 3.3.9. *The inverse of the Bergman function B_k admits an asymptotic expansion given by*

$$B_k^{-1} = k^{-n}\left(1 - \frac{S}{8\pi}k^{-1} + \left(\frac{S^2}{64\pi^2} - b_2\right)k^{-2} + O\left(k^{-3}\right)\right).$$

Proof. This readily follows from the asymptotic expansion of the Bergman function given in theorem 2.3.4. □

Lemma 3.3.10. *There is an asymptotic expansion*

$$\operatorname{Tr}\left(\left(Q_{k,f}\bar{\mu}_k\right)^2\right) = \eta_0(f)\,k^{-n} + \eta_1(f)\,k^{-n-1} + \eta_2(f)\,k^{-n-2} + O\left(k^{-n-3}\right),$$

where

$$\begin{aligned}
\eta_0(f) &= \int_X f^2 \frac{\omega^n}{n!},\\
\eta_1(f) &= \int_X \left(-\frac{1}{4\pi} f\Delta f - \frac{1}{8\pi} S f^2\right)\frac{\omega^n}{n!},\\
\eta_2(f) &= \int_X \Bigg(\frac{1}{32\pi^2} f\Delta^2 f + \frac{3}{32\pi^2} S f\Delta f - \frac{1}{16\pi^2} S|df|^2\\
&\qquad + \frac{1}{8\pi^2} f\left(Ric, i\bar{\partial}\partial f\right) + \frac{1}{64\pi^2} S^2 f^2 - b_2 f^2\Bigg)\frac{\omega^n}{n!}.
\end{aligned}$$

Proof. By the composition of the kernels of Toeplitz operators, we can write

$$\operatorname{Tr}\left(\left(Q_{k,f}\bar{\mu}_k\right)^2\right) = \int_X B_k^{-1} K_{k,f,B_k^{-1},f} \frac{\omega^n}{n!}.$$

Applying lemma 3.3.9 this can be rewritten as

$$\operatorname{Tr}\left(\left(Q_{k,f}\bar{\mu}_k\right)^2\right) = k^{-n}\int_X \left(1 - \frac{S}{8\pi}k^{-1} + \left(\frac{S^2}{64\pi^2} - b_2\right)k^{-2} + O\left(k^{-3}\right)\right) K_{k,f,B_k^{-1},f}\frac{\omega^n}{n!}$$

and hence we must compute the following 3 terms,

$$k^{-n}\int_X K_{k,f,B_k^{-1},f}, \quad -k^{-n-1}\int_X \frac{S}{8\pi} K_{k,f,B_k^{-1},f}, \quad \text{and} \quad k^{-n-2}\int_X \left(\frac{S^2}{64\pi^2} - b_2\right) K_{k,f,B_k^{-1},f}. \tag{3.3.5}$$

To compute the first one, we apply a trick we learned from [11], saying that for $f,g,h \in C^\infty(X,\mathbb{R})$, $\int K_{k,f,g,h} = \int g K_{k,f,h}$. Hence,

$$\int_X K_{k,f,B_k^{-1},f} = \int_X B_k^{-1} K_{k,f,f}.$$

Now,

$$\begin{aligned}
&B_k^{-1} K_{k,f,f}\\
&= \left(1 - \frac{S}{8\pi}k^{-1} + \left(\frac{S^2}{64\pi^2} - b_2\right)k^{-2}\right)\left(b_{0,f,f} + b_{1,f,f}k^{-1} + b_{2,f,f}k^{-2}\right) + O\left(k^{-3}\right)\\
&= b_{0,f,f} + \left(b_{1,f,f} - \frac{S}{8\pi} b_{0,f,f}\right)k^{-1}\\
&\qquad + \left(b_{2,f,f} - \frac{S}{8\pi} b_{1,f,f} + \left(\frac{S^2}{64\pi^2} - b_2\right) b_{0,f,f}\right)k^{-2} + O\left(k^{-3}\right).
\end{aligned}$$

Since the trick we used before clearly also holds for the *coefficients* of Toeplitz kernels, i.e. for example $\int b_{2,f,f} = \int f b_{2,f}$, we get

$$\int_X B_k^{-1} K_{k,f,f} \frac{\omega^n}{n!} = \int_X f b_{0,f} \frac{\omega^n}{n!} + k^{-1} \int_X \left(f b_{1,f} - \frac{S}{8\pi} b_{0,f,f} \right) \frac{\omega^n}{n!}$$
$$+ k^{-2} \int_X \left(f b_{2,f} - \frac{S}{8\pi} b_{1,f,f} + \left(\frac{S^2}{64\pi^2} - b_2 \right) b_{0,f,f} \right) \frac{\omega^n}{n!} + O\left(k^{-3}\right).$$

Introducing the exact values for the coefficients from section 2.3.2, we see after a short computation that the first of the 3 terms in (3.3.5) is given by

$$\begin{aligned}
& k^{-n} \int_X K_{k,f,B_k^{-1},f} \\
=& k^{-n} \int_X f^2 \frac{\omega^n}{n!} - k^{-n-1} \int_X \frac{1}{4\pi} f \Delta f \frac{\omega^n}{n!} \\
& + k^{-n-2} \int_X \left(\frac{1}{32\pi^2} f \Delta^2 f + \frac{1}{32\pi^2} S f \Delta f - \frac{1}{32\pi^2} S |df|^2 + \frac{1}{8\pi^2} f \left(\mathrm{Ric}, i \bar{\partial} \partial f \right) \right) \frac{\omega^n}{n!} \\
& + O\left(k^{-n-3}\right).
\end{aligned}$$

We now compute to the second term in (3.3.5), namely $-k^{-n-1} \int_X \frac{S}{8\pi} K_{k,f,B_k^{-1},f}$. First observe that by lemma 3.3.9 we have

$$\begin{aligned}
K_{k,f,B_k^{-1},f} &= k^{-n} K_{k,f,1,f} - \frac{k^{-n-1}}{8\pi} K_{k,f,S,f} + O\left(k^{-2}\right) \\
&= b_{0,f,f} + k^{-1} \left(b_{1,f,f} - \frac{1}{8\pi} b_{0,f,S,f} \right) + O\left(k^{-2}\right).
\end{aligned}$$

Introducing the exact values of the coefficients, we can rewrite this as

$$f^2 + k^{-1} \left(-\frac{1}{2\pi} f \Delta f + \frac{1}{4\pi} |df|^2 \right) + O\left(k^{-2}\right).$$

Therefore the second term in (3.3.5) is given by

$$\begin{aligned}
& -k^{-n-1} \int_X \frac{S}{8\pi} K_{k,f,B_k^{-1},f} \\
=& k^{-n-1} \int_X -\frac{1}{8\pi} S f^2 \frac{\omega^n}{n!} + k^{-n-2} \int_X \left(\frac{1}{16\pi^2} S f \Delta f - \frac{1}{32\pi^2} S |df|^2 \right) \frac{\omega^n}{n!} + O\left(k^{-n-3}\right).
\end{aligned}$$

We now come to the third term in (3.3.5), namely $k^{-n-2}\int_X \left(\frac{S^2}{64\pi^2}-b_2\right) K_{k,f,B_k^{-1},f}$. Applying lemma 3.3.9, we see that it can be written as

$$k^{-n-2}\int_X \left(\frac{S^2}{64\pi^2}-b_2\right) f^2\frac{\omega^n}{n!}+O\left(k^{-n-3}\right).$$

Putting the asymptotics of the three terms together, one sees that only the first one contributes to the k^{-n}-term. It follows that

$$\eta_0(f)=\int_X f^2\frac{\omega^n}{n!}.$$

The term in k^{-n-1} is composed by a contribution of the first and the second term. We get,

$$\eta_1(f)=-\int_X\left(\frac{1}{4\pi}f\Delta f+\frac{1}{8\pi}Sf^2\right)\frac{\omega^n}{n!}.$$

Finally the term in k^{-n-2} is a combination of the asymptotics of all three terms in (3.3.5). Explicitely,

$$\begin{aligned}\eta_2(f)=\int_X\Big(&\frac{1}{32\pi^2}f\Delta^2 f+\frac{3}{32\pi^2}Sf\Delta f-\frac{1}{16\pi^2}S|df|^2\\&+\frac{1}{8\pi^2}f\left(\mathrm{Ric},i\bar\partial\partial f\right)+\frac{1}{64\pi^2}S^2f^2-b_2f^2\Big)\frac{\omega^n}{n!}.\end{aligned}$$

□

Lemma 3.3.11. *There is an asymptotic expansion*

$$\int_X \mathrm{Tr}\left(Q_{k,f}\bar\mu_k\mu_k(x)\right)H_{Q_{k,f}}(x)\frac{\omega^n}{n!}=\eta_0(f)k^{-n}+\eta_1(f)k^{-n-1}+\eta_2(f)k^{-n-2}+O\left(k^{-n-3}\right)$$

where

$$\begin{aligned}\eta_0(f)&=\int_X f^2\frac{\omega^n}{n!}\\ \eta_1(f)&=\int_X\left(-\frac{1}{2\pi}f\Delta f-\frac{1}{8\pi}Sf\right)\frac{\omega^n}{n!}\\ \eta_2(f)&=\int_X\Bigg(\frac{1}{16\pi^2}f\Delta^2 f+\frac{1}{32\pi^2}f^2\Delta S+\frac{1}{16\pi^2}Sf\Delta f+\frac{1}{16\pi^2}(\Delta f)^2\\&\quad+\frac{1}{64\pi^2}f^2S^2-\frac{1}{32\pi^2}f(dS,df)+\frac{1}{4\pi^2}f\left(Ric,i\bar\partial\partial f\right)-f^2b_2\Bigg)\frac{\omega^n}{n!}.\end{aligned}$$

