Edward E. Roskam (Ed.)

Mathematical Psychology in Progress

Universitas Bruxellensis
SILO

081
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n° 11

Springer-Verlag

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n° 11

Editor

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ISBN 3-540-51686-7 Springer-Verlag Berlin Heidelberg New York

ISBN 0-387-51686-7 Springer-Verlag New York Berlin Heidelberg

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Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2126/3140-543210 - Printed on acid-free paper

An Impossibility Theorem for Fair Bidimensional Representation:
Towards a Biproportional Solution¹.

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Abstract

This paper concerns fair (proportional) representation of a population which is divided into categories according to two criteria as is the case, for example, when an electoral system takes into account the geographical as well as the political group of the elector. Our first model leads us to the conclusion that, on the one hand, there can be no fair representation - under our very weak conditions - as soon there are more than two constituencies or more than two parties, but on the other hand, a two by two situation can always be fairly represented. As these results are mainly negative, we go on to a second model and we give a necessary and sufficient condition for the existence of a biproportional delegation.

I. Introduction

It has now become classical, in mathematical voting and representation theory to seek normative or even ethical conditions for social and political processes. In this spirit, the concrete problem of a Dutch election led us to conceive equity conditions in the search for a two-dimensional fair representation.

When a population, classified according to a given criterion (such as belonging to a region, to a political party,...), is to elect delegates in order to form an assembly, the question of the fairness of the representation arises; the fundamental barrier is that the ideal apportionment, from the point of view of fairness, should be proportional (in the mathematical sense)

¹Several conversations with G. De Meur, X. Hubaut and Ph. Vincke stimulated a revision of a first draft of this paper, including shortcuts in the proof of our possibility theorem.

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to the respective populations of the classes, which is generally impossible to achieve in integers.

If many elections take into account a division of the population based upon one criterion only (geographical, for example), some others are based upon several criteria simultaneously (for example geographical and partisan). In the latter case, the same problem of fairness arises for each of the criteria taken separately; but an extra difficulty is created by the interference of the criteria with each other; a kind of "multi-dimensional" balance must be found.

Political scientists and mathematicians have broadly examined the one-dimensional problem (a complete survey of the subject can be found in Balinski and Young, 1982), but the only formalized study of a multi-criteria electoral system that we encountered, is one of a regional election in the Netherlands (Anthonisse, 1984). In this study, the electors of a region are ranked according to two criteria: the commune in which they vote and the political party of their choice. Among the communal councilors of a region, a delegation must be chosen which will form the regional council. This must be done while respecting as accurately as possible the political and geographical (parties, communes) distribution of the communal mandatories. The method described by Anthonisse, while respecting legal texts, cannot avoid certain impossibilities and, furthermore, is contrary to some aspects of fairness which seem fundamental to us (De Meur, Gassner, Hubaut, 1984 and 1985).

In § 2, we set ourselves in an institutional frame derived from real-life situations i.e. the total number of delegates will be predetermined, as will their distribution according to one of the two criteria; these numbers are not arbitrarily chosen, as they must obey some conditions. We chose, among the different fairness aspirations one could reasonably have, some conditions which we judged "minimal" in order to meet those demands. Even though these requirements were quite modest, we had to prove that it is impossible to satisfy the constraints when one of the two criteria offers more than two modalities. In § 3 however, we prove the existence of a fair representation, as we define it, when each of the dimensions offers two options only.

We judged that the latter results were not satisfactory, at least as far as democracy was concerned, on the one hand because of the unrealistic size of the only possible case (very few real-life election situations, for example, are limited to two constituencies and two political parties!); on the other hand, even when a "fair" delegation does exist, the total numbers of delegates per party can be forced very far away from those obtained by dealing out the

total number of seats available to the parties proportionally to their scores by a one-dimensional rule of apportionment (Hamilton's or d'Hondt's method, for example); in § 4, following these remarks, we considered it most important that the total number of seats available should be allotted to constituencies and to parties proportionally to their respective strengths (i.e. by a classical one-dimensional rule of apportionment); we show that an integer delegation which sums up to these a priori determined fixed numbers and is "near" global proportionality does not necessarily exist and we give a necessary and sufficient condition for the existence of a biproportional delegation i.e. one which is "near" the classical econometric RAS biproportional solution.

