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Integral equation formulation of the spinless Salpeter equation

F. Brau[a),b)]
Université de Mons-Hainaut, Place du Parc 20, B-7000 Mons, Belgique

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The spinless Salpeter equation presents a rather particular differential operator. In this paper we rewrite this equation into integral and integro-differential equations. These kinds of equations are well known and can be more easily handled. We also present some analytical results concerning the spinless Salpeter equation and the action of the square-root operator. © *1998 American Institute of Physics.* [S0022-2488(98)00403-4]

I. INTRODUCTION

The Schrödinger equation is a very well defined partial derivative equation since its differential operator is a Laplacian. Moreover it reduces to a simple differential equation when the interaction is central. This equation has been intensively studied and it is well understood. A simple relativistic version of the Schrödinger equation, the spinless Salpeter equation (SSE), presents more difficulties. This equation is

$$\left(\sqrt{\mathbf{p}^2+m_1^2}+\sqrt{\mathbf{p}^2+m_2^2}\right)\Psi(\mathbf{r})=(E-V(\mathbf{r}))\Psi(\mathbf{r}), \tag{1}$$

where m_1, m_2 are the masses of the particles, $\mathbf{p}$ is their relative momentum, $V(\mathbf{r})$ is the potential interaction and E the eigenenergy of the stationary state $\Psi(\mathbf{r})$ $(\hbar=c=1)$. $\mathbf{p}$ and $\mathbf{r}$ are conjugate variables. Actually, this last equation is not so well defined since the kinetic energy operator is a nonlocal one. Its action on a function $f(\mathbf{r})$ is known only if $f(\mathbf{r})$ is an eigenfunction of the operator $\mathbf{p}^2$. In this case we obtain

$$\sqrt{\mathbf{p}^2+m^2}f(\mathbf{r})=\sqrt{\alpha+m^2}f(\mathbf{r}), \tag{2}$$

where α is the corresponding eigenvalue of $\mathbf{p}^2$. Consequently this operator is difficult to handle. However the SSE is not a marginal equation. The correct description of the bound states of two particles is achieved with the Bethe–Salpeter equation.[1] This last reduces to the SSE (Ref. 2, p. 297) when the following occurs.

- The elimination of any dependences on timelike variables is performed (which leads to the Salpeter equation[3]).
- Any references to the spin degrees of freedom of particles are neglected as well as negative energy solutions.

Despite the presence of a so particular operator, the SSE is often used (see for instance Refs. 4–10) since its numerical resolution is very easy for bound states (see for example Refs. 9,11–13). However, it would be interesting to reformulate this equation in a way to better understand it or even to better use it. The main problem is the nonlocality hidden inside the kinetic energy operator. The idea is thus to extract this nonlocality to put it into evidence. That is why in this work we rewrite the SSE as integral or integro-differential equations where the nonlocality is then explicit. Moreover these kinds of equations are well known and defined. Another integral equation, different from those presented here, was found in Ref. 11. This formalism allows us to define the action of the square-root operator on wave functions and calculate the resulting functions.

[a)]Chercheur I.I.S.N.
[b)]Electronic mail: fabian.brau@umh.ac.be

This paper is organized as follows. In Sec. II, we present the integral and integro-differential equations and we calculate their kernels. In Sec. III, we find some analytical results concerning the action of the square-root operator and solutions of the SSE. At last, we present, in Sec. IV, a summary of this work.

II. INTEGRAL EQUATION FORMULATION

It is possible to rewrite the SSE into different forms. The first one that we present is an integral equation. It is obtained using the Green functions of the kinetic energy operator. The corresponding kernels are different depending on whether the particles are identical or not. The second one is an integro-differential equation. It is obtained taking the square of the square-root operator and is only valid for two identical particles. For these formulations of the SSE only local interactions can be used contrary to the formulation proposed in Ref. 11.

