Cross-Ownership:  
A Device for Management Entrenchment?  

Marc Levy and Ariane Szafarz  

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JEL Classifications: G32, G34, C71, D72, C44, D74  

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Abstract

By artificially inflating capital and creating own shares, cross-ownership can be a key device for managerial entrenchment. This paper proposes a game-theoretical method to measure the extent of shareholder expropriation through cross-ownership. By properly accounting for cross-ownership linkages, we show how managers can seize indirect voting rights, and so insulate their firms from outside control. Significant examples of cross-ownership are found not only in civil law countries, but also in the U.S. mutual fund industry. We apply our method to Germany’s Allianz Group. This paper paves the way to better regulatory appraisal of management entrenchment through cross-ownership.
1. Introduction

In the simple cross-ownership situation firm A has voting rights in firm B and at the same time firm B has voting rights in firm A. \(^1\) In complex ownership groups, however, circular ownership patterns—or cross-ownership rings—are hardly detectable. Consequently, their impact on corporate control is probably underestimated. This paper argues that cross-ownership can represent a key entrenchment device for the management of the firms in the ring. By properly accounting for cross-ownership linkages, we show how managers can seize indirect voting rights, and so insulate their firms from outside control.

Although the worldwide extent of cross-ownership remains unknown, significant examples of actual rings can be found in Korean *chaebols* (Almeida et al., 2011) and in the European and Japanese banking sectors (Temurshoev and Stakhovych, 2009).\(^2\) Evidence of cross-ownership is reported by La Porta et al. (1999), who cite Germany’s Allianz Group as a case in point (see Figure 1). Cross-ownership can also be found in the U.S. mutual fund industry. For instance,

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\(^1\) This situation corresponds to a two-firm ownership ring or mutual control (Karos and Peters, 2013). In fact, the length of a cross-ownership ring may take any positive integer value. The value of one corresponds to own voting shares.

\(^2\) Additional examples of cross-ownership concern the Czech banking sector (Turnovec, 1999) and the Colombian brewery industry (Gutiérrez et al., 2008). Possibly, cross-ownership is under-reported because scholars ignore how to deal with it.
State Street Corporation (STT) owns minority shares in other funds in a way that creates within-industry cross-ownership rings of various lengths.3

< Insert Figure 1 here >

In civil law countries, cross-ownership is often combined with pyramidal ownership (Thesmar, 2001), which further obscures the underlying control linkages. Both pyramidal ownership and cross-ownership are mechanisms for exerting control over firms with low cash-flow rights.4 However, the two mechanisms are quite different. Pyramids are structures including top-down relationships only while cross-ownership structures include closed loops (Weidenbaum, 1996; Ritzberger and Shorish, 2002). In particular, cross-ownership should not be confused with cross-holding, i.e. multiple control chains in pyramids (Claessens et al., 2000; Faccio and Lang, 2002).5

So far, the literature has identified three possible motivations for using cross-ownership. First, Perotti (1992) suggests that collaboration between firms can be enforced if they exchange control rights. In this context, cross-ownership acts as a commitment device. Second, it is well-known that passive acquisitions of shares in rival firms by Cournot oligopolists make them act

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3 First, STT holds 4.80% of its own shares (one-firm ring). Second, STT owns a 4.77% share in T. Rowe Price Group, Inc. (TROW), which holds a 3.28% share in STT (two-firm ring). Third, STT is a shareholder of Black Rock, Inc. (3.05%), which in turn owns 2.68% of TROW (three-firm ring). Source: http://finance.yahoo.com/, data retrieved on May 13, 2013.

4 See Bebchuk et al. (2000), Allen et al. (2008), Adams and Ferreira (2008), and Burkart and Lee (2008). The interplay of within-industry cross-ownership and collusion is analyzed by Malueg (1992), Gilo et al. (2006), and Temurshoev and Stakhovych (2009).

5 Appendix A clarifies the terminology further.
less competitively (Flath, 1991). Amundsen and Bergman (2002) push this argument one step further and show that cross-ownership can create implicit collusion and increase market power. Third, Bebchuk et al. (2000) mention that cross-ownership rings are attractive to external shareholders because the firms in the ring control one other artificially.

Our contention is that cross-ownership is primarily a way to artificially create own shares and expropriate external shareholders. To make our case, we need a reliable method for measuring shareholder control stake in complex corporate groups involving cross-ownership. The chain-based methods proposed by Claessens et al. (2000) and Faccio and Lang (2002) successfully address pyramidal ownership but fall short of dealing with cross-ownership. Therefore, we propose a new approach combining the use of a power index with Markov chains.

The rest of this paper is organized as follows. Section 2 presents the literature review. Section 3 introduces voting games in corporate structures including cross-ownership rings. Our measure of voting power is defined in Section 4, and applied to the Allianz Group in Section 5. Section 6 concludes.

2. Literature Review

Cross-ownership breaks the traditional rule of one-sided control. This section illustrates the features of cross-ownership using simple examples and motivates the need for the new mathematical approach proposed in Section 3. First, we take a fresh look at the symmetrical two-firm case introduced by Bebchuk et al. (2000) to show how cross-ownership obscures control
mechanisms. Second, we explain how cross-ownership brings multiple equilibria to voting games.

To simplify terminology, we assume away non-voting shares as well as multiple voting shares. Hence, direct control equals direct ownership. Likewise, we use “control” to designate voting rights that are exerted either directly or indirectly.

