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Light meson spectra and instanton-induced forces

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The spinless Salpeter equation supplemented by an instanton-induced force is used to describe the spectra of light mesons, including the pseudoscalar ones. The coupling constants of the instanton-induced potential, as well as the quark constituent masses, are not treated as simple free parameters but are calculated from the underlying instanton theory. Quite good results are obtained provided the quarks are considered as effective degrees of freedom with a finite size. A further test of the model is performed by calculating the electromagnetic mass differences between S-wave mesons. [S0556-2821(98)05815-9]

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I. INTRODUCTION

Numerous papers has been devoted to the study of meson spectra in the framework of the potential model. In most of these works, it is assumed that the quark interaction is dominated by a linear confinement potential, and that a supplementary short-range potential stems from the one-gluon exchange mechanism (see, for instance, Refs. [1,2]). The results obtained with these models are generally in good agreement with the experiments but the mesons η and η' cannot be described without adding an appropriate flavor mixing procedure with supplementary parameters.

On the other hand, Blask *et al.* [3] have developed a nonrelativistic quark model which describes quite well all mesons (including η and η') and baryons composed of u, d, or s quarks. The long-range part of their interaction is the usual linear confinement potential, but their short-range part is a pairing force stemming from instanton effects. This force presents the peculiarities to act only on quark-antiquark states with zero spin and zero angular momentum, and to generate constituent masses for the light quarks. The main problem of this model, and more generally of all nonrelativistic models, is that the velocity of a light quark inside a meson is not small compared with the speed of light. This makes the interpretation of the parameters of such models questionable [2] [p. 164].

At present, several works have been devoted to the study of mesons with the instanton-induced forces in the framework of relativistic or semirelativistic models [4–6], but in all cases the constituent masses and coupling constants of the instanton-induced forces have been considered as free parameters fitted to reproduce at best meson spectra. Actually, these quantities can be calculated from instanton theory. In this work, our purpose is to develop a semirelativistic model for meson spectra including the instanton-induced forces, but with parameters calculated, as far as possible, with the underlying theory. We will show that such a procedure is possible and gives very good results provided the quarks are considered as effective degrees of freedom with a finite size.

Our paper is organized as follows. In Sec. II, the model is constructed and the various parameters are presented, including the quark size parameters. The numerical techniques and the fitting procedure for the parameters are described in Sec. III, where the results are analyzed and a further test of the model is achieved with the calculation of the electromagnetic mass differences. Concluding remarks are given in Sec. IV.

II. MODEL

A. Spinless Salpeter equation

Our models rely on the spinless Salpeter equation. This equation is not a covariant one, but it takes into account relativistic kinematics. It can be deduced from the Bethe-Salpeter equation by neglecting retardation effects, the mixing with negative energy states, and the spinor structure of the eigenstates. The spinless Salpeter equation has been often used to describe meson spectra (see, for instance, Refs. [1,6,7]). This equation has the following form:

$$H=\sqrt{\vec{p}^{\,2}+m_1^2}+\sqrt{\vec{p}^{\,2}+m_2^2}+V(\vec{r}), \tag{1}$$

where V is the potential between the particles and where $\vec{p}$ is their relative momentum. The vector $\vec{p}$ is the conjugate variable of the interdistance $\vec{r}$.

As usual, we assume that the isospin symmetry is not broken, that is to say, that the u and d quarks have the same mass. In the following, these two quarks will be named by the symbol n (for normal or nonstrange quark).

B. Funnel potential

It is now well accepted that the long-range part of the interquark interaction is dominated by the confinement. The best way to simulate this phenomenon in a semirelativistic equation is to use a potential increasing linearly with the distance. With such an interaction, the Regge trajectories of light mesons are well reproduced [2] [p. 137]. Moreover, lattice calculations also find that the confinement is roughly proportional to $r=|\vec{r}|$. The short-range part is very often a Coulomb-like interaction stemming from the one-gluon exchange process. The idea is that, once the confinement is taken into account, other contributions to the potential energy can be treated as residual interactions. It is worth noting that the contribution of a constant potential is always necessary to obtain good spectra. Finally, the central potential considered in our models is the so-called funnel potential [2]

$$V(r) = -\frac{\kappa}{r} + ar + C. \tag{2}$$

Despite its simple form, this potential was the first one which is able to reproduce the charmonium spectrum quite well [2]. Moreover, it gives very good results even in the light meson sector (see, for instance, Ref. [7]).

Experiment shows that vibrational and orbital excitations give many more contributions to the meson masses than variations of spin S or total angular momentum J in a multiplet. In this case, a spinless Salpeter equation, only supplemented by the funnel potential, can yield very satisfactory results. Nevertheless, the situation is completely different for the $L=0$ mesons for which the mass differences between $S=0$ and $S=1$ states are extremely large. For these mesons, another interaction must be taken into account in the model.

C. Instanton interaction

The instanton-induced interaction provides a suitable formalism to reproduce well the pseudoscalar spectrum. Indeed it is possible to explain the masses of the pion and the kaon and to describe states with flavor mixing as η and η' mesons. The form of the interaction depends on the quantum numbers of the state [3].

For $L \neq 0$ or $S \neq 0$,

$$V_{\rm Inst} = 0. \tag{3}$$

For $L=S=0$ and $I=1$,

$$V_{\rm Inst} = -8g\,\delta(\vec{r}). \tag{4}$$

For $L=S=0$ and $I=1/2$,

$$V_{\rm Inst} = -8g'\,\delta(\vec{r}). \tag{5}$$

For $L=S=0$ and $I=0$,

$$V_{\rm Inst} = 8\begin{pmatrix} g & \sqrt{2}g' \\ \sqrt{2}g' & 0 \end{pmatrix}\delta(\vec{r}). \tag{6}$$