Proof. Using notation from section 2.3.2, we can rewrite

$$\mathrm{Tr}\left(Q_{k,f}\bar{\mu}_k\mu_k(x)\right) = B_k^{-1}(x)K_{k,f,B_k^{-1}}(x),$$

and

$$H_{Q_{k,f}}(x) = B_k^{-1}(x)K_{k,f}(x).$$

Hence the integrand is given by

$$\mathrm{Tr}\left(Q_{k,f}\bar{\mu}_k\mu_k(x)\right)H_{Q_{k,f}}(x) = B_k^{-2}(x)\,K_{k,f,B_k^{-1}}(x)\,K_{k,f}(x) \tag{3.3.6}$$

By lemma 3.3.9 we easily see that

$$B_k^{-2} = k^{-2n}\left(1 - \frac{1}{4\pi}Sk^{-1} + \left(\frac{3}{64\pi^2}S^2 - 2b_2\right)k^{-2} + O\left(k^{-3}\right)\right).$$

Moreover, by theorem 2.3.6

$$\begin{aligned}K_{k,f} = k^{-n}\Bigg(& f + k^{-1}\left(\frac{1}{8\pi}Sf - \frac{1}{4\pi}\Delta f\right)\\ &+ k^{-2}\left(b_2(\omega)f + \frac{1}{32\pi^2}\Delta^2 f - \frac{1}{32\pi^2}S(\omega)\Delta f + \frac{1}{8\pi^2}(Ric, i\bar{\partial}\partial f)\right) + O\left(k^{-3}\right)\Bigg).\end{aligned}$$

Multiplying B_k^{-2} by $K_{k,f}$ we get

$$B_k^{-2}K_{k,f} = \sigma_0(f)k^{-n} + \sigma_1(f)k^{-n-1} + \sigma_2(f)k^{-n-2} + O\left(k^{-n-3}\right), \tag{3.3.7}$$

where

$$\begin{aligned}\sigma_0(f) &= f,\\ \sigma_1(f) &= -\frac{1}{8\pi}Sf - \frac{1}{4\pi}\Delta f,\\ \sigma_2(f) &= \frac{1}{64\pi^2}S^2 f + \frac{1}{32\pi^2}S\Delta f + \frac{1}{32\pi^2}\Delta^2 f - fb_2 + \frac{1}{8\pi^2}(Ric, i\bar{\partial}\partial f).\end{aligned}$$

Let us now have a look at $K_{k,f,B_k^{-1}}$. Using lemma 3.3.9, write

$$\begin{aligned}K_{k,f,B_k^{-1}} &= k^{-n}\left(K_{k,f} + k^{-1}K_{k,f,-\frac{S}{8\pi}} + k^{-2}K_{k,f,\frac{S^2}{64\pi^2}-b_2} + O\left(k^{-n-3}\right)\right)\\ &= \gamma_0(f) + \gamma_1(f)k^{-1} + \gamma_2(f)k^{-2} + O\left(k^{-3}\right),\end{aligned}$$

where

$$\begin{aligned}
\gamma_0(f) &= C_{0,f}, \\
\gamma_1(f) &= C_{1,f} + C_{0,f,-\frac{S}{8\pi}}, \\
\gamma_2(f) &= C_{2,f} + C_{1,f,-\frac{S}{8\pi}} + C_{0,f,\frac{S^2}{64\pi^2}-b_2}.
\end{aligned}$$

Using the explicit values of the coefficients from theorem 2.3.6, we get

$$\begin{aligned}
\gamma_0(f) &= f, \\
\gamma_1(f) &= -\frac{1}{4\pi}\Delta f, \\
\gamma_2(f) &= \frac{1}{32\pi^2}\Delta^2 f + \frac{1}{32\pi^2} f\Delta S - \frac{1}{32\pi^2}(df, dS) + \frac{1}{8\pi^2}\left(\mathrm{Ric}, i\bar{\partial}\partial f\right).
\end{aligned}$$

Combining this with (3.3.7) and integrating over X we get

$$\int_X B_k^{-2}(x) K_{k,f,B_k^{-1}}(x) K_{k,f}(x) \frac{\omega^n}{n!} = \eta_0(f) k^{-n} + \eta_1(f) k^{-n-1} + \eta_2(f) k^{-n-2} + O\left(k^{-n-3}\right)$$

where

$$\begin{aligned}
\eta_0(f) &= \int_X f^2 \frac{\omega^n}{n!} \\
\eta_1(f) &= \int_X \left(-\frac{1}{2\pi} f\Delta f - \frac{1}{8\pi} S f\right) \frac{\omega^n}{n!} \\
\eta_2(f) &= \int_X \left(\frac{1}{16\pi^2} f\Delta^2 f + \frac{1}{32\pi^2} f^2 \Delta S + \frac{1}{16\pi^2} S f\Delta f + \frac{1}{16\pi^2}(\Delta f)^2\right. \\
&\qquad \left. + \frac{1}{64\pi^2} f^2 S^2 - \frac{1}{32\pi^2} f(dS, df) + \frac{1}{4\pi^2} f\left(\mathrm{Ric}, i\bar{\partial}\partial f\right) - f^2 b_2\right) \frac{\omega^n}{n!}.
\end{aligned}$$

□

Lemma 3.3.12. *There is an asymptotic expansion*

$$\begin{aligned}
&\int_{X\times X} H_{Q_{k,f}}(x) H_{Q_{k,f}}(y) H_{\mu_k(x)}(y) \frac{\omega^n \wedge \omega^n}{(n!)^2} \\
&\qquad = \eta_0(f) k^{-n} + \eta_1(f) k^{-n-1} + \eta_2(f) k^{-n-2} + O\left(k^{-n-3}\right),
\end{aligned}$$

where

$$\eta_0(f) = \int f^2 \frac{\omega^n}{n!},$$
$$\eta_1(f) = \int \left(-\frac{3}{4\pi} f\Delta f - \frac{1}{8\pi} S f^2\right) \frac{\omega^n}{n!},$$
$$\eta_2(f) = \int_X \left(\frac{5}{32\pi^2} f\Delta^2 f + \frac{1}{32\pi^2} f\Delta(Sf) + \frac{1}{16\pi^2} Sf\Delta f + \frac{1}{8\pi^2}(\Delta f)^2 + \frac{1}{64\pi^2} S^2 f^2 + \frac{3}{8\pi^2} f\left(Ric, i\bar\partial\partial f\right) - f^2 b_2\right) \frac{\omega^n}{n!}.$$

Proof. First observe that

$$H_{Q_{k,f}} = \mathrm{Tr}\left(Q_{k,f}\mu_k\right) = B_k^{-1} K_{k,f}.$$

Furthermore,

$$\begin{aligned}\int_X H_{Q_{k,f}}(y) H_{\mu_k(x)}(y) \frac{\omega^n}{n!} &= \int_X H_{Q_{k,f}}(y) \mathrm{Tr}\left(\mu_k(x)\mu_k(y)\right) \frac{\omega^n}{n!} \\ &= B_k^{-1}(x) K_{k, B_k^{-1} H_{Q_{k,f}}}(x) \\ &= B_k^{-1}(x) K_{k, B_k^{-2} K_{k,f}}(x).\end{aligned}$$

Hence, we can rewrite

$$\int_{X\times X} H_{Q_{k,f}}(x) H_{Q_{k,f}}(y) H_{\mu_k(x)}(y) \frac{\omega^n \wedge \omega^n}{(n!)^2} = \int_X B_k^{-2} K_{k,f} K_{k, B_k^{-2} K_{k,f}} \frac{\omega^n}{n!}. \tag{3.3.8}$$

From the proof of lemma 3.3.11 we already know that

$$B_k^{-2} K_{k,f} = \sigma_0(f) k^{-n} + \sigma_1(f) k^{-n-1} + \sigma_2(f) k^{-n-2} + O\left(k^{-n-3}\right),$$

where

$$\sigma_0(f) = f,$$
$$\sigma_1(f) = -\frac{1}{8\pi} Sf - \frac{1}{4\pi}\Delta f,$$
$$\sigma_2(f) = \frac{1}{64\pi^2} S^2 f + \frac{1}{32\pi^2} S\Delta f + \frac{1}{32\pi^2}\Delta^2 f - f b_2 + \frac{1}{8\pi^2}(Ric, i\bar\partial\partial f).$$

Therefore,

$$\begin{aligned}K_{k,B_k^{-2}K_{k,f}} &= k^{-n}\left(K_{\sigma_0(f)} + k^{-1}K_{\sigma_1(f)} + k^{-2}K_{\sigma_2(f)}\right)\\ &= \gamma_0(f) + \gamma_1(f)k^{-1} + \gamma_2(f)k^{-2} + O\left(k^{-3}\right)\end{aligned}$$

where

$$\begin{aligned}\gamma_0(f) &= b_{0,\sigma_0(f)},\\ \gamma_1(f) &= b_{1,\sigma_0(f)} + b_{0,\sigma_1(f)},\\ \gamma_2(f) &= b_{2,\sigma_0(f)} + b_{1,\sigma_1(f)} + b_{0,\sigma_2(f)}.\end{aligned}$$

Putting in the exact values yields after some calculations

$$\begin{aligned}\gamma_0(f) &= f,\\ \gamma_1(f) &= -\frac{1}{2\pi}\Delta f,\\ \gamma_2(f) &= \frac{1}{8\pi^2}\Delta^2 f - \frac{1}{32\pi^2}S\Delta f + \frac{1}{32\pi^2}\Delta(Sf) + \frac{1}{4\pi^2}\left(\mathrm{Ric}, i\bar{\partial}\partial f\right).\end{aligned}$$

