Several Lagrangians describing the SU(2) Yang–Mills (YM) field interacting with matter are considered, which support both instantons (in four Euclidean dimensions) and sphaleron (in three dimensions, static) solutions. The matter fields are the complex Higgs doublet for the Weinberg–Salam (WS) model, and a $(2 \times 4)$ Grassmannian model. These Lagrangians feature Skyrme-like extensions to enable the existence of the instantons, which decay as pure gauge at infinity. For two of these models, we have numerically integrated the one-dimensional system arising from the imposition of radial ansatz.

1. **Introduction**

In the semiclassical treatment of quantum field theories, instantons\textsuperscript{1–3} play the important role of providing the tunneling between topologically inequivalent vacua in an essentially nonperturbative framework. Instantons are classical solutions to the Euler–Lagrange equations with Euclidean time, whence comes the tunneling interpretation. Another, closely related mechanism for vacuum to vacuum transitions but now at nonzero temperature, is provided by the (Euclidean) static solutions of the field equations called sphalerons.\textsuperscript{4} In contrast to instantons, sphalerons are unstable solutions and provide a classical rather than quantum (tunneling) transition over the energy barrier separating the two vacua. In this context it is very useful to consider another class of classical solutions, which are periodic in
(Euclidean) time, with the period being identified with the inverse of the temperature. These are the periodic instantons as defined in Ref. 5. Thus at zero temperature the period of the periodic instanton becomes infinite, rendering it nonperiodic, which can be identified as the instanton itself.

Now a given field theoretical model supporting sphalerons and hence also periodic instantons (with larger period and lower temperature than the former), may or may not support (zero-temperature or infinite period) instanton. In the case where a finite-size instanton exists, it can be arrived at as the period of the periodic instanton tends to infinity. In the case however where no finite size instanton exists, the situation is more complicated and the so-called constrained instantons must be employed. The consequences of a theory being of one type or of the other are thought to be potentially important, insofar as the respective contributions to the transition amplitudes may be enhanced or suppressed relatively, in the respective theories.

There has been some exploratory work done in this direction, in the context of the (1 + 1)-dimensional scale-breaking $O(3)$ sigma model which does not support finite size instantons, and its skyrmed version which does so. The study of the periodic instantons in these two models from the viewpoint described in the previous paragraph was carried out in Refs. 9 and 10 respectively.

The purpose of the present work is to take the preliminary steps towards such a study for the Weinberg–Salam (WS) model. This consists of proposing some extensions of the WS model that support finite size instantons, in analogy with the corresponding work of Ref. 8. In addition, we will consider an SU(2) gauged Grassmannian model which shares some qualitative features with the WS model.

2. Skyrme-Extended WS Models

It is well known that the WS model does not support finite size instantons because Derrick’s scaling requirement is violated. To prevent the size of the instanton from shrinking to zero, a constraint may be imposed on it as shown by Affleck, which is rather complicated. An alternative procedure, which is what we pursue in the present work, is to modify the model such that the deficiency in its scaling properties is corrected. This is nothing but the addition of Skyrme-like terms.

Skyrme-like terms have higher scaling characteristics than the Yang–Mills (YM) and the covariant derivative kinetic terms. They are higher order terms in gauge covariant quantities, i.e. the curvature and the covariant derivative. Most importantly, they are the gauge and Lorentz invariant norms of higher order tensor field densities, that are totally antisymmetric tensors. This property results in the canonical requirement that all the kinetic terms are restricted to the squares of the velocity fields and no higher powers of the latter. This is their analogy with the Skyrme model, and otherwise have nothing else in common with the latter which happens to be an $O(4)$ sigma model, while the WS theory is a Higgs model. In this paper we consider only the SU(2) part of the WS model.
With these restrictions, there is a limited number of such terms available for
skyrming the WS model. As it happens, these terms were constructed previously via
dimensional descent in a somewhat different context. What is important here for us
is that in Ref. 13 the consistency of the ansatz for the WS field multiplet in
four Euclidean dimensions, which we shall also use here, was tested.

Since the standard electroweak model features the YM system, the topological
charge of the instanton will be the usual second Chern–Pontryagin class. Adding
to the YM term the positive definite kinetic term quadratic in the covariant
derivative of the Higgs field, the Higgs self-interaction potential, and some suitable
Skyrme-like terms, leaves the original lower bound valid. Such Lagrangians satisfy
Derrick’s scaling requirement in four dimensions. In three dimensions, namely in the static
limit where we seek the sphaleron solution, the Skyrme-like term is not necessary
for the scaling requirement but may be included.