The last potential acts in the flavor space $(1/\sqrt{2}(|u\bar{u}\rangle + |d\bar{d}\rangle), |s\bar{s}\rangle)$. The parameters g and g' are two dimensioned coupling constants [4] defined as

$$g = \frac{3}{8} g_{\rm eff}(s), \tag{7}$$

$$g' = \frac{3}{8} g_{\rm eff}(n), \tag{8}$$

$$g_{\rm eff}(i) = \left(\frac{4}{3}\pi^2\right)^2 \int_0^{\rho_c} d\rho \;\; d_0(\rho)\rho^2 \times (m_i^0 - \rho^2 c_i), \tag{9}$$

where m_i^0 is the current mass of the flavor i and $c_i = (2/3)\pi^2\langle \bar{q}_i q_i\rangle$, $\langle \bar{q}_i q_i\rangle$ being the quark condensate for this flavor. The function $d_0(\rho)$ is the instanton density as a function of the instanton size ρ. For three colors and three flavors this quantity is given by [4]

$$d_0(\rho) = 3.63\times 10^{-3}\left(\frac{8\pi^2}{g^2(\rho)}\right)^6 \exp\left(-\frac{8\pi^2}{g^2(\rho)}\right), \tag{10}$$

where

$$\left(\frac{8\pi^2}{g^2(\rho)}\right) = 9\,\ln\left(\frac{1}{\Lambda\rho}\right) + \frac{32}{9}\ln\left[\ln\left(\frac{1}{\Lambda\rho}\right)\right], \tag{11}$$

within two-loop accuracy [4]. The quantity Λ is the QCD scale parameter and ρ_c is the maximum size of the instanton. This is a cutoff value for which the ln-ln term in Eq. (11) is still reasonably small compared with the ln term.

An interesting property of the instanton-induced interaction is the renormalization of quark masses, as it gives contributions to the constituent masses. The expression of these contributions are given by [4]

$$\Delta m_n = \frac{4}{3}\pi^2 \int_0^{\rho_c} d\rho \;\; d_0(\rho)(m_n^0 - \rho^2 c_n)(m_s^0 - \rho^2 c_s) \tag{12}$$

and

$$\Delta m_s = \frac{4}{3}\pi^2 \int_0^{\rho_c} d\rho \;\; d_0(\rho)(m_n^0 - \rho^2 c_n)^2. \tag{13}$$

The instanton interaction is not necessarily the only source for the constituent masses [3]. Actually we introduce two supplementary terms δ_n and δ_s which can be added to the running masses. These terms are free parameters and are not dependent on the instanton parameters. Results with vanishing and nonvanishing δ_n and/or δ_s are given in Table II, below. Finally the constituent masses in our models are given by

$$m_n = m_n^0 + \Delta m_n + \delta_n, \tag{14}$$

$$m_s = m_s^0 + \Delta m_s + \delta_s. \tag{15}$$

We can rewrite expressions (9), (12), and (13) in a more interesting form for numerical calculations by setting a dimensionless instanton size

$$x = \Lambda\rho, \tag{16}$$

and defining another dimensionless quantity

$$\alpha_n(x_c) = \int_0^{x_c} dx\,\left\{9\ln\left(\frac{1}{x}\right) + \frac{32}{9}\ln\left[\ln\left(\frac{1}{x}\right)\right]\right\}^6 x^n\left[\ln\left(\frac{1}{x}\right)\right]^{-32/9}, \tag{17}$$

where $x_c = \Lambda\rho_c$. So we obtain

$$g = \frac{\delta\pi^2}{2}\frac{1}{\Lambda^3}\left[m_s^0\alpha_{11}(x_c) - \frac{c_s}{\Lambda^2}\;\;\alpha_{13}(x_c)\right], \tag{18}$$

$$g' = \frac{\delta\pi^2}{2}\,\frac{1}{\Lambda^3}\left[m_n^0 \alpha_{11}(x_c) - \frac{c_n}{\Lambda^2}\;\alpha_{13}(x_c)\right], \tag{19}$$

$$\Delta m_n = \delta\frac{1}{\Lambda}\left[m_n^0 m_s^0 \alpha_9(x_c) - \frac{(c_n m_s^0 + c_s m_n^0)}{\Lambda^2}\;\alpha_{11}(x_c) + \frac{c_n c_s}{\Lambda^4}\alpha_{13}(x_c)\right], \tag{20}$$

$$\Delta m_s = \delta\frac{1}{\Lambda}\left[(m_n^0)^2 \alpha_9(x_c) - 2\frac{c_n m_n^0}{\Lambda^2}\alpha_{11}(x_c) + \frac{(c_n)^2}{\Lambda^4}\alpha_{13}(x_c)\right], \tag{21}$$

with $\delta = 3.63\times 10^{-3}\times 4\pi^2/3$. Except the quantity x_c, all parameters involved in Eqs. (18)–(21) have expected values from theoretical and/or experimental considerations. The integration in Eq. (17) must be carried out until the ratio of the ln term on the ln-ln term in Eq. (11) stays small. This ratio, called R here, increases with x from zero at $x = x_1 = 1/e$ to very large values (see Fig. 1). At $x = x_2 \approx 0.683105$, the value of this ration is 1. This last value corresponds to the minimum of the instanton density (see Fig. 1). Thus we define the parameter ϵ by

$$x_c = x_1 + \epsilon(x_2 - x_1) \quad \text{with} \quad \epsilon \in [0,1]. \tag{22}$$

In this work ϵ is a pure phenomenological parameter whose value must be contained between 0 and 1. To save calculation times, we have calculated some values of $\alpha_n(x_c)$ functions for some given values of ϵ, and we use a spline algorithm to find other values. A good accuracy can be obtained since these functions and their derivatives are known. We have verified that the spline procedure allows an accuracy better than 10^{-3} on values of quark masses and instanton coupling constants. So no numerical integration is necessary. The functions $\alpha_9(x)$, $\alpha_{11}(x)$, and $\alpha_{13}(x)$ between x_1 and x_2 are given in Fig. 2.

D. Effective quarks

The quark masses used in our model are the constituent masses and not the current ones. It is then natural to suppose that a quark is not a pure pointlike particle, but an effective degree of freedom which is dressed by the gluon and quark-antiquark pair cloud. As a correct description of this effect is far from being obvious, we use a phenomenological *Ansatz*, as is the case in many other works (see, for instance, Refs. [1,6]). It seems natural to consider that the probability density of a quark in the configuration space is a peaked function around its average position. The form that we retain is a Gaussian function

$$\rho_i(\vec{r}) = \frac{1}{(\gamma_i\sqrt{\pi})^{3/2}}\exp(-r^2/\gamma_i^2). \tag{23}$$

TABLE I. Centers of gravity (c.o.g.) of L and I multiplets in GeV for mesons chosen to fix the parameters of the models. The values of the c.o.g. and their corresponding errors are given by formula (33). The symbol ‘‘mf ’’ means ‘‘mixed flavor.’’ A meson name used to represent a multiplet in Figs. 4, 5, and 6 is underlined.

