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Analytical solution of the relativistic Coulomb problem with a hard core interaction for a one-dimensional spinless Salpeter equation

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In this paper, we construct an analytical solution of the one-dimensional spinless Salpeter equation with a Coulomb potential supplemented by a hard core interaction, which keeps the particle in the x positive region. © *1999 American Institute of Physics.* [S0022-2488(99)02003-4]

I. INTRODUCTION

A simple relativistic version of the Schrödinger equation is the Spinless Salpeter Equation (SSE). For the one-dimensional case we have

$$\sqrt{-d_x^2+m^2}\Psi(x)=(E-V(x))\Psi(x), \tag{1}$$

where m is the mass of the particle, $V(x)$ is the potential interaction, E the eigenenergy of the stationary state $\Psi(x), d_x^2=d^2/dx^2=-p^2$ and p is the relative momentum of the particle ($\hbar=c=1$). p and x are conjugate variables. The differential operator of the Schrödinger equation is well defined because it is a second derivative. To solve a physical problem, we must just solve an ordinary eigenvalue differential equation. The situation is more complicated with the SSE because the associated differential operator is a nonlocal one. Its action cannot be calculated directly from its operator form. Indeed, its action on a function $f(x)$ is known only if $f(x)$ is an eigenfunction of the operator d_x^2. In this case we obtain

$$\sqrt{-d_x^2+m^2}f(x)=\sqrt{-\alpha+m^2}f(x), \tag{2}$$

where α is the corresponding eigenvalue of d_x^2. That is why we need first to rewrite the SSE into a form easier to handle. Since the operator is a nonlocal one, this form could be an integral equation. This have been done for the three-dimensional case in Refs. 1, 2. We present the one-dimensional corresponding form in the next section. With the method used to obtain this form, it is possible to rewrite the SSE as an integro-differential equation (see Ref. 2 for the three-dimensional case). But the kernel is really complicated and the resulting equation seems to be very difficult to treat. We will use, here, another method to obtain the solution of the equation.

To solve the relativistic Coulomb problem we do not solve any differential equation. We calculate the action of the square-root operator on the functions $x^n e^{-\beta x}$. Because the result is analytical and because the wave functions of the Coulomb problem with a hard core interaction are an exponential multiplied by a polynomial, which is also the form of the Schrödinger and Klein–Gordon wave functions, a complete solution of Eq. (1), with $V(x)\propto 1/x$ and $x>0$, can be found.

The paper is organized as follow. In Sec. II, we give some useful mathematical results concerning the square-root operator. In Sec. III, we solve the one-dimensional Coulomb problem

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with a hard core interaction. In Sec. IV, we compare our results to those obtained with the Schrödinger equation[3–6] and with the Klein–Gordon equation.[7,8] At last, we give our conclusion in Sec. V.

II. MATHEMATICAL FRAMEWORK

In this section we give some results concerning the square-root operator which we use to solve the Coulomb problem supplemented by a hard core interaction.

A. Integral representation of the square-root operator

To obtain the integral representation of the square-root operator we use the Fourier transform of the one-dimensional delta function. We have

$$\sqrt{-d_x^2+m^2}\Psi(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dp\,dq\,\sqrt{p^2+m^2}e^{-i(q-x)p}\Psi(q). \tag{3}$$

Extracting the operator $-d_x^2+m^2$ and integrating over the momentum p (see Ref. 1) we obtain

$$\begin{aligned}\sqrt{-d_x^2+m^2}\Psi(x)&=\frac{1}{\pi}(-d_x^2+m^2)\int_{-\infty}^{+\infty}dq\,K_0(m|q-x|)\Psi(q)\\&=\frac{1}{\pi}(-d_x^2+m^2)\int_{0}^{+\infty}dq\,K_0(mq)[\Psi(x+q)+\Psi(x-q)],\end{aligned} \tag{4}$$

where $K_0(x)$ is the modified Bessel function of order 0 (Ref. 9, p. 952).