2. An Impossibility Theorem

The distribution into classes of a population ranked by two criteria can be described by a matrix (c_{ij}) of non-negative integers where element c_{ij} is the number of individuals associated with option i of the first criterion and option j of the second. Examining real-life cases of two-dimensional settings (see Anthonisse, 1984, for example) convinced us of the interest to make a distinction between the two dimensions, and not to award them systematically identical properties.

In this manner, we retain the possibility of imposing "institutional" constraints on one of the criteria and purely logical ones on the other: for example, the size of the global representation of a constituency (first criterion) can be predetermined by an electoral law, while the size of the global representation of a party (second criterion) is a consequence of the considered political and geographical setting. In the same way, the ordinal structure induced by the initial sizes of the populations will be respected differently for the two dimensions. More precisely, we formalize the problem as follows:

from a given n by m matrix (c_{ij}) of non-negative integers, one must derive a matrix (r_{ij}) of non-negative integers of same dimensions, the row sums of which are predetermined (rows being associated with the first criterion, columns with the second). These given values and the unknown elements of (r_{ij}) must obey the following constraints of fairness:

Condition 1 (C1)

$$\forall i, 0 < r_i < c_i \text{ except if } c_i = 1 \text{ in which case } r_i = 0 \text{ or } 1. \quad (i)$$

$$\forall i, j: \begin{cases} c_i < c_j \Rightarrow r_i \leq r_j \text{ and } r_j - r_i \leq c_j - c_i \\ c_i = c_j \Rightarrow |r_i - r_j| \leq 1 \end{cases} \quad (ii)$$

Condition 2 (C2)

$$\forall i, j: 0 \leq r_{ij} \leq c_{ij}$$

$$\forall i, j, k: \begin{cases} c_{ij} < c_{ik} \Rightarrow r_{ij} \leq r_{ik} \text{ and } r_{ik} - r_{ij} \leq c_{ik} - c_{ij} \\ c_{ij} = c_{ik} \Rightarrow |r_{ij} - r_{ik}| \leq 1 \end{cases} \quad (iii)$$

Condition 3 (C3)

$$\forall i, j: \begin{cases} c_i < c_j \Rightarrow r_i \leq r_j \text{ and } r_j - r_i \leq c_j - c_i \\ c_i = c_j \Rightarrow |r_i - r_j| \leq 1 \end{cases}$$

Commentary concerning the formalization

a) C1, which restricts the number of allowed given values for the communes' delegations, increases the possibility of its existence as opposed to C2 and C3, which tend to reduce it.

b) Generally, one wishes a "true" delegation, meaning non zero and lower than the initial size for each option of the first criterion (i). The non zero condition justifies itself by avoiding immediate impossibilities, even in the simple case of 2 by 2 matrices:

$$\text{If } (c_{ij}) = \begin{pmatrix} 0 & 2 \\ 4 & 3 \end{pmatrix} \begin{matrix} 2 \\ 7 \\ 4 & 5 & 9 \end{matrix} \text{ then } (r_{ij}) = \begin{pmatrix} a \leq b \\ c \geq d \end{pmatrix} \begin{matrix} 0 \\ 1 \\ a+csb+d & 1 \end{matrix}$$

which is clearly impossible to obtain.

On the other hand, the exception to this rule, in the elementary case where the initial size is 1, allows to prevent an immediate cul-de-sac.

c) One will see to it that the order induced by the marginal comparisons between c_i and c_j not be reversed in the delegation and that the increase of

the initial difference between these two numbers is severely restricted: for a positive initial difference, no increase will be allowed; an initial zero difference will be allowed to reach one (ii).

d) The same kind of concern appears in the conditions involving elements r_{ij} ; nevertheless, the little sense contained in comparing any two elements of the matrix, and the institutional character of the first criterion, made us limit internal comparisons to elements within a same row (iii).

We call fair representation rule, a rule which associates any matrix (c_{ij}) of $N^{n \times m}$ and any n-uple $\bar{r} = (r_1, r_2, \dots, r_n)$ of N^n which obeys C1, with a matrix (r_{ij}) of $N^{n \times m}$ such that, for each i , $r_i = \sum_{j=1}^m r_{ij}$ and which obeys C2 and C3.

THEOREM 1: if n or m is larger than 2, no such rule exists.