Consider the symmetrical two-firm case shown in Figure 2. Firm A owns 40% of each of its subordinated firms C and D, while these two firms are linked by a 20% symmetrical cross-ownership. If the remaining ownership of firm C is dispersed, the total control share of firm A in firm C is 60%, namely its direct 40% share plus its indirect 20% share via firm B. For symmetry reasons, firm A also controls 60% of firm C. In sum, cross-ownership allows firm A to exert majority control over both firm C and D. This leads Bebchuk et al. (2000) to conclude that cross-ownership is profitable to external shareholders.

However, actual control over a cross-ownership ring can also be exerted by the firms belonging to the ring. In Figure 3, firm C owns 60% of firm D, and firm D owns 60% of firm C. Hence, firms C and D share the control over themselves. The ultimate shareholder, firm A is expropriated of their minority control. This example stresses that in corporate structures with

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6 This standard convention does not affect the generality of our derivations and arguments. If direct control differs from direct ownership, we should use direct control links as the starting point.
cross-ownership, full control may be exerted by any firm involved, not only by ultimate shareholders.

Figure 3 shows that cross-ownership can make subsidiaries actively participate in the decision-making process. Hence, the top managers of these firms enjoy voting rights without necessarily having cash-flow rights. Interestingly, in Figure 3 a subsidiary, say firm C, has significant voting powers in both firms C and D. The same is true for firm D. Not only does cross-ownership create own-share-like situations; it also expropriates the ultimate shareholders of all the firms embedded in cross-ownership rings.\textsuperscript{7}

Our examples illustrate the specific features of cross-ownership. But how can these features be addressed in a general way? There exists a wide variety of methods for measuring shareholder control power in complex corporate groups. Scholars in mainstream finance typically use the chain-based methods proposed by Claessens et al. (2000) and Faccio and Lang (2002). Inspired by graph theory, these methods successfully address pyramidal ownership. However, they do not apply to cross-ownership rings because cycles generate infinitely many control chains, some of which are infinitely long.

Within the field of operational research, a related stream of literature has added two alternative measures to the toolbox. Under the first, control stakes are consolidated through input-output matrix algebra (Brioschi et al., 1989; Chapelle and Szafarz, 2005 and 2007). But again, this method fails to address the cyclic features of cross-ownership. In fact, methods based on

\textsuperscript{7} Additional examples in Appendix B show how top managers with no cash-flow rights can gain voting rights from cross-ownership.
graph theory and matrix algebra rely on the assumption that control is a one-sided relationship, which sits at odds with cross-ownership.

The second measure is based on game theory and power indices (Leech, 1988 and 2003; Crama and Leruth, 2007; Levy, 2009). Power indices relate shareholder control to the probability of being the decisive voter. Surprisingly few scholars in finance use power indices. Two factors could explain this lack of success. First, computation is demanding, especially when dealing with large corporate structures. Second, the corporate governance literature has rapidly adopted more intuitive methods inspired by graph theory. Power indices are, however, suitable for estimating the voting power of shareholders in a capitalistic environment. According to Crama and Leruth (2007, p. 880), these indices have “key features that should be expected from any realistic measure of control: first, such measure must take the whole distribution of shares into account, since the amount of control owned by a shareholder typically depends on the holdings of the other shareholders; moreover, a sound measure of control will probably not grow in direct proportion to shareholdings, and, in any case, the measure will undergo a sharp discontinuity when a majority of the shares are transferred to a unique owner”. These features contrast with those of typical graph-based methods, which determine each shareholder’s share of control individually but fail to account for the dispersion of control rights among the other shareholders. In addition, technological progress makes the computational complexity of power indices less and less relevant.

Here, we follow Crama and Leruth (2007) and opt for a power index inspired by the deterministic index proposed by Banzhaf (1965). Originally, the Banzhaf index was designed for measuring the influence of the U.S. states in the Electoral College. Associated with a voter, the index represents the probability for him or her to be decisive, under the assumption that each
participant votes 0 or 1 with equal probability. Applied to one shareholder and one subordinated firm, the index represents the probability that the shareholder has a decisive influence on the firm. As such, the index cannot deal with cross-ownership structures properly. In section 4, however, we circumvent the awkward multiple-equilibria problem by combining the Banzhaf index with stochastic voting. To measure voting power, we then introduce a generalized version of the standard Banzhaf power index presented in Section 3.

3. Voting Games in Corporate Structures with Cross-Ownership

Mathematically, cross-ownership rings are represented by cycles—or circuits—in ownership graphs. Ownership graphs including cycles are more complex to deal with than acyclic ones. Logically, a few authors (La Porta et al., 1999; Aminadav et al., 2011; Crama and Leruth, 2013) mention cross-ownership as a problem to be addressed. However, existing theories on corporate control concentrate—explicitly or not—on top-down relationships, thus excluding cross-ownership. Starting from the standard presentation of voting games in corporate structures, this section introduces our novel approach combining a generalization of the Banzhaf power index with Markov voting chains.