B. Invariant space functions of the one-dimensional square-root operator

In this section we calculate the action of the square-root operator on the functions $x^n e^{-\beta x}$. We obtain that the result is equal to a polynomial of order n, $M_n(m,\beta,x)$, multiplied by the same exponential. Thus, the space of functions $P_n(x)e^{-\beta x}$ is the invariant space functions of this operator. Using formula (4) we have

$$\sqrt{-d_x^2+m^2}x^n e^{-\beta x}=\frac{1}{\pi}(-d_x^2+m^2)e^{-\beta x}\int_{0}^{+\infty}dq\,K_0(mq)[(x+q)^n e^{-\beta q}+(x-q)^n e^{\beta q}]. \tag{5}$$

This leads to (Ref. 9, p. 712)

$$\sqrt{-d_x^2+m^2}x^n e^{-\beta x}=\frac{1}{\sqrt{\pi}}(-d_x^2+m^2)e^{-\beta x}\sum_{k=0}^{n}\binom{n}{k}G_k(m,\beta)x^{n-k}. \tag{6}$$

The coefficients $G_k(m,\beta)$ are given by

$$G_k(m,\beta)=\frac{\Gamma(k+1)^2}{\Gamma(k+3/2)}\left(\frac{1}{(m+\beta)^{k+1}}F\left(k+1,1/2;k+3/2;-\frac{m-\beta}{m+\beta}\right)+(-)^k(\beta\to-\beta)\right), \tag{7}$$

where $F(\alpha,\beta;\gamma;x)$ is the hypergeometric function (Ref. 9, p. 1039). Performing the derivation in Eq. (6) and rearranging the obtained relation, we have

$$\begin{aligned}\sqrt{-d_x^2+m^2}x^n e^{-\beta x}=\frac{1}{\sqrt{\pi}}e^{-\beta x}\sum_{k=0}^{n}\binom{n}{k}\{&(m^2-\beta^2)G_k(m,\beta)\\&+2\beta kG_{k-1}(m,\beta)-k(k-1)G_{k-2}(m,\beta)\}x^{n-k}.\end{aligned} \tag{8}$$

It is possible to write the coefficients $G_k(m,\beta)$ into a more useful form. This form will allow us to find a recursion relation between the coefficients $G_k(m,\beta)$ and to simplify the expression (8). We will be able to construct the polynomial, $M_n(m,\beta,x)$, for each value of n. We have the relation (Ref. 10, p. 562)

$$F(a,1/2;a+1/2;-x)=\Gamma(a+1/2)\frac{x^{(1-2a)/4}}{\sqrt{1+x}}P_{-1/2}^{1/2-a}\left(\frac{1-x}{1+x}\right), \tag{9}$$

with $x>0$. The functions $P_\nu^\mu(x)$ are the associated Legendre functions for x real and $|x|<1$. With this relation, we find that

$$G_{k-1}(m,\beta)=\frac{\Gamma(k)^2}{\sqrt{2m}(m^2-\beta^2)^{(2k-1)/4}}[P_{-1/2}^{1/2-k}(\beta/m)+(-)^{k-1}P_{-1/2}^{1/2-k}(-\beta/m)]. \tag{10}$$

Now, using the recursion relation of the associated Legendre functions (Ref. 9, p. 1005),

$$P_\nu^{\mu+2}(x)=-2(\mu+1)\frac{x}{\sqrt{1-x^2}}P_\nu^{\mu+1}(x)+(\mu-\nu)(\mu+\nu+1)P_\nu^\mu(x), \tag{11}$$

and the explicit expression of $P_{-1/2}^{-1/2}(x)$ and $P_{-1/2}^{1/2}(x)$ (Ref. 9, p. 1008), we can write the following relations:

$$G_{k+2}(m,\beta)=\frac{1}{m^2-\beta^2}[(k+1)^2G_k(m,\beta)-(2k+3)\beta G_{k+1}(m,\beta)], \tag{12}$$

with

$$G_0(m,\beta)=\sqrt{\frac{\pi}{m^2-\beta^2}}, \tag{13}$$

$$G_1(m,\beta)=-\frac{\sqrt{\pi}\beta}{(m^2-\beta^2)^{3/2}}. \tag{14}$$

At last, one can find, using Eq. (12), that the general coefficient of the sum of Eq. (8) becomes

$$F_{k,n}(m,\beta)=\frac{1}{\sqrt{\pi}}\binom{n}{k}[\beta G_{k-1}(m,\beta)-(k-1)G_{k-2}(m,\beta)], \quad \text{with } k\geqslant 1. \tag{15}$$