Proof:

It is sufficient to exhibit two counter examples: one (a) of $N^{3 \times 2}$ and the other (b) of $N^{2 \times 3}$.

$$(a) \text{ Let } (c_{ij}) = \begin{pmatrix} 4 & 1 \\ 2 & 3 \\ 2 & 3 \end{pmatrix} \text{ and } \bar{r} = (1, 1, 1) \text{ which obeys C1.}$$

$$\text{To obey C2, } (r_{ij}) \text{ must be } \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \text{ which does not respect C3.}$$

$$(b) \text{ Let } (c_{ij}) = \begin{pmatrix} 3 & 4 & 1 \\ 3 & 1 & 4 \end{pmatrix} \text{ and } \bar{r} = (1, 1) \text{ which obeys C1.}$$

$$\text{To obey C2, } (r_{ij}) \text{ must be } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ which does not respect C3.}$$

3. A Possibility Theorem for the Smallest Case

THEOREM 2: If $m=n=2$, at least one fair representation rule exists

Proof:

For any given matrix $(c_{ij}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $N^{2 \times 2}$ and any two non-negative

integers k_1, k_2 obeying C1, we build a matrix (r_{ij}) of $N^{2 \times 2}$, the row sums of which equal respectively k_1 and k_2 and which respects C2 and C3.

(r_{ij}) can thus be written as $\begin{pmatrix} a & k_1 - a \\ k_2 - d & d \end{pmatrix}$

Without loss of generality, we suppose

$$A + C \geq B + D \quad \text{and} \quad A \geq B$$

We thus have to examine four cases:

- 1) $A > B$ and $C < D$
- 2) $A > B$ and $C > D$
- 3) $A > B$ and $C = D$ (identical to $A = B$ and $C > D$)
- 4) $A = B$ and $C = D$

CASE 1

$A > B$ and $C < D$

First step

$A + C > B + D$

Condition C2 imposes the following bounds for a and d :

$$\alpha = \max \left\{ \frac{k_1}{2}, k_1 - B \right\} \leq a \leq \beta = \min \left\{ k_1, \frac{(k_1 + A - B)}{2} \right\} \quad [1]$$

$$\gamma = \max \left\{ \frac{k_2}{2}, k_2 - C \right\} \leq d \leq \delta = \min \left\{ k_2, \frac{(k_2 + D - C)}{2} \right\} \quad [2]$$

Condition C3 implies

$$d - a \leq \frac{k_2 - k_1}{2} \quad [3]$$

and

$$d - a \geq \frac{k_2 - k_1 - (A + C) + (B + D)}{2} \quad [4]$$

In this case, we thus have to prove the existence of two non-negative integers a and d which obey [3], [4], [5] and [6].

Each of the intervals $[\alpha, \beta]$ and $[\gamma, \delta]$ obviously contains at least one non-negative integer. Let us use a geometrical representation in a real plane coordinated by axes a and d .

In this way, rectangle $\mathcal{R} = \{ (a, d) : \alpha \leq a \leq \beta \text{ and } \gamma \leq d \leq \delta \}$ contains at least one point with non-negative integer coordinates. [7]

[5] and [6] determine two half-planes:

$$Z_1 = d - a \leq \frac{k_2 - k_1}{2}$$

and

$$Z_2 = d - a \geq \frac{k_2 - k_1 - (A + C) + (B + D)}{2}$$

Their intersection $\mathcal{S} = Z_1 \cap Z_2$ is a strip parallel to the bisectrix of axes a and d ; the width of $\mathcal{S}$ is at least $\frac{1}{2}$. [8]

FIGURE 1

It is thus sufficient to prove the existence, in $\mathcal{R} \cap \mathcal{Y}$, of at least one point with non-negative integer coordinates.

Lemma 1: $\mathcal{R} \cap \mathcal{Y}$ is non-empty.

Proof:

FIGURE 2

We prove that (α, δ) is in Z_2 and (β, γ) is in Z_1 .

$(\alpha, \delta) \in Z_2$ is equivalent to

$$\delta \geq \alpha + \frac{k_2 - k_1 - (A + C) + (B + D)}{2}$$

1) Suppose $\alpha = \frac{k_1}{2}$ and $\delta = k_2$

We must prove

$$k_2 \geq \frac{k_1}{2} + \frac{k_2 - k_1 - (A + C) + (B + D)}{2}$$

which is equivalent to $k_2 \geq (B + D) - (A + C)$.

As $k_2 \geq 0$ by C_1 , by [2], we have $k_2 \geq (B + D) - (A + C)$.