In the next two subsections, we will study the instanton solutions of the four-
dimensional Euclidean theories, and the sphaleron solutions of the static limits of
these theories in three dimensions, respectively.

2.1. Instantons

We start by introducing the Skyrme extended WS models in four Euclidean
dimensions. We denote the complex Higgs field doublet by \( \varphi \), and in terms of the anti-
Hermitian SU(2) YM connection \( A_\mu \), the covariant derivative is \( D_\mu \varphi = \partial_\mu \varphi + A_\mu \varphi \).

We will use the following notations, \( |\varphi|^2 = \langle \varphi, \varphi \rangle \) and \( |D\varphi|^2 = \langle D_\mu \varphi, D_\mu \varphi \rangle \) for the
inner product of the isodoublets. The two Skyrme-like terms we will consider are

\[
\text{Tr} F_{\mu \nu \rho \sigma}^2 = \{ F_{[\mu} \varphi, F_{\rho \sigma]} \}^2, \\
|F \wedge D\varphi|^2 = \langle F_{[\mu \nu} D_{\rho]} \varphi, F_{[\mu \nu} D_{\rho]} \varphi \rangle
\]  

(1)

here the symbol \([\nu \rho \sigma]\) implies cyclic symmetrization, in the following \([\nu \mu]\) means
antisymmetrization. There is yet another candidate for a Skyrme-like term, namely

\(\langle D_{[\mu} \varphi, D_{\nu]} \varphi \rangle^2\)

which we shall eschew here. This is because apart from other problems this term
scales exactly as the YM term does and hence it does not play an essential role as
a Skyrme term. It is therefore excluded in the first place on grounds of economy.
Indeed, criteria of economy dictate that only one of the candidates (1) for a Skyrme-
like term should be employed in a given extended model.

The Lagrangians of the two candidates for the bosonic part of the extended WS
models are

\[
\mathcal{L}_1^{\text{WS}} = \lambda_0 (\eta^2 - |\varphi|^2)^2 + \lambda |D\varphi|^2 + |F|^2 + \kappa^4 |F \wedge F|^2, \\
\mathcal{L}_2^{\text{WS}} = \lambda_0 (\eta^2 - |\varphi|^2)^2 + \lambda |D\varphi|^2 + |F|^2 + \kappa^2 |F \wedge D\varphi|^2
\]  

(2)

where the constant \( \kappa \) has the dimension of a length, \( \lambda_0 \) and \( \lambda \) are dimensionless.

The constant \( \eta \) is the vacuum expectation value of the Higgs field.
We will look for spherically symmetric solutions in four Euclidean dimensions and employ the ansatz
\[ A_{\mu} = \frac{2}{r}(1 - k(r))\sigma^{(\pm)}_{\mu}x_{\nu}, \quad \varphi = \eta h(r)\sigma^{(\pm)}_{\mu}x_{\nu} \varphi_0, \]
where \( \varphi_0 \) is a constant complex doublet, \( \sigma^{(\pm)}_{\mu} = (\pm i\sigma_1, 1) \) in terms of the three Pauli matrices \( \sigma_i \), and \( \sigma^{(\pm)}_{\mu} = -\frac{1}{2}\sigma^{(\pm)}_{\mu} \sigma_{1\nu} \) are the two chiral SO(4) generators in spinor representation variable \( r \) in (4) is \( r = \sqrt{x_{\mu}x^{\mu}} \).

Note that the ansatz for the Higgs doublet \( \varphi \) in (4) is not strictly spherically symmetric, but that its consistency in the case of both the systems (2) and (3) has been verified in Ref. 13. The ansatz (4) leads to reduced radial one-dimensional subsystems of (2) and (3), which solve the full Euler–Lagrange equations of the two systems (2) and (3).

In the present work, we will restrict to the numerical integration of the simpler one of the two systems (2) and (3), namely to (2).

