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UPPER AND LOWER LIMITS ON THE EIGENVALUES OF THE SPINLESS SALPETER EQUATION

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In the context of relativistic quantum mechanics, we obtain a nonlinear first order differential equation for the energy as a function of the coupling constant of a central potential. This differential equation is only exact for power law and logarithmic potentials in the massless limit. For other potentials, we discuss under which conditions the differential equation yields rigorous upper and lower limits on the value of energy levels. These results are applied to the Cornell potential used in meson spectroscopy. We also show that the method applies to noncentral potentials.

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1. Introduction

Completely exactly solvable problems in nonrelativistic quantum mechanics of one (or two) particle are exceptional cases. This number of exceptional cases decreases drastically in the context of the relativistic quantum mechanics where the Schrödinger equation is replaced by the spinless Salpeter equation. In this case, only the problem of a particle with a vanishing mass bound by an harmonic oscillator potential can be solved exactly. Since very few potentials for which an exact treatment can be achieved exist, some methods and theorems have been found to cast some light, qualitatively or quantitatively, on the properties of the system without having to solve explicitly the corresponding wave equation. These methods and theorems are particularly interesting when they yield analytical formula from which conclusions about the properties of the system can be more easily inferred. Semiclassical methods, like WKB, yields in some cases accurate quantitative information about various observables (see for instance Ref. 1). Unfortunately, it is difficult to control the error introduced by these approximations. Indeed, the approximate energy levels can be, in some favorable cases, very close to the exact energy levels but it is not known if they yield upper or lower limits. Various bounds on some observables, the number of bound states[2,3] or the binding energy,[4–7] can also be

used to get information on the system (see also Refs. 8–12 for results concerning the Coulomb potential).

In this article, we obtain quantitative information about the evolution of the energy levels with a variation of the coupling constant of the potential for a system of two spinless particles. To obtain this evolution, we should know the value of the energy level for one value of the coupling constant. This additional input can be removed (with some loss of accuracy) since for any value of the coupling constant smaller than its critical value (for which the state has a vanishing energy) the position of the energy level is known: $E = 0$. Two lower limit on this critical value of the coupling constant for the ground state can be found in the literature.[2,3] The results of this work apply to relativistic quantum systems and are obtained essentially with the help of the Virial and of the Hellmann–Feynman theorems.

2. Upper and Lower Limits on the Eigenvalues

We consider systems for which the dynamics is governed by the Hamiltonian ($\hbar = c = 1$)

$$H = \alpha\sqrt{\mathbf{p}^2 + m^2} + V(r)\,, \tag{1}$$

where $r = |\mathbf{r}|$ and $\alpha = 1$ or 2 for one or two (identical) particle problems respectively. In the case of a two-body problem, the center of mass frame is chosen. If we write $V(r) = gr^{-1}v(r/R)$ (the formulation we present below holds for noncentral potentials, see Sec. 4), the expression for the energy is given by

$$M = \alpha\langle\sqrt{\mathbf{p}^2 + m^2}\,\rangle + gR^{-1}\langle v(r/R)\rangle\,, \tag{2}$$

where $M = E + \alpha m$ is the mass of the eigenstate and E is the binding energy.

The Hellmann–Feynman theorem, for a self-adjoint Hamiltonian, gives a relation between the derivative of the energy with respect to a given parameter, say λ, and the expectation value of the derivative of the Hamiltonian with respect to this parameter:

$$\frac{dM}{d\lambda} = \left\langle \frac{dH}{d\lambda} \right\rangle. \tag{3}$$

The Hellmann–Feynman theorem yields the following relations:

$$\frac{dM}{dg} = R^{-1}\langle v(x)\rangle\,, \tag{4}$$

where $x = r/R$.