2) Suppose $\alpha = k_1 - B$ and $\delta = k_2$

We must prove

$$k_2 \geq k_1 - B + \frac{k_2 - k_1 - (A + C) + (B + D)}{2}$$

which is equivalent to $k_1 - k_2 \leq A + C + B - D$.

If $k_2 \geq k_1$, this is obviously true by [2].

If $k_2 < k_1$, two cases arise:

(i) $A + B > C + D$; then, by C_1 , we have:

$$k_1 - k_2 \leq (A + B) - (C + D) \leq A + C + B - D$$

(ii) $A + B = C + D$; then, by C_1 , we have $k_1 = k_2 + 1$ and we must prove

$k_1 - k_2 \leq A + C + B - D$ which is equivalent to $1 \leq 2C$; this is true unless $C = 0$; however, in that case, $A + B = D$ and, by [2],

$A > B + D$, a contradiction.

3) Suppose $\alpha = \frac{k_1}{2}$ and $\delta = \frac{(k_2 + D - C)}{2}$

We must prove

$$\frac{k_2 + D - C}{2} \geq \frac{k_1}{2} + \frac{k_2 - k_1 - (A + C) + (B + D)}{2}$$

which is equivalent to $0 \geq B - A$, obvious by [1].

4) Suppose $\alpha = k_1 - B$ and $\delta = \frac{(k_2 + D - C)}{2}$

We must prove

$$\frac{k_2 + D - C}{2} \geq k_1 - B + \frac{k_2 - k_1 - (A + C) + (B + D)}{2}$$

which is equivalent to $k_1 \leq A + B$; by C_1 , in this case, $k_1 < A + B$ as [1] and [2] do not allow $A + B = 1$.

Using the same arguments, (β, γ) is in Z_1 . $\square$

In the four cases considered hereabove, it should be noted that (α, δ) is in fact in $\dot{Z}_2$ (interior of Z_2) (resp. (β, γ) is in $\dot{Z}_1$) except eventually if $\alpha = k_1 - B$ and $\delta = k_2$ (resp. $\gamma = k_1$ and $\beta = k_2 - C$); however, in that case, (α, δ) (resp. (β, γ)) is in N^2 and is thus a solution of the problem.

Let us note that all intersection points of the borders of $\mathcal{R}$ and $\mathcal{Y}$ have integer or half-integer coordinates. [9]

If $\mathcal{R}$ is not degenerated (i.e. $\alpha \neq \beta$ and $\gamma \neq \delta$ or, equivalently, neither B nor C is zero), in general, by [8], [9] and lemma 1, $\mathcal{Y} \cap \mathcal{R}$ contains one of the following configurations

FIGURE 3

With P_1 's (and thus all P_i 's) coordinates being non-negative integers or half-integers.

Obviously, one of the P_i 's has non-negative integer coordinates. The only exception is when $\mathcal{Y} \cap \mathcal{R}$ is a right isosceles triangle, the edges of the right angle measuring $\frac{1}{2}$.

FIGURE 4

$$\text{Euclidean distance } d(P, Q) = d(P, R) = \frac{1}{2}$$

The only case to be examined is that where P 's coordinates are both half-integer, as otherwise one of points P, Q, R is a solution. If α and δ (resp. β and γ) were both half-integer, then α would be $\frac{k_1}{2}$ and δ would be $\frac{(k_2 + D - C)}{2}$ (resp. β would be $\frac{(k_1 + A - B)}{2}$ and γ would be $\frac{k_2}{2}$).

Let us compute R 's coordinates by two different ways:

on the one hand, as a point at distance $\frac{1}{2}$ from P , R 's coordinates are $\left(\frac{k_1}{2}, \frac{(k_2 + D - C - 1)}{2} \right)$ (resp. $\left(\frac{(k_1 + A - B)}{2}, \frac{(k_2 + 1)}{2} \right)$);

on the other hand, as an intersection of the borders of $\mathcal{Y}$ and $\mathcal{R}$, R 's coordinates are

$$\left(\frac{k_1}{2}, \frac{(k_2 - (A + C) + (B + D))}{2} \right) \text{ (resp. } \left(\frac{(k_1 + A - B)}{2}, \frac{(k_2 + A - B)}{2} \right))$$

This implies $\frac{(A - B)}{2} = \frac{1}{2}$ or $A = B + 1$, a contradiction by [1] and [2].