Using \( x = \eta r \) as variable and the ansatz (4) leads to the following action and one-dimensional radial Lagrangian
\[ A(\lambda_0, \lambda, \xi) = \int L_{1}^{WS} d^4 x, \]
\[ \xi = \frac{k^4}{\eta^2} = 2\pi^2 \int L_{1}^{WS} d x, \]
with
\[ L_{1}^{WS} = \lambda_0 x^3(1 - h^2)^2 + \lambda x[x^2 h^2 + 3k^2 h^2] + x \left[ k^2 + \frac{4}{x^2}k^2(k - 1)^2 \right] + \frac{\xi}{x^3}(k - 1)^2k^2k'^2, \]
and the prime means derivative with respect to \( x \).

We seek solutions satisfying the following asymptotic values of the functions \( h(x) \) and \( k(x) \)
\[ \lim_{x \to 0} h(x) = 0, \quad \lim_{x \to \infty} h(x) = 1, \]
\[ \lim_{x \to 0} k(x) = 1, \quad \lim_{x \to \infty} k(x) = 0, \]
necessary for the solution to be regular at the origin and to have a finite action. The second of (9) implies that the gauge field at infinity behaves like a pure gauge, on the BPST instanton.\(^1\)

In the \( x \gg 1 \) region, we have found that the functions \( 1 - h \) and \( k \) decay exponentially. In the \( x \ll 1 \) region, we have found the following analytic behaviors
\[ h(x) = h_1 x + \mathcal{O}(x^3), \quad k(x) = 1 - k_2 x^2 + \mathcal{O}(x^4), \]
where \( h_1, k_2 \) are constants. The equations of motion arising from the reduced Lagrangian (7) were integrated numerically, subject to the boundary conditions (8) and (9).
The dependence of the action $A$ on the Skyrme coupling strength $\xi$ is presented in Fig. 1 for the two values $\lambda_0 = 0$ and $\lambda_0 = 1$. We note that, as $\xi$ tends to zero, the value of the action sinks to the value $A = 4\pi^2/3$ corresponding to the action of the pure SU(2) YM instanton. The reason is that as the stabilizing Skyrme-like term disappears, it becomes impossible to sustain a nontrivial Higgs field and the WS system reverts to the pure SU(2) YM model.

Fig. 1. The action (in units $2\pi^2$) of the instanton solution of (5) is shown as a function of the dimensionless parameter $\xi$ for $\lambda_0 = 0$, $\lambda_0 = 1$ and $\lambda = 1$.

Fig. 2. The same as Fig. 1 for the quantity $h'(0)$. 
The trivialization of the Higgs field in the limit of vanishing \( \xi \) can be seen more clearly from Fig. 2. Here we have plotted the first derivative of the function \( h \) at the origin against \( r \). The profile of the Higgs field function \( h(r) \) starts as usual from zero at \( r = 0 \) and increases to one as \( r \to \infty \). As the Skyrme coupling constant goes to zero the slope of the function \( h \) at the origin tends to infinity, resulting in a trivial Higgs field for which the function \( h(x) = 1 \) in \( \mathbb{R}^3 \). This demonstrates the role of the Skyrme-like term in enabling a finite size instanton in the WS model and the trivialization of the Higgs field when the Skyrme-like term disappears.

### 2.2. Sphalerons

The content of this subsection is already known since the static Hamiltonian arising from (7) is identical to that of the WS model. This is because the four form \( F_{\mu\nu\rho\sigma} \) in four dimensions vanishes in the static limit due to antisymmetry. Thus the sphaleron of this model, (2), are identical to the sphalerons of the WS model. The sphaleron of the model (3) on the other hand are different, but we restrict to model (2) here. For the completeness of the paper we just mention that the classical energy of the sphaleron is a monotonic function of the parameter \( \lambda \) and that

\[
E_{\text{sph}}(\lambda = 0) \approx 3.04(2\pi \eta), \quad E_{\text{sph}}(\lambda = \infty) \approx 5.41(2\pi \eta).
\]  

### 3. SU(2) Grassmannian Models

Next we carry out the same analysis as done above with the extended WS model to certain extensions of the SU(2) gauged Grassmannian model in four dimensions. The \( 4p \)-dimensional \( SO_{\pm}(4p) \) Grassmannian models introduced in Ref. 11 are scale-invariant, and support self-dual instantons. Thus their static Hamiltonians in \( (4p - 1) \) dimensions would necessarily violate the scaling requirement for finite energy and hence would not support sphalerons. This necessitates the modification of the scale-invariant Lagrangian suitably, such that the system supports both instantons and sphalerons.