A. First form

In the Introduction we gave the operator expression of the SSE. We are going now to formulate the integral expression of this equation. To obtain it, we use the Green functions of the kinetic energy operator. We first present the equal masses case. Let us consider the following function:

$$G(\Delta)=\frac{1}{(2\pi)^3}\int\frac{\exp(-i\mathbf{p}.\mathbf{\Delta})}{2\sqrt{p^2+m^2}}d\mathbf{p}, \tag{3}$$

with $\mathbf{\Delta}=\mathbf{r}-\mathbf{r}'$ and $\Delta=|\mathbf{\Delta}|$. Then, $G(\Delta)$ is the Green function of the equal masses square-root operator since

$$2\sqrt{\mathbf{p}^2+m^2}G(\Delta)=\frac{1}{(2\pi)^3}\int \exp(-i\mathbf{p}\cdot\mathbf{\Delta})d\mathbf{p}=\delta^3(\mathbf{r}-\mathbf{r}'). \tag{4}$$

The solution of the SSE can be written as

$$\Psi(\mathbf{r})=\Psi_0(\mathbf{r})+\int G(\Delta)(E-V(\mathbf{r}'))\Psi(\mathbf{r}')d\mathbf{r}', \tag{5}$$

with $\sqrt{\mathbf{p}^2+m^2}\Psi_0(\mathbf{r})=0$. The solutions of this last equation are $\Psi_0(\mathbf{r})=0$ and $\Psi_0(\mathbf{r})=i_l(mr)Y_{lm}(\hat{r})$, where $i_l(x)=\sqrt{(\pi/2x)}\,I_{l+1/2}(x)$, $I_\nu(x)$ is a modified Bessel function (Ref. 14, p. 952), $Y_{lm}(\hat{r})$ is a spherical harmonic, $r=|\mathbf{r}|$ and $\hat{r}=\mathbf{r}/r$. In Appendix B we show that $\Psi_0(\mathbf{r})=k_l(mr)Y_{lm}(\hat{r})$ is not a solution, the function $k_l(x)$ being defined as equal to $\sqrt{(2/\pi x)}K_{l+1/2}(x)$, with $K_\nu(x)$ a modified Bessel function (Ref. 14, p. 952). Thus only the vanishing solution is relevant for physical problems. One sees immediately that the utilization of nonlocal potentials is impossible if we want to obtain a 1-dimensional integral equation. Calculating the 3-dimensional integral (3) we obtain

$$G(\Delta)=\frac{m}{4\pi^2\Delta}K_1(m\Delta). \tag{6}$$

If we consider only the case of central potentials, the wave function can be written in the form $\Psi(\mathbf{r})=R_l(r)Y_{lm}(\hat{r})$ where $R_l(r)$ is the radial part of the wave function. If, moreover, we place the z axis along $\mathbf{r}$, the angular dependence of Δ is just given by the angle between the vector $\mathbf{r}'$ and the z axis. Then knowing that (Ref. 15, p. 158)

$$Y_{lm}(0,\varphi)=\sqrt{\frac{2l+1}{4\pi}}\delta_{m0}, \tag{7}$$

Eq. (5) is written as

$$R_l(r)\sqrt{\frac{2l+1}{4\pi}}\delta_{m0}=\frac{m}{4\pi^2}\int_0^\infty r'^2(E-V(r'))R_l(r')dr'\int_0^\pi \sin\,\theta'\,\frac{K_1(m\Delta)}{\Delta}d\theta'\int_0^{2\pi}Y_{lm}(\hat{r}')d\varphi'. \tag{8}$$

We can now perform the integration over the angular variables. We have the well known relations

$$\int_0^{2\pi}Y_{lm}(\hat{r}')d\varphi'=2\pi Y_{lm}(\hat{r}')\,\delta_{m0}, \tag{9}$$

and (Ref. 15, p. 133)

$$Y_{lm}(\theta,\varphi)=\exp(im\varphi)\sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\,\theta), \tag{10}$$

where the functions $P_l^m(x)$ are the Legendre functions (Ref. 14, p. 998). Finally, using Eqs. (8)–(10) and the regularized function $u_l(r)=rR_l(r)$, we can write the spinless Salpeter equation on the form of the following integral equation:

$$u_l(r)=\frac{1}{2\pi}\int_0^\infty \mathcal{G}_l(mr,mr')(E-V(r'))u_l(r')dr', \tag{11}$$

with

$$\mathcal{G}_l(mr,mr')=mrr'\int_0^\pi \sin\,\theta'\,\frac{K_1(m\Delta)}{\Delta}P_l(\cos\,\theta')d\theta', \tag{12}$$

where the functions $P_l(x)$ are the Legendre polynomials (Ref. 14, p. 1025). The functions $\mathcal{G}_l(mr,mr')$ are analytic for each value of l. Performing the transformation $y=\sqrt{r^2+r'^2-2rr'\,\cos\,\theta'}$, we obtain

$$\mathcal{G}_l(mr,mr')=m\int_{|r-r'|}^{r+r'}K_1(my)P_l\left(\frac{r^2+r'^2-y^2}{2rr'}\right)dy. \tag{13}$$

For each value of l, we can find the primitive and thus calculate the kernel. Indeed, we have the following relations (Ref. 16, p. 87):

$$\int x^{\nu+1}K_\nu(x)dx=-x^{\nu+1}K_{\nu+1}(x), \tag{14a}$$

$$\int x^{-\nu+1}K_\nu(x)dx=-x^{-\nu+1}K_{\nu-1}(x). \tag{14b}$$

With these it is easy to show that

$$\int x^\nu K_\mu(x)dx=-x^\nu K_{\mu-1}(x)-(\nu+\mu-1)x^{\nu-1}K_\mu(x)+(\nu+\mu-1)(\nu-\mu-1)\int x^{\nu-2}K_\mu(x)dx. \tag{15}$$

This last relation is useful when one calculates the kernel for $l\geqslant 2$. The modified Bessel functions recursion relation is (Ref. 14, p. 970)

$$K_{\nu+1}(x)=\frac{2\nu}{x}K_\nu(x)+K_{\nu-1}(x). \tag{16}$$

With Eq. (13) we see that the polynomial under the integral is composed only of even powers of the integration variable. Then with relation (15), we can calculate analytically the kernel for each value of l. Another way (more simple) to calculate the kernel is to use the relation derived in Ref. 11,

$$\mathscr{G}_l(x,x')=2^l z^{l+1}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^l \frac{1}{z}[(y-z)^{l/2}K_l(\sqrt{y-z})-(y+z)^{l/2}K_l(\sqrt{y+z})], \tag{17}$$

with $y=x^2+x'^2, z=2xx'$. The kernel has as an expression for $l=0$ and $l=1$,

$$\mathscr{G}_0(x,x')=K_0(|x-x'|)-K_0(x+x'), \tag{18}$$

$$\mathscr{G}_1(x,x')=K_0(|x-x'|)+K_0(x+x')+\frac{1}{xx'}[(x+x')K_1(x+x')-|x-x'|K_1(|x-x'|)]. \tag{19}$$

The expressions (18) and (19) show that kernels decrease exponentially when arguments are important enough ($(x+x')$ and $|x-x'|\gg 1$) since $K_\nu(x)\sim\sqrt{\pi/(2x)}\exp(-x)$ when $|x|\gg 1$ (Ref. 14, p. 961). This behavior indicates that when the masses of particles are large, the nonlocality becomes negligible, the SSE becomes almost local and its spectrum coincides with the Schrödinger spectrum. However the explicit limit cannot be done in this formalism. One can remark that these kernels present a logarithmic singularity for $r=r'$.

We treat now the unequal masses case. After the previous calculations, it is obvious that the Green function for the unequal masses kinetic energy operator is given by

$$\widetilde{G}(\Delta)=\frac{1}{(2\pi)^3}\int\frac{\exp(-i\mathbf{p}.\boldsymbol{\Delta})}{\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}}d\mathbf{p}. \tag{20}$$

This equation becomes, if $m_1\neq m_2$,

$$\widetilde{G}(\Delta)=\frac{1}{(2\pi)^3}\frac{1}{m_1^2-m_2^2}\int(\sqrt{p^2+m_1^2}-\sqrt{p^2+m_2^2})\exp(-i\mathbf{p}.\boldsymbol{\Delta})d\mathbf{p}. \tag{21}$$