To end the proof of this first step, we must examine the cases in which $\mathcal{R}$ is degenerated:

1') $B = C = 0$

In this case, $\mathcal{R}$ becomes $\{(k_1, k_2)\}$ which is, by lemma 1, the unique solution.

2') $B = 0 \neq C$

In this case, $\alpha = \beta = k_1$

FIGURE 5

Lemma 1 shows that (k_1, γ) is in $\dot{Z}_1$ and (k_1, δ) is in $\dot{Z}_2$. It is thus obvious that $\mathcal{R} \cap \mathcal{Y}$ contains a point with integer coordinates.

3*) $B \neq 0 = C$

Same argumentation as in 2*).

Second step

$$A + C = B + D$$

[2']

[3] and [4] again give bounds for a and d .

C3 and [2'] imply that one of the following must hold:

$$d - a - \frac{k_2 - k_1}{2} = 0 \text{ or } \frac{1}{2} \text{ or } -\frac{1}{2}$$

$$\text{Let } D_1 = d = a + \frac{(k_2 - k_1)}{2}, D_2 = d = a + \frac{(k_2 - k_1 + 1)}{2}$$

$$\text{and } D_3 = d = a + \frac{(k_2 - k_1 - 1)}{2}$$

It is sufficient, in this case, to prove, rectangle $\mathcal{R}$ having been defined in the first step, the existence of at least one point with non-negative coordinates in $\mathcal{R} \cap (D_1 \cup D_2 \cup D_3)$.

Lemma 2: $\mathcal{R} \cap D_1$ is non-empty

Proof:

Let

$$Z_4 = d - a \leq \frac{(k_2 - k_1)}{2} + \frac{1}{2}$$

and

$$Z_5 = d - a \geq \frac{(k_2 - k_1)}{2} - \frac{1}{2}$$

By the same arguments used in the proof of lemma 1, we can prove that (α, δ) is in $\dot{Z}_5$ and (β, γ) is in $\dot{Z}_4$. $\square$

If $\mathcal{R}$ is not degenerated:

(i) $D_1 \cap \mathcal{R}$ is not a singleton, then $D_2 \cap \mathcal{R}$ and $D_3 \cap \mathcal{R}$ are non-empty and

$\mathcal{R} \cap (D_1 \cup D_2 \cup D_3)$ thus obviously contains a solution.

(ii) $D_1 \cap \mathcal{R}$ is a singleton $\{P(\alpha, \delta)\}$ (or $\{P(\beta, \gamma)\}$), let us prove that it is impossible for both of P 's coordinates to be half-integer.

Indeed, if this was the case, α would be $\frac{k_1}{2}$ and δ would be $\frac{(k_2 + D - C)}{2}$ (resp. β would be $\frac{(k_1 + A - B)}{2}$ and γ would be $\frac{k_2}{2}$)

with $\frac{(k_2 + D - C)}{2} = \frac{k_1}{2} + \frac{(k_2 - k_1)}{2}$ which implies $D = C$ (resp.

with $\frac{k_2}{2} = \frac{(k_1 + A - B)}{2} + \frac{(k_2 - k_1)}{2}$ which implies $A = B$), a

contradiction by [1].

If $\mathcal{R}$ is degenerated, the same arguments as in the first step hold.

CASE 2

$A > B$ and $C > D$ which implies $A + C > B + D$

Condition C2 implies, in this case:

$$\alpha = \max \left\{ \frac{k_1}{2}, k_1 - B \right\} \leq a \leq \beta = \min \left\{ k_1, \frac{(k_1 + A - B)}{2} \right\} \quad [3']$$

$$\gamma = \max \left\{ \frac{(k_2 + D - C)}{2}, 0 \right\} \leq d \leq \delta = \min \left\{ \frac{k_2}{2}, D \right\} \quad [4']$$

Condition C3 is obviously, in this case, a consequence of [3'] and [4'], and it is thus sufficient to notice that there is at least one point with non-negative integer coordinates in $\mathcal{R}$ as defined in case 1.

CASE 3

$A > B$ and $C = D$ (and thus $A + C > B + D$)

Condition C2 implies, in this case:

$$\alpha = \max \left\{ \frac{k_1}{2}, k_1 - B \right\} \leq a \leq \beta = \min \left\{ k_1, \frac{(k_1 + A - B)}{2} \right\}$$

and

$$d = \frac{k_2}{2} \text{ (} k_2 \text{ even) or } d = \frac{(k_2 - 1)}{2} \text{ or } \frac{(k_2 + 1)}{2} \text{ (} k_2 \text{ odd)}$$

Condition C₃ is again a consequence of the latter and it is thus sufficient to notice that $[\alpha, \beta]$ contains at least one non-negative integer.