The four-dimensional \( SO_{\pm}(4) = SU(2) \) scale-invariant gauged Grassmannian model is described by the Lagrangian

\[
L^{\text{grass}} = \text{Tr}(F_{\mu\nu}^2 + G_{\mu\nu}^2),
\]

where \( F_{\mu\nu} \) is the YM curvature and \( G_{\mu\nu} = D_{[\mu} z^{\dagger}_{\nu]} \), defined in turn in terms of the covariant derivative

\[
D_\mu z = \partial_\mu z - z A_\mu.
\]

The \((4 \times 2)\) Grassmann valued field \( z = (z_\alpha^i, z^a_i) \), with \( i = 1, 2, \alpha = 1, 2 \) and \( a = 3, 4 \), is subject to the \( 2 \times 2 \) condition \( z^{\dagger}_\alpha z = 1 \). We note here that the Grassmannian field \( z \) and its covariant derivative \( D_\mu z \) transform as

\[
z \to zg, \quad D_\mu z \to (D_\mu z)g,
\]

where \( g \) is the \( 2 \times 2 \) complex representation of \( SO_{\pm}(4) = SU(2) \).
The system (12) supports self-dual instantons satisfying the first order equations $F_{\mu\nu} = G_{\mu\nu}$, which are localized to an arbitrary scale. Here we will modify the system (12) such that in the static limit, in three dimensions, it supports sphaleron solutions. This necessitates the addition of the conventional quadratic kinetic term

$$\text{Tr} D_\mu z^\dagger D_\mu z = \text{Tr}|D_\mu z|^2.$$  

This in turn will cause the scaling requirement for finite action in four dimensions to be violated, so there is the need to augment the system further by a suitably scaling Skyrme-like term. Thus the system will be extended both by the quadratic kinetic term, as well as by a Skyrme-like term. In the event, when self-duality is not an issue, we eliminate the term $G_{\mu\nu}^2$ since it plays no indispensable role for scaling. We are motivated in this by economy.

Before proceeding to present the models which support instantons in the next subsection, and the sphalerons in the following one, we point out that the topological charge that stabilizes the instanton is the second Chern–Pontryagin class. This is because as in the WS case above, the Lagrangian features the usual YM term and the instantons will behave like pure gauges at infinity.

### 3.1. Instantons

In this case there are three candidates for Skyrme-like terms that enable the existence of instantons by compensating the scaling of the quadratic kinetic term.

$$\text{Tr}|F \wedge F|^2 = \text{Tr} F_{\mu\nu\rho\sigma}^2,$$

$$\text{Tr}|F \wedge Dz|^2 = \text{Tr}|F_{\mu\nu}D_\rho z|^2,$$

$$\text{Tr}|Dz \wedge Dz \wedge Dz|^2 = \text{Tr}|D_\mu z D_\nu z^\dagger D_\rho z|^2,$$

where we have used the same notation for a norm as that used in (15). The most economical models satisfying our requirements are those combining the YM term and quadratic kinetic term (15), with one of the Skyrme-like terms (16)–(18). To keep in the same spirit as the WS model, we also include a potential term in the three models

$$\mathcal{L}_1^{\text{grass}} = \text{Tr}[\tau^4(1 - z^\dagger \gamma_5 z) + \mu^2|Dz|^2 + |F|^2 + \kappa^4|F \wedge F|^2],$$

$$\mathcal{L}_2^{\text{grass}} = \text{Tr}[\tau^4(1 - z^\dagger \gamma_5 z) + \mu^2|Dz|^2 + |F|^2 + \kappa^2|G \wedge Dz|^2],$$

$$\mathcal{L}_3^{\text{grass}} = \text{Tr}[\tau^4(1 - z^\dagger \gamma_5 z) + \mu^2|Dz|^2 + |F|^2 + \kappa^2|F \wedge Dz|^2],$$

where the constants $\tau$ and $\mu$ have the dimensions of $L^{-1}$ and the latter of these two would play the role of the vector boson mass in an eventual symmetry breaking, while the constant $\kappa$ has the dimensions of $L$.

We will look for spherically symmetric solutions in four Euclidean dimensions. The spherically symmetric ansatz for the SU(2) gauge field is already given by the
first member of (4). The spherically symmetric ansatz for the Grassmannian field $z$ is

$$z^a_i = \sin \left( \frac{f(r)}{2} \right) \delta^a_i, \quad z^a_i = \cos \left( \frac{f(r)}{2} \right) \delta_{\mu} \left( \Sigma^a_\mu \right) i.$$

As in (4), the radial variable $r$ here is $r = \sqrt{x^\mu x^\mu}$.