And with this form, a recursion relation for $F_{k,n}(m,\beta)$ can be easily found. Thus, to conclude this section, we are able now to rewrite Eq. (8) into a simple form:

$$\sqrt{-d_x^2+m^2}x^ne^{-\beta x}=M_n(m,\beta,x)e^{-\beta x}=\left[\sum_{k=0}^n F_{k,n}(m,\beta)x^{n-k}\right]e^{-\beta x}, \tag{16}$$

$$F_{0,n}(m,\beta)=\sqrt{m^2-\beta^2}, \tag{17}$$

$$F_{1,n}(m,\beta)=\frac{n\beta}{\sqrt{m^2-\beta^2}}, \tag{18}$$

$$F_{k+2,n}(m,\beta)=\frac{n-k-1}{(k+2)(m^2-\beta^2)}[(k-1)(n-k)F_{k,n}(m,\beta)-(2k+1)\beta F_{k+1,n}(m,\beta)], \tag{19}$$

$$F_{k,n+1}=\frac{n+1}{n+1-k}F_{k,n}. \tag{20}$$

We can see that we obtain the expected relation [from Eq. (2)] for $n=0$. And thus we see that we must have $\beta<m$. With the relations (17)–(20) the polynomial $M_n(m,\beta,x)$ is completely defined and we can construct it for each value of n. This result will allow us to find, with few calculations, the solution of the one-dimensional relativistic Coulomb problem with a hard core interaction. We give below the polynomials, as an example, for $n=0\rightarrow 4$,

$$M_0(m,\beta,x)=S, \tag{21}$$

$$M_1(m,\beta,x)=Sx+\frac{\beta}{S}, \tag{22}$$

$$M_2(m,\beta,x)=Sx^2+\frac{2\beta}{S}x-\frac{m^2}{S^3}, \tag{23}$$

$$M_3(m,\beta,x)=Sx^3+\frac{3\beta}{S}x^2-\frac{3m^2}{S^3}x+\frac{3m^2\beta}{S^5}, \tag{24}$$

$$M_4(m,\beta,x)=Sx^4+\frac{4\beta}{S}x^3-\frac{6m^2}{S^3}x^2+\frac{12m^2\beta}{S^5}x-\frac{3m^2}{S^7}(m^2+4\beta^2), \tag{25}$$

with

$$S=\sqrt{m^2-\beta^2}. \tag{26}$$

Note that these last relations can be simply checked by acting the square-root operator on each side of Eq. (16). For $n=1$, we see that we have an identity if we use the relation for $n=0$. Now, knowing these two relations we see that the relation for $n=2$ is also an identity, and so on for each value of n.

III. THE ONE-DIMENSIONAL RELATIVISTIC COULOMB PROBLEM WITH A HARD CORE INTERACTION

The equation to solve is

$$\sqrt{-d_x^2+m^2}\Psi(x)=\left(E+\frac{\kappa}{x}\right)\Psi(x). \tag{27}$$

We just consider here the case $x>0$ (we will discuss after the extension to the whole x axis). Physically this means that we have a hard core interaction for $x\leqslant 0$. Then the wave functions will possess the following asymptotic behavior: $\Psi(x)=0$ for $x\leqslant 0$ and for $x=+\infty$. Suppose that the wave functions have the following form:

$$\begin{gathered}\Psi(x)\propto\sum_{k=1}^{n}\gamma_{k,n}x^k e^{-\beta x},\quad \text{for}\quad x>0\quad \text{and}\quad n=1,2,...,\\ \Psi(x)=0\quad \text{for}\quad x\leqslant 0.\end{gathered} \tag{28}$$

We do not consider the normalization of the functions here. Thus, replacing Eq. (28) into Eq. (27) and using Eq. (16), we obtain

$$\sum_{k=1}^{n} \gamma_{k,n} \sum_{p=0}^{k} F_{p,k}(m,\beta) x^{k-p} = E \sum_{k=1}^{n} \gamma_{k,n} x^{k} + \kappa \sum_{k=1}^{n} \gamma_{k,n} x^{k-1}. \tag{29}$$

Now equaling order by order we will determine the solution. The term of order n gives

$$E = F_{0,n}(m,\beta) = \sqrt{m^2 - \beta^2}. \tag{30}$$

From the term of order $n-1$, we have

$$\kappa = F_{1,n}(m,\beta), \tag{31}$$

which leads to

$$\beta = \frac{\kappa m}{n\sqrt{1+(\kappa/n)^2}}. \tag{32}$$

We can remark that we have as well the necessary relation $\beta < m$. We are now already able to determine the energy spectrum. Using Eq. (30) and Eq. (32) we have