CASE 4

$A = B$ and $C = D$ (and thus $A + C = B + D$)

Condition C₂ implies, in this case:

$$a = \frac{k_1}{2} \text{ (} k_1 \text{ even) or } a = \frac{(k_1 - 1)}{2} \text{ or } \frac{(k_1 + 1)}{2} \text{ (} k_1 \text{ odd)}$$

and

$$d = \frac{k_2}{2} \text{ (} k_2 \text{ even) or } d = \frac{(k_2 - 1)}{2} \text{ or } \frac{(k_2 + 1)}{2} \text{ (} k_2 \text{ odd)}$$

Condition C₃ implies:

$$d - a - \frac{k_2 - k_1}{2} = 0 \text{ or } \frac{1}{2} \text{ or } -\frac{1}{2}$$

Whether k_1 and k_2 are even or odd, it is obvious that a and d can be chosen in a way which satisfies C₂ and C₃ simultaneously.

4. Towards a Biproportional Solution

The latter results are not satisfactory, at least as far as democracy is concerned, on the one hand, and mainly, because of the unrealistic size of the only possible case: indeed, very few real-life election situations are limited to two constituencies and two political parties; on the other hand, even when a "fair" delegation does exist, the total numbers of delegates per party can be forced very far away from those obtained by dealing out the total number of seats available proportionally to the parties' scores by a one-dimensional rule of apportionment³.

The main feature in this type of representation problem is the necessity to take into account both dimensions simultaneously. Obviously, priorities must be set: on which parts of the delegation is it most important to set fairness constraints (knowing that it is impossible to demand too much), and what conditions should be chosen? The answer is not, in our opinion, strait

³Hamilton's or d'Hondt's method, for example.

forward. Different problems trigger different priorities. However, most existing electoral systems of the type described⁴ need the a priori determination of the numbers of delegates due to each constituency. This is achieved by dealing out a fixed number of seats to constituencies "proportionally" to their numbers of voters⁵ using a classical rule of apportionment; secondly, "proportionality" should also hold globally between political parties with respect to the amount of votes they received (otherwise, why not simply consider each constituency separately, dividing the two-dimensional problem into several one-dimensional ones).

Following the latter remarks concerning existing systems, we found it was appropriate to award the seats to constituencies and to parties proportionally to their respective magnitudes by some one-dimensional rule of apportionment. The problem remaining is thus to find an integer delegation which sums up to these predetermined total margins. Recalling results of § 2 and § 3, one can obviously set up examples of situations for which no such delegation can respect condition C₂; a natural question is then whether there always exists such a delegation, each element of which is an integer adjacent to the exact proportion of seats the corresponding number of votes is worth⁶ (a sort of "global" proportionality). Unfortunately, the answer is "no". Indeed, consider the following matrix of vote results, each row representing a constituency, each column a political party. Suppose 16 seats are to be dealt out.

3960	750	4710
1960	800	2760
1950	700	2650
940	1700	2640
1610	10	1620
1610	10	1620
12030	3970	16000

Global exact proportionality yields:

⁴Such as those used for parliamentary elections in Belgium (Gassner, 1987) or for some regional elections in the Netherlands (Anthonisse, 1984).

⁵Or their populations.

⁶This means quantity $\frac{v_{ij} \cdot S}{\sum_j v_{ij}}$ if v_{ij} is the number of votes for party j in constituency i and S is the total number of seats available.

$$\begin{pmatrix} 3.96 & 0.75 \\ 1.96 & 0.80 \\ 1.95 & 0.70 \\ 0.94 & 1.70 \\ 1.61 & 0.01 \\ 1.61 & 0.01 \end{pmatrix} \begin{matrix} 4.71 \\ 2.76 \\ 2.65 \\ 2.64 \\ 1.62 \\ 1.62 \end{matrix}$$

2.03 3.97 16

Rounding row sums by either Hamilton's or d'Hondt's rule of apportionment yields (5, 3, 3, 3, 1, 1). As the rounded values of the first column must add up to a minimum of twelve, obviously at least one of the two last rows must be rounded up, making it impossible to obtain the imposed row marginal sums. This example thus shows that if one wishes to stay "near" exact proportions, it may be impossible for the rounded values to sum up marginally to predetermined numbers of delegates. As this is of major importance in many electoral systems we are concerned with here, we tried to find an alternative solution.