The solution of the SSE can be written as a form similar to Eq. (5) and the function $\Psi_0(\mathbf{r})$ is here always null. One can find it easily by acting the operator $(\sqrt{\mathbf{p}^2+m_1^2}-\sqrt{\mathbf{p}^2+m_2^2})$ on the equation $(\sqrt{\mathbf{p}^2+m_1^2}+\sqrt{\mathbf{p}^2+m_2^2})\Psi_0(\mathbf{r})=0$. Extracting the operator $(-\Delta+m^2)$ from Eq. (21), performing the integration over the momenta and using the Green's theorem (see Ref. 11), one can show that

$$\Psi(\mathbf{r})=\frac{m_1}{2\pi^2(m_1^2-m_2^2)}\int\frac{K_1(m_1\Delta)}{\Delta}(-\Delta_{r'}+m_1^2)(E-V(r'))\Psi(\mathbf{r}')d\mathbf{r}'+(m_1\to m_2). \tag{22}$$

Again, placing the z axis along $\mathbf{r}$ and repeating the previous calculation we obtain

$$u_l(r)=\frac{1}{\pi(m_1^2-m_2^2)}\int_0^\infty\widetilde{\mathscr{G}}_l(m_1r,m_1r')(E-V(r'))u_l(r')dr'+(m_1\to m_2), \tag{23}$$

with

$$\widetilde{\mathscr{G}}_l(mr,mr')=\mathscr{G}_l(mr,mr')\left[-\frac{d^2}{dr'^2}+\frac{l(l+1)}{r'^2}+m^2\right]. \tag{24}$$

B. Second form

This second rewriting of the SSE leads to an integro-differential equation. By acting the kinetic energy operator on the left of the equal masses SSE we obtain

$$4(\mathbf{p}^2+m^2)\Psi(\mathbf{r})=E(E-V(\mathbf{r}))\Psi(\mathbf{r})-2\sqrt{\mathbf{p}^2+m^2}V(\mathbf{r})\Psi(\mathbf{r}). \tag{25}$$

Thus, we must calculate the action of the square-root operator on the product of functions $V(\mathbf{r})\Psi(\mathbf{r})$. If we consider only central potentials this product gives a function for which the angular dependence is a spherical harmonic. One can show, with the formalism developed in Ref. 11, that the radial part $g(r)$ of the function $g(r)Y_{lm}(\hat{r})$, resulting from the action of the kinetic energy operator on the function $h(r)Y_{lm}(\hat{r})$, is given by

$$g(r)=\frac{1}{\pi r}\int_0^\infty \tilde{\mathscr{G}}_l(mr,mr')\tilde{h}(r')dr', \tag{26}$$

with $\tilde{h}(x)=xh(x)$. When the right-hand side integral is relevant, this rewriting gives meaning to the action of the kinetic energy operator on a central problem wave function. With this result, one obtains from Eq. (25),

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-m^2+\frac{E(E-V(r))}{4}\right]u_l(r)=\frac{1}{2\pi}\int_0^\infty \tilde{\mathscr{G}}_l(mr,mr')V(r')u_l(r')dr'. \tag{27}$$

This last equation is interesting because this is a kind of nonlocal Klein–Gordon Equation (KGE). This shows how the SSE differs from KGE when a nonvanishing interaction is introduced (the free SSE leads to a the free KGE with a different wave number k). Thus the difference is partially given by the nonlocal right-hand side of Eq. (27).