Keeping in mind relation (3.3.8), we next compute

$$\int_X B_k^{-2} K_{k,f} K_{k,B_k^{-2}K_{k,f}} \frac{\omega^n}{n!} = \eta_0(f)k^{-n} + \eta_1(f)k^{-n-1} + \eta_2(f)k^{-n-2} + O\left(k^{-n-3}\right),$$

where

$$\begin{aligned}\eta_0(f) &= \int \sigma_0(f)\gamma_0(f),\\ \eta_1(f) &= \int \sigma_0(f)\gamma_1(f) + \sigma_1(f)\gamma_0(f),\\ \eta_2(f) &= \int \sigma_0(f)\gamma_2(f) + \sigma_1(f)\gamma_1(f) + \sigma_2(f)\gamma_0(f).\end{aligned}$$

Using the explicit values yields

$$\eta_0(f) = \int f^2 \frac{\omega^n}{n!},$$
$$\eta_1(f) = \int \left(-\frac{3}{4\pi} f\Delta f - \frac{1}{8\pi} S f^2\right) \frac{\omega^n}{n!},$$
$$\eta_2(f) = \int_X \left(\frac{5}{32\pi^2} f\Delta^2 f + \frac{1}{32\pi^2} f\Delta(Sf) + \frac{1}{16\pi^2} Sf\Delta f + \frac{1}{8\pi^2}(\Delta f)^2 \right.$$
$$\left. + \frac{1}{64\pi^2} S^2 f^2 + \frac{3}{8\pi^2} f\left(\mathrm{Ric}, i\bar{\partial}\partial f\right) - f^2 b_2\right) \frac{\omega^n}{n!}.$$

□

Having computed all the necessary asymptotics, the proof of proposition 3.3.6 is now just a matter of putting the right terms together.

Recall from lemma 3.3.8 that for all $f \in C^\infty(X, \mathbb{R})$,

$$\mathrm{Tr}\left(\left(P_k^* P_k(Q_{k,f})\right)^2\right) = \mathrm{Re}\,\mathrm{Tr}\left(\left(Q_{k,f}\bar{\mu}_k\right)^2\right) - 2\,\mathrm{Re}\int_X \mathrm{Tr}\left(Q_{k,f}\bar{\mu}_k \mu_k(x)\right) H_{Q_{k,f}}(x) \frac{\omega^n}{n!}$$
$$+ \int_{X\times X} H_{Q_{k,f}}(x) H_{Q_{k,f}}(y) H_{\mu_k(x)}(y) \frac{\omega^n \wedge \omega^n}{(n!)^2}.$$

By lemmas 3.3.10, 3.3.11 and 3.3.12 we immediately see that the k^{-n}-contribution vanishes. The term in k^{-n-1} is given by

$$\int_X \left(-\frac{1}{4\pi} f\Delta f - \frac{1}{8\pi} S f^2\right) - 2\left(-\frac{1}{2\pi} f\Delta f - \frac{1}{8\pi} Sf\right) + \left(-\frac{3}{4\pi} f\Delta f - \frac{1}{8\pi} S f^2\right) \frac{\omega^n}{n!}$$

and hence it vanishes too.

Let us rewrite the full k^{-n-2} term.

$$\int_X \left(\frac{1}{32\pi^2} f\Delta^2 f + \frac{3}{32\pi^2} Sf\Delta f - \frac{1}{16\pi^2} S|df|^2 + \frac{1}{8\pi^2} f\left(\mathrm{Ric}, i\bar{\partial}\partial f\right) + \frac{1}{64\pi^2} S^2 f^2 - b_2 f^2 \right) \frac{\omega^n}{n!}.$$

$$-2\int_X \left(\frac{1}{16\pi^2} f\Delta^2 f + \frac{1}{32\pi^2} f^2 \Delta S + \frac{1}{16\pi^2} Sf\Delta f + \frac{1}{16\pi^2} (\Delta f)^2 \right.$$
$$\left. + \frac{1}{64\pi^2} f^2 S^2 - \frac{1}{32\pi^2} f(dS, df) + \frac{1}{4\pi^2} f\left(\mathrm{Ric}, i\bar{\partial}\partial f\right) - f^2 b_2 \right) \frac{\omega^n}{n!}$$
$$+\int_X \left(\frac{5}{32\pi^2} f\Delta^2 f + \frac{1}{32\pi^2} f\Delta(Sf) + \frac{1}{16\pi^2} Sf\Delta f + \frac{1}{8\pi^2} (\Delta f)^2 \right.$$
$$\left. + \frac{1}{64\pi^2} S^2 f^2 + \frac{3}{8\pi^2} f\left(\mathrm{Ric}, i\bar{\partial}\partial f\right) - f^2 b_2 \right) \frac{\omega^n}{n!}.$$

After simplifying the *obvious* terms, the integrand becomes

$$\frac{1}{16\pi^2} f\Delta^2 + \frac{1}{32\pi^2} Sf\Delta f - \frac{1}{16\pi^2} S|df|^2 - \frac{1}{16\pi^2} f^2 \Delta S + \frac{1}{32\pi^2} f\Delta(Sf) + \frac{1}{16\pi^2} f(dS, df).$$

Now using the fact that

$$\Delta(Sf) = f\Delta S + S\Delta f - 2(dS, df)$$

and that

$$\frac{1}{2}\int_X f^2 \Delta S \frac{\omega^n}{n!} = \int_X S\left(f\Delta f - |df|^2\right)$$

one sees that all the terms vanish, except

$$\frac{1}{16\pi^2} \int_X f\Delta^2 f \frac{\omega^n}{n!}.$$

This concludes the proof of proposition 3.3.6 and hence of proposition 3.3.3.

Chapter 4

Geometric Quantisation and the Derivative of Hilb_k

4.1 Introduction

The fourth chapter of this thesis is dedicated to an intriguing link between geometric quantisation and a program initiated by Donaldson to study the geometry of the space of Kähler metrics in a fixed cohomology class using finite dimensional approximations.

For the sake of completeness we start recalling some notation and definitions we already used in the previous chapters. Let $L \to X$ be an ample line bundle over a compact complex manifold of complex dimension n. Write $\mathscr{H}$ for the space of all positively curved Hermitian metrics on L. Any element $h \in \mathscr{H}$ induces a Kähler metric ω_h on X by $\omega_h = \frac{i}{2\pi} F_h$. Furthermore let $\mathscr{B}_k$ be the *Bergman space of level k*, i.e. the space of Hermitian inner products on $H^0\left(X, L^k\right)$. $\mathscr{H}$ carries the structure of an infinite dimensional Riemannian manifold and its tangent space at a point h is naturally identified with the space of smooth, real valued functions on X. On the other hand the tangent space to $\mathscr{B}_k$ at b is identified with the space V_k of Hermitian endomorphisms on $H^0\left(X, L^k\right)$, see section 4.2 or [6] for further information.

Choosing a metric on L induces an L^2-inner product on $H^0\left(X, L^k\right)$ and hence a map $\mathrm{Hilb}_k : \mathscr{H} \to \mathscr{B}_k$. Explicitly,

$$\mathrm{Hilb}_k(h)\,(s,t) = \int_X h^k\left(s(x), t(x)\right) \frac{\omega_h^n}{n!}.$$

Remark 4.1.1. Note that this Hilb_k-map differs from the one we used in the chapters before. This time the volume form depends on the point $h \in \mathscr{H}$, whereas before it was fixed.

There is also a map in the other direction called the Fubini–Study map. Given $b \in \mathscr{B}_k$, we define $\mathrm{FS}_k(b)$ as follows: any b-orthonormal basis $\underline{s} = s_1, \ldots, s_{N_k}$ of $H^0(X, L^k)$ embeds X into $\mathbb{C}P^{N_k - 1}$. Pulling back the Fubini–Study metric from $O(1)$ via this embedding defines a Hermitian metric on L^k. Then take the k-th root to get a genuine metric on L which we denote by $\mathrm{FS}_k(b)$. Hence we get a map $\mathrm{FS}_k : \mathscr{B}_k \to \mathscr{H}$. Note that $\mathrm{FS}_k(b)$ does not depend on the b-orthonormal basis we choose.

A lot of research has been devoted to the geometric quantisation of Kostant [16] and Souriau [21]. It explains how to naturally associate to every classical observable (i.e. a smooth function on X), a quantum observable (i.e. a Hermitian operator on a Hilbert space). The Hilbert space in question is nothing else than $H^0\left(X, L^k\right)$ together with the $\mathrm{Hilb}_k(h)$-inner product. To define the quantum observables, one first associates to every function $f \in C^\infty(X, \mathbb{R})$ the so-called pre-quantum operator $\tilde{\sigma}_{k,f}$ acting on the space $C^\infty\left(X, L^k\right)$ of smooth sections into L^k. Explicitly

$$\tilde{\sigma}_{k,f} = 2\pi k f + i\nabla^{(k)}_{X_f}.$$

Here X_f denotes the Hamiltonian vector field associated to f and ∇^k is the Chern connection on $\left(L^k, h^k\right)$. The genuine quantum operators $\sigma_{k,f}$ are then given by taking the holomorphic part of the pre-quantum operators by composing $\tilde{\sigma}_{k,f}$ with the orthogonal projection onto the sub-space of *holomorphic* sections in $C^\infty\left(X, L^k\right)$. In this way we get a map

$$f \in C^\infty(X, \mathbb{R}) \mapsto \sigma_{k,f} \in V_k.$$

The starting observation is that *this map is nothing else than the derivative of* Hilb_k, see section 4.4. In this sense one can think of the Hilb_k-map as a *curved version of geometric quantisation*. It is then natural to ask of what use the higher derivatives of the Hilb_k-map might be and if they have some physical interpretation. Furthermore one might think whether the differential of the FS_k-map also has an interpretation in terms of some *dequantisation*. And indeed, in section 4.5 we show that it is nothing else than *Berezin's covariant symbol*. Using expansions of Toeplitz operators one easily sees that the composition $d\mathrm{FS}_k \circ d\mathrm{Hilb}_k$ tends to the identity as k goes to infinity and hence, at least asymptotically, Berezin's covariant symbol can be interpreted as the inverse of geometric quantisation. This sheds new light on the results obtained by Cahen, Gutt and Rawnsley in [20].