Consider set $V = \{1, \ldots, n\}$ representing $n$ firms connected by ownership links. The corresponding ownership structure is the directed graph of (direct) shareholdings: $H = (V, R, A)$, where $V$ is the vertex set, $R \subseteq V \times V$ is the arc set, and $A = (a_{ij})$ is the $n \times n$ matrix of direct ownership, with $a_{ij} \in [0,1]$ being the share that firm $i$ owns in firm $j$. The set of ultimate shareholders—or sources—of $V$, denoted by $S = \{1, 2, 3, \ldots, s\} \subset V$, includes all firms/individuals.
having no shareholders in $V$. We denote by $F = \{s+1, s+2, \ldots, n\}$ the set of subsidiaries—or non-source firms.

For expositional convenience, we restrict the presentation to complete ownership structures, meaning that all the shareholders of non-source firms are identified: $\forall j \in F : \sum_{i=1}^{n} a_{ij} = 1$. The algorithm in Appendix G lifts this assumption.

Cross-ownership refers to the presence of cycles in the ownership structure. More precisely, a cycle is a set of vertices $\{i_1, i_2, \ldots, i_k\} \subset F$ such that:

$$\forall j \in \{1, 2, \ldots, k-1\} : a_{i_ji_{j+1}} > 0 \text{ and } a_{i_1i_k} > 0 \quad (1)$$

In particular, own shares are cycles (for $k = 1$). In contrast, pyramids are defined as cycle-free, or acyclic, graphs.\(^8\)

Consider a binary voting game taking place in $H = (V, R, A)$. Each firm in $V$ votes 1 or 0, meaning yes or no to a given proposal. A voting state is an n-uple $X = (x_1, \ldots, x_n) \in \{0,1\}^n$, where $x_j$ represents the vote of firm $j$. For instance, the voting state $X = (1, \ldots, 1)$ means that all firms vote in favor of the proposal.

To differentiate the votes of ultimate shareholders and subsidiaries, we will sometimes write the voting state $X$ as:

\(^8\) Actually, the term “pyramids” is often loosely used to refer to ownership structures including chains of shareholdings. We introduce here a stricter definition to make a clear distinction between pyramids and cross-ownership-inclusive structures.
\[ X = (X_s, X_F), \text{ where } X_s = (x_1, \ldots, x_s) \text{ and } X_F = (x_{s+1}, \ldots, x_n) \]  

A voting state is said to be *admissible* if each firm in \( V \) votes in accordance with the votes of a majority of its direct shareholders. In other words, only admissible voting states are consistent with the majority voting rule. Non-admissible voting states cannot be observed in the real world. Since sources have no shareholders, their votes are unrestricted. For instance, in a pyramidal group where \( X_s = (1, \ldots, 1) \in \{0,1\}^s \), all the firms vote 1 because the votes are dictated by direct shareholders, and the process starts from the top of the pyramid where the ultimate shareholders are located. As a result, the admissible voting state corresponding to \( X_s = (1, \ldots, 1) \in \{0,1\}^s \) is unique and given by \( X = (1, \ldots, 1) \in \{0,1\}^n \).

An *equilibrium* \( \Omega \) of the binary voting game is defined as a function that maps any s-uple of sources’ votes to an admissible voting state:

\[ \Omega : \{0,1\}^s \rightarrow \{0,1\}^n : X_s \rightarrow X^* \text{ such that: } X^* = (X_s^*, X_F^*) \text{ and } X^* = g(X^*). \]  

Intuitively, \( \Omega \) summarizes how the decisions of the subsidiaries in \( F \) depend (directly or indirectly) on those of the group’s ultimate shareholders in \( S \).

Gambarelli and Owen (1994) show that equilibrium exists in pyramids and is unique. In a pyramidal group, the votes of the ultimate shareholders fully determine the votes of the other firms. Consequently, subsidiaries have no voting power. In contrast, Figure 4 in Section 2 shows that in cross-ownership-inclusive structures equilibrium is no longer unique. The ultimate shareholders may prove unable to secure a voting state, and subsidiaries can enjoy voting power,
hidden away from their ultimate shareholders. Cross-ownership can thus generate residual voting power that sits in the hands of the top management of some non-source firms.

Computing Banzhaf indices is computationally intensive, especially when the number of firms involved is large. Crama and Leruth (2007) propose an efficient Monte-Carlo algorithm based on Equation (3). This algorithm effectively computes voting powers in pyramids. The authors admit, however, that applying their approach to cyclic corporate structures is problematic because the algorithm relies on one-sided recursive progression. Referring to Gambarelli and Owen (1994), they suggest addressing this issue with a heuristic stopping rule that prevents the algorithm from diverging. Levy (2009) simplifies the Crama and Leruth (2007) algorithm by introducing a so-called “coincidence matrix” that leads to the Banzhaf index with a single simulation. But when it comes to cross-ownership, the simplified algorithm presents the same limitation as the original one. More generally, any algorithm built from Equation (3) would have to deal with the possible occurrence of multiple equilibria, which causes divergence in the recursive calculation of the Banzhaf index.\(^9\) In an attempt to circumvent this problem, Crama, Leruth, and Wang (2009) and Wang (2009) suggest making the float vote in a stochastic way that does not guarantee convergence toward equilibrium. Hence, the voting process they build can be trapped into a cycle in the graph. In that case, it fails to deliver a well-defined Banzhaf index. The solution we propose in this paper takes another avenue and provides a convergent algorithm based on stochastic voting processes.