To show how the square-root operator can be particular, we give a simple example in which we show how the hidden nonlocality of the kinetic energy operator could lead to some problems. Let us consider a square well with a range r_c and a depth $-V_0$. In this case, according to r is less than r_c or not, the right-hand side of equal masses SSE is a constant (E or $E+V_0$) times the wave function. Then one can think that in each area this equation leads to

$$[\Delta+\mathrm{Sgn}(r_c-r)k^2]\Psi(\mathbf{r})=0, \tag{28}$$

with

$$k^2=\frac{(E+V_0)^2}{4}-m^2, \quad \text{if } r<r_c,$$

$$k^2=m^2-\frac{E^2}{4}, \quad \text{if } r>r_c.$$

The continuity conditions, for the interior and exterior solutions, at distance $r=r_c$, fix the wave function and the energy. But this method to resolve the SSE does not take into account the nonlocality of the kinetic operator and we obtain the same kind of solution as the KGE. Actually, the problem is the discontinuity. The potential is not a constant for all values of r and the kinetic operator does not commute with it. Thus the equal masses SSE does not lead to Eq. (28). However, in Eq. (27) the nonlocality is contained in the potential part and we can now write an equation for each area ($r<r_c$ and $r>r_c$). We remark that this equation does not reduce to the free KGE when $r>r_c$ because the right-hand side does not vanish. Thus Eq. (27) leads to different solutions that these obtained with KGE for the same square well.

We conclude this section with some remarks concerning these integral equations. The equation derived in Ref. 11 is, for the equal masses case,

$$[E-V(r)]u_l(r)=\frac{2}{\pi}\int_0^\infty \tilde{\mathscr{G}}_l(mr,mr')u_l(r')dr'. \tag{29}$$

Thus if the potential is just a constant, $V(r)=V$, using this equation and Eq. (27) we obtain

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-m^2+\frac{(E-V)^2}{4}\right]u_l(r)=0. \tag{30}$$

The nonlocality disappears and we find the corresponding Klein–Gordon equation.

It is easy to show, with the operator expression of the SSE, that the free radial solutions are $R_l(r)\propto j_l(kr)$ where the function $j_l(kr)$ is the regular spherical Bessel function defined as

$$j_l(x)=\sqrt{\frac{\pi}{2x}}J_{l+1/2}(x), \tag{31}$$

where the function $J_\nu(x)$ is the Bessel function of the first kind (Ref. 14, p. 951). The wave number k is given by

$$k^2=\left(\frac{1}{4E^2}(m_1^2-m_2^2+E^2)^2-m_1^2\right). \tag{32}$$

In Appendix A we show that

$$\int_0^\infty \mathcal{G}_l(mr,mr')\gamma r' j_l(\gamma r')dr'=\frac{\pi}{\sqrt{\gamma^2+m^2}}\gamma r j_l(\gamma r). \tag{33}$$

Then one can verify that Eqs. (11), (23) and (27) are true for a vanishing potential when γ is replaced by k. In Appendix B, we show that the situation is rather different if one considers the irregular spherical Bessel functions.

III. ANALYTICAL RESULTS

In this section we present some analytical results concerning both the action of the square-root operator and solutions of the SSE. We first calculate the action of the kinetic energy operator on the functions $r^n \exp(-mr)$ ($n\geqslant -1$ and n integer). The Fourier transform, $\tilde{f}(p)$, of these functions are

$$\tilde{f}(p)=(-)^{n+1}\sqrt{\frac{2}{\pi}}\frac{\partial^{n+1}}{\partial m^{n+1}}\frac{1}{p^2+m^2}. \tag{34}$$

Then for each value of n we can calculate the action of the square-root operator. We have, for $n=-1$ and $n=0$,

$$\sqrt{\mathbf{p}^2+m^2}\ \frac{\exp(-mr)}{r}=\frac{2m}{\pi r}K_1(mr), \tag{35}$$

$$\sqrt{\mathbf{p}^2+m^2}\ \exp(-mr)=\frac{4m}{\pi}K_0(mr). \tag{36}$$

We can get two more relations knowing the following integrals (Ref. 14, p. 429):

$$\int_0^\infty \frac{x^{2b+1}\sin(ax)}{(m^2+x^2)^{n+1/2}}\,dx=(-)^{b+1}\frac{\sqrt{\pi}}{2^n\, m^n\, \Gamma\left(n+\frac{1}{2}\right)}\frac{\partial^{\,2b+1}}{\partial\, a^{2b+1}}[a^n\, K_n(ma)], \tag{37}$$

