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Two-Dimensional Rounding for a Quasi-Proportional Representation

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1. Two-Dimensional Proportional Representation

Problems of finding a fair delegation for a given "population" are encountered in numerous situations, from composing the House of Representatives of a country through national elections, to selecting the members of, for example, a school committee. Many representation systems are designed to translate simultaneously two (or more) aspects of the initial population; several countries' electoral systems take into account the voters' political preferences *and* their geographical distribution; in such a case, vote results can be summarized in a matrix, the rows and columns of which represent respectively constituencies and political parties; a fair¹ delegation (which can be viewed as a matrix of same dimensions) should be proportional to both constituency sizes and party strengths. Evidently, this is generally not possible in integers, so a compromise must be found.

In De Meur, Gassner and Hubaut (1985), De Meur and Gassner (1987) and Gassner (1986), we examined this problem from several viewpoints, setting fairness constraints which seemed very weak (for example, we imposed that a party which received less votes than another should not be awarded more delegates); unfortunately, these conditions turned out to be impossible to respect simultaneously. One of the models we elaborated consisted in computing a proportion of the different categories of the population to be delegated, resulting in a matrix of real numbers (exact proportions), and rounding each of these numbers to an adjacent integer; in some cases, this method could not yield any delegation respecting very weak constraints of fairness similar to those described above at every level of the delegation (globally between parties and constituencies, in each constituency between parties and vice-versa). Interestingly, it seems we imposed here "too much and too little": obviously, constraints at all

levels (though they seem minimal) along with the desire to stay "near" global proportionality (i.e. rounding restricted to integers adjacent to exact proportions) is too much to ask for; on the other hand, even when this method does yield a fair solution, the total numbers of delegates per constituency (and per party) can be forced very far from those obtained by a one-dimensional rule of apportionment (Hamilton's or d'Hondt's method, for example); this is, in our opinion, quite crucial, as most existing electoral systems of this type² need the a priori determination of the numbers of delegates due to each constituency. This is achieved by dealing out a fixed number of seats to constituencies "proportionally" to their numbers of voters³ using a classical rule of apportionment; furthermore, "proportionality" should also hold globally between political parties with respect to the amount of votes they received (otherwise, why not simply consider each constituency separately, dividing the two-dimensional problem into several one-dimensional ones).

2. Rounding Problems

Following the latter remarks concerning existing systems, we found it was appropriate to award the fixed number of seats to constituencies and to parties proportionally to their respective magnitudes by some one-dimensional rule of apportionment. The problem remaining is thus to find an integer delegation which sums up to these predetermined total margins. Going back to the idea of a representation "near" exact global proportionality, the question arises whether there exists such a delegation, each element of which is an integer adjacent to the exact proportion of seats the corresponding number of votes is worth⁴. The answer is "no" when the margins are predetermined. Indeed, consider the following matrix of vote results, each row representing a constituency, each column a political party. Suppose 16 seats are to be dealt out.

$$\left[\begin{array}{cc} 3960 & 750 \\ 1960 & 800 \\ 1950 & 700 \\ 940 & 1700 \\ 1610 & 10 \\ 1609 & 11 \\ 12029 & 3971 \end{array} \right] \begin{array}{l} 4710 \\ 2760 \\ 2650 \\ 640 \\ 1620 \\ 1620 \\ 16000 \end{array} \quad (1)$$

Global exact proportionality (1/1000) yields:

$$\left(\begin{array}{cc} 3.96 & 0.75 \\ 1.96 & 0.80 \\ 1.95 & 0.70 \\ 0.94 & 1.70 \\ 1.61 & 0.01 \\ 1.609 & 0.011 \\ 12.029 & 3.971 \end{array} \right) \begin{array}{l} 4.71 \\ 2.76 \\ 2.65 \\ 2.64 \\ 1.62 \\ 1.62 \\ 16 \end{array} \quad (2)$$

Rounding row sums by either Hamilton's or d'Hondt's rule of apportionment yields (5, 3, 3, 3, 1, 1). As the rounded values of the first column must add up to a minimum of twelve, obviously at least one of the two last rows must be rounded up, making it impossible to obtain the imposed row marginal sums. This example thus shows that if one wishes to stay "near" exact proportions, it may be impossible for the rounded values to sum up marginally to predetermined numbers of delegates. As this is of major importance in many electoral systems we are concerned with here, we tried to find an alternative solution.

It can be easily shown that there exist integer matrices which add up to given integer row and column sums (as long as the sum of the row sums equals that of the column sums). If we impose, as before, that row sums be (5, 3, 3, 3, 1, 1) and the column sums be 12 and 4, we have, for example:

$$\left(\begin{array}{cc} 5 & 0 \\ 3 & 0 \\ 3 & 0 \\ 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{array} \right) \begin{array}{l} 5 \\ 3 \\ 3 \\ 3 \\ 1 \\ 1 \\ 12 \quad 4 \quad 16 \end{array} \quad (3)$$

This "solution" is obviously not optimal, with, for example, the second party represented in the two last constituencies where it obtained particularly low scores.

We must somehow include the notion of proportionality between the initial matrix of votes and an acceptable delegation matrix.

3. Rounding the Biproportional Solution

The latter sort of problem will undoubtedly remind economists of the well-known econometric "RAS" model designed to estimate the behavior of input-output relations through time: an initial matrix is known, and one must derive from it another matrix, the marginal sums of which are given. Bacharach (1970) precised what he calls the *biproportional constrained matrix problem* as follows (a similar approach was used by Balinski and Demange (1986)—though involving a much more sophisticated mathematical presentation):
given A , a non-negative $m \times n$ matrix and marginal sums

$$\begin{cases} l = (l_1, l_2, \dots, l_m) > 0 \\ c = (c_1, c_2, \dots, c_n) > 0 \end{cases}$$

find $B = (b_{ij})$, a non-negative $m \times n$ matrix with

$$\sum_i b_{ij} = c_j \quad \forall j \in \{1, \dots, n\}$$

$$\sum_j b_{ij} = l_i \quad \forall i \in \{1, \dots, m\}$$

and $B = \lim_{t \rightarrow \infty} \langle R^t \rangle A \langle S^t \rangle$ for some sequences $\{R^t\}$, $\{S^t\}$

of vectors, $R_i^t = 1 \quad \forall t$ ($\langle M \rangle$ is a diagonal matrix)

Bacharach proved that the problem has a unique solution if and only if:

$$a_{ij} = 0 \quad \forall i \in I', \quad \forall j \in J \Rightarrow \begin{cases} \sum_{i \in I'} l_i \leq \sum_{j \in J'} c_j \\ \sum_{j \in J} c_j \leq \sum_{i \in I} l_i \end{cases}$$

where I and J are respectively subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, I' is $\{1, \dots, m\} \setminus I$ and J' is $\{1, \dots, n\} \setminus J$.

This *possibility condition* prevents the following immediate impossibility due to the eventual presence of zeroes in the initial matrix: obviously $a_{ij} = 0 \Rightarrow b_{ij} = 0$ so, if A is of the following form:

$$\begin{array}{c} J' \quad J \\ I' \left[\begin{array}{c|c} A_{I'J'} & 0 \\ \hline A_{IJ'} & A_{IJ} \end{array} \right] \\ I \end{array}$$

then B must be such that block B_{ij} contains only zeroes; thus

$$B = \begin{array}{c} J' \quad J \\ I' \left[\begin{array}{c|c} B_{I'J'} & 0 \\ \hline B_{IJ'} & B_{IJ} \end{array} \right] \\ I \end{array} \left. \begin{array}{l} \sum_{i \in I'} l_i \\ \sum_{i \in I} l_i \end{array} \right\} \begin{array}{l} \sum_{j \in J'} c_j \\ \sum_{j \in J} c_j \end{array}$$

the possibility condition prevents the total sum of the elements of block $B_{I'J'}$ (resp. B_{IJ}) from exceeding that of those of block $B_{(I \cup I')J'}$ (resp. $B_{IJ \cup J'}$), an immediate impossibility.

Bacharach also describes a "natural" algorithm yielding the solution i.e.

$$A^0 = A$$

$$A^{2t+1} = \langle R^{t+1} \rangle A^{2t}$$

$$A^{2t+2} = \langle R^{t+1} \rangle A^{2t} \langle S^{t+1} \rangle$$

$$\text{where } R_i^{t+1} = \frac{l_i}{\sum_j a_{ij}^{2t}} \text{ and } S_j^{t+1} = \frac{c_j}{\sum_i a_{ij}^{2t+1}} \quad t = 0, 1, \dots$$

This process is simply the multiplication of each row by a scalar in order for it to add up to its prescribed sum, then the same for each column, and so forth.

It should be noted that the biproportional problem is strikingly similar to the search for a delegation matrix in a two-dimensional voting setting, principally by the following points:

1) *Non-negativity:*

the initial and solution matrices are non-negative in both problems.

2) *Predetermined margins:*

the marginal row and column sums of the biproportional solution matrix are pre-imposed as are those of the delegation matrix⁵; their strict positivity is not in contradiction: rows or columns required to add up to zero can be discarded.

3) *Biproportionality:*

The type of solution matrix to be found ($B \lim_{t \rightarrow \infty} \langle R^t \rangle A \langle S^t \rangle$), which

can be seen as a "double-proportionality" to the initial matrix, is a compromise vis-à-vis strict proportionality. It should be noted, however, that if the imposed marginal sums *are* proportional (in the strict sense) to those of the initial matrix, then the solution matrix is strictly proportional to the initial one.

Unfortunately, the crucial point in the representation problem, the fact that the delegation matrix is required to be integral, is obviously generally not respected by the biproportional solution; so, once again, can the elements of the solution matrix be rounded to adjacent integers while preserving the imposed marginal sums?

This time, the answer is "yes"; let us call such a rounding a *biproportional delegation*.

Theorem:

Let $A = (a_{ij})$ be an $m \times n$ matrix of votes and S , the total number of seats to be filled.

Let $l = (l_1, \dots, l_m) \in \mathbb{N}^m$ (resp. $c = (c_1, \dots, c_n) \in \mathbb{N}^n$) be the distribution of the S seats among the constituencies (resp. the parties) by a one-dimensional rule of apportionment. If the possibility condition of the biproportional constrained matrix problem holds for initial matrix A and imposed marginal sums l and c , and if $B = (b_{ij})$ is the corresponding solution, then a *biproportional delegation* exists for A i.e. a matrix $D = (d_{ij}) \in \mathbb{N}^{m \times n}$ such that

$$\begin{aligned}
 d_{ij} &= b_{ij} \quad \text{if } b_{ij} \text{ is integer} \\
 d_{ij} &= [b_{ij}] \text{ or } [b_{ij}] + 1 \text{ otherwise} \\
 \text{and } \forall i, d_{i.} &= l_i, \forall j, d_{.j} = c_j
 \end{aligned}$$

Proof:

Cox and Ernst (1982) proved the following result:

a *controlled rounding* of a given matrix of real numbers always exists i.e. each entry of the matrix (including marginal sums and grand total) can be rounded to an adjacent integer while preserving the additive structure of the array; furthermore, this existence result remains valid if *zero-restrictedness* is imposed i.e. if integers must be rounded to themselves. An example of the latter is the following:

$$\text{Let } A = \begin{bmatrix} 2.17 & 3.75 \\ 1.33 & 2.43 \\ 0.75 & 3.01 \\ 4.25 & 9.19 \end{bmatrix} \begin{matrix} 5.92 \\ 3.76 \\ 3.76 \\ 13.44 \end{matrix}$$

A controlled rounding of A is, for example:

$$R(A) = \begin{bmatrix} 2 & 4 \\ 2 & 2 \\ 0 & 4 \\ 4 & 10 \end{bmatrix} \begin{matrix} 6 \\ 4 \\ 4 \\ 14 \end{matrix}$$

Using this result, a zero-restricted controlled rounding of the biproportional solution matrix exists; since the marginal sums of the latter are integers, zero-restrictedness ensures that some controlled rounding sums up exactly to the pre-imposed margins.

We have thus proven the existence, under a very weak condition, of a biproportional delegation matching a priori determined total numbers of delegates per constituency and per political party. Of course, as unicity of a biproportional delegation is not guaranteed, the possibility of finding an eventual best solution among the existing controlled roundings of the biproportional solution must be discussed.

In this direction, Cox (1987) presents an algorithm for unbiased controlled rounding of a matrix of real numbers; going back fictive example (1) of vote results presented above let us determine a biproportional delegation by this method.

As before, row sums are required to be respectively 5, 3, 3, 3, 1, 1, imposed column sums being 12 and 4; the corresponding biproportional solution is:

$$\left[\begin{array}{cc|c} 4.31 & 0.69 & 5 \\ 2.23 & 0.77 & 3 \\ 2.30 & 0.70 & 3 \\ 1.18 & 1.82 & 3 \\ 0.99 & 0.01 & 1 \\ 0.99 & 0.01 & 1 \\ \hline 12 & 4 & 16 \end{array} \right] \quad (4)$$

An unbiased controlled rounding obtained by Cox's algorithm is:

$$\left[\begin{array}{cc|c} 5 & 0 & 5 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \hline 12 & 4 & 16 \end{array} \right] \quad (5)$$

This distribution seems to be a rather good compromise as, for example, upper quota⁶, though impossible to respect, is "almost" satisfied.

Biproportional delegations seem to be an acceptable answer to the questions raised by two-dimensional P.R. The next essential step in the study of this important problem is to thoroughly examine which of the classical desired properties of apportionment methods are satisfied or not by the method described; one should not expect too much: we have already shown how much more difficult this double-balance is to achieve than the corresponding one-dimensional equilibrium. One must also examine in which circumstances the possibility condition of Bacharach's theorem is violated when dealing out seats by a one-dimensional rule of apportionment; can this dead-end be avoided by some marginal apportionment rules, and why? If not, what should be done? (For example, should the problem be divided into two or more sub-problems?)

Notes:

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¹ With respect to the distribution of vote results.

² Such as those used for parliamentary elections in Belgium (Gassner (1987)) or for some regional elections in the Netherlands (Anthonisse (1984)).

³ Or to their populations.

⁴ This means quantity $v_{ij} \cdot S / \sum_j v_{ij}$ if v_{ij} is the number of votes for party j in constituency i and S is the total number of seats available.

⁵ By dealing out the total number of seats by a one-dimensional rule of apportionment.

⁶ In fact, upper "global" quota.

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