It can be easily shown that there exist integer matrices which add up to given integer row and column sums (as long as the sum of the row sums equals that of the column sums). If we impose, for example, that row sums be (5, 3, 3, 3, 1, 1) and the column sums be 12 and 4, we have, for example:

$$\begin{pmatrix} 5 & 0 \\ 3 & 0 \\ 2 & 1 \\ 0 & 3 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{matrix} 5 \\ 3 \\ 3 \\ 3 \\ 1 \\ 1 \end{matrix}$$

12 4 16

However, we must somehow include the notion of proportionality between the initial matrix of votes and an acceptable delegation matrix.

The latter sort of problem will undoubtedly remind economists of the well-known econometric "RAS" model designed to estimate the behavior of input-output relations through time: an initial matrix is known, and one must derive from it another matrix, the marginal sums of which are given.

Bacharach (1970) precised what he calls the biproportional constrained matrix problem as follows: given A , a non-negative $m \times n$ matrix and

$$\begin{cases} I = (I_1, I_2, \dots, I_m) > 0 \\ c = (c_1, c_2, \dots, c_n) > 0 \end{cases}$$

find $B = (b_{ij})$, a non-negative $m \times n$ matrix with

$$\begin{cases} \sum_i b_{ij} = c_j \quad \forall j \in \{1, \dots, n\} \\ \sum_j b_{ij} = I_i \quad \forall i \in \{1, \dots, m\} \end{cases}$$

and $B = \lim \langle R^t \rangle A \langle S^t \rangle$ for some sequences $\langle R^t \rangle$, $\langle S^t \rangle$ of vectors, $R_i^t = 1 \quad \forall t$ ($\langle M \rangle$ is a diagonal matrix).

He proved the problem has a unique solution if and only if:

$$a_{ij} = 0 \quad \forall i \in I', \quad \forall j \in J \Leftrightarrow \begin{cases} \sum_{i \in I'} I_i \leq \sum_{j \in J'} c_j \\ \sum_{j \in J} c_j \leq \sum_{i \in I} I_i \end{cases}$$

where I and J are respectively subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, I' is $\{1, \dots, m\} \setminus I$ and J' is $\{1, \dots, n\} \setminus J$.

This possibility condition prevents the following immediate impossibility due to the eventual presence of zeros in the initial matrix. Indeed, obviously $a_{ij} = 0 \Rightarrow b_{ij} = 0$ so, if A is of the following form:

$$\begin{matrix} & \begin{matrix} J' & J \end{matrix} \\ \begin{matrix} I' \\ I \end{matrix} & \left(\begin{array}{c|c} A_{I', J'} & 0 \\ \hline A_{IJ'} & A_{IJ} \end{array} \right) \end{matrix}$$

then B must be such that block $B_{I'J}$ contains only zeros; thus

$$B = \begin{matrix} & \begin{matrix} J' & J \end{matrix} \\ \begin{matrix} I' \\ I \end{matrix} & \left(\begin{array}{c|c} B_{I', J'} & 0 \\ \hline B_{IJ'} & B_{IJ} \end{array} \right) \begin{matrix} \sum_{i \in I'} I_i \\ \sum_{i \in I} I_i \end{matrix} \end{matrix}$$

$\sum_{j \in J'} c_j \quad \sum_{j \in J} c_j$

the possibility condition prevents the total sum of the elements of block

B_{ij} (resp. B_{ij}) from exceeding that of those of block $B_{i'j'}$ (resp. $B_{i'j'}$), an immediate impossibility.

Bacharach also describes a "natural" algorithm yielding the solution i.e.

$$A^0 = A$$

$$A^{2t+1} = \langle R^{t+1} \rangle A^{2t}$$

$$A^{2t+2} = \langle R^{t+1} \rangle A^{2t} \langle S^{t+1} \rangle$$

$$\text{where } R_i^{t+1} = \frac{l_i}{\sum_j a_{ij}^{2t}} \text{ and } S_j^{t+1} = \frac{c_j}{\sum_i a_{ij}^{2t+1}} \quad t = 0, 1, \dots$$

This process is simply the multiplication of each row by a scalar in order for it to add up to its prescribed sum, then the same for each column, and so forth.