$$E = \frac{m}{\sqrt{1+(\kappa/n)^2}}. \tag{33}$$

To obtain a complete solution, we must now find all the $\gamma_{k,n}$, and prove that the system of equations which gives these quantities always has a solution. Obviously we can fix $\gamma_{n,n}=1$. We see that the term of order $n-j$ determines the coefficient $\gamma_{n-j+1,n}$ if the previous $\gamma_{k,n}$ are known. Beginning with the term of order $n-2$, we obtain directly $\gamma_{n-1,n}$. And now we can get $\gamma_{n-2,n}$ from the term of order $n-3$. The independent term will fix the last factor $\gamma_{1,n}$. Thus, we have a triangular system of $n-1$ algebraic equations with $n-1$ unknowns. This system will always possess a solution if the determinant of the coefficient matrix is non-null. As this is a triangular matrix, the determinant is the product of the diagonal elements. The expression of these elements is $\kappa - F_{1,n-j}(m,\beta)$ which is equal to $j\beta/S$. These quantities are always non-null since $j>0$. The general form of $\gamma_{k,n}$ is obtained from the term of order $n-j-1$. We have

$$\gamma_{n-j,n} = \frac{S}{j\beta} \sum_{k=0}^{j-1} \gamma_{n-k,n} F_{j-k+1,n-k}(m,\beta). \tag{34}$$

We can inverse the summation to finally obtain

$$\gamma_{n-j,n} = \sum_{p_1=0}^{j-1} \sum_{p_2=0}^{p_1-1} \cdots \sum_{p_j=0}^{p_{j-1}-1} \tilde{F}(n,p_1,j)\tilde{F}(n,p_2,p_1)\ldots\tilde{F}(n,p_j,p_{j-1}), \tag{35}$$

with

$$\tilde{F}(n,k,j) = \frac{S}{j\beta} F_{j-k+1,n-k}(m,\beta). \tag{36}$$

For the summation in Eq. (35), we must use the following rule: If in a summation over p_α, α being arbitrary, the bound $p_{\alpha-1}-1$ is negative, all the $\tilde{F}(n,k,j)$ containing the indices $p_{\beta \geqslant \alpha}$ are equal to 1. With the formula (35), we are able to construct the wave functions for the Coulomb problem with a hard core interaction. As an example we give the three first wave functions:

$$\Psi(x) \propto x Q_n(m,\kappa,x) e^{-\beta x}, \tag{37}$$

with β given by Eq. (32), and

$$Q_1(m,\kappa,x)=1, \tag{38}$$

$$Q_2(m,\kappa,x)=x-\frac{m^2}{S^2\beta}, \tag{39}$$

$$Q_3(m,\kappa,x)=x^2-\frac{3m^2}{S^2\beta}x+\frac{3m^2}{2\beta^2S^4}(\beta^2+m^2), \tag{40}$$

with S defined by Eq. (26). Again, we can perform a simple verification by putting these solutions into Eq. (27) and using Eq. (16).

Contrary to the Schrödinger or Klein–Gordon equation, the extension of the solution to the whole x axis is really more complicated. We can try to use $\exp(-\beta|x|)$ instead of $\exp(-\beta x)$ in our solution. But the situation is quite more difficult. Indeed, the construction of the solution was based on the fact that $\exp(-\beta x)$ was an eigenfunction of the square-root operator and that the invariant space functions of this operator was $P_n(x)\exp(-\beta x)$, where $P_n(x)$ is a polynomial of order n. But it is easy to show, with Eq. (4), that

$$\sqrt{-d_x^2+m^2}\,\exp(-m|x|)=\frac{2m}{\pi}K_0(m|x|). \tag{41}$$

This is non-null, as this is the case in the Eq. (16). Thus, $\exp(-\beta|x|)$ is not an eigenfunction of the square-root operator and $P_n(x)\exp(-\beta|x|)$ is not an invariant space function of this operator. So it seems that the pure Coulomb problem has quite different solutions for the wave functions and certainly for the spectrum.

IV. DISCUSSION

The one-dimensional Coulomb problem has been treated by many authors, both nonrelativistically[3–6] and relativistically.[7,8] But in these works the whole x axis is considered. As a consequence, the ground state gives some difficulties.