State	Flavor	I	$J^{P(C)}$	$N\;^{2S+1}L_J$	c.o.g.
π	$n\bar{n}$	1	0^{-+}	$1\,^1S_0$	0.138 ± 0.003
ω	$n\bar{n}$	0	1^{--}	$1\,^3S_1$	0.772 ± 0.001
ρ	$n\bar{n}$	1	1^{--}	$1\,^3S_1$	
$h_1(1170)$	$n\bar{n}$	0	1^{+-}	$1\,^1P_1$	1.265 ± 0.013
$b_1(1235)$	$n\bar{n}$	1	1^{+-}	$1\,^1P_1$	
$f_1(1285)$	$n\bar{n}$	0	1^{++}	$1\,^3P_1$	
$a_1(1260)$	$n\bar{n}$	1	1^{++}	$1\,^3P_1$	
$f_2(1270)$	$n\bar{n}$	0	2^{++}	$1\,^3P_2$	
$a_2(1320)$	$n\bar{n}$	1	2^{++}	$1\,^3P_2$	
$\pi_2(1670)$	$n\bar{n}$	1	2^{-+}	$1\,^1D_2$	1.681 ± 0.012
$\omega(1600)$	$n\bar{n}$	0	1^{--}	$1\,^3D_1$	
$\rho(1700)$	$n\bar{n}$	1	1^{--}	$1\,^3D_1$	
$\omega_3(1670)$	$n\bar{n}$	0	3^{--}	$1\,^3D_3$	
$\rho_3(1690)$	$n\bar{n}$	1	3^{--}	$1\,^3D_3$	
$f_4(2050)$	$n\bar{n}$	0	4^{++}	$1\,^3F_4$	2.039 ± 0.022
$a_4(2040)$	$n\bar{n}$	1	4^{++}	$1\,^3F_4$	
$\pi(1300)$	$n\bar{n}$	1	0^{-+}	$2\,^1S_0$	1.300 ± 0.100
$\omega(1420)$	$n\bar{n}$	0	1^{--}	$2\,^3S_1$	1.454 ± 0.026
$\rho(1450)$	$n\bar{n}$	1	1^{--}	$2\,^3S_1$	
K	$s\bar{n}$	1/2	0^-	$1\,^1S_0$	0.496 ± 0.002
$K^*(892)$	$s\bar{n}$	1/2	1^-	$1\,^3S_1$	0.892 ± 0.001
$K_1(1270)$	$s\bar{n}$	1/2	1^+	$1\,^1P_1$	1.382 ± 0.005
$K_0^*(1430)$	$s\bar{n}$	1/2	0^+	$1\,^3P_0$	
$K_1(1400)$	$s\bar{n}$	1/2	1^+	$1\,^3P_1$	
$K_2^*(1430)$	$s\bar{n}$	1/2	2^+	$1\,^3P_2$	
$K_2(1770)$	$s\bar{n}$	1/2	2^-	$1\,^1D_2$	1.774 ± 0.012
$K^*(1680)$	$s\bar{n}$	1/2	1^-	$1\,^3D_1$	
$K_2(1820)$	$s\bar{n}$	1/2	2^-	$1\,^3D_2$	
$K_3^*(1780)$	$s\bar{n}$	1/2	3^-	$1\,^3D_3$	
ϕ	$s\bar{s}$	0	1^{--}	$1\,^3S_1$	1.019 ± 0.001
$h_1(1380)$	$s\bar{s}$	0	1^{+-}	$1\,^1P_1$	1.482 ± 0.009
$f_1(1510)$	$s\bar{s}$	0	1^{++}	$1\,^3P_1$	
$f_2'(1525)$	$s\bar{s}$	0	2^{++}	$1\,^3P_2$	
$\phi_3(1850)$	$s\bar{s}$	0	3^{--}	$1\,^3D_3$	1.854 ± 0.007
$\phi(1680)$	$s\bar{s}$	0	1^{--}	$2\,^3S_1$	1.680 ± 0.020
$f_2(2010)$	$s\bar{s}$	0	2^{++}	$2\,^3P_2$	2.011 ± 0.080
η	mf	0	0^{-+}	$1\,^1S_0$	0.547 ± 0.001
η'	mf	0	0^{-+}	$1\,^1S_0$	0.958 ± 0.001

It is generally assumed that the quark size γ_i depends on the flavor. So we consider two size parameters γ_n and γ_s for quarks n and s, respectively. Any operator which depends on the quark positions $\vec{r}_i$ and $\vec{r}_j$ must be replaced by an effective one which is obtained by a double convolution of the origi-

TABLE II. Optimal parameters found for various models (see Sec. III C). Values between square brackets are fixed quantities in the model. When available, the expected value of a parameter is also given in the column ''Expt.'' The *Ansatz* to calculate $\gamma_{\rm mf}$ (see Sec. II D) is indicated and the parameters m_n, m_s, g, g', and $\rho_{\rm c}$ are calculated. The quantity χ^2(light) is the value of the χ^2 given by relation (32) for the set of mesons from Table I. The corresponding quantity for an extended set of mesons (see Sec. III C) is indicated by χ^2(all).