We note here that the ansatz (22) is not strictly spherically symmetric since a four-dimensional space rotation cannot be counteracted by a gauge transformation on the field $z$, as can be seen from the action of the gauge group element $g$ in (14). We have verified that this ansatz leads to one-dimensional equations for the functions $k(r)$ and $f(r)$ which solve the Euler–Lagrange equations of each of the three Lagrangians (19)–(21). The one-dimensional equations are also arrived at by substituting this ansatz in those Lagrangians first, establishing consistency.

In the present work, we will restrict to the numerical integration of the simpler one of the three systems (2) and (19)–(21), namely to (19). Using the variable $x = \mu r$, the action reads in terms of a one-dimensional radial Lagrangian $L^{\text{grass}} \sim r^3 L^{\text{grass}}_1$

$$A(\Lambda, \xi) = \int L^{\text{grass}}_1 d^4 x, \quad \xi \equiv \kappa^4 \mu^4, \quad \Lambda = \frac{\tau^4}{\mu^4},$$

$$= 2\pi^2 \int L^{\text{grass}} dx,$$

with

$$L^{\text{grass}} = \Lambda x^3 (1 - \cos f)$$

$$+ \frac{x}{2} \left( \frac{x^2}{2} f'^2 + 3[k^2(1 + \cos f) + (k - 1)^2(1 - \cos f)] \right)$$

$$+ \left[ xk'^2 + \frac{4}{x} k^2 (k - 1)^2 \right] + \frac{\xi}{x^3} (k - 1)^2 k^2 k'^2.$$

We seek solutions of (25) satisfying the following boundary conditions,

$$\lim_{x \to 0} f(x) = \pi, \quad \lim_{x \to \infty} f(x) = 0,$$

$$\lim_{x \to 0} k(x) = 1, \quad \lim_{x \to \infty} k(x) = 0,$$

which are the asymptotic values of the anti-self-dual solutions\(^\text{15}\) to the scale invariant system (12). Since we are seeking solutions to the second order equations of motion here, we could just as well have chosen instead the asymptotic values of the self-dual solutions,\(^\text{15}\) namely $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to \infty} f(x) = \pi$ in place of (26), together with $\lim_{x \to 0} k(x) = 0$, $\lim_{x \to \infty} k(x) = 1$ in place of (27). We choose here (26), and hence also (27), since the former is the more conventional choice for various skyrmions.
We have verified that in the \( x \gg 1 \) region the functions \( f \) and \( k \) approach their asymptotic values given in (26)–(27) exponentially. In the \( x \ll 1 \) we have found the following asymptotic solutions

\[
 f(x) = \pi - f_1 x + O(x^3), \quad k(x) = 1 - k_2 x^2 + O(x^4),
\]

which are both consistent with the requirement of differentiability at the origin of the fields \((A_\mu, z)\) given by the first member of (4) and (22) respectively.

The equations of motion arising from the reduced Lagrangian (25) were integrated numerically, subject to the boundary conditions (26) and (27).

![Graph showing the action of the instanton solution of (23) as a function of the dimensionless parameter \( \xi \) and for \( \Lambda = 1 \) and \( \Lambda = 0.01 \) (the curves are identical).](image)

The dependence of the action on the Skyrme coupling strength (here \( \xi \)) is presented in Fig. 3 for the two values \( \Lambda = 0.01 \) and \( \Lambda = 1 \). It appears that the value of the action is not at all sensitive to the value if the coupling strength of the self-interaction potential of the Grassmann field. We note that as \( \xi \) tends to zero, the value of the action sinks again to the value \( 4\pi^2/3 \), i.e. the action of the pure SU(2) YM instanton. As in the previous case, when stabilizing Skyrme-like term disappears, it becomes impossible to sustain a nontrivial \( z \) field and the SU(2) gauged Grassmannian system (22) reverts to the pure SU(2) YM model. We demonstrate this here by plotting \( f'(0) \), the modulus of the slope of the function \( f(r) \), at the origin against \( \xi \). This is exhibited in Fig. 4, from which we conclude that the Grassmann field function \( f(r) \) drops infinitely fast from its value \( \pi \) at the origin to its asymptotic value of 0, namely that it vanishes everywhere rendering the Grassmannian field \( z \) trivial. This is just what happens in the WS model, where the Higgs field trivializes in the same limit.
There is another interesting limit in the less economical version of the present model (16). This is the model in which we have included the Skyrme-like term