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\(^9\) To pinpoint the divergence problem that can arise in their algorithm, Crama and Leruth (2007) consider the simple (but unrealistic) case of a cycle made of \(n\) firms with full ownership (firm 1 owns 100% of the shares of firm 2, firm 2 owns 100% of the shares of firm 3, and so on until firm \(n\) owns 100% of the shares of firm 1).
Providing a precise and comprehensive picture of voting rights in a cross-ownership-inclusive corporate structure is complex for two reasons. First, we need to identify all the firms (regardless of whether they are ultimate shareholders or subsidiaries) that can hold voting rights in any firm of the structure, including themselves. Second, we need a measure of the voting power, which applies not only to ultimate shareholders (like existing measures do), but also to subsidiaries. The next section addresses these two issues.

4. Voting Power

This section generalizes in three steps the Banzhaf index to compute the voting power of any firm over any firm, be they equal or different. The starting point is an ownership structure that possibly includes cross-ownership. To identifying subsidiaries with voting power, the first step lists all admissible voting states. In the second step, we associate probabilities of occurrence with the admissible voting states. For instance, voting states supported by stronger coalitions occur with higher probabilities than others, all things equal. The admissible voting states and their probabilities make it possible to define a new notion of stochastic equilibrium that restores existence and uniqueness for cross-ownership-inclusive corporate structures. Last, we will generalize the Banzhaf index and introduce a measure of voting power to fit our probabilistic framework. In particular, a positive voting power of a subsidiary over itself will signal shareholder expropriation through cross-ownership.

Like Crama and Leruth (2007), we use sequential voting. However, in Crama and Leruth (2007), each step simultaneously replaces the votes of all the firms by the majority vote of their predecessors in the previous turn. By contrast, in our approach, the vote of only one firm, chosen
randomly, is replaced by the vote of its predecessor in the previous turn. The reason for this difference is that our objective is to identify all the admissible voting states accessible from any given initial voting state. Contrary to methods based on simultaneous voting, the stochastic environment allows our sequential approach to circumvent the divergence problem associated with multiple equilibria. Importantly, sequential voting is simply a technical trick to consistently evaluate probabilities. We do not actually assume that firms vote in random sequences.

Intuitively, sequential voting works as follows. Let us consider an arbitrary—possibly non-admissible—initial voting state, i.e. any vector $X^0 \in \{0,1\}^n$. Next, a randomly picked subsidiary, say firm A, is asked to vote according to the majority rule under the assumption that the votes of firm A’s shareholders are determined by $X^0$. Since $X^0$ is arbitrary, the resulting vote of the firm A may differ from its initial vote in $X^0$. By replacing the initial vote of firm A by this new vote, we obtain voting state $X^1 \in \{0,1\}^n$, where at most one vote has been changed. From there, a second randomly picked subsidiary is asked to vote, and so on. With the help of Markov chains, we prove that the resulting sequence of voting states ends up reaching an admissible voting state with probability one. As a consequence, considering all the possible voting sequences allows us to exhaust the set of admissible voting states and determine their probabilities of occurrence.

Let us rephrase sequential voting in mathematical terms. Consider initial voting state $X^0 \in \{0,1\}^n$ and a sequence of randomly picked subsidiaries $W = (w_k) \in F^\omega$. We refer to $W$ as the voting order. The transition from voting state $X^{k-1} (k \geq 1)$ to voting state $X^k$ is obtained by making the direct shareholders of firm $w_k$ vote under the majority rule. We obtain $X^k = (x^k_1, \ldots, x^k_n)$ where:
\[ \forall j = 1, \ldots, n: \quad x^k_j = \begin{cases} g_j(X^{k-1}) & \text{if } w_k = j \\ x^{k-1}_j & \text{if } w_k \neq j \end{cases} \quad (4) \]

where \( g_j(.) \) expresses the majority voting rule:

\[
g_j : \{0,1\}^n \rightarrow \{0,1\} : X \rightarrow \begin{cases} x_j & \text{if } j \in S \\ 1 & \text{if } j \in F \text{ and } \sum_{i=1}^n a_{ij} x_i + \frac{\bar{a}_j}{2} > 0.5 \\ 0 & \text{if } j \in F \text{ and } \sum_{i=1}^n a_{ij} x_i + \frac{\bar{a}_j}{2} \leq 0.5 \end{cases} \quad (5) \]

Theorem 1 establishes conditions under which the process defined by Equations (4) and (5) leads to an admissible voting state in a finite number of steps.

**THEOREM 1.** \( \exists K \in \mathbb{R}, \forall X^0 \in \{0,1\}^n, \exists W \in F^\cup^n \) such that \( X^K \) is an admissible voting state, where \( (X^k) \) is the process defined by Equations (4) and (5).

**Proof.** See Appendix C.\(^{10}\)

According to Theorem 1, there is an upper limit to the number of steps needed to reach an admissible voting state. Remarkably, this upper limit holds uniformly for all initial voting states \( X^0 \). However, this is not true for all voting orders. Different voting orders may lead to different admissible voting states or even to none at all. For instance, some voting orders drive periodic, and hence diverging, sequences of voting states.