Motivated by the fact that the linearisation of Hilb_k gives geometric quantisation we compute in section 4.6 its next order approximation, namely its Hessian. To state the result, define $\mathscr{D} : C^\infty(X,\mathbb{R}) \to \Omega^{0,1}(TX)$ to be the operator given by

$$\mathscr{D}(f) = \bar{\partial}\left(X_f\right).$$

$\mathscr{D}(f)$ measures the failure of the Hamiltonian vector field X_f of being holomorphic and the operator $\mathscr{D}$ is called the *Lichnerowicz operator*, see also section 2.3.10.

Theorem 4.1.2. *The Hessian of $Hilb_k : \mathscr{H} \to \mathscr{B}_k$ admits an asymptotic expansion in which the leading order term is given by the leading order of the Toeplitz operator associated to the function $(\mathscr{D}f, \mathscr{D}g)$. More precisely, as $k \to \infty$, one has*

$$(\nabla dHilb_k)_\phi(f,g) = T_{(\mathscr{D}f,\mathscr{D}g)} + O\left(k^{n-1}\right).$$

As a corollary we reprove a result by Fine saying that the Hessian of balancing energy converges to the Hessian of Mabuchi energy, see theorem 2 in [11].

4.2 The geometric structures on $\mathscr{H}$ and $\mathscr{B}_k$

Fixing $h \in \mathscr{H}$, every other element in $\mathscr{H}$ can be written as $h_\phi = e^{4\pi\phi}h$ for some smooth function ϕ and the associated Kähler metric ω_ϕ is given by $\omega_\phi = \omega_0 + 2i\bar{\partial}\partial\phi$. Explicitly, we can write

$$\mathscr{H} = \{\phi \in C^\infty(X,\mathbb{R}) \,|\, \omega_\phi = \omega_0 + 2i\bar{\partial}\partial\phi > 0\}.$$

Hence $\mathscr{H}$ can be thought of as an open subset of the vector space $C^\infty(X,\mathbb{R})$ and we identify its tangent space $T_\phi\mathscr{H}$ with the space of smooth functions on X.

Remark 4.2.1. Note that one can equally well think of $\mathscr{H}$ as the space of Kähler potentials with respect to ω_0. Two elements ϕ and ψ in $\mathscr{H}$ give the same Kähler form if and only if they differ by an additive constant. The space of Kähler forms in $c_1(L)$ can then be identified with $\mathscr{H}/\mathbb{R}$ by modding out these constants.

There is a natural Riemannian structure on $\mathscr{H}$ known as the Donaldson–Mabuchi–Semmes metric. We start recalling its definition and some of its properties. For further information and proofs, we refer the reader to [6]. At each point $\phi \in \mathscr{H}$ one defines an inner product

$$\langle f,g\rangle_\phi := \int_X fg\,\frac{\omega_\phi^n}{n!}$$

where f and g are two smooth, real-valued functions thought of as tangent vectors to $\mathscr{H}$ at ϕ. It happens that the Donalson–Mabuchi–Semmes metric admits a Levi-Civita connection (recall that the existence of a metric, torsion free connection is not guaranteed in infinite dimensions) which can be described as follows. Let $h_t = e^{4\pi t f} h$ be a path in $\mathscr{H}$ and g_t a path of tangent vectors along h_t which amounts to a function on $X \times [0,1]$. One defines the covariant derivative $D_t^{LC} g_t$ of g_t along h_t by

$$D_t^{LC} g_t = \frac{\partial g_t}{\partial t} + (df, dg_t)_{\omega_t}.$$

One checks that this connection is symmetric and compatible with the metric.

Similar to the fact that $\mathscr{H}$ can be thought of as an open subset of $C^\infty(X,\mathbb{R})$ we can embedd $\mathscr{B}_k$ into the space of Hermitian forms on $H^0(X,\mathbb{R})$. Denote by N_k the dimension of the space $H^0(X,L^k)$. Note that $\mathrm{GL}(N_k,\mathbb{C})$ acts transitively on $\mathscr{B}_k$ with stabilizer $U(N_k)$ so that if we choose a point $b \in \mathscr{B}_k$ we can identify $\mathscr{B}_k$ with the symmetric space $\mathrm{GL}(N_k,\mathbb{C})/U(N_k)$.

There is a natural Riemannian structure on $\mathscr{B}_k$ given by

$$(A,B)_b := \mathrm{Tr}\left(b^{-1}Ab^{-1}B\right)$$

where $b \in \mathscr{B}_k$ and $A, B \in T_b\mathscr{B}_k \cong i\mathfrak{u}(N_k)$. The Levi-Civita connection associated to this metric can be written as

$$\nabla = d + a$$

where the connection 1-form $a \in \Omega^1(\mathscr{B}_k, \mathrm{End}(T\mathscr{B}_k))$ is explicitly given by

$$a_b(A)(B) = -\frac{1}{2}\left(Ab^{-1}B + Bb^{-1}A\right).$$

4.3 The derivatives of Hilb_k and FS_k

Recall that we defined $\mathrm{Hilb}_k : \mathscr{H} \to \mathscr{B}_k$ by

$$\mathrm{Hilb}_k(\phi) = \int_X h_\phi^k(\cdot,\cdot)\,\frac{\omega_\phi^n}{n!}.$$

To compute its differential at a point ϕ in the direction f, we differentiate Hilb_k along the path $e^{4\pi f t} h_\phi$. One gets

$$\left((d\mathrm{Hilb}_k)_\phi(f)\right)_{\alpha\beta} = \int_X \left(4\pi k f + \Delta_\phi f\right) h_\phi^k(s_\alpha, s_\beta)\,\frac{\omega_\phi^n}{n!}$$

where $\underline{s} = s_1, \ldots, s_{N_k}$ is a $\mathrm{Hilb}_k(\phi)$-orthonormal basis of $H^0(X,L^k)$. The Laplacian term in this expression comes from the variation of the volume form $\frac{\omega_\phi^n}{n!}$ and follows from the well-known formula

$$2ni\bar{\partial}\partial f \wedge \omega^{n-1} = \Delta f \omega^n$$

from Kähler geometry.

Recall that for k large enough the Kodaira embedding theorem tells us that for any basis $\underline{s} = s_1, \ldots, s_{N_k}$ of $H^0(X, L^k)$, $\iota_{\underline{s}} : X \to \mathbb{C}P^{N_k - 1}$ given by

$$\iota_{\underline{s}}(x) = [s_1(x), \ldots, s_{N_k}(x)]$$

embeds X holomorphically into $\mathbb{C}P^{N_k}$. The Fubini–Study map was given by the k-th root of the pull-back of the Fubini–Study metric from $O(1)$ to L^k. Explicitly if $\underline{s}$ is any b-orthonormal basis of $H^0\left(X, L^k\right)$ and h any auxilliary Hermitian inner product,

$$h_b = \frac{h_x}{\sum_{i=0}^{N_k} |s_i(x)|_h^2}$$

and $\mathrm{FS}_k(b) = (h_b)^{1/k}$. To compute the differential of FS_k, we differentiate FS_k along the path $\underline{s}(t) = e^{-\frac{1}{2}At} \cdot \underline{s}$ where $A \in i\mathfrak{u}(N_k + 1)$. One gets

$$\frac{d}{dt}\left(-\frac{1}{4\pi k}\log\sum_{i=0}^{N_k}|s_i(t)(x)|_h^2\right)\Bigg|_{t=0} = \frac{1}{4\pi k}\frac{\sum_{i,j=0}^{N_k} A_{ij} h\left(s_j(x), s_i(x)\right)}{\sum_{i=0}^{N_k}|s_i(x)|_h^2}.$$

For every positive integer m, consider the map $\mu : \mathbb{C}P^m \to i\mathfrak{u}(m+1)$ given in homogeneous coordinates by

$$\left(\mu\left([Z_1, \ldots, Z_m]\right)\right)_{\alpha\beta} = \frac{Z_\alpha \bar{Z}_\beta}{4\pi \sum_\gamma |Z_\gamma|^2}.$$

Moreover, if $\underline{s} = s_1, \ldots, s_{N_k}$ is a basis of $H^0(X, L^k)$, we denote by $\mu_{\underline{s}}$ the composition $\mu \circ \iota_{\underline{s}}$.

Remark 4.3.1. At this point the conventions we use in this chapter differ from those used in the previous chapters in the sense that we multiply the map μ defined in 2.2.2 by $\frac{1}{4\pi}$. The reason why we do this is because we recover Fine's quantisation of the Hessian of Mabuchi energy from [11] in section 4.6.1 and we want to use his conventions.