Parameter		Model I	Model II	Model III	Model IV	Model V	Expt.
m_n^0	(GeV)	0.015	0.013	0.004	0.015	0.015	0.002–0.015 [10]
m_s^0	(GeV)	0.215	0.196	0.199	0.271	0.166	0.100–0.300 [10]
Λ	(GeV)	0.245	0.299	0.223	0.231	0.262	$0.209^{+0.039}_{-0.033}$ [10]
$\langle \overline{n}n \rangle$	(GeV^3)	$(-0.243)^3$	$(-0.223)^3$	$(-0.236)^3$	$(-0.250)^3$	$(-0.240)^3$	$(-0.225\pm0.025)^3$ [12]
$\langle \overline{s}s \rangle / \langle \overline{n}n \rangle$		0.706	0.834	0.704	0.700	0.703	0.8 ± 0.1 [12]
a	(GeV^2)	0.212	0.225	0.207	0.201	0.235	0.20 ± 0.03 [13]
κ		0.440	0.283	0.366	0.367	[0.000]	
C	(GeV)	-0.666	-0.781	-0.593	-0.551	-0.860	
γ_n	(GeV^{-1})	0.736	0.730	0.764	0.773	0.586	
γ_s	(GeV^{-1})	0.515	0.186	0.552	0.524	0.409	
δ_n	(GeV)	0.120	0.147	[0.000]	[0.000]	0.120	
δ_s	(GeV)	0.173	0.228	0.118	[0.000]	0.194	
ϵ		0.031	0.162	0.011	0.046	0.003	
$\gamma_{\rm mf}$		$\sqrt{2\gamma_n\gamma_s}$	$\sqrt{\gamma_n^2+\gamma_s^2}$	$\sqrt{2\gamma_n\gamma_s}$	$\sqrt{2\gamma_n\gamma_s}$	$\sqrt{2\gamma_n\gamma_s}$	
m_n	(GeV)	0.192	0.206	0.061	0.090	0.176	
m_s	(GeV)	0.420	0.443	0.351	0.312	0.385	
g	(GeV^{-2})	2.743	2.767	3.117	3.336	2.002	
g'	(GeV^{-2})	1.571	1.213	1.846	1.807	1.241	
$\rho_{\rm c}$	(GeV^{-1})	1.541	1.402	1.664	1.598	1.461	
χ^2(light)		14.9	16.7	10.9	73.1	43.9	
χ^2(all)		35	95.5				

nal bare operator with the density functions ρ_i and ρ_j. As a double convolution is a heavy procedure which generates a very complicated form for the convoluted potentials, we assume that the dressed expression $\tilde{O}_{ij}(\vec{r})$ of a bare operator $O_{ij}(\vec{r})$, which depends only on the relative distance $\vec{r}=\vec{r}_i-\vec{r}_j$ between the quarks q_i and q_j, is given by

$$\tilde{O}_{ij}(\vec{r})=\int d\vec{r}\,' O_{ij}(\vec{r}\,')\rho_{ij}(\vec{r}-\vec{r}\,'), \qquad (24)$$

where ρ_{ij} is also a Gaussian function of type (23) with the size parameter γ_{ij} given by

$$\gamma_{ij}=\sqrt{\gamma_i^2+\gamma_j^2}. \qquad (25)$$

This formula is chosen because the convolution of two Gaussian functions, with size parameters γ_i and γ_j, respectively, is also a Gaussian function with a size parameter given by Eq. (25).

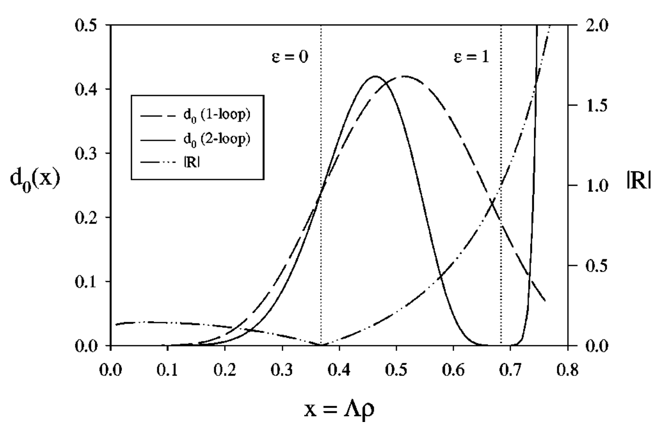

FIG. 1. One-loop and two-loop approximations of the instanton density [see Eq. (10)] as a function of the dimensionless instanton size $x=\Lambda\rho$. The absolute value of the ratio R (see Sec. II C) and the values of x corresponding to $\epsilon=0$ and $\epsilon=1$ are indicated.

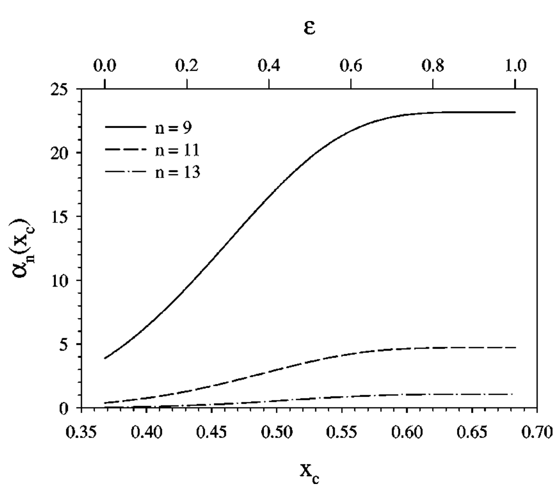

FIG. 2. Values of the function $\alpha_n(x_{\rm c})$ [see Eq. (17)] as a function of $x_{\rm c}$ and ϵ for the three interesting values of n. The curves begin at $x_{\rm c}=x_1=1/e$ $(\epsilon=0)$, and end at $x_{\rm c}=x_2$ $(\epsilon=1)$.

After convolution with the quark density, the funnel dressed potential has the form

$$\tilde{V}(r) = -\kappa \frac{\mathrm{erf}(r/\gamma_{ij})}{r} + ar\left[\frac{\gamma_{ij}\exp(-r^2/\gamma_{ij}^2)}{\sqrt{\pi}r} + \left(1+\frac{\gamma_{ij}^2}{2r^2}\right)\mathrm{erf}(r/\gamma_{ij})\right] + C, \quad (26)$$

while the Dirac distribution is transformed into a Gaussian function

$$\tilde{\delta}(\vec{r}) = \frac{1}{(\gamma_{ij}\sqrt{\pi})^3}\exp(-r^2/\gamma_{ij}^2). \quad (27)$$

Despite this convolution, we will consider, for simplicity, that the instanton-induced forces act only on $L=0$ states.