\(^{10}\) Appendix C proves Theorem 1 with \( K = 2 \times (n - s) \). However, for many corporate structures, smaller values of \( K \) exist.
In the next step, we establish the probabilities of occurrence of the admissible voting states. Two variables drive the process in Equations (4) and (5): The initial voting state, $X^0$, and the voting order, $W$. Theorem 1 shows that convergence to an admissible voting state is ensured for any $X^0$. In contrast, convergence may fail for some voting orders $W$. To address this issue, we introduce uncertainty into the picture. We assume that all the firms have the same probability of being asked to vote at any point in time.\footnote{This is the most natural assumption. Proceeding otherwise would assume the existence of an agenda setter who is able to influence the organization of the corporate structure prior to any shareholder meeting. In addition, our methodology is easily adaptable to any probability distribution of the voting sequences.} This implies that admissible voting states reached via fewer and/or longer voting sequences have lower probabilities. Accordingly, we define the stochastic voting order as follows.

**DEFINITION 1.** The stochastic voting order, $\tilde{W} = (\tilde{w}_k)_{k \in 0}$, is the independent and identically distributed (i.i.d.) stochastic process such that the probability distribution of random variable $\tilde{w}_k = \tilde{w}$ is:

$$\forall k \in 0, \forall j \in F : \Pr(\tilde{w}_k = j | \tilde{w}_{k-1}, \tilde{w}_{k-2}, ..., \tilde{w}_1) = \Pr(\tilde{w}_k = j) = \frac{1}{n-s}$$

(6)

While there are infinitely many deterministic voting orders $W$, there is only one stochastic voting order process, $\tilde{W}$. Actually, each realization of $\tilde{W}$ is a deterministic $W$. As a consequence, each (deterministic) initial voting state $X^0 \in \{0,1\}^n$ is attached to a single stochastic voting process defined as follows.

**DEFINITION 2.** The stochastic voting process starting from $X^0 \in \{0,1\}^n$ is $(\tilde{X}^k)_{k \in 0}$ such that:
\[ \forall k \in \mathbb{N}_0, \forall X, Y \in \{0,1\}^n : \]

\[
\Pr[\bar{X}^k = X | \bar{X}^{k-1} = Y] = \begin{cases} 
0 & \text{if } X \notin B_i(Y) \\
\frac{1}{n-s} & \text{if } X \in B_i(Y) \text{ and } X \neq Y \\
\frac{\# \{ j \in F : y_j = g_j(Y) \}}{n-s} & \text{if } X \in B_i(Y) \text{ and } X = Y 
\end{cases}
\tag{7}
\]

with \( \Pr[\bar{X}^0 = X^0] = 1 \),

and \( B_i(Y) = \{ X \in \{0,1\}^n : X \text{ is accessible from } Y \text{ in 1 step} \} \).

Equation (7) gives the transition probabilities driven by \( \tilde{W} \). Three cases are possible depending on the one-step accessibility of voting state \( X \) from voting state \( Y \). In the first case, \( X \) is not accessible in one step from \( Y \). The transition from \( Y \) to \( X \) is thus impossible. In the second case, \( X \) is accessible in one step from \( Y \), and \( X \) is different from \( Y \). Since there is only one firm \( i \) in \( F \) such that \( y_i \neq x_i = g_i(Y) \), the transition probability is equal to \( \frac{1}{n-s} \). In the last case, \( X \) is accessible in one step from \( Y \), and \( X \) is equal to \( Y \). The equalities \( x_j = y_j = g_j(Y) \) may then be met for several firms in \( F \). This explains the numerator of the transition probability, while the denominator, \( \frac{1}{n-s} \), is the probability of any firm being the \( k \)-th voting firm.

It turns out that the stochastic voting process defined by Equation (7) is a Markov chain with \( 2^n \) states, each being an \( n \)-dimensional vector with binary components. The next theorem states the convergence of this Markov chain.

THEOREM 2. \( \forall X^0 \in \{0,1\}^n \), the Markov chain defined by Equation (7) is such that:
Pr[∃K ∈ I, ∀k > K : \tilde{X}^k is an admissible voting state] = 1 \tag{8}

\textit{Proof.} See Appendix D.

Theorem 2 is the cornerstone of our approach. It states that the probability associated with diverging voting orders is zero in the limit. For any initial voting state, Theorem 2 ensures the existence and uniqueness of a stochastic admissible voting state reached by the Markov chain in a finite number of steps. This result allows us to consistently extend the definition of equilibrium in Equation (3) to any corporate structure while keeping the key properties of existence and uniqueness. These properties are vital to assess the voting power of shareholders.

**DEFINITION 3.** The \textit{equilibrium of the voting game} in corporate structure \( H = (V, R, A) \) is:

\[ \tilde{\Omega}: \{0,1\}^n \rightarrow \mathcal{Z}_{\{0,1\}}: X^0 \rightarrow \tilde{X}^* \]

where \( \mathcal{Z}_{\{0,1\}} \) is the set of random variables taking values in \( \{0,1\}^n \), and \( \tilde{X}^* \) is the limit state of the Markov chain defined in Equation (7).

When \( H = (V, R, A) \) is a pyramid, Definition 3 boils down to its deterministic counterpart, Equation (3). Definition 3 is however significantly more general. Indeed, \( \tilde{\Omega} \) maps any initial voting state to a (stochastic) admissible voting state. In contrast, \( \Omega \) in Equation (3) only maps the votes of the sources to a (determinist) admissible voting state.\(^{12}\) This extension is relevant not only to properly address the multiple-equilibrium issue driven by Equation (3), but also to

\(^{12}\) The votes of the sources remain constant in sequential voting.
coherently define the notion of equilibrium in structures without ultimate shareholders, such as the one featured in Figure B2 in Appendix B.