The Virial theorem gives a relation between the expectation value of the kinetic energy and the directional derivative of the potential as follow:[13]

$$\langle\sqrt{\mathbf{p}^2 + m^2}\,\rangle = \frac{1}{\alpha}\langle\mathbf{r}\cdot\boldsymbol{\nabla}V(\mathbf{r})\rangle + \left\langle \frac{m^2}{\sqrt{\mathbf{p}^2 + m^2}} \right\rangle. \tag{5}$$

This relativistic Virial theorem used for the system described by the Hamiltonian (1) yields the formula

$$\langle \sqrt{\mathbf{p}^2 + m^2} \rangle = \frac{gR^{-1}}{\alpha} \left\langle x \frac{dv(x)}{dx} \right\rangle + \left\langle \frac{m^2}{\sqrt{\mathbf{p}^2 + m^2}} \right\rangle . \tag{6}$$

We consider the relation (2) together with (4) and (6) to obtain

$$\mathcal{M} = g \left\langle x \frac{dv(x)}{dx} \right\rangle + \alpha \left\langle \frac{\tilde{m}^2}{\sqrt{\mathbf{p}^2 + \tilde{m}^2}} \right\rangle + g \frac{d\mathcal{M}}{dg} , \tag{7}$$

where $\mathcal{M} = MR$ and $\tilde{m} = mR$. In the following, we restrict our attention to the ultrarelativistic regime ($m = 0$). We consider that the energy depend only on g; the other parameters, if any, are supposed to be are fixed. Suppose now that it is possible to write the following relation:

$$x \frac{dv(x)}{dx} = F(v(x)) . \tag{8}$$

The condition that $F(y)$ must satisfy in order to have rigorous upper and lower limits on the eigenvalues follows then from the Jensen inequality (see for example Refs. 14 and 15). Suppose that the $F(y)$ is a convex function in the following interval:

$$\frac{d^2F(y)}{dy^2} \geq 0 \quad \text{for} \quad y_m \leq y \leq y_M , \tag{9}$$

with y_m and y_M defined by the relation

$$y_m \leq v(x) \leq y_M \quad \text{for} \quad 0 \leq x < \infty . \tag{10}$$

The quantities y_m and y_M are not restricted to be finite. The Jensen inequality implies for a convex function $F(y)$ in the interval $y_m \leq y \leq y_M$ the following relation:

$$\langle F(v(x)) \rangle \geq F(\langle v(x) \rangle) = F\left(\frac{d\mathcal{M}}{dg} \right) . \tag{11}$$

Conversely, if $F(y)$ is a concave function in the same interval, then the Jensen inequality implies the following relation:

$$\langle F(v(x)) \rangle \leq F(\langle v(x) \rangle) = F\left(\frac{d\mathcal{M}}{dg} \right) . \tag{12}$$

Let us first consider that $F(y)$ is a convex function for $\min[v(x)] \leq y \leq \max[v(x)]$. We then obtain from (7) a closed nonlinear first order differential equation which reads

$$\mathcal{M} \geq gF\left(\frac{d\mathcal{M}}{dg} \right) + g \frac{d\mathcal{M}}{dg} \equiv gG^{-1}\left(\frac{d\mathcal{M}}{dg} \right) . \tag{13}$$

Suppose now that the derivative of $G(y)$ has a given sign. If $dG(y)/dy \geq 0$, we can write from (13),

$$G\left(\frac{\mathcal{M}}{g} \right) \geq \frac{d\mathcal{M}}{dg} . \tag{14}$$

We then define the quantity $\tilde{\mathcal{M}}$ by the differential equation

$$G\left(\frac{\tilde{\mathcal{M}}}{g}\right) = \frac{d\tilde{\mathcal{M}}}{dg}\,, \tag{15a}$$

with the boundary condition

$$\tilde{\mathcal{M}}(g = g_0) = \mathcal{M}(g = g_0)\,. \tag{15b}$$

Then we obtain the following upper and lower limits on the energy:

$$\tilde{\mathcal{M}} \geq \mathcal{M} \quad \text{for} \quad g \geq g_0 \quad \text{and} \quad \tilde{\mathcal{M}} \leq \mathcal{M} \quad \text{for} \quad g \leq g_0\,. \tag{16}$$

Indeed, at $g = g_0$, from (15b) we have the equality $\tilde{\mathcal{M}}(g = g_0) = \mathcal{M}(g = g_0)$ and the comparison between (14) and (15a) indicates that we have

$$\frac{d\mathcal{M}}{dg}(g = g_0) \leq \frac{d\tilde{\mathcal{M}}}{dg}(g = g_0)\,. \tag{17}$$

Consequently, the relations (16) certainly hold for g contained in some neighborhood around g_0. Now, it is obvious that we have only one crossing point between $\mathcal{M}$ and $\tilde{\mathcal{M}}$ (at $g = g_0$) because at another crossing point, says at $g = g_1$, we would have $\tilde{\mathcal{M}}(g = g_1) = \mathcal{M}(g = g_1)$ and $\frac{d\mathcal{M}}{dg}(g = g_1) \geq \frac{d\tilde{\mathcal{M}}}{dg}(g = g_1)$; the comparison between (14) and (15a) indicates that this is impossible. Since we have only one crossing point, the relation (16) holds for all values of g. The relation (16) can also be considered as a consequence of the Chaplygin theorem (see for example Refs. 16 and 17).