In the nonrelativistic case the solution is

$$\Psi(x)=x\,\exp(-\kappa m|x|/n)L_{n-1}^{(1)}(2\kappa m|x|/n), \tag{42}$$

$$E=m-\frac{m\kappa^2}{2n^2},\quad \text{with}\ \ n=1,2,\ldots\,. \tag{43}$$

But we see that for $n=1$, the wave function has a node at the origin. So this is not the wave function for the ground state. In fact, it is found to be infinitely bounded and the wave function is a delta function.[3,8]

In the Klein–Gordon case the solution is

$$\Psi(x)=x^S\exp(-\beta|x|/2)L_{n-1}^{(\gamma)}(\beta|x|), \tag{44}$$

$$E=m\Bigg/\sqrt{1+\frac{\kappa^2}{(n-1+S)^2}}, \tag{45}$$

with

$$\beta=2m\kappa/\sqrt{(n-1+S)^2+\kappa^2},\quad \text{with}\ \ n=1,2,\ldots, \tag{46}$$

and

$$S=\tfrac{1}{2}(1+\gamma)=\tfrac{1}{2}(1\pm\sqrt{1-4\kappa^2}). \tag{47}$$

Thus, we see that we have two distinct solutions according the sign for S. Actually, the spectrum with the minus sign for S is not acceptable. Indeed, a reason is that, for $n=1$, when we perform the limit $\kappa\to 0$, we obtain $E=0$. This means that the particle is still bounded when the interaction vanishes. Thus, the problem for the ground state persists (see Ref. 8 for a complete discussion).

In this paper we do not consider the whole x axis and we have no problem with the ground state. We consider a hard core interaction, for $x\leqslant 0$, which gives $\Psi(x)=0$ in this region. Thus, $x\exp(-\beta x)$ is the wave function for the ground state. Actually the purpose of these work was to solve a particular kind of differential equations with a difficult to handle nonlocal operator. Indeed, any analytical solutions are known for the spinless Salpeter equation. Thus, we do not discuss the problem of the ground state of the one-dimensional Coulomb problem. In Sec. III, we have shown that the extension to the whole x axis is not easy. Moreover, the ground state problem could persist.

To compare our result to the results of previous works, we can consider the Schrödinger and the Klein–Gordon equation for the Coulomb potential supplemented by a hard core interaction. The spectra and the wave functions remain unchanged but the ground state problem has disappeared. In the three cases we have the same kind of wave functions: an exponential (with different arguments) multiplied by a polynomial (with different coefficients). For the spectrum we have, in the limit of small κ,

$$E_{\mathrm{Sch}}=m\left(1-\frac{\kappa^2}{2n^2}\right), \tag{48}$$

$$E_{\mathrm{KG}}=m\left(1-\frac{\kappa^2}{2n^2}-\frac{\kappa^4}{n^3}+\frac{3\kappa^4}{8n^4}\right), \tag{49}$$

$$E_{\mathrm{Sal}}=m\left(1-\frac{\kappa^2}{2n^2}+\frac{3\kappa^4}{8n^4}\right). \tag{50}$$

Thus we see that in the expansion of the Salpeter spectrum the term in n^3 is missing compared to the Klein–Gordon spectrum. So the difference between these two spectra is rather important. For an electron in an electromagnetic Coulomb potential, the splitting is about 10^{-3} eV.

Another characteristic of the spinless Salpeter spectrum is that κ can grow up without limit. This could come from the fact that we have another kinetic operator than in the Klein–Gordon equation and that the result could be quite different. But the main explanation is certainly that we do not solve the real Coulomb problem and that this spectrum could be different contrary to the Klein–Gordon equation, which keeps the same spectrum in both cases. Indeed, for the SSE, there exists a limit value for κ in three dimensions.[11]

V. CONCLUSION

The purpose of this work was to find an analytical solution of a particular kind of differential equations containing a nonlocal differential operator. The equation considered in this paper, the one-dimensional spinless Salpeter equation (SSE), is a simple relativistic version of the one-dimensional Schrödinger equation. The SSE is not a marginal equation. For three dimensions, this equation comes from the Bethe–Salpeter equation (Refs. 12, 13, p. 297), which gives the correct description of bound states of two particles. Moreover, despite the presence of a so particular operator, the SSE is often used in the potential models (see, for instance, Refs. 14–20), which give a phenomenological description of hadrons.

To find this analytical solution, we calculate, in Sec. II B, the action of the square-root operator on a polynomial multiplied by an exponential and we show that this constitutes the invariant space functions of this operator. To be able to perform this calculation, we have constructed, in Sec. II A, an integral representation of the square-root operator. In Sec. III, we have obtained, without solving any differential equation, a complete solution of the SSE with a Coulomb potential and a hard core interaction. This last interaction is introduced to keep the particle in the x positive region. We remark that the SSE wave functions have the same form than the Schrödinger and the Klein–Gordon wave functions. We remark also that the splitting between the SSE and the Klein–Gordon spectrum is rather important. Indeed, it is of the same order of the first relativistic correction given by these two equations.

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