With Definition 3, we are now equipped to introduce a game-theoretic measurement of voting power in cross-ownership-inclusive corporate structures. This measurement will be applicable to any shareholder, ultimate or not, in any firm. In pyramids, the Banzhaf index measures the voting power of a source in a non-source firm. It is consistently defined because the deterministic equilibrium is unique. The notion of equilibrium in Definition 3 allows us to generalize the Banzhaf index and make it applicable to any corporate structure. Specifically, we replace the deterministic equilibrium by its stochastic counterpart and take mathematical expectations to obtain the following definition.

DEFINITION 4. The voting power of shareholder \( i \) in firm \( j \) is:

\[
Z_{ij} = \frac{1}{2^{n-1}} \sum_{X^0 \in \{0,1\}^n} \left[ E \left( \tilde{x}^*_j \middle| x^0_j = 1 \right) - E \left( \tilde{x}^*_j \middle| x^0_j = 0 \right) \right], \quad i \in V, \ j \in F
\]

where \( \tilde{x}^*_j \) is the \( j \)th component of the stochastic equilibrium vector \( \tilde{X}^* \).

According to this definition, the voting power of shareholder \( i \) in firm \( j \) is the expected frequency with which firm \( i \)’s vote is pivotal in firm \( j \). Note that firm \( i \) denotes any firm in \( V \), and not necessarily a source. Moreover, the sum in the right-hand side of Equation (9) is taken over the initial voting states of all firms, acknowledging that non-source firms may influence the votes in other non-source firms, and even in themselves. The next proposition shows that in a pyramidal structure, Definition 4 boils down to the usual Banzhaf index.
THEOREM 3. If ownership structure $H = (V, R, A)$ is a pyramid, then Equation (9) boils down to:

$$Z_{ij} = \frac{1}{2^{n-1}} \left\{ \sum_{x^*_j \in \{0,1\}^n : x^*_j = 1} x^*_j \left( X^*_S \right) - \sum_{x^*_0 \in \{0,1\}^n : x^*_0 = 0} x^*_j \left( X^*_S \right), i \in S, j \in F \right\}$$

where $\left( X^*_S \right) = \left( x^*_0 \left( X^*_S \right), x^*_1 \left( X^*_S \right), ..., x^*_n \left( X^*_S \right) \right)$ represents the unique equilibrium accessible from $X^0 = (X^0_S, X^0_F)$, and $X^0_F$ is arbitrary.

Proof. See Appendix E.

Contrasting with the approaches of Bennedsen and Wolfenzon (2000) and Gambarelli and Owen (1994) for pyramidal structures, our probabilistic approach does not require us to identify winning coalitions. This is a substantial advantage. Given the circularity of cross-ownership, the same coalition may be winning in some situations and losing in others. In addition to enlarging the scope of the Banzhaf index to include corporate structures with cross-ownership, Theorem 3 demonstrates that, for pyramids, the enlarged Definition 4 will automatically provide the same figures as the algorithms that compute the classical Banzhaf index for acyclic graphs (Crama and Leruth, 2007; Levy, 2011).

In practice, however, computing the voting power in Definition 4 may prove to be tedious because the combinatorial complexity of Equation (9) is $O(2^n)$. To address this issue for pyramids, Crama and Leruth (2007) use Monte-Carlo simulations. Alternatively, Levy’s (2011) algorithm uses a so-called coincidence matrix to compute all Banzhaf indices in a single step. The next theorem makes Definition 4 operational by adapting Levy’s approach to Equation (9).
THEOREM 4.

\[ \forall i \in V, \forall j \in F : Z_{ij} = 2C_{ij} - 1 \]  \hspace{1cm} (11)

where:

\[ C_{ij} = \frac{1}{2^n} \sum_{x^i \in \{0,1\}^n} \left( x^i \cdot \Pr[\bar{x}_j^i = 1] + (1 - x^i) \cdot \Pr[\bar{x}_j^i = 0] \right) \]  \hspace{1cm} (12)

Proof. See Appendix F

Entry \( c_{ij} \) measures the expected occurrence of firms \( i \) and \( j \) voting alike. The sum in Equation (12) is taken over all possible voting states of the sources, and then divided by the number of such possibilities. Theorem 4 is a key piece of our algorithm. The intuition is the following. First, if firms \( i \) and \( j \) vote independently, their votes are identical in 50% of the cases and \( C_{ij} = 0.5 \). According to Equation (11), this produces a zero control power of firm \( i \) over firm \( j \) \( (Z_{ij} = 0) \). Second, if firm \( i \) fully controls firm \( j \), both firms always vote alike and \( C_{ij} = 1 \). In this case, Equation (10) gives a control power of firm \( i \) over firm \( j \) equal to one \( (Z_{ij} = 1) \). Third, in the middle scenario, where firm \( i \) exerts partial influence over firm \( j \), we have \( C_{ij} \in (0.5,1) \) and \( Z_{ij} \in (0,1) \). For instance, \( C_{ij} = 0.7 \) means that, on top of the 50% of cases where firms \( i \) and \( j \) vote alike by chance, there is an additional 20% of active control of firm \( i \) over firm \( j \). However, this 20% level of influence is to be understood with respect to a maximal value of 50% (not 100%). Therefore, on a 0-100% scale the influence of firm \( i \) on firm \( j \) is \( 2 \times 20\% = 40\% \), which is precisely the value of \( Z_{ij} \) resulting from Equation (11) for \( C_{ij} = 0.7 \).
Consider for instance the symmetrical two-firm case with two ultimate shareholders featured in Figure 4. The coincidence matrix of the ownership structure is given by \( C = \left( C_{ij} \right)_{4 \times 4} : \\
C = \begin{pmatrix}
1 & 0.5 & 0.75 & 0.75 \\
0.5 & 1 & 0.75 & 0.75 \\
0.5 & 0.5 & 0.625 & 0.625 \\
0.5 & 0.5 & 0.625 & 0.625 \\
\end{pmatrix}