with $a>0$, Re $m>0$, $-1\leqslant b\leqslant n$ and

$$\int_0^\infty \frac{x^{2b+1}\sin(ax)}{(m^2+x^2)^{n+1}}\,dx=(-)^{b+n}\frac{\pi}{2\,n!}\left[\frac{\partial^n}{\partial\,\gamma^n}(\gamma^b\exp(-a\sqrt{\gamma}))\right]_{\gamma=m^2}, \tag{38}$$

with $a>0$, $0\leqslant b\leqslant n$, $|\arg(m^2)|<\pi$. Rewriting these integrals as 3-dimensional integrals, extracting the square-root operator and integrating the resulting integrals one obtains the following formulas:

$$\sqrt{\mathbf{p}^2+m^2}\left\{\frac{1}{r}\left[\frac{\partial^n}{\partial\,\gamma^n}(\gamma^b\,\exp(-r\sqrt{\gamma}))\right]_{\gamma=m^2}\right\}=(-)^{n+1}\frac{2\,n!}{\pi\,m^n\,(2n-1)!!\,r}\frac{\partial^{\,2b+1}}{\partial\,r^{2b+1}}[r^n\,K_n(mr)], \tag{39}$$

and

$$\sqrt{\mathbf{p}^2+m^2}\left\{\frac{1}{r}\frac{\partial^{\,2b+1}}{\partial\,r^{2b+1}}(r^{n+1}\,K_{n+1}(mr))\right\}=(-)^{n+1}\frac{\pi\,m^{n+1}(2n+1)!!}{2\,n!\,r}\times\left[\frac{\partial^n}{\partial\,\gamma^n}(\gamma^b\,\exp(-r\sqrt{\gamma}))\right]_{\gamma=m^2}. \tag{40}$$

These two equations are valid for $0\leqslant b\leqslant n$.

With these results we can find easily some analytical solutions of the SSE with a nonlocal interaction. Let us consider the following equation:

$$2\sqrt{\mathbf{p}^{\,2}+m^2}\,R_0(r)=E\,R_0(r)-\int_0^\infty W(r,r')\,R_0(r')\,dr'. \tag{41}$$

If we choose

$$W(r,r')=\left[\alpha\,\exp(-mr)-\frac{8m}{\pi}\,(\beta+m)\,K_0(mr)\right]\exp(-\beta r'), \tag{42}$$

then using Eq. (36) we obtain the solution

$$R_0(r)\propto\exp(-mr), \tag{43}$$

$$E=\frac{\alpha}{\beta+m}. \tag{44}$$

Thus it is easy to construct a set of solutions with the operator expression of the SSE, we must just consider a well chosen interaction. We can also find analytical solutions using Eq. (29) with a nonlocal interaction:

$$\frac{2}{\pi}\int_0^\infty\mathscr{G}_l(mr,mr')\,u_l(r')\,dr'=E\,u_l(r)-\int_0^\infty W(r,r')\,u_l(r')\,dr'. \tag{45}$$

Choosing, for example, $l=0$ and

$$W(r,r')=-\frac{2}{\pi}\,\mathscr{G}_0(mr,mr')\left[\frac{2\beta}{r'}+(m^2-\beta^2)\right]+\gamma r\,\exp(-\beta r-\alpha r'), \tag{46}$$

then we get the solution

$$u_0(r)\propto r\,\exp(-\beta r), \tag{47}$$

$$E=\frac{\gamma}{(\alpha+\beta)^2}. \tag{48}$$

Again we can construct with this representation of the SSE a set of analytical solutions corresponding to a set of well chosen interactions.