The modification of the confinement potential seems very important but, actually, only its short-range part is modified. It has little effect for a potential whose essential role is to govern the long-range dynamics. It has been verified [6] that the convolution of the linear potential could be neglected without changing sensibly the results, but we have nevertheless used the form (26) to be consistent.

The Coulomb part of the interaction is transformed into a potential with a finite value at the origin. This can be seen as a means to simulate asymptotic freedom. It has also been noticed that the use of an error function removes the singularity of the spin-spin interaction, when this correction is taken into account [2] [p. 162].

The introduction of a quark size is only necessary, from a mathematical point of view, to avoid collapse of the eigenvalues due to the presence of a Dirac distribution in the instanton-induced interaction. The definition of the size parameter γ_{ij} is not obvious in the case of the nondiagonal term of the instanton-induced interaction (6). This matrix element mixes $|n\bar{n}\rangle$ and $|s\bar{s}\rangle$ flavor states. So, what definition do we choose for γ_{ij} that we will note γ_{mf} in this case? A first possibility is to take

$$\gamma_{\mathrm{mf}} = \sqrt{\gamma_n^2 + \gamma_s^2}, \quad (28)$$

as in the case of $|s\bar{n}\rangle$ mesons. But we can also choose

$$\gamma_{\mathrm{mf}} = \sqrt{2\gamma_n\gamma_s}, \quad (29)$$

for instance. Other choices are possible, but we only take into account these two definitions in the following.

III. NUMERICAL RESULTS

A. Numerical techniques

The eigenvalues and the eigenvectors of our Hamiltonian are obtained in expanding trial states in a harmonic oscillator basis $|nlm\rangle$. In such a basis, matrix elements of the potential are given by

$$\langle n'lm|V(r)|nlm\rangle = \sum_{p=l}^{l+n+n'} B(n',l,n,l,p)I_p, \quad (30)$$

with

$$I_p = \frac{2}{\Gamma(p+3/2)}\int_0^\infty dx\ x^{2p+2}\exp(-x^2)V(bx). \quad (31)$$

The quantities I_p are the Talmi integrals, while the coefficients $B(n',l,n,l,p)$ are geometric factors [8] which can be calculated once for all. The parameter b is the oscillator length which fixes the scale of the basis states. With this parameter a dimensionless length $x=b/r$ can be defined. In our model all the Talmi integrals for the potential part of the Hamiltonian are given by analytical expressions [6]. The matrix elements of the kinetics part can be calculated with numerical integrations, but we prefer to use another technique, much more accurate and less time consuming. This method is described in Ref. [7]. We just give here the main points of the procedure. We calculate the matrix elements of the operator $\vec{p}^2+m^2$ in the oscillator basis (this is an analytical calculation). Let us assume that P is the corresponding matrix, D the diagonal matrix of the eigenvalues of P, and U the transformation matrix $(D=U^{-1}PU)$. Then the matrix elements of the operator $\sqrt{\vec{p}^2+m^2}$ are contained in the matrix $UD^{1/2}U^{-1}$. This procedure is about 6 times faster than a numerical integration and gives a better accuracy. The utilization of this procedure and the analyticity of the Talmi integrals lead to a very short time of calculation. A minimization with 13 parameters lasts about 10–20 min on a Pentium 200 workstation.

At last, it is worth noting that all the results obtained with the technique described above have been verified with the three-dimensional Fourier grid Hamiltonian method [9].

B. Fitting procedure

The purpose of this work is to try to reproduce the spectrum of light mesons with a quite simple model. Indeed we use only a central potential supplemented by an instanton-induced interaction to describe the pseudoscalar sector. We need 13 parameters to obtain a satisfactory theoretical spectrum. The instanton interaction is defined by six parameters: The current quark masses, the quark condensates for the flavors n and s, the QCD scale parameter Λ, and the maximum size ρ_{c} of the instanton. Three parameters are used for the spin-independent part of the potential: the slope of the confinement a, the strength κ for the Coulomb-like part, and the constant C which renormalizes the energy. Four supplementary parameters are introduced: the effective size of the quarks n and s, and two terms δ_n and δ_s, which contribute to the constituent quarks masses.

In our model, the quantum numbers L, S, and I are good quantum numbers. For $L=0$ states, the instanton-induced interaction raises the degeneracy between $S=0$ (pseudoscalar) and $S=1$ (vector) mesons, but $S=1$ $n\bar{n}$ mesons, which differ only by isospin, have the same mass. This is in quite good agreement with the data. In the sector $L\neq 0$, the instanton-induced interaction vanishes and the potential is spin independent. Consequently, states which differ only by S, J, and I are degenerate. In a first approximation, this corresponds also quite well to the experimental situation.

To find the value of the parameters, we minimize a χ^2 function based on the masses of 18 centers of gravity (c.o.g.) of multiplets containing well-known mesons (see Table I):

$$\chi^2 = \sum_i \left[\frac{M_i^{\text{theor.}} - M_i^{\text{expt.}}}{\Delta M_i^{\text{expt.}}} \right]^2, \tag{32}$$

where the quantity $\Delta M_i^{\text{expt.}}$ is the error on the experimental masses (it is fixed at the minimum value of 10 MeV). The quantum numbers v (vibrational or radial quantum number), L, and S of these mesons are determined on an assignment made in the Particle Data Group tables [10]. It is worth noting that all members of several multiplets considered here are not known. So we calculate the center of gravity with only the known mesons; no attempt is made to estimate the masses of the missing states. The usual procedure to define a center of gravity is

$$M_{\text{c.o.g.}} = \frac{\sum_{J,I} (2I+1)(2J+1) M_{J,I}}{\sum_{J,I} (2I+1)(2J+1)}. \tag{33}$$

In the following, a center of gravity will be indicated by the name of the state of the multiplet with the higher quantum numbers J and I. At last, note that we do not include the mesons $a_0(980)$ and $f_0(980)$ in the $L=1$ multiplet since experimental considerations and some theoretical works [10] [pp. 99, 557] suggest that they are not $q\bar{q}$ mesons.

To perform the minimization, we use the most recent version of the MINUIT code from the CERN library [11].