Consequently, the voting powers in the corporate structure in question are summarized in matrix \( Z = \left( Z_{ij} \right)_{4 \times 4} : \\
Z = \begin{pmatrix}
1 & 0 & 0.5 & 0.5 \\
0 & 1 & 0.5 & 0.5 \\
0 & 0 & 0.25 & 0.25 \\
0 & 0 & 0.25 & 0.25 \\
\end{pmatrix}

In particular, Matrix \( Z \) indicates that both firms A and B have a 50% voting power in firm C. More interestingly, the voting power of firm D in firm C is 25%, and the voting power of C in itself is also 25%. When a binary vote takes place in firm C, and firms A and B disagree, which happens with probability 0.5, the outcome depends on the firms in the ring, i.e. firms C and D, each enjoying a 25% probability to be decisive.

In sum, this section has emphasized that voting power is not restricted to top-down relationships. Subsidiaries can enjoy actual voting power either over their own company or over other companies belonging to the same cross-ownership ring. To evaluate the extent of this oft-unnoticed possibility, we suggest using a power index approach. From now on, voting power in
any ownership structure can be fully elucidated. We present in Appendix G the algorithm that makes our approach implementable in practice. This is an important step since the main problem associated with the use of power indices is computational complexity.

5. Who Controls Allianz?

The case of Allianz, a German insurance company, is taken from La Porta et al. (1999). The ownership situation of Allianz in 1998, featured in Figure 1, has become a typical example of cross-ownership.\(^{13}\) Table I shows that the Allianz ownership structure has two ultimate shareholders, Bayerische Vereinsbank (BV) and the Finck Family (FF). Both own 5% of the company’s shares. Four other shareholders are involved in cross-ownership with Allianz. The group’s largest direct shareholder, Münchener Rückversicherung (MR), holds 25% of Allianz’s shares while Allianz owns 25% of MR’s shares. Moreover, Allianz also has indirect control over MR via Dresdner Bank (DrB).

< Insert Table I here >

We have applied our algorithm to two different scenarios. Scenario 1 corresponds to a hypothetical situation without cross-ownership. More precisely, starting from the real data, we have we zeroed the shares owned by Allianz (25% in MR, 22.5% in DrB, 22.6% in Bayerische

\(^{13}\) Franks and Mayer (2001) and Gugler and Yurtoglu (2003) mention Allianz to illustrate the complexity of determining actual control stakes when cross-ownership is involved. However, only Dorofeenko et al. (2008) really try to identify Allianz’s actual controller, starting from the assumption that control is held by one ultimate shareholder at most. Strikingly, the authors conclude that all ultimate shareholders could be the controller of Allianz, so that actual control is probably hidden among the firms with which Allianz has cross holdings.
Hypotheken- und Wechsel-Bank (BHWB), and 5% in Deutsche Bank (DeB)) as well as the 10% share DrB holds in MR. Scenario 2 is simply the actual cross-ownership-inclusive situation reported in Table I. In both scenarios, we assume that the float of each non-source firm is split, with equal parts voting 0 and 1 respectively.

In both scenarios, Table II provides the voting powers in Allianz of any firm in the group including Allianz itself. The total of each column in Table II can exceed 100%, which is a feature of the Banzhaf index.

The second scenario, i.e. the real situation, delivers striking results. While cross-ownership does not significantly decrease the ultimate shareholders’ voting power, Allianz has the highest voting power (59.4%) over itself. The actual voting power of its ultimate shareholders, BV and FF, equals their direct ownership share (5%). Moreover, the voting power of the other Allianz shareholders is substantially lower in scenario 2 than in scenario 1, and even lower than their direct ownership shares, except for MR.

Two effects concur to explain the results. First, in scenario 2, each shareholder of Allianz has to share control with Allianz itself. To better understand how this situation affects voting powers, let us consider the extreme—and fictitious—situation where two firms, say A and B, own 100% of each other’s shares. At first sight, it might seem that each firm fully controls the other. In fact, this is not the case. The voting power in A is to be split between A and B, each having a voting power of 50% .

14 Thus, although unrealistic and counter-intuitive, this extreme situation provides a method to understand the effects of cross-ownership. In this extreme case, there are four possible initial voting states and two equilibrium voting states: (0,0) and (1,1). Voting states (0,1) (1,0) both lead to equilibrium (0,0) with probability 0.5, and to equilibrium (1,1) with probability...
case illustrates how cross-ownership insulates a firm from outside control, even when exerted by a dominant shareholder.

The second effect of cross-ownership stems from the last-mover’s advantage. If an Allianz shareholder wishes to challenge the company, then Allianz can use cross-ownership to force other firms to vote in its favor, provided that the votes are organized first in these other firms. Interestingly, using our algorithm shows that in all admissible voting states, the votes of the firms embedded in Allianz’s cross-ownership ring, are the same as that of Allianz. This means that after a few iterations all the firms start acting as a coalition. This prevents external shareholders from taking control over Allianz.