IV. SUMMARY

In this work our purpose was to rewrite the spinless Salpeter equation into forms more easy to handle, since its differential operator is so particular. To perform it we have extracted the hidden nonlocality of the kinetic energy operator to get it really explicit. Thus we have obtained in Sec. II the following equations:

• For $m_1 = m_2$,

$$u_l(r) = \frac{1}{2\pi}\int_0^\infty \mathscr{G}_l(mr,mr')\ (E - V(r'))\ u_l(r')\ dr',$$

and

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - m^2 + \frac{E(E-V(r))}{4}\right] u_l(r) = \frac{1}{2\pi}\int_0^\infty \tilde{\mathscr{G}}_l(mr,mr')\ V(r')\ u_l(r')\ dr',$$

with

$$\tilde{\mathscr{G}}_l(mr,mr') = \mathscr{G}_l(mr,mr')\left[-\frac{d^2}{dr'^2} + \frac{l(l+1)}{r'^2} + m^2\right].$$

• And for $m_1 \neq m_2$,

$$u_l(r) = \frac{1}{\pi(m_1^2 - m_2^2)}\int_0^\infty \tilde{\mathscr{G}}_l(m_1 r, m_1 r')\ (E - V(r'))\ u_l(r')\ dr' + (m_1 \to m_2).$$

The kernel $\mathscr{G}_l(mr,mr')$ was first derived in Ref. 11. It is analytic for each value of angular momentum and is given in Sec II. There are two integral equations and one integro-differential equation. These kind of equations are well known. The problem of a definition, connected to the particular kinetic energy operator, are removed when the spinless Salpeter equation is rewritten into these forms or into the form (29) (derived in Ref. 11). Moreover with this last one we can calculate the action of the square-root operator on any functions of the form $h(r)\ Y_{lm}(\hat{r})$ and find the radial part of the resulting functions. If $g(r)$ is this radial part, we have

$$g(r) = \frac{1}{\pi r}\int_0^\infty \tilde{\mathscr{G}}_l(mr,mr')\ \tilde{h}(r')\ dr',$$

with $\tilde{h}(x) = x\ h(x)$. When the integral in the right-hand side is relevant, this expression gives meaning to the action of the square-root operator on central problem wave functions. Indeed, this last relation allows us to explicitly calculate the resulting functions.

In Sec. III we have found some analytical results concerning the action of the square-root operator on some particular functions. With these results and the integral formalism we have shown how to construct a set of analytical solutions of the SSE with well chosen interactions.

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APPENDIX A: FREE SOLUTIONS OF THE INTEGRAL FORM OF THE SPINLESS SALPETER EQUATION

In this section we show, with the integral equation formalism, that the free radial solutions are the regular spherical Bessel functions. It is easy to prove it with the operator expression of the SSE because one can rewrite it as

$$(\Delta + k^2)\ \Psi(\mathbf{r}) = 0, \tag{A1}$$

where k is given by Eq. (32).

To prove it with the integral formulation, we use the integral expression of the kernel $\mathscr{G}_l(mr,mr')$ [see Eq. (25), Ref. 11],

$$\mathscr{G}_l(mr,mr') = \frac{m^2}{2}\, r\, r' \int_0^\infty \exp\left[-\frac{1}{u} - \frac{m^2}{4}(r^2+r'^2)u\right] i_l\left(\frac{1}{2}m^2 rr'u\right) du. \tag{A2}$$

Then knowing that

$$j_l(x) = \sqrt{\frac{\pi}{2x}}\, J_{l+1/2}(x), \quad i_l(x) = \sqrt{\frac{\pi}{2x}}\, I_{l+1/2}(x), \tag{A3}$$

and (Ref. 14, p. 718)

$$\int_0^\infty x \exp(-\alpha x^2)\, I_\nu(\beta x)\, J_\nu(\gamma x)\, dx = \frac{1}{2\alpha} \exp\left(\frac{\beta^2-\gamma^2}{4\alpha}\right) J_\nu\left(\frac{\beta\gamma}{2\alpha}\right) \tag{A4}$$