\[ \text{Tr} \, G_{\mu\nu}^2 = \text{Tr} |Dz \wedge Dz|^2, \]

which we had omitted in (16) because it scales exactly the same as the YM term. Its inclusion however enables the study of the limit of this model

\[ \mathcal{L}_{4}^{\text{grass}} = \text{Tr}[\alpha \tau^4(1 - z^\dagger \gamma_5 z) + \alpha \mu^2 |Dz|^2 + |F|^2 + |Dz \wedge Dz|^2 + \alpha \kappa^4 |F \wedge F|^2], \]

in the limit \( \alpha \to 0 \) when it coincides with the scale-invariant model (12), which supports analytically evaluated self-dual solutions. In (29), \( \alpha \) is a dimensionless constant introduced to effect this limit.

The spherically symmetric one-dimensional reduced action density of (29), \( L_4 \sim r^3 \mathcal{L}_{4}^{\text{grass}} \), is

\[
L_4 = \alpha \Delta x^3 (1 - \cos f)
+ \frac{\alpha}{2} \left( \frac{x}{2} f'^2 + 3[k^2(1 + \cos f) + (k - 1)^2(1 - \cos f)] \right)
+ \left[ xk'^2 + \frac{4}{x} k^2 (k - 1)^2 \right] + \frac{1}{4} x f'^2 \sin^2 f
+ \frac{4}{x} \left[ k^2 + \frac{1}{2}(2k - 1)(\cos f - 1) \right] + \alpha \frac{\xi}{x^3} (k - 1)^2 k^2 k'^2. \]

In Fig. 5, we plot the action pertaining to the model (29) against the parameter \( \alpha \) and see that it decreases to the value of twice the action of the BPST instanton as
Fig. 5. The action (in units $2\pi^2$) of the instanton solution of (29), (30) in function of $\alpha$ without ($\tau = 0$, solid line) and with ($\tau = 1$, dashed line) potential. The quantity $\Delta_m$ of Eq. (31) is superposed on the figure.

In this limit, the $z$ field does not trivialize and the solution coincides with the known anti-self-dual solution where the function $k$ coincides with the corresponding function of the BPST solution, and the function $f$ is given by

$$k(x) - \sin^2 \frac{f(x)}{2} = 0.$$  

The quantity

$$\Delta_m \equiv \max_{0 \leq x \leq \infty} \left( k(x) - \sin^2 \frac{f(x)}{2} \right),$$  

is superposed on Fig. 5. It clearly indicates that $\Delta_m$ vanishes in the limit $\alpha \to 0$.

### 3.2. Sphalerons

We shall construct the sphaleron solution only for the model (19), whose instanton was constructed numerically in the previous subsection. Moreover, we shall restrict to the sphaleron solution itself and will not construct the noncontractible loops either using a geometric construction like that given by Manton\(^4\) or the finite energy path like that given by Akiba et al.,\(^16\) for the WS model. Both these constructions can be systematically carried out in the present models, but we do not carry them out here because the Grassmannian models at hand are not of immediate physical interest.
In the temporal gauge $A_0 = 0$, the static Hamiltonian corresponding to the model (19) reduces to

$$
H_{\text{grass}} = \frac{4}{\tau^4} \text{Tr}(1 - z^\dagger \gamma_5 z) + \mu^2 |Dz|^2 + |F|^2.
$$

(32)

The one-dimensional radial sphaleron ansatz for the $z$ field is formally similar to that of the instanton (22), and for the asymptotic properties of the field $z$ in this static limit we shall adopt the values given by (26) for the (Euclidean) time dependent theories. The ansatz for the static SU(2) gauge connection $A_i$, with $A_0 = 0$, can be inferred by requiring that asymptotically the covariant derivative of $z$ at infinity vanishes, i.e. that $A_i^\infty = (z^\infty)^\dagger \partial_i z^\infty$. This yields

$$
A_i = \frac{1 - f_A}{2r} i \varepsilon_{ijk} \sigma_j \hat{x}_k, \quad z^\alpha_i = \sin \frac{f}{2} \delta^\alpha_i, \quad z^a_i = \cos \frac{f}{2}(\hat{x} \cdot \sigma)^a_i.
$$