Conversely, if $dG(y)/dy \leq 0$, we can write from (13),

$$G\left(\frac{\mathcal{M}}{g}\right) \leq \frac{d\mathcal{M}}{dg}\,. \tag{18}$$

We again define the quantity $\tilde{\mathcal{M}}$ by the differential equation (15) to obtain the following upper and lower limits on the energy:

$$\tilde{\mathcal{M}} \leq \mathcal{M} \quad \text{for} \quad g \geq g_0 \quad \text{and} \quad \tilde{\mathcal{M}} \geq \mathcal{M} \quad \text{for} \quad g \leq g_0\,. \tag{19}$$

The differential equation (15) can be solved explicitly:

$$\tilde{\mathcal{M}} = g\Psi\left[\frac{g}{g_0}\Psi^{-1}\left(\frac{\mathcal{M}_0}{g_0}\right)\right] \tag{20a}$$

with

$$\Psi^{-1}(y) = \exp\left(\int \frac{dy}{G(y) - y}\right). \tag{20b}$$

A similar derivation is possible when $F(y)$ is a concave function. We are then in position to formulate the following theorem which summarize the results obtained above.

Theorem. *Let a one- or two-body relativistic quantum system be governed by the Hamiltonian* (1) *with* $m = 0$ *and* $V(r) = gR^{-1}v(r/R)$. *Let* M *be any eigenvalue*

of this Hamiltonian. Let $F(y)$ *be defined by the relation* $F(v(x)) = x\,dv(x)/dx$. *Let* $G(y)$ *be defined by the relation* $G^{-1}(y) = F(y) + y$. *Let the quantity* $\tilde{\mathcal{M}}$ *defined by the following nonlinear first order differential equation*:

$$G\left(\frac{\tilde{\mathcal{M}}}{g}\right) = \frac{d\tilde{\mathcal{M}}}{dg} \quad \text{with} \quad \tilde{\mathcal{M}}(g = g_0) = \mathcal{M}(g = g_0)\,. \tag{21}$$

Let $d^2F(y)/dy^2$ *and* $dG(y)/dy$ *have a given sign for* $\min[v(x)] \leq y \leq \max[v(x)]$. *If* $d^2F(y)/dy^2\, dG(y)/dy \geq 0$ *then*

$$M \leq \frac{\tilde{\mathcal{M}}}{R} \quad \text{for} \quad g \geq g_0\,,$$

$$M \geq \frac{\tilde{\mathcal{M}}}{R} \quad \text{for} \quad g \leq g_0\,.$$

If $d^2F(y)/dy^2\, dG(y)/dy \leq 0$ *then*

$$M \geq \frac{\tilde{\mathcal{M}}}{R} \quad \text{for} \quad g \geq g_0\,,$$

$$M \leq \frac{\tilde{\mathcal{M}}}{R} \quad \text{for} \quad g \leq g_0\,.$$

To conclude this section, we discuss the case where the differential equation (13) is exact. This occurs when the inequalities (11) and (12) are saturated, that is to say when $F(y) = 1$ or py. The corresponding potentials are obtained by solving the differential equation (8) and are simply $v(x) = \ln(x)$ or x^p. It is indeed well known that for these potential we have exact scaling relations.