C. Meson spectra

A great number of parameter sets have been found for our model Hamiltonian. We present here only five among the most interesting ones, denoted from I to V. Parameters for these models are presented in Table II. All these models have common features. For instance, it is always possible to obtain a quite good fit with a low current quark mass m_n^0, in agreement with the bounds expected [10]. The strange quark mass m_s^0 varies more significantly following the model, but its value is always reasonable [10]. The QCD scale parameter Λ is a very sensible parameter of our model since a small change can largely modify the quantities deduced from the instanton theory [see Eqs. (18) to (21)]. Nevertheless, the values found are always in agreement with usual estimations [10]. Values for the quark condensates are also reasonable [12]. There is no direct measurement of the string tension parameter a, but lattice calculations favor the value 0.20 GeV^2 with about a 30% error [13]. Our values are in good agreement with this prediction.

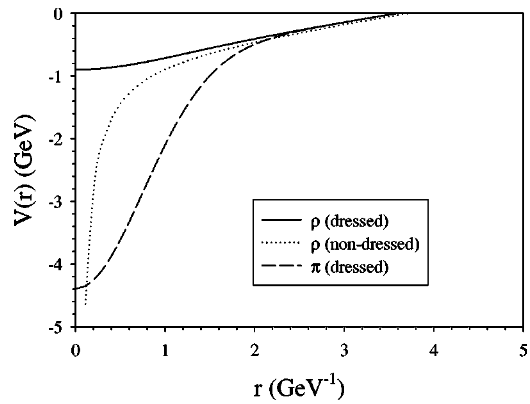

FIG. 3. Interquark potential $V(r)$ in GeV as a function of r in GeV^{-1} for the ρ and π mesons. The potential for the ρ meson without taking into account the effect of the quark sizes (non-dressed) is also presented. The corresponding potential for the π meson cannot be shown because the presence of a Dirac distribution coming from the instanton-induced interaction.

The strength κ of the Coulomb-like potential can strongly vary from one model to another (see, for instance, Ref. [2]). The values found in this paper are always less than 0.5, which can be considered as an upper limit for relevant values. In model V, we have fixed $\kappa=0$ as in the model of Ref. [3]. But the χ^2 obtained is not good; this is essentially due to radial excitations $\rho(1450)$, $\pi(1300)$, and $\phi(1680)$. We conclude that the Coulomb-like potential must be taken into account and that instanton-induced forces cannot explain alone the short-range part of the potential. Actually, this remark is already mentioned in Ref. [3]. As we can see from Table II, the constant potential is always necessary to obtain good spectra. Its origin is not simple to explain. It can be considered as a mechanism linked to the flux-tube model of mesons [5]. It has also been suggested that $C \approx -2\sqrt{a}$ [2] [p. 190]. Our models are not in very good agreement with this last prediction, as we can note a deviation as large as 40%. But we do not consider that it is as an important drawback of the models.

There is no real estimation of the quark sizes. They are purely phenomenological parameters whose role is to take into account relativistic effects [1] as well as sea-quark contributions. In all our models the size of the n quarks, γ_n, is nearly a constant, around 0.7–0.8 GeV^{-1}. The n-quark size drops down to 0.6 GeV^{-1} in model V without the Coulomb-like potential. This shows the strong influence of this interaction in determining γ_n. It is worth noting that the introduction of quark sizes modifies deeply the structure of the potential. This is illustrated in Fig. 3 where the dressed and nondressed potential for the ρ meson are presented. The s-quark size depends on the *Ansatz* chosen to calculate γ_{mf}. When $\gamma_{\text{mf}} = \sqrt{\gamma_n^2 + \gamma_s^2}$, γ_s can take vanishing values without generating collapse of the eigenenergies. This is no longer true if $\gamma_{\text{mf}} = \sqrt{2\gamma_n\gamma_s}$. The first definition of γ_{mf} is chosen in model II and we can remark that in this case the value of γ_s is significantly smaller than in other models, where the second definition is used.

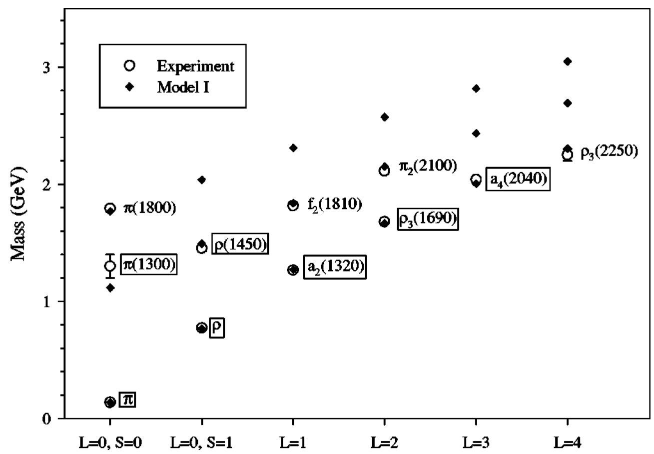

FIG. 4. Comparison of experimental (open circles with the corresponding error bars) and calculated (solid diamonds) spectra of $n\bar{n}$ mesons for model I. Framed names indicate centers of gravity of multiplets used to fix the parameters.

Table II shows that the contributions of parameters δ_n and δ_s can be quite large when they are not fixed in the minimization. The origin of these terms is not clear but their values indicate that the instanton effects cannot generate solely the constituent masses in our model Hamiltonian. We have found sets of parameters which give reasonable χ^2 values with vanishing δ_n and/or δ_s but, in this case, the constituent quark masses can be very small. Model III has actually the lowest χ^2 value that we have found but the constituent quark masses are so small that a generalization of this model to baryon spectra appears very problematic [14].

In all sets of parameters that we have determined, the value of ϵ is always close to zero. This is consistent with the fact that the cutoff radius for the integration over instanton density must be small enough in order that the ln-ln term in Eq. (11) must be reasonably small compared with the ln term. Actually, we can fix arbitrarily $\epsilon=0$ without spoiling the results, while values close to unity yield bad results.