(33)

Note that the radial variable $r$ in (33) is $r = \sqrt{x_i x_i}$, unlike in the previous subsection. Again, we note here that the ansatz (33) is not strictly spherically symmetric since a three-dimensional space rotation cannot be counteracted by a gauge transformation on the field $z$, as can be seen from the action of the SU(2) gauge group element $g$ in (14). We have verified that this ansatz leads to one-dimensional equations for the functions $f_A(r)$ and $f(r)$ which solve the Euler–Lagrange equations of static Hamiltonian (32). The one-dimensional equations are also arrived at by substituting this ansatz in the Hamiltonian first, establishing consistency.

The classical energy can then be formulated in terms of a one-dimensional radial Hamiltonian

$$
E(\Xi) = \int H_{\text{grass}}^1 d^3 x, \quad \Xi = \frac{\tau^4}{\mu^4}
$$

(34)

$$
= 4\pi \mu \int H_{\text{grass}}^1 dx,
$$

(35)

where the variable $x = \mu r$ is used, and

$$
H_{\text{grass}} = \Xi x^2 (1 - \cos f) + \left( \frac{x^2}{4} f'^2 + \frac{1}{4} (f_A - 1)^2 (1 - \cos f) 
\right.

+ (f_A + 1)^2 (1 + \cos f) \left. \right) + \frac{1}{4} [f_A^2 + \frac{2}{x^2} (f_A^2 - 1)^2].
$$

(36)

We integrate the Euler–Lagrange equations arising from the one-dimensional density (36) subject to the following asymptotic conditions

$$
\lim_{x \to \pi} f(x) = \pi, \quad \lim_{x \to 0} f(x) = 0,
$$

(37)

$$
\lim_{x \to 0} f_A(x) = 1, \quad \lim_{x \to \infty} f_A(x) = -1,
$$

(38)

which are necessary for the solution to be regular at the origin and to have a finite energy.
We have verified that in the $x \gg 1$ region the functions $f$ and $f_A$ approach their asymptotic values stated in (37)–(38). In the $x \ll 1$ we have found the following asymptotic solutions for

$$f(x) = \pi - Bx + \mathcal{O}(x^3), \quad f_A(x) = 1 - Cx^2 + \mathcal{O}(x^4),$$

(39)

where $B, C$ are constants.

The result of our numerical computations are given in Fig. 6 where we have plotted the classical energy of the solution against the coupling constant $\Xi$. From the deep analogy between the model under investigation with the WS model we strongly believe that the static solution we have constructed is the counterpart of the Klinkhamer–Manton sphaleron. The demonstration of this statement would need, in particular, the construction of a noncontractible loop whose energy culminates at the solution presented above. We expect that such a noncontractible loop will excite the degrees of freedom corresponding to the full spherically symmetric ansatz (see e.g. Ref. 16), with a suitable extension of the ansatz of the Grassmannian fields.

**Fig. 6.** The energy (in units $4\pi \mu$) of the sphaleron solution of (31) as a function of the dimensionless parameter $\Xi$.

4. **Summary**

We have considered the two possible extensions of the Weinberg–Salam model, such that the extended models support instanton solutions, subject to the requirement that the Lagrangian density does not feature higher powers than the square of any velocity field. We have carried out the integrations numerically in the simplest model and have verified the existence of the instanton, and in addition have shown how the Higgs field trivializes in the limit when the Skyrme-like term extending the
WS model vanishes. In this limit, the instanton reduces to the BPST instanton.\textsuperscript{1} In the static limit, the Hamiltonian in the temporal gauge coincides with that of the WS model, so the sphaleron of this model is the same as the well studied case of the WS model.

We have carried out the same study for the corresponding extensions of a SU(2) gauged Grassmannian model.\textsuperscript{11} In this case there are three candidates for such a model. Again we have integrated the simplest such model numerically to find the instanton, and have examined two different limits of the parameters of the model, for which the instanton reduces to the BPST instanton.\textsuperscript{1} In one case, the Grassmannian field $z$ trivializes as does the Higgs field of the WS model, while in the other case it does not. Finally, we have integrated the equations of the static Hamiltonian of this model to find a regular, finite energy solution that we interpret as the sphaleron of the model.

References