Using these notations, the differential of FS_k at a point b in the direction A can be written as

$$(d\mathrm{FS}_k)_b(A) = \frac{1}{k}\mathrm{Tr}(A\mu_b).$$

4.4 Relation between geometric quantisation and the $\mathbf{Hilb}_k$-map.

We start recalling some of the general theory from geometric quantisation. Let $L \to X$ be a ample line bundle over a compact complex manifold of complex dimension n. Fix an Hermitian metric h on L inducing a Kähler form ω on X. To every smooth function $f \in C^\infty(X,\mathbb{R})$ we associate an Hermitian endomorphism σ_f of the Hilbert space $H^0\left(X,L^k\right)$. This is done by composing the Lie algebra map $\tilde{\sigma} : C^\infty(X,\mathbb{R}) \to Op(\Gamma(X,L))$ given by

$$\tilde{\sigma}_f = 2\pi f + i\nabla_{X_f}.$$

with the orthogonal projection $\Pi : \Gamma(X,L) \to H^0(X,L)$. Here we use the L^2-Hermitian inner product on $\Gamma(X,L)$ given by

$$\langle s_1, s_2\rangle = \int_X h(s_1,s_2)\,\frac{\omega^n}{n!}.$$

X_f denotes the Hamiltonian vector field with respect to ω and ∇ denotes the Chern connection on L.

In the following lemma (known as Tuynman's lemma) we compute the matrix of σ_f in terms of an L^2-orthonormal basis of $H^0(X,L)$.

Lemma 4.4.1. *Given an orthonormal basis $\underline{s} = s_1,\dots,s_N$ of $H^0(X,L)$, the matrix of the operator σ_f with respect to $\underline{s}$ is given by*

$$\left(\sigma_f\right)_{\alpha\beta} = \int_X (2\pi f + \frac{1}{2}\Delta f) h(s_\alpha, s_\beta)\frac{\omega^n}{n!}.$$

Proof. Using the definitions we see that

$$\begin{aligned}
\left(\sigma_f\right)_{\alpha\beta} &= \langle s_\alpha, \tilde{\sigma}_f s_\beta\rangle \\
&= \int_X h(s_\alpha, \tilde{\sigma} s_\beta)\,\frac{\omega^n}{n!} \\
&= \int_X 2\pi f\, h(s_\alpha, s_\beta)\,\frac{\omega^n}{n!} - i\int_X h(s_\alpha, \nabla_{X_f} s_\beta)\,\frac{\omega^n}{n!}.
\end{aligned}$$

Hence it suffices to compute the second term of the right-hand side. Let v be some vector field on X. The holomorphic, resp. anti-holomorphic part of v is given by $\frac{1}{2}(v \mp iIv)$. Using the fact that the covariant derivative of a holomorphic section s in the direction of a vector field of type $(0,1)$ vanishes we see that

$$\nabla_{\frac{1}{2}(v+iIv)} s_\beta = 0.$$

and hence

$$\nabla_{Iv} s_\beta = i\nabla_v s_\beta .$$

Furthermore we know that $X_f = I \operatorname{grad} f$ and so

$$h\left(s_\alpha, \nabla_{X_f} s_\beta\right) = -ih\left(s_\alpha, \nabla_{\operatorname{grad} f} s_\beta\right).$$

On the other hand we can write

$$h\left(s_\alpha, \nabla_{X_f} s_\beta\right) = d\left(h\left(s_\alpha, s_\beta\right)\right)\left(X_f\right) - ih\left(\nabla_{\operatorname{grad} f} s_\alpha, s_\beta\right).$$

Adding up the last two equations we get

$$\begin{aligned} 2h\left(s_\alpha, \nabla_{X_f} s_\beta\right) &= d\left(h\left(s_\alpha, s_\beta\right)\right)\left(X_f\right) - i\big(h\left(\nabla_{\operatorname{grad} f} s_\alpha, s_\beta\right) + h\left(s_\alpha, \nabla_{\operatorname{grad} f} s_\beta\right)\big) \\ &= d\left(h\left(s_\alpha, s_\beta\right)\right)\left(X_f\right) - i\,d\left(h\left(s_\alpha, s_\beta\right)\right)(\operatorname{grad} f) \end{aligned}$$

Integrating over X and using the fact that $\operatorname{div} X_f = 0$ and $\operatorname{div}\operatorname{grad} f = -\Delta f$ implies

$$\int_X h\left(s_\alpha, \nabla_{X_f} s_\beta\right) \frac{\omega^n}{n!} = \frac{i}{2}\int_X \Delta f\, h\left(s_\alpha, s_\beta\right) \frac{\omega^n}{n!}$$

Putting everything together we get

$$\begin{aligned} \left(\sigma_f\right)_{\alpha\beta} &= \int_X 2\pi f\, h(s_\alpha, s_\beta)\frac{\omega^n}{n!} - i\int_X h(s_\alpha, \nabla_{X_f} s_\beta)\frac{\omega^n}{n!} \\ &= \int_X (2\pi f + \frac{1}{2}\Delta f) h(s_\alpha, s_\beta)\frac{\omega^n}{n!}. \end{aligned}$$

□

Even if the map $\tilde{\sigma}$ was a Lie algebra map, there is no reason for σ to be one. However the property is satisfied asymptotically, in the so-called semi-classical limit. More precisely we will look at higher and higher tensor powers of the line bundle L. The curvatures $F^{(k)}$ on L^k and F on L are related by

$$F^{(k)} = kF$$

and we get a new Kähler form $\omega^{(k)} = k\omega$. The Hamiltonian vector field with respect to the Kähler form $\omega^{(k)}$ is then given by

$$X_f^{(k)} = \frac{1}{k} X_f$$

and we define associated pre-quantum operators by

$$\tilde{\sigma}_{k,f} = k\left(2\pi f + i\nabla^{(k)}_{X_f^{(k)}}\right) = 2\pi k f + i\nabla^{(k)}_{X_f}.$$

This operator acts on the space of smooth sections of L^k and we need to compose it with the orthogonal projection $\Pi^{(k)}$ onto the space of holomorphic sections with respect to the inner product given by

$$\langle s_1, s_2\rangle_k = \int_X h^k\left(s_1, s_2\right) \frac{\omega^n}{n!}.$$

This allows us to define quantum operators

$$\sigma_{k,f} = \Pi^{(k)} \circ P_{k,f}.$$

The next proposition tells us that the Hilb_k-map itself can be thought of as some *curved version* of geometric quantisation.

Proposition 4.4.2. *Let $f \in C^\infty(X, \mathbb{R})$ and $\phi \in \mathscr{H}$. Then*

$$d\left(Hilb_k\right)_\phi (f) = 2\,\sigma_{k,f}$$

where all the objects in $\sigma_{k,f}$ are computed with respect to ϕ.

Proof. From lemma 4.4.1 we readily see that the matrix of $\sigma_{k,f}$ with respect to an $\mathrm{Hilb}_k(\phi)$-orthonormal basis $\underline{s}$ of $H^0\left(X, L^k\right)$ is given by

$$\left(\sigma_{k,f}\right)_{\alpha\beta} = \int_X (2\pi k f + \frac{1}{2}\Delta f)\, h^k(s_\alpha, s_\beta) \frac{\omega^n}{n!}.$$

Comparing this with the differential of Hilb_k yields the result. □

4.5 Relation between Berezin quantisation and the FS_k-map

We start recalling the definition of Berezin's covariant symbol. We follow the presentation of Cahen, Gutt and Rawnsley in [20].

Let q be a non-zero point in the fibre of L over $x \in X$. Evaluation of a section $s \in H^0(X,L)$ defines a multiple $\ell_q(s)$ of q given by

$$s(x) = \ell_q(s)q.$$

Fixing $b \in \mathscr{B}_k$, ℓ_q is a continuous linear form on the Hilbert space $\left(H^0(X,L), b\right)$ so that by the Riesz representation theorem, there exist an element $e_q \in H^0(X,L)$ such that for any $s \in H^0(X,L)$,

$$\ell_q(s) = \langle s, e_q \rangle_b$$

The section e_q is called a *coherent state*.

Definition 4.5.1. To every endomorphism A of $H^0(X,L)$ one associates a *covariant symbol* $\hat{A} \in C^\infty(X,\mathbb{C})$ by

$$\hat{A}(x) = \frac{\langle Ae_q, e_q \rangle}{\|e_q\|^2}.$$

This definition is actually independent of the point q we chose in the fibre over x. In fact, for $c \in \mathbb{C}^*$,

$$\ell_{cq}(s) = c^{-1}\ell_q(s) \quad \text{and} \quad e_{cq} = c^{-1}\bar{e}_q.$$

As before, we can do this construction for tensor powers of L and get covariant symbols $\hat{A}_k$.

Proposition 4.5.2. *The covariant symbol $\hat{A}_k$ of an Hermitian endomorphism A of $H^0\left(X,L^k\right)$ is given by the derivative of the Fubini–Study map at the point b in the direction A. Explicitly*

$$\hat{A}(x) = 4\pi k \left(dFS_k\right)_b (A)(x).$$

Proof. Pick a b-orthonormal basis $\underline{s}$ of $H^0\left(X,L^k\right)$ and write

$$e_q = \sum_\alpha e_q^\alpha s_\alpha.$$

Clearly $e_q^\alpha = \langle s_\alpha, e_q \rangle = \ell_q(s_\alpha)$ and thus the coordinates of e_q is the basis $\underline{s}$ define a point $[s_1(x), \ldots, s_{N_k}(x)]$ in projective space. Since the formula defining $\hat{A}$ does not depend on the point q in the fibre over x, we can use the notation of the $s_\alpha(x)$ to do the following computation:

$$\begin{aligned}
\hat{A}(x) &= \frac{\langle Ae_q, e_q \rangle}{\|e_q\|^2} \\
&= \frac{\langle A \sum_\alpha s_\alpha(x) s_\alpha, \sum_\beta s_\beta(x) s_\beta \rangle}{\sum_\gamma |s_\gamma(x)|^2} \\
&= \frac{\sum_{\alpha\beta} \langle As_\alpha, s_\beta \rangle s_\alpha(x) \overline{s_\beta(x)}}{\sum_\gamma |s_\gamma(x)|^2} \\
&= \frac{\sum_{\alpha\beta} A_{\alpha\beta} s_\alpha(x) \overline{s_\beta(x)}}{\sum_\gamma |s_\gamma(x)|^2} \\
&= 4\pi \mathrm{Tr}\left(A\mu_{\underline{s}}(x)\right)
\end{aligned}$$

□

4.6 The Hessian of Hilb$_k$

Guided by the observation that the derivative of Hilb$_k$ is geometric quantisation, we compute the next order approximation of the Hilb$_k$-map in this section.