3. Applications

In this section, we apply the method explained in Sec. 2 to meson spectroscopy. The simplest potential which is used to describe the main properties of spin-averaged vector mesons is the Cornell potential (see for example Refs. 18–20)

$$V(r) = -\frac{\kappa}{r} + ar\,. \tag{22}$$

The dimensionless parameter κ is related to strong coupling constant and its value is around 0.5. The parameter a is the string tension and its value is around 0.2 GeV^2. This potential can be written in, we think, a nice form

$$V(r) = V\,\sinh[\ln(r/R)]\,. \tag{23}$$

The advantage of the expression (23) is that it features parameters with more familiar units (energy and length). The link between the two set of parameters is simply

$$V = 2\sqrt{\kappa a} \cong 0.6 \text{ GeV} \quad \text{and} \quad R = \sqrt{\frac{\kappa}{a}} \cong 0.3 \text{ fm}\,. \tag{24}$$

This leads to

$$g = VR = 2\kappa. \tag{25}$$

It is easy to verify from the definition (8) that the function $F(y)$ takes the form

$$F(y) = \sqrt{1+y^2}\,. \tag{26}$$

The function $G(y)$ is found to be

$$G(y) = \sinh[\ln(y)]\,. \tag{27}$$

Since $F(y)$ is convex and the first derivative of $G(y)$ is positive, we can now use the theorem to obtain the following limits on the mass of the system:

$$\begin{aligned} M &\leq \frac{1}{R}\sqrt{gC_0 - g^2} \quad \text{for} \quad g \geq g_0\,, \\ M &\geq \frac{1}{R}\sqrt{gC_0 - g^2} \quad \text{for} \quad g \leq g_0\,, \end{aligned} \tag{28}$$

where

$$C_0 = \frac{1}{g_0}\left[\mathcal{M}_0^2 + g_0^2\right], \tag{29}$$

where $\mathcal{M}_0 = \mathcal{M}(g = g_0)$.

From the limits (28) we see that there exists a maximal value for g. This is a well-known result: there exists a critical value, $\kappa_c = 2\alpha/\pi$, for the strength of the coulomb potential when a relativistic kinematic operator is used[8] (this also appears for the Klein–Gordon and the Dirac equations). From this property we can obtain a nice lower bound on the mass of the system.

We note that, if we consider only the Coulomb potential ($a = 0$) and a massless particle ($m = 0$), the mass of the system is zero. Indeed, the lower bound on the mass of the eigenstate found by Herbst[8] and a variational upper bound on this mass[21] tend to zero in the limit of a vanishing mass of the particles. Consequently, if a (positive) confining potential is added to the interaction, the ground state have a positive mass when κ is below its critical value.

To obtain a lower limit on the mass of the ground state for a Cornell potential, we choose g_0 below the critical value but otherwise arbitrary. The upper bound (28) indicates that there exist a maximal value, $C_0 > g_0$, for g and from the relation (25) between g and κ, this yields a maximal value for κ. This maximal value is then, for any choice of g_0, an upper bound on the critical value. Indeed, it cannot be a lower bound because in this case for $2\kappa \equiv g = C_0$, the mass of the system would certainly be positive which contradicts the upper limit (28). We then obtain the upper bound on the critical value $2\kappa_c \leq C_0$, and since this upper bound is true for any g_0, we obtain, using (29), a lower limit on the mass of the system

$$M \geq 2\sqrt{a\left(\frac{2\alpha}{\pi} - \kappa\right)}\,, \tag{30}$$

where we have used the value of κ_c. Such a simple and explicit analytical lower bound on the ground state energy of the spinless Salpeter equation are rare in the literature, especially for a potential which is not a pure power law.

4. Noncentral Potentials

We conclude this article by showing briefly that the method explained above applies to noncentral potentials. It is easy to show that the relation (7) with $m = 0$ is replaced, for noncentral potentials, by the equation

$$\mathcal{M} = g\langle \mathbf{x} \cdot \boldsymbol{\nabla} v(\mathbf{x}) \rangle + g \frac{d\mathcal{M}}{dg} \,. \tag{31}$$

We now define the function $F(y)$ as a generalization of the definition (8):

$$\mathbf{x} \cdot \boldsymbol{\nabla} v(\mathbf{x}) = F(v(\mathbf{x})) \,. \tag{32}$$

With this definition, the theorem applies exactly in the same way. The difficulty in general is to solve the partial differential equation (32). This equation can be solved, for example, for $F(y) = 1$ and $F(y) = \alpha y$. For these choices of $F(y)$, we obtain an equality between $\mathcal{M}$ and $\tilde{\mathcal{M}}$ and thus an exact scaling property.

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