In Figs. 4–6, we present the meson spectra of model I. We consider this model as the best we have obtained. Model III is characterized by a lowest χ^2 value but, as mentioned above, the corresponding constituent quark masses are too small to hope to obtain good baryon spectra. The χ^2 value of model II is also good but the strange quark size seems very small with respect to the usual estimations found in the literature. In order to decide between models I and II, we have tested these two potentials in the heavy meson sector. By fitting the masses and the sizes of the quarks c and b on 10 c.o.g. of $c\bar{c}$, $b\bar{b}$, $\bar{s}n$, $c\bar{s}$, and $\bar{b}n$ mesons, we have recalculated a χ^2 value for the all 28 c.o.g. considered in this paper (18 for light mesons and 10 for heavy mesons). The results are indicated in Table II. Clearly, model I is preferable. The bad value obtained for model II is due to an interplay between the small strange quark size and the low Coulomb strength.

The partial χ^2 for each meson (actually, each c.o.g. of a meson multiplet) of model I is below 1.6, except for the $\pi(1300)$, $\rho(1450)$, and $a_4(2040)$ mesons. In Fig. 4, we can see that the $n\bar{n}$ mesons are quite well reproduced. The larger error is obtained for the $\pi(1300)$, but the uncertainty on this meson is very large. We can remark on a slight deviation from the linear Regge behavior, but it can be very well reproduced within semirelativistic kinematics when only a linear potential is used.

In the $\bar{s}n$ sector, vibrational excitations of the $L=0$ mesons are not satisfactorily reproduced (these states are not taken into account in the minimization procedure). From the experimental point of view, the situation is not clear [10]. For instance, internal quantum numbers of $K^*(1410)$ and of $K^*(1680)$ are not well defined yet. This problem was also revealed in some previous works [5,6]. But all the ground states are quite well reproduced.

The $s\bar{s}$ sector is poor in experimental data but all the states obtained in our model are in quite good agreement with these data. In the mixed-flavor mesons, we reproduce the η and η' states. The two vibrational excitations of these mesons can be identified quite well with the $\eta(1295)$ and the $\eta(1760)$. Note that a calculated state lies between the $\eta(1295)$ and $\eta(1440)$, which is a non-$q\bar{q}$ meson candidate [10]. At last, we find a supplementary state between the $\eta(1760)$ and the $\eta(2225)$, the last one being a not-well-established state.

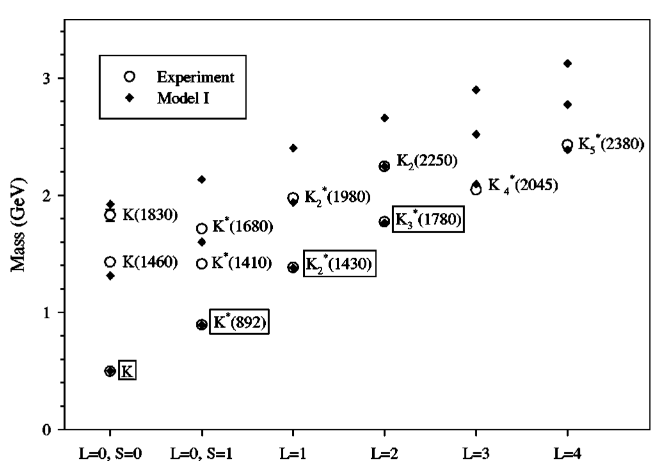

FIG. 5. Same as for Fig. 4 but for $\bar{s}n$ mesons.

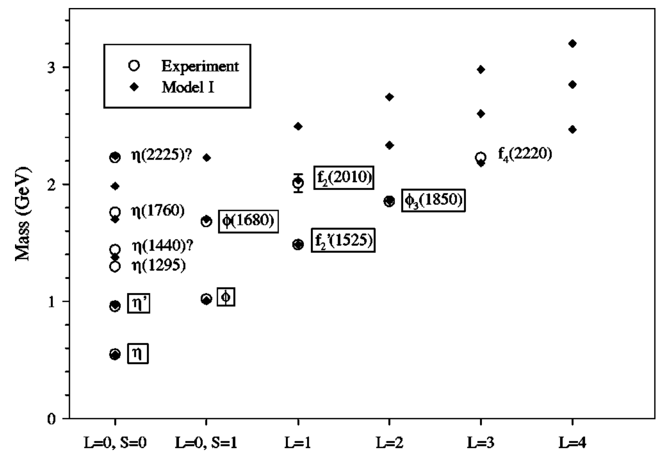

FIG. 6. Same as for Fig. 4 but for $s\bar{s}$ and mixed-flavor mesons.

D. Electromagnetic corrections

The electromagnetic mass differences between mesons are usually ignored in studies of meson spectra. The reason is that these mass differences are very small, of the order of some MeV, compared with the orbital and vibrational excitations due to the strong interaction, which can amount to about 1 GeV. Nevertheless, it is interesting to calculate the electromagnetic splittings as they provide further tests of the model. In particular, the mass differences are very sensitive to the short-range part of the wave functions.

The electromagnetic mass differences between mesons are due to two distinct effects. A first contribution is provided by the mass difference between the u and d quarks. In our models we have assumed that these two quarks have the same mass, but it is no longer relevant to calculate electromagnetic phenomena. In the following, we will assume that $m_n=(\overline{m}_d+\overline{m}_u)/2$, where $\overline{m}_i$ is the real constituent quark mass. We define $\epsilon_d=\overline{m}_d-m_n$ and $\epsilon_u=\overline{m}_u-m_n$. For the strange quark, we have $\overline{m}_s=m_s$ and $\epsilon_s=0$. The second contribution is due to the electromagnetic potential existing between quarks. In a first order approximation, this interaction has the following form:

$$V_{\rm em}(r)=\alpha\frac{Q_iQ_j}{r}, \tag{34}$$

where α is the electromagnetic fine-structure constant, and Q_i is the quark charge for the flavor i in units of e. This approximation is too crude since the first relativistic corrections of $V_{\rm em}$ and $V_{\rm em}$ have similar contributions to the electromagnetic mass differences [2]. We will consider only S-wave mesons, that is to say, that the spin-spin interaction is the only nonvanishing correction. As we work in the framework of semirelativistic models, we will use a relativized version of the spin-spin potential. A method to obtain such a potential is suggested in Ref. [2] [p. 198] and has been used in Ref. [1]. The idea is to replace the factors $1/m_i$ by the operators $1/\sqrt{\vec{p}^{\,2}+m_i^2}$ in the interaction expression.