(with Re $\alpha > 0$, Re $\nu > -1$), we have

$$\int_0^\infty \mathscr{G}_l(mr,mr')\, \gamma r'\, j_l(\gamma r')\, dr' = \frac{\sqrt{\pi}}{m}\, \gamma r\, j_l(\gamma r) \int_0^\infty \frac{\exp[(-1/u)(1+\gamma^2/m^2))]}{u^{3/2}}\, du$$

$$= \frac{\pi}{\sqrt{\gamma^2+m^2}}\, \gamma r\, j_l(\gamma r). \tag{A5}$$

APPENDIX B: SINGULARITY AND THE RELATIVISTIC COULOMB PROBLEM

We know that the relation

$$\sqrt{-\Delta+m^2}\, G(r) = \sqrt{\pm\beta^2+m^2}\, G(r), \tag{B1}$$

leads to $G(r) \propto j_l(\beta r)$ (plus sign) or $G(r) \propto i_l(\beta r)$ (minus sign). The situation is rather different for the irregular spherical Bessel functions $n_l(x)$ and $k_l(x)$ $(n_l(x) = \sqrt{(\pi/2x)} N_{l+1/2}(x)$, where $N_\nu(x)$ is the Bessel function of the second kind). To illustrate, we just consider the function $k_0(x) = \exp(-x)/x$. We see that the right-hand side of Eq. (35) is different from zero as we could expect from Eq. (B1). Then we can write

$$\sqrt{\mathbf{p}^{\,2}+m^2}\, \frac{\exp(-\gamma r)}{r} = \sqrt{m^2-\gamma^2}\, \frac{\exp(-\gamma r)}{r} + f(r), \tag{B2}$$

where

$$(\sqrt{\mathbf{p}^{\,2}+m^2} + \sqrt{m^2-\gamma^2}) f(r) = 4\pi\delta(\mathbf{r}), \tag{B3}$$

since

$$(\Delta-\gamma^2)\frac{\exp(-\gamma r)}{r} = -4\pi\delta(\mathbf{r}). \tag{B4}$$

From Eq. (B3) we deduce

$$f(r) = \frac{2}{\pi r}\int_0^\infty \frac{p \sin(pr)}{\sqrt{p^2+m^2}+\sqrt{m^2-\gamma^2}}\, dp. \tag{B5}$$

Thus we see that the singularity at the origin of the functions $n_l(x)$ and $k_l(x)$ is very important and very annoying. Indeed, if $f(r)$ was null it would be easy to solve the equal masses relativistic Coulomb problem for $l=0$. In this case one can show that

$$\sqrt{\mathbf{p}^{\,2}+m^2}\exp(-\gamma r)=\sqrt{m^2-\gamma^2}\exp(-\gamma r)+\frac{\gamma}{\sqrt{m^2-\gamma^2}}\frac{\exp(-\gamma r)}{r}\tag{B6}$$

and

$$\sqrt{\mathbf{p}^{\,2}+m^2}r\ \exp(-\gamma r)=\sqrt{m^2-\gamma^2}r\ \exp(-\gamma r)+\frac{2\gamma}{\sqrt{m^2-\gamma^2}}\exp(-\gamma r)-\frac{m^2}{(m^2-\gamma^2)^{3/2}}\frac{\exp(-\gamma r)}{r}.\tag{B7}$$

Acting as the square-root operator on the left of these relations one can verify them. The equation to solve is

$$2\ \sqrt{\mathbf{p}^2+m^2}R_l(r)=\left(E+\frac{\kappa}{r}\right)R_l(r).\tag{B8}$$

Using Eq. (B6), we find that the ground state is

$$R_0(r)\propto\exp\left(-\frac{\kappa m}{\sqrt{4+\kappa^2}}r\right),\tag{B9}$$

with

$$E=\frac{2m}{\sqrt{1+(\kappa/2)^2}}.\tag{B10}$$

Using Eqs. (B6) and (B7), the first excited state is given by

$$R_0(r)\propto\left(1-\frac{16\kappa m}{(16+\kappa^2)^{3/2}}r\right)\exp\left(-\frac{\kappa m}{\sqrt{16+\kappa^2}}r\right),\tag{B11}$$

with

$$E=\frac{2m}{\sqrt{1+(\kappa/4)^2}}.\tag{B12}$$

This spectrum is a rather good approximation. For example, the error is $\approx$ 0.25%, for the ground state, if $\kappa=0.456$ and $m=1$ GeV (see Ref. 13). These energies were first obtained in Ref. 17. Let us remark that the wave functions are Schrödinger-like wave functions.

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