It should be noted that the biproportional problem is closely akin to the search for a delegation matrix in a two-dimensional voting setting by the following points:

1) Non-negativity:

the initial and solution matrices are non-negative in both problems.

2) Predetermined margins:

the marginal row and column sums of the biproportional solution matrix are pre-imposed as are those of the delegation matrix⁷; their strict positivity is not in contradiction: rows or columns required to add up to zero can be discarded.

3) Biproportionality:

The type of solution matrix to be found ($B = \lim_{t \rightarrow \infty} \langle R^t \rangle A \langle S^t \rangle$), which can be seen as a "double-proportionality" to the initial matrix, is a compromise vis-à-vis strict proportionality. It should be noted, however, that if the imposed

⁷By dealing out the total number of seats by a one-dimensional rule of apportionment.

marginal sums are proportional (in the strict sense) to those of the initial matrix, then the solution matrix is strictly proportional to the initial one.

Unfortunately, the crucial point in the representation problem, the fact that the delegation matrix is required to be integral, is obviously generally not respected by the biproportional solution; so, once again, can the elements of the solution matrix be rounded to adjacent integers while preserving the imposed marginal sums?

This time, the answer is "yes"; let us call such a rounding a **biproportional delegation**.

THEOREM:

Let $A = (a_{ij})$ be a $m \times n$ matrix of votes and S , the total number of seats to be filled.

Let $l = (l_1, \dots, l_m) \in \mathbb{N}^m$ (resp. $c = (c_1, \dots, c_n) \in \mathbb{N}^n$) be the distribution of the S seats among the constituencies (resp. the parties) by a one-dimensional rule of apportionment.

If the possibility condition of the biproportional constrained matrix problem holds for initial matrix A and imposed marginal sums l and c , and if $B = (b_{ij})$ is the corresponding solution, then a biproportional delegation exists for A i.e. a matrix $D = (d_{ij}) \in \mathbb{N}^{m \times n}$ such that

$$\begin{aligned} d_{ij} &= b_{ij} && \text{if } b_{ij} \text{ is integer} \\ d_{ij} &= \lfloor b_{ij} \rfloor \text{ or } \lfloor b_{ij} \rfloor + 1 && \text{otherwise} \\ \text{and } \forall i, d_{i.} &= l_i, \forall j, d_{.j} &= c_j \end{aligned}$$

Proof:

Cox and Ernst (1982) proved the following result:

a controlled rounding of a given matrix of real numbers always exists i.e. each entry of the matrix (including marginal sums and grand total) can be rounded to an adjacent integer while preserving the additive structure of the array; furthermore, this existence result remains valid if zero-restrictedness is imposed i.e. if integers must be rounded to themselves. An example of the latter is the following:

$$\text{Let } A = \begin{pmatrix} 2.17 & 3.75 \\ 1.33 & 2.43 \\ 0.75 & 3.01 \\ 4.25 & 9.19 \end{pmatrix} \begin{matrix} 5.92 \\ 3.76 \\ 3.76 \\ 13.44 \end{matrix}$$

A controlled rounding of A is, for example:

$$R(A) = \begin{pmatrix} 2 & 4 \\ 2 & 2 \\ 0 & 4 \\ 4 & 10 \end{pmatrix} \begin{matrix} 6 \\ 4 \\ 4 \\ 14 \end{matrix}$$

Using this result, a zero-restricted controlled rounding of the biproportional solution matrix exists; since the marginal sums of the latter are integers, zero-restrictedness ensures that some controlled rounding sums up exactly to the pre-imposed margins. $\square$

We have thus proven the existence of a biproportional delegation matching a priori determined total numbers of delegates per constituency and per political party. Of course, as unicity of a biproportional delegation is not guaranteed, the possibility of finding an eventual best solution among the existing controlled roundings of the biproportional solution must be discussed.

One must also examine in which circumstances the possibility condition of Bacharach's theorem is violated when dealing out seats to constituencies and to parties by a one-dimensional rule of apportionment; can this dead-end be avoided by some marginal apportionment rules, and why? If not, what should be done? (for example, should the problem be divided into two or more sub-problems?)

Finally, the priorities awarded to constraints to be set on different aspects of the delegation are not the same in every representation problem; a classification of such problems should thus be attempted in that direction.

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