Definition 4.6.1. Let (M, g_M) and (N, g_N) be two Riemannian manifolds and let $\varphi : M \to N$ be a smooth map between M and N. The *Hessian* of φ is defined to be $\nabla d\varphi$.

A first thing to note about this definition is that $d\varphi \in \Gamma(M, T^*M \otimes \varphi^*TN)$ and ∇ is the connection on $T^*M \otimes \varphi^*TN$ induced by the Levi-Civita connections ∇^M and ∇^N on M and N respectively. Explicitely,

$$(\nabla d\varphi)(X, Y) = \nabla^\varphi_X (d\varphi(Y)) - d\varphi\left(\nabla^M_X Y\right), \tag{4.6.1}$$

where ∇^φ denotes the pull-back connection.

Theorem 4.6.2. *The Hessian of Hilb$_k$: $\mathscr{H} \to \mathscr{B}_k$ admits an asymptotic expansion in which the leading order term is given by the leading order of the Toeplitz operator associated to the function $(\mathscr{D}f, \mathscr{D}g)$. More precisely, as $k \to \infty$,*

$$(\nabla dHilb_k)_\phi (f, g) = T_{(\mathscr{D}f, \mathscr{D}g)} + O\left(k^{n-1}\right).$$

We postpone the proof to section 4.6.2 and first discuss one of its applications.

4.6.1 Application

Our starting point is the following lemma.

Lemma 4.6.3. *Let $\varphi : (M, g_M) \to (N, g_N)$ and $\psi : (N, g_N) \to (P, g_P)$ be two smooth functions between Riemannian manifolds. The Hessian of the composition $\psi \circ \varphi$ is given by*

$$\nabla d(\psi \circ \varphi) = d\psi(\nabla d\varphi) + \nabla d\psi(d\varphi, d\varphi).$$

Proof.

$$\begin{aligned}(\nabla d(\psi \circ \varphi))(X,Y) &= \nabla_X (d(\psi \circ \varphi)(Y)) - d(\psi \circ \varphi)(\nabla_X Y) \\ &= \left(\nabla_{d\varphi(X)}(d\psi)\right)(d\varphi(Y)) + d\psi(\nabla_X (d\varphi(Y))) - d\psi(d\varphi(\nabla_X Y)) \\ &= (\nabla d\psi)(d\varphi(X), d\varphi(Y)) + d\psi(\nabla d\varphi(X,Y))\end{aligned}$$

□

Let $F_k : \mathscr{B}_k \to \mathbb{R}$ be *any* functional on Bergman space. Using the Hilb_k-map, F_k can be pulled-back to $\mathscr{H}$ to get a functional $E_k : \mathscr{H} \to \mathbb{R}$. Using lemma 4.6.3, we can compute the pullback by Hilb_k of the Hessian of F_k in terms of the Hessians of Hilb_k and E_k.

$$\nabla dF_k(d\mathrm{Hilb}_k, d\mathrm{Hilb}_k) = \nabla dE_k - dF_k(\nabla d\mathrm{Hilb}_k).$$

A very interesting functional on $\mathscr{B}_k$ is *balancing energy*. It can be defined via its derivative by putting for $b \in \mathscr{B}_k$ and $A \in i\mathfrak{u}(N_k+1)$,

$$(dF_k)_b(A) = \mathrm{Tr}(A\bar{\mu}_b)$$

where $\bar{\mu}_b = \int_X \mu_b \frac{\omega_{FS}^n}{n!}$. The left hand-side of this expression actually defines a 1-form on $\mathscr{B}_k$ and one needs to check that it is closed so that it can be integrated to get a function on $\mathscr{B}_k$ (well defined up to a constant). Balancing energy can be thought-off as a finite dimensional analogue of yet another functional defined on $\mathscr{H}$ which is called *Mabuchi energy*. Similar to what we did for balancing energy, Mabuchi energy can be defined via its derivative by

$$(dE)_\phi(f) = \int_X fS(\phi)\frac{\omega_\phi^n}{n!}.$$

where $\phi \in \mathscr{H}$ and $f \in C^\infty(X, \mathbb{R})$. Again one has to check that the left-hand side defines a closed 1-form to get a function on $\mathscr{H}$. Using the asymptotic expansion for the Hessian of Hilb_k we got in theorem 4.6.2, we recover the following theorem proved by Fine in [11].

Theorem 4.6.4. *Under the map $Hilb_k : \mathscr{H} \to B_k$ the pull-back of the Hessian of balancing energy admits an asymptotic expansion in which the leading order term is the Hessian of Mabuchi energy. More precisely, for any $f, g \in C^\infty(X, \mathbb{R})$ there is an asymptotic expansion as $k \to \infty$ given by*

$$(\nabla dF_k)_{Hilb_k(\phi)}\left((dHilb_k)_\phi(f), (dHilb_k)_\phi(g)\right) = \frac{k^n}{4\pi}\int_X f\mathscr{D}^*\mathscr{D}g\frac{\omega_\phi^n}{n!} + O\left(k^{n-1}\right).$$

Proof. Using theorem 4.6.2 and the formula for the differential of balancing energy, the second term of the right-hand side of formula 4.6.1 is given by,

$$\begin{aligned}(dF_k)_{\mathrm{Hilb}_k(\phi)}(\nabla d\mathrm{Hilb}_k)_\phi(f,g) &= \mathrm{Tr}\left((\nabla d\mathrm{Hilb}_k)_\phi(f,g)\,\bar{\mu}_k\right)\\ &= \sum_{\alpha\beta}\int_{X\times X}(\mathscr{D}f, \mathscr{D}g)\frac{(s_\alpha, s_\beta)(s_\beta, s_\alpha)}{4\pi\rho_k}\frac{\omega^n\wedge\omega_k^n}{n!^2}.\end{aligned}$$

Using the asymptotics for the Bergman kernels from section 2.3.2 and the fact that the operator $\mathscr{D}$ is self-adjoint, one easily sees that the leading order term in this expression is given by

$$\frac{k^n}{4\pi}\int_X f\mathscr{D}^*\mathscr{D}g\frac{\omega^n}{n!} \tag{4.6.2}$$

On the other hand, we know from lemma 2 in Donaldson's paper [8] that the differential of $E_k = F_k \circ \mathrm{Hilb}_k$ is given by

$$(dE_k)_\phi(f) = \int_X (4\pi k\rho_k + \Delta\rho_k) f\frac{\omega^n}{n!}.$$

This expression admits an asymptotic expansion

$$(dE_k)_\phi(f) = (\alpha_0)_\phi(f)k^{n+1} + (\alpha_1)_\phi(f)k^n + O\left(k^{n-1}\right).$$

Using the asymptotic expansion of the Bergman function from section 2.3.2, one sees that

$$\begin{aligned}(\alpha_0)_\phi(f) &= 4\pi k^{n+1}\int_X f\frac{\omega^n}{n!}\\ (\alpha_1)_\phi(f) &= \frac{k^n}{2\pi}\int_X fS\frac{\omega^n}{n!}\end{aligned}$$

These two quantities define 1-forms on $\mathscr{H}$ which by construction can be integrated up and define functionals AY and E up to constants. AY is sometimes called the Aubin–Yau functional and E is nothing else that Mabuchi energy (up to the $k^n/2\pi$-factor). The Hessian of AY computed with respect to the Donaldson–Mabuchi–Semmes metric vanishes, whereas the Hessian of Mabuchi energy is given by $\mathscr{D}^*\mathscr{D}$. To sum up, we get that the leading order term of the right-hand side in 4.6.1 is given by

$$\frac{k^n}{2\pi}\int_X f\mathscr{D}^*\mathscr{D}g\,\frac{\omega^n}{n!} \tag{4.6.3}$$

Putting the terms 4.6.2 and 4.6.3 together yields the result.

□

Remark 4.6.5. Note that in our example, we used balancing energy as functional on B_k, but the technique works in principle for any other functional on $\mathscr{B}_k$ as well.

4.6.2 Proof of theorem 4.6.2

We start recalling some standard formulas from Kähler geometry of which a proof can be found for example in [22].

Lemma 4.6.6. *Let $h_t = e^{4\pi f t}h$ be a path in $\mathscr{H}$. We have:*

1. $\omega_t = \omega_0 + t2i\bar{\partial}\partial f$;
2. $2ni\bar{\partial}\partial f\wedge\omega^{n-1} = \Delta f\omega^n$;
3. $n(n-1)\,2i\bar{\partial}\partial f\wedge 2i\bar{\partial}\partial g\wedge\omega^{n-2} = \Delta f\Delta g - \langle 2i\bar{\partial}\partial f, 2i\bar{\partial}\partial g\rangle$.

Throughout the proof, we work at a point $\phi\in\mathscr{H}$ and all quantities are computed with respect to ϕ except mentioned otherwise. We fix $\underline{s} = s_1,\dots,s_{N_k}$ to be a $\mathrm{Hilb}_k(\phi)$-orthonormal basis of $H^0\left(X,L^k\right)$. For $\psi\in\mathscr{H}$, the matrix of the inner product $\mathrm{Hilb}_k(\psi)$ with respect to $\underline{s}$ is given by

$$\left(\mathrm{Hilb}_k(\psi)\right)_{\alpha\beta} = \int_X h^k_\psi\left(s_\alpha,s_\beta\right)\frac{\omega^n_\psi}{n!}.$$

Note that by definition $\left(\mathrm{Hilb}_k(\phi)\right)_{\alpha\beta} = \delta_{\alpha\beta}$.