This example reveals that the main impact of cross-ownership is not the capture of control by outsiders with low cash-flow rights, as Bebchuk et al. (2000) argue, but rather empowerment of management. Indeed, while the control of Allianz over itself is trivially zero in the absence of cross-ownership, it amounts to 59.4% when cross-ownership is taken into account. Evidence thus shows that cross-ownership turns Allianz into its own dominant controller. The figures in Table II should, nevertheless, be treated with caution. Indeed, the control powers in scenario 2 are likely over-estimated because Allianz’s shareholders have owners who are not reported in the data. Therefore, our algorithm puts all these owners in the float, and neutralizes their votes. In this way, the voting power of Allianz in its shareholders is inflated.

This evidence contradicts the intuition that only ultimate shareholders can control a firm. While considering only top-down control relationships makes perfect sense in pyramidal

\[ \text{0.5. As a consequence, the coincidence index of any firm with itself is 0.75, and the corresponding voting power is 0.5. The same holds true for the voting powers of A in B, and of B in A.} \]
structures, cross-ownership changes the picture dramatically. Indeed, taking cross-ownership into
account reveals how Allianz has managed to be its own majority controller.

6. Conclusion

This paper argues that firms embedded in cross-ownership rings should be viewed as potential
controllers of themselves. Since cross-ownership artificially creates own shares by leveraging the
firm’s shareholdings, it insulates firms from outside control and can lead to external shareholders
being expropriated. But who ultimately controls such firms? In Bebchuk et al. (2000), there is a
dominant shareholder who is assumed to be also the manager or, at least, to control the manager
(by controlling the board and thus being able to fire the manager). In our framework, cross-
ownership enables the controller-manager to lock in control with a much lower fraction of the
cash-flow rights. Our analysis suggests, however, that when the dominant shareholder does not
tightly control the manager or when there is no dominant shareholder, cross-ownership favors the
manager. Cross-ownership is thus a fragile instrument of shareholder entrenchment compared
with, say, dual-class shares or pyramids. More plausibly, cross-ownership represents an
entrenchment device for insiders, namely the management of the firms in the ring. In addition,
cross-ownership can go far beyond own shares, so that the subsequent voting power conferred on
managers is not limited to their own firm.

In line with the general theory developed by Aghion and Tirole (1997), the literature on
corporate governance typically considers that shareholders hold “formal” (i.e. rights-based)
authority, while managers exert “real” (i.e. effective) authority (Burkart et al., 1997; Becht et al.,
Moreover, the impact on managers of separating shareholders’ cash-flow rights from their voting rights remains controversial (Burkart and Lee, 2008). Surprisingly, though, the specific expropriation mechanism generated by cross-ownership rings is hardly explored. With respect to the formal v. real authority classification, cross-ownership may be viewed as a kind of in-between situation since it can give formal authority (i.e. voting rights) to managers with no cash-flow rights.

Countries are often classified according to their corporate governance systems. The standard split opposes the U.S. system, supposedly characterized by dispersed ownership and powerful managers, to the civil-law system involving powerful shareholders who control pyramidal structures. Holderness (2009) challenges this conventional wisdom by showing that ownership concentration of U.S. firms is similar to that observed in other countries. Our findings push the argument one step further by stressing that cross-ownership rings embedded in pyramidal structures can significantly empower managers. In this respect, our new approach to managerial entrenchment contributes to the ongoing conversation on corporate governance systems.

Moreover, the literature lacks appropriate tools for inferring the practical consequences of cross-ownership on governance. This paper fills the gap by offering a new measure of voting power that generalizes the Banzhaf index from game theory. In line with this theory, we propose an algorithm to compute the voting powers of all shareholders (ultimate or not) in all subsidiary firms through Monte-Carlo simulations. As a result, researchers are now equipped to compute control stakes in any corporate structure including cross-ownership rings. The fundamentals of our

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15 See also Hamdani and Yafeh (2013) on the voting behavior of minority institutional investors.
model could also be applied to other fields involving voting issues in cyclic graphs, such as political science.

Our findings may also be instrumental for regulators for at least two reasons. First, cross-ownership, like share buyback, artificially inflates capital, and gives a distorted image of the real financial situation (Bøhren and Michalsen, 1994). Second, it can increase cost-inefficient entrenchment. Although most countries restrict companies from purchasing their own shares, cross-ownership is mostly disregarded by regulators, likely because they lack a rigorous method of measuring the actual consequences. The method we propose could pave the way to better regulatory appraisal of management entrenchment through cross-ownership.

Our paper opens additional avenues for research. Recent literature documents the impact of governance variables on different dimensions of corporate performance. However, cross-ownership is rarely taken into consideration as a relevant aspect of governance. This ought to be changed, especially regarding management entrenchment. However, further research is needed to assess this claim in terms of the corporate characteristics typically associated with the presence of managerial entrenchment, such as private benefits of control and vulnerability to takeovers or control contests.

Future work could also assess the robustness of our results with respect to endogeneity in the voting process. A first source of potential endogeneity is the vote of the float. Our algorithm builds on the assumption that this vote is neutral (i.e. equally split). This assumption may prove overstated, especially in situations such as mergers and acquisitions, where small investors come to shareholder meetings with a concerted agenda (Matvos and Ