



**Blanchard and Kahn's (1980) Solution for a Linear Rational  
Expectations Model with One State Variable and One Control  
Variable: the Correct Formula**

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# Blanchard and Kahn's (1980) solution for a linear rational expectations model with one state variable and one control variable: the correct formula

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## Abstract

This note corrects Blanchard and Kahn's (1980) solution for a linear dynamic rational expectations model with one state variable and one control variable.

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## 1. Introduction

In their classical paper, Blanchard and Kahn (1980) [BK] derived the solution for an important class of dynamic linear rational expectations models. The BK algorithm has become a standard tool for economic modelers.<sup>2</sup> In general, the model solution is analytically intractable. However, as pointed out by BK, models with one predetermined and one non-predetermined endogenous variable can be handled analytically (which may facilitate an intuitive understanding of the model solution). That special case is important as it includes, e.g., the basic Real Business Cycle model with fixed labor (King and Rebelo (1999)). In this note, we show that the formula provided by BK, for this key special case, is incorrect; we also provide the correct formula.

## 2. A linear rational expectations model with one state and one control

Consider the following model (the notation follows BK):

$$\begin{bmatrix} x_{t+1} \\ E_t p_t \end{bmatrix} = A \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} Z_t, \quad (1)$$

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<sup>2</sup>The BK algorithm is e.g. often used to solve linearized dynamic general equilibrium models, the workhorses of modern macroeconomics (King and Rebelo (1999)). Google Scholar records 2342 cites (03/2016) for the BK paper.

where  $x_t$  is a predetermined variable ('state'), and  $p_t$  is a non-predetermined variable ('control').

$Z_t$  is a  $(k \times 1)$  vector of exogenous variables.  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  is a  $(2 \times 2)$  matrix, and  $\gamma_1, \gamma_2$  are  $(1 \times k)$

vectors. Let  $\lambda_1, \lambda_2$  be the eigenvalues of  $A$ , and let  $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$  be the matrix of eigenvectors of

$A$ , i.e.  $AB = BJ$ , with  $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ . Finally, let  $C \equiv B^{-1}$ ,  $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ . Note that  $A = BJC$ .

Proposition 1 of BK (p.1308) shows that model (1) has a unique (non-exploding) solution if and only if one eigenvalue of  $A$  is outside the unit circle, while the other eigenvalue is inside (or on) the unit circle. Assume that this condition holds, and let  $|\lambda_1| \leq 1, |\lambda_2| > 1$ . BK (p.1309) state that then the solution of (1) is:

$$x_t = \lambda_1 x_{t-1} + \gamma_1 Z_{t-1} + \mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_{t-1} Z_{t+i-1}, \quad (2)$$

$$p_t = a_{12}^{-1} [(\lambda_1 - a_{11})x_t + \mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_t Z_{t+i}], \quad (3)$$

$$\text{with } \mu \equiv (\lambda_1 - a_{11})\lambda_1 - a_{12}\lambda_2. \quad (4)$$

Comment: When  $\mu$  is defined by (4), then  $\mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_t Z_{t+i-1}$  is a  $(k \times 1)$  vector. This implies that (2) and (3) cannot hold for  $k > 1$  when quantity  $\mu$  is given by (4) (as  $x_t$  and  $p_t$  are scalars). This suggests that the formula for  $\mu$  is incorrect.

We now derive the correct formula for  $\mu$ .

Equations (2) and (3) are special cases of the solution for general linear difference models (with arbitrary numbers of states and controls) given in Proposition 1 of BK (p.1308). For convenience, the general case is shown in the Appendix. The general solution for predetermined variable  $x_t$  indicates that the correct expression for the vector  $\mu$  in equation (2) above is

$$\mu = -(b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22})c_{22}^{-1}(c_{21}\gamma_1 + c_{22}\gamma_2).$$

Write this as  $\mu = \phi_1 \gamma_1 + \phi_2 \gamma_2$ , with  $\phi_1 \equiv -(b_{11}\lambda_1 c_{12} c_{22}^{-1} c_{21} + b_{12}\lambda_2 c_{21})$  and  $\phi_2 \equiv -(b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22})c_{22}^{-1}$ .  $A = BJC$  implies that  $a_{11} = b_{11}\lambda_1 c_{11} + b_{12}\lambda_2 c_{21}$  and  $a_{12} = b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22}$ . We thus see that  $\phi_2 = -a_{12}$  holds.

Substituting  $b_{12}\lambda_2c_{21}=a_{11}-b_{11}\lambda_1c_{11}$  into the definition of  $\phi_1$  gives  $\phi_1=-(b_{11}\lambda_1c_{12}c_{22}^{-1}c_{21}+a_{11}-b_{11}\lambda_1c_{11})$   
 $\Leftrightarrow \phi_1=-(a_{11}+b_{11}\lambda_1[c_{12}c_{22}^{-1}c_{21}-c_{11}])$ .  $B=C^{-1}$  implies  $b_{11}=c_{22}/(c_{11}c_{22}-c_{12}c_{21})$  and  $c_{12}c_{22}^{-1}c_{21}-c_{11}=-b_{11}^{-1}$ .  
 Thus  $\phi_1=\lambda_1-a_{11}$ . In summary, the correct formula for  $\mu$  is:

$$\mu \equiv (\lambda_1 - a_{11})\gamma_1 - a_{12}\gamma_2. \quad (5)$$

It can readily be verified from the general solution for the non-predetermined variable  $p_t$  (see Appendix) that equation (3) above holds when the quantity  $\mu$  is defined by (5).

## References

- Blanchard, O. and C. Kahn, 1980. The Solution of Linear Difference Models Under Rational Expectations. *Econometrica* 48, 1305-1311.
- King, R. and S. Rebelo, S., 1999. Resuscitating Real Business Cycles, in: *Handbook of Macroeconomics* (J. Taylor and M. Woodford, eds.), Elsevier, Vol. 1B, pp. 927-1007.

## Appendix

### Blanchard and Kahn (1980): the general model

Consider the model

$$\begin{bmatrix} X_{t+1} \\ E_t P_t \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t, \quad (\text{A1})$$

where  $X_t$  is an  $n \times 1$  vector of predetermined variable, and  $p_t$  is an  $m \times 1$  vector of non-predetermined variables;  $Z_t$  is a  $(k \times 1)$  vector of exogenous variables.  $A$  is an  $(n+m) \times (n+m)$  matrix, and  $\gamma$  is an  $(n+m) \times k$  matrix. Consider the Jordan canonical form  $A = C^{-1} J C$ , where  $C$  and  $J$  are  $(n+m) \times (n+m)$  matrices. Let the diagonal elements of  $J$  (i.e. the eigenvalues of  $A$ ) be ordered by increasing absolute value. Let  $\bar{n}$  ( $\bar{m}$ ) denote the number of eigenvalues of  $A$  that are on or inside the unit circle (outside the unit circle).

Partition  $J$  as  $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$ , where  $J_1$  and  $J_2$  are matrices of dimensions  $(\bar{n} \times \bar{n})$  and  $(\bar{m} \times \bar{m})$ , respectively. Decompose  $C$ ,  $B \equiv C^{-1}$  and  $\gamma$

as  $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ ,  $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$  and  $\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$ , where  $C_{11}, C_{12}, C_{21}, C_{22}$  are matrices of

dimensions  $(\bar{n} \times \bar{n}), (\bar{n} \times \bar{m}), (\bar{m} \times \bar{n})$  and  $(\bar{m} \times \bar{m})$ , respectively;  $B_{11}, B_{12}, B_{21}, B_{22}$  have dimensions  $(\bar{n} \times \bar{n}), (\bar{n} \times \bar{m}), (\bar{m} \times \bar{n})$  and  $(\bar{m} \times \bar{m})$ , respectively, while  $\gamma_1$  and  $\gamma_2$  have dimensions  $(\bar{n} \times k)$  and  $(\bar{m} \times k)$ , respectively.

Proposition 1 in Blanchard and Kahn (1980) states that the model (A1) has a unique (non-explosive) solution if and only if the number of non-predetermined variables equals the number of eigenvalues of  $A$  outside the unit circle:  $m = \bar{m}$ . If that condition is met, then the solution is:

$$X_t = B_{11} J_1 B_{11}^{-1} X_{t-1} + \gamma_1 Z_{t-1} - (B_{11} J_1 C_{12} + B_{12} J_2 C_{22}) C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1} (C_{21} \gamma_1 + C_{22} \gamma_2) E_{t-1} Z_{t+i-1},$$

$$P_t = -C_{22}^{-1} C_{21} X_t + C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1} (C_{21} \gamma_1 + C_{22} \gamma_2) E_t Z_{t+i}.$$