Using formula (4.6.1) and the expression of the connection on $\mathscr{B}_k$ described in section 4.2 we get for $f,g\in T_\phi\mathscr{H}\cong C^\infty(X,\mathbb{R})$,

$$\nabla\left(d\mathrm{Hilb}_k\right)_\phi(f,g) = f\cdot\left(d\mathrm{Hilb}_k\right)_\phi(g) - \frac{1}{2}[[\left(d\mathrm{Hilb}_k\right)_\phi(f),\left(d\mathrm{Hilb}_k\right)_\phi(g)]] \\ -\left(d\mathrm{Hilb}_k\right)\phi\left(D^{LC}_f(g)\right).$$

Here $[[\cdot,\cdot]]$ denotes the anti-commutator $[[A,B]] = AB + BA$.

We first compute the three terms of the right-hand side separately and then combine them.

Lemma 4.6.7. *Let $f,g \in C^\infty(X,\mathbb{R})$ and put $\phi(s,t) = fs + gt$. The second variation*

$$\partial_t \partial_s \left(Hilb_k \left(\phi(s,t) \right) \right)|_{s=0}|_{t=0}$$

of $Hilb_k(\phi(s,t))$ is given by

$$16\pi^2 k^2\, T_{k,fg} + 4\pi k\, T_{k,f\Delta g + g\Delta f} + T_{k,\Delta f \Delta g - \left(2i\bar\partial\partial f, 2i\bar\partial\partial g\right)}.$$

Proof. First recall that given $\phi(s,t) \in \mathscr{H}$, the associated family of Hermitian metrics on L is given by

$$h_{\phi(s,t)} = e^{4\pi(fs+gt)} h_\phi.$$

We compute

$$\begin{aligned}
&\partial_t \partial_s \left(\mathrm{Hilb}_k \left(\phi(s,t) \right) \right)|_{s=0}|_{t=0} \\
&= \partial_t \partial_s \left(\int_X h^k_{\phi(s,t)} \left(s_\alpha, s_\beta \right) \frac{\omega^n_{\phi(s,t)}}{n!} \right) \Big|_{s=0} \Big|_{t=0} \\
&= \partial_t \left(\int_X 4\pi k f h^k_{\phi(0,t)} \left(s_\alpha, s_\beta \right) \frac{\omega^n_{\phi(0,t)}}{n!} + \int h^k_{\phi(0,t)} \left(s_\alpha, s_\beta \right) \frac{2ni\bar\partial\partial f \wedge \omega^{n-1}_{\phi(0,t)}}{n!} \right) \Big|_{t=0} \\
&= \int_X 16\pi^2 k^2 fg\, h^k_\phi \left(s_\alpha, s_\beta \right) \frac{\omega^n_\phi}{n!} \\
&\qquad + \int_X 4\pi k f\, h^k_\phi \left(s_\alpha, s_\beta \right) \frac{2ni\bar\partial\partial g \wedge \omega^{n-1}_\phi}{n!} + \int_X 4\pi k g\, h^k_\phi \left(s_\alpha, s_\beta \right) \frac{2ni\bar\partial\partial f \wedge \omega^{n-1}_\phi}{n!} \\
&\qquad\quad + \int_X h^k_\phi \left(s_\alpha, s_\beta \right) \frac{n(n+1) 2i\bar\partial\partial f \wedge 2i\bar\partial\partial g \wedge \omega^{n-2}_\phi}{n!}.
\end{aligned}$$

Using lemma 4.6.6 this can be rewritten as

$$\begin{aligned}
16\pi^2 k^2 \int_X fg\, h^k_\phi \left(s_\alpha, s_\beta \right) \frac{\omega^n_\phi}{n!} + 4\pi k \int_X \left(f\Delta g + g\Delta f \right) h^k_\phi \left(s_\alpha, s_\beta \right) \frac{\omega^n_\phi}{n!} \\
+ \int_X \left(\Delta f \Delta g - \langle 2i\bar\partial\partial f, 2i\bar\partial\partial g \rangle \right) h^k_\phi \left(s_\alpha, s_\beta \right) \frac{\omega^n_\phi}{n!}
\end{aligned}$$

which proves the lemma.

$\square$

Lemma 4.6.8. *For $f, g \in C^\infty(X, \mathbb{R})$, we have*

$$(dHilb_k)_\phi \left(D_f^{LC}(g)\right) = 4\pi k\, T_{k,(df,dg)} + T_{k,\Delta(df,dg)}$$

Proof. This follows from the expression of the differential of Hilb_k and the Levi-Civita connection on $\mathscr{H}$. In fact since f and g can be thought of as constant vector fields on $\mathscr{H}$ we have that $D_f^{LC}(g) = (df, dg)$.

□

Lemma 4.6.9. *Let $f, g \in C^\infty(X, \mathbb{R})$. There is an asymptotic expansion of the anti-commutator $[[(dHilb_k)_\phi(f), (dHilb_k)_\phi(g)]]$ given by*

$$k^2\, T_{k,\eta_0(f,g)} + k\, T_{k,\eta_1(f,g)} + T_{k,\eta_2(f,g)} + O(k^{-1})$$

where

$$\begin{aligned}
\eta_0(f,g) &= 32\pi^2 fg \\
\eta_1(f,g) &= 8\pi\left(f\Delta g + g\Delta f)\right) - 8\pi\left(\langle\partial f, \bar\partial g\rangle + \langle\partial g, \bar\partial f\rangle\right) \\
\eta_2(f,g) &= 2\left(\langle D^{1,0}\partial f, D^{0,1}\bar\partial g\rangle + \langle D^{1,0}\partial g, D^{0,1}\bar\partial f\rangle\right) + 4\langle Ric, i\partial f \wedge \bar\partial g + i\partial g \wedge \bar\partial f\rangle \\
&\quad - 2\left(\langle\partial f, \bar\partial(\Delta g)\rangle + \langle\partial(\Delta f), \bar\partial g\rangle + \langle\partial g, \bar\partial(\Delta f)\rangle + \langle\partial(\Delta g), \bar\partial f\rangle\right) + 2\Delta f \Delta g.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&(d\mathrm{Hilb}_k)_\phi(f) \circ (d\mathrm{Hilb}_k)_\phi(g) \\
&\qquad = T_{k,4\pi k f + \Delta f} \circ T_{k,4\pi k g + \Delta g} \\
&\qquad = 16\pi^2 k^2\, T_{k,f} \circ T_{k,g} + 4\pi k\left(T_{k,f} \circ T_{k,\Delta g} + T_{k,\Delta f} \circ T_{k,g}\right) + T_{k,\Delta f} \circ T_{k,\Delta g}
\end{aligned}$$

Using the notation from theorem 2.3.8 we can rewrite this as

$$\begin{aligned}
&16\pi^2 k^2\left(T_{k,C_0(f,g)} + k^{-1} T_{k,C_1(f,g)} + k^{-2} T_{k,C_2(f,g)} + O(k^{-3})\right) \\
&\qquad + 4\pi k\left(T_{k,C_0(f,\Delta g)+C_0(\Delta f,g)} + k^{-1} T_{k,C_1(f,\Delta g)+C_1(\Delta f,g)} + O(k^{-2})\right) \\
&\qquad\qquad + T_{k,C_0(\Delta f,\Delta g)} + O(k^{-1})
\end{aligned}$$

Putting the terms of the same order together gives

$$k^2\, T_{k,\sigma_0(f,g)} + k\, T_{k,\sigma_1(f,g)} + T_{k,\sigma_3(f,g)} + O(k^{-1}),$$

where

$$\sigma_0(f,g) = 16\pi^2 C_0(f,g)$$
$$\sigma_1(f,g) = 16\pi^2 C_1(f,g) + 4\pi\left(C_0(f,\Delta g) + C_0(\Delta f, g)\right)$$
$$\sigma_2(f,g) = 16\pi^2 C_2(f,g) + 4\pi\left(C_1(f,\Delta g) + C_1(\Delta f, g)\right) + C_0(\Delta f,\Delta g).$$

Using the exact values of the coefficients from theorem 2.3.8 these can be rewritten as

$$\sigma_0(f,g) = 16\pi^2 fg$$
$$\sigma_1(f,g) = -8\pi\langle\partial f,\bar\partial g\rangle + 4\pi\left(f\Delta g + g\Delta f)\right)$$
$$\sigma_2(f,g) = 2\langle D^{1,0}\partial f, D^{0,1}\bar\partial g\rangle + 4\langle \mathrm{Ric}, i\partial f\wedge\bar\partial g\rangle - 2\left(\langle\partial f,\bar\partial(\Delta g)\rangle + \langle\partial(\Delta f),\bar\partial g\rangle\right) + \Delta f\Delta g.$$

By symmetry we deduce that the term $[[(d\mathrm{Hilb}_k)_\phi(f),(d\mathrm{Hilb}_k)_\phi(g)]]$ is given by

$$k^2 T_{k,\eta_0(f,g)} + kT_{k,\eta_1(f,g)} + T_{k,\eta_3(f,g)} + O(k^{-1})$$

where

$$\eta_0(f,g) = 32\pi^2 fg$$
$$\eta_1(f,g) = 8\pi\left(f\Delta g + g\Delta f)\right) - 8\pi\left(\langle\partial f,\bar\partial g\rangle + \langle\partial g,\bar\partial f\rangle\right)$$
$$\eta_2(f,g) = 2\left(\langle D^{1,0}\partial f, D^{0,1}\bar\partial g\rangle + \langle D^{1,0}\partial g, D^{0,1}\bar\partial f\rangle\right) + 4\langle \mathrm{Ric}, i\partial f\wedge\bar\partial g + i\partial g\wedge\bar\partial f\rangle$$
$$-2\left(\langle\partial f,\bar\partial(\Delta g)\rangle + \langle\partial(\Delta f),\bar\partial g\rangle + \langle\partial g,\bar\partial(\Delta f)\rangle + \langle\partial(\Delta g),\bar\partial f\rangle\right) + 2\Delta f\Delta g.$$

□

End of the proof of theorem 4.6.2.