The contribution of the electromagnetic Hamiltonian $H_{\rm em}$ can be calculated within first order in perturbation theory [15]. Assuming that this Hamiltonian is the difference between the total Hamiltonian (strong plus electromagnetic) and the Hamiltonian used in our models, the electromagnetic contribution, is, in a first order approximation,

$$\langle H_{\rm em}\rangle = m_i\epsilon_i\left\langle\frac{1}{\sqrt{\vec{p}^{\,2}+m_i^2}}\right\rangle + m_j\epsilon_j\left\langle\frac{1}{\sqrt{\vec{p}^{\,2}+m_j^2}}\right\rangle + \alpha\left\langle\frac{Q_iQ_j}{r}\right\rangle - \alpha\frac{8\pi}{3}\left\langle Q_iQ_j\vec{S}_i\cdot\vec{S}_j\left[\frac{1}{\sqrt{\vec{p}^{\,2}+m_i^2}}\delta(\vec{r})\frac{1}{\sqrt{\vec{p}^{\,2}+m_j^2}}\right]\right\rangle, \tag{35}$$

where the symbol $[O]$ is used to indicate the symmetrization of the operator O. The expression (35) reduces to the usual nonrelativistic form when quark masses tend toward infinity [2] [p. 169]. Mean values of operators can be evaluated as well by using the harmonic oscillator development of the wave functions as well directly in configuration space. The isospin breaking between the u and d quarks is a free parameter in our models, and so this quantity has been fitted to reproduce the K^+-K^0 mass difference. The results for model I are shown in Table III. Other models give similar results since in all cases the pseudoscalar states are well described. In magnitude, all results are around 2 times the experimental data, but the hierarchy of mass differences is preserved. In particular the quantity m_d-m_u has a good sign. This is not always obvious to obtain in potential models [16]. It is worth noting that our models are optimized to reproduce meson mass spectra with an accuracy of around 10 MeV, and not to obtain the best possible wave function. So we consider that the results found are quite well satisfactory.

TABLE III. Electromagnetic mass differences for some mesons in MeV. The results of our model I are compared with the data.

	Model I	Experiment [10]
$m_{\pi^+}-m_{\pi^0}$	7.184	4.594 ± 0.001
$m_{\rho^+}-m_{\rho^0}$	1.361	-0.300 ± 2.200
$m_{K^+}-m_{K^0}$	fitted	-3.995 ± 0.034
$m_{K^{*+}}-m_{K^{*0}}$	-11.047	-6.700 ± 1.200
m_d-m_u	26.610 [a]	$\lesssim 15$

[a]Fitted to reproduce the $m_{K^+}-m_{K^0}$ difference.

We also tried to calculate the electromagnetic mass differences with the nonrelativistic equivalent of Eq. (35), but the results obtained are very bad. For instance, isospin breaking between d and u quark can be found as large as 50 MeV. We have found that the nonrelativistic form of the electromagnetic spin-spin interaction is responsible for so poor results. This indicates that relativized forms of operators are important to obtain coherent results in semirelativistic models. Since the Coulomb-like strong interaction is of pure vector type, a strong spin-spin interaction must be introduced if one wants to take into account the relativistic corrections of the strong potential. We can guess that the use of a relativized form will be highly preferable than the usual nonrelativistic form.

IV. CONCLUDING REMARKS

It is well known that the spinless Salpeter equation supplemented by the so-called funnel potential can describe quite well the main features of the meson spectra [7]. The mass differences between members of a L multiplet are generally small with respect to vibrational and orbital excitations. Except in the pseudoscalar meson sector, the spinless approximation is then generally good. The instanton-induced forces provide a satisfactory way to describe the structure of light S-wave mesons, including the annihilation phenomenon.

Several works are devoted to the study of meson spectra within relativistic or semirelativistic models using instanton effects [4–6]. In these works, the instanton parameters are simply considered as free parameters to be fitted on data. In our paper, we try to calculate these parameters from the underlying theory. Our model relies on the spinless Salpeter equation supplemented by the usual funnel potential and an instanton-induced force. Thirteen parameters are necessary to completely fix the Hamiltonian. This number could appear large, but six parameters are strongly constrained by experimental or theoretical considerations, namely, the current quark masses, the QCD scale parameter, the quark condensates and the string tension. The Coulomb strength and the constant potential are unavoidable parameters of potential models. Actually, the Coulomb-like potential could be replaced by an interaction taking into account the asymptotic freedom, but it is a complication especially necessary when describing heavy meson spectra.

The relevance of the remaining five parameters is most questionable. If quarks are considered as effective degrees of freedom, then a size as well as a constituent mass can be associated with a quark. Generally, the quark size is described with a two-parameter function of the constituent mass. As we consider only two different flavors, it is useless to introduce such a quark size parametrization. To our knowledge, there is no reliable estimation of the quark size, but our values are in good agreement with values found in other works [1,6]. The radial part of the instanton-induced potential is a Dirac distribution, and so it is necessary to replace this form by a short-range potential peaked at the origin. The introduction of quark sizes offers a natural way to realize that, without introducing a new free parameter which is the instanton range.

The two-dimensioned coupling constants of the instanton-induced force are calculated from the instanton theory. This interaction generates also constituent masses for the light quarks. Our work shows that other contributions must be added to explain the large constituent masses of n and s quarks. These contributions which lie between 100 and 200 MeV are quite large. It is possible to cancel them, but the price to pay is then to obtain unrealistic very small constituent masses. We consider that it is better to keep the large values of these contributions. Their origin is not clear but sea-quark or relativistic effects could explain their presence.

The parameter ϵ reflects the uncertainties about the form of the instanton density, which is known within the two-loop approximation. The values found in all models are consistent with the fact that the cutoff radius of the instanton size must be small.

We have shown that the light meson spectra can be described by a semirelativistic model including an instanton-induced force whose characteristics are calculated from the underlying theory. Moreover, calculation of electromagnetic mass differences between mesons indicates that the wave functions obtained are quite satisfactory. The next steps are to calculate meson widths and to generalize the model to baryons. Such a work is in progress.

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