Putting lemma 4.6.7, 4.6.8 and 4.6.9 together we get

$$\begin{aligned}(\nabla d\mathrm{Hilb}_k)_\phi(f,g) &= f\cdot(d\mathrm{Hilb}_k)_\phi(g) - (d\mathrm{Hilb}_k)_\phi\left(D^{LC}_f(g)\right) - \frac{1}{2}[[(d\mathrm{Hilb}_k)_\phi(f),(d\mathrm{Hilb}_k)_\phi(g)]]\\ &= 16\pi^2k^2 T_{k,fg} + 4\pi k T_{k,f\Delta g + g\Delta f} + T_{k,\Delta f\Delta g - \langle 2i\bar\partial\partial f, 2i\bar\partial\partial g\rangle}\\ &\quad - 4\pi k T_{k,(df,dg)} - T_{k,\Delta(df,dg)}\\ &\quad - 16\pi^2k^2 T_{k,fg} + 4\pi k T_{\langle\partial f,\bar\partial g\rangle + \langle\partial g,\bar\partial f\rangle} - 4\pi k T_{k,f\Delta g + g\Delta f} - \frac{1}{2}T_{k,\eta_2(f,g)}.\end{aligned}$$

One readily sees that the terms in k^2 vanish. Furthermore since

$$(df,dg) = \langle \partial f, \bar\partial g\rangle + \langle \partial g, \bar\partial f\rangle$$

we see that the term in k vanishes too so that the leading order is the leading order of the Toeplitz operator applied to the function

$$\begin{aligned}
&\Delta f \Delta g - \langle 2i\bar\partial\partial f, 2i\bar\partial\partial g\rangle - \Delta(df,dg)\\
&- \left(\langle D^{1,0}\partial f, D^{0,1}\bar\partial g\rangle + \langle D^{1,0}\partial g, D^{0,1}\bar\partial f\rangle\right) - 2\langle \mathrm{Ric}, i\partial f\wedge\bar\partial g + i\partial g\wedge\bar\partial f\rangle\\
&\quad + \left(\langle \partial f, \bar\partial(\Delta g)\rangle + \langle \partial(\Delta f), \bar\partial g\rangle + \langle \partial g, \bar\partial(\Delta f)\rangle + \langle \partial(\Delta g), \bar\partial f\rangle\right) - \Delta f\Delta g.
\end{aligned}$$

Formula (5.79) in [19] tells us that

$$\Delta\langle \partial f, \bar\partial g\rangle = \langle \partial\Delta f, \bar\partial g\rangle + \langle \partial f, \bar\partial\Delta g\rangle - 2\langle \nabla^{T^*X}\partial f, \nabla^{T^*X}\bar\partial g\rangle - 2\langle \mathrm{Ric}, i\partial f\wedge\bar\partial g\rangle.$$

The curvature term in the right-hand side comes from the non-commutativity of the Bochner-Laplacian with the ∂ and $\bar\partial$-operators.

Applying this formula to the term

$$\Delta(df,dg) = \Delta\left(\langle \partial f, \bar\partial g\rangle + \langle \partial g, \bar\partial f\rangle\right)$$

enables us to rewrite the leading order as

$$\begin{aligned}
&-\langle 2i\bar\partial\partial f, 2i\bar\partial\partial g\rangle + 2\left(\langle \nabla^{T^*X}\partial f, \nabla^{T^*X}\bar\partial g\rangle + \langle \nabla^{T^*X}\partial g, \nabla^{T^*X}\bar\partial f\rangle\right)\\
&\quad - \left(\langle D^{1,0}\partial f, D^{0,1}\bar\partial g\rangle + \langle D^{1,0}\partial g, D^{0,1}\bar\partial f\rangle\right)
\end{aligned}$$

Observing that

$$\langle \nabla^{T^*X}\partial f, \nabla^{T^*X}\bar\partial g\rangle = \langle D^{1,0}\partial f, D^{0,1}\bar\partial g\rangle + \langle D^{0,1}\partial f, D^{1,0}\bar\partial g\rangle$$

and

$$\langle 2i\bar\partial\partial f, 2i\bar\partial\partial g\rangle = 2\left(\langle D^{0,1}\partial f, D^{1,0}\bar\partial g\rangle + \langle D^{0,1}\partial g, D^{1,0}\bar\partial f\rangle\right)$$

the leading order term simplifies to

$$\langle D^{1,0}\partial f, D^{0,1}\bar\partial g\rangle + \langle D^{1,0}\partial g, D^{0,1}\bar\partial f\rangle.$$

In terms of the Lichnerowicz operator $\mathscr{D} : C^\infty(X, \mathbb{R}) \to \Omega^{0,1}(TX)$ this can be written as

$$(\mathscr{D} f, \mathscr{D} g).$$

To see this, note that $D^{0,1}\bar{\partial} f$ is a section of $(T^*X)^{0,1} \otimes (T^*X)^{0,1}$. Using the metric, we can identify $\bar{\partial} f$ with an element of the holomorphic tangent bundle $(TX)^{1,0}$. Furthermore, on $(TX)^{1,0}$ the $(0,1)$-part of the covariant derivative coincides with the usual $\bar{\partial}$-operator. Hence under this identification, $D^{0,1}\bar{\partial} f = \bar{\partial}\left(\mathrm{grad}^{1,0} f\right)$. Moreover, $(TX)^{1,0}$ is canonically isomorphic to (TX, J) and hence we can think of $D^{0,1}\bar{\partial} f$ as a section of $\Omega^{0,1}(TX)$. Finally, under the isomorphism $(TX)^{1,0} \cong (TX, J)$, one has that

$$(\cdot, \cdot) = 2\langle \cdot, \cdot \rangle|_{(TX)^{1,0}}.$$

Observing that $\bar{\partial}$ commutes with J since X is Kähler concludes the proof.

References

[1] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[2] T. Bouche. Convergence de la métrique de Fubini-Study d'un fibré linéaire positif. *Ann. Inst. Fourier (Grenoble)*, 40(1):117–130, 1990.

[3] D. Catlin. The Bergman kernel and a theorem of Tian. In *Analysis and geometry in several complex variables (Katata, 1997)*, Trends Math., pages 1–23. Birkhäuser Boston, Boston, MA, 1999.

[4] J.-P. Demailly. *Complex analytic and differential geometry*. https://www-fourier.ujf-grenoble.fr/ $\sim$ demailly/books.html.

[5] S. K. Donaldson. Remarks on gauge theory, complex geometry and 4-manifold topology. 5:384–403, 1997.

[6] S. K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 13–33. Amer. Math. Soc., Providence, RI, 1999.

[7] S. K. Donaldson. Scalar curvature and projective embeddings. I. *J. Differential Geom.*, 59(3):479–522, 2001.

[8] S. K. Donaldson. Scalar curvature and projective embeddings. II. *Q. J. Math.*, 56(3):345–356, 2005.

[9] S. K. Donaldson. Some numerical results in complex differential geometry. *Pure Appl. Math. Q.*, 5(2, Special Issue: In honor of Friedrich Hirzebruch. Part 1):571–618, 2009.

[10] J. Fine. Calabi flow and projective embeddings. *J. Differential Geom.*, 84(3):489–523, 2010. With an appendix by Kefeng Liu and Xiaonan Ma.

[11] J. Fine. Quantization and the Hessian of Mabuchi energy. *Duke Math. J.*, 161(14):2753–2798, 2012.

[12] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[13] E. L. Grinberg. Spherical harmonics and integral geometry on projective spaces. *Trans. Amer. Math. Soc.*, 279(1):187–203, 1983.

[14] J. Keller, J. Meyer, and R. Seyyedali. Quantization of the Laplacian operator on vector bundles, I. *Math. Ann.* To appear.

[15] S. Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987. Kanô Memorial Lectures, 5.

[16] B. Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[17] Z. Lu. On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch. *Amer. J. Math.*, 122(2):235–273, 2000.

[18] X. Ma and G. Marinescu. *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[19] X. Ma and G. Marinescu. Berezin-Toeplitz quantization on Kähler manifolds. *J. Reine Angew. Math.*, 662:1–56, 2012.

[20] J. Rawnsley, M. Cahen, and S. Gutt. Quantization of Kähler manifolds. I. Geometric interpretation of Berezin's quantization. *J. Geom. Phys.*, 7(1):45–62, 1990.

[21] J.-M. Souriau. *Structure des systèmes dynamiques*. Maîtrises de mathématiques. Dunod, Paris, 1970.

[22] G. Székelyhidi. *An introduction to extremal Kähler metrics*, volume 152 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2014.

[23] X. Wang. Balance point and stability of vector bundles over a projective manifold. *Math. Res. Lett.*, 9(2-3):393–411.

[24] X. Wang. Canonical metrics on stable vector bundles. *Comm. Anal. Geom.*, 13(2):253–285.

[25] S. Zelditch. Szego kernels and a theorem of Tian. *Internat. Math. Res. Notices*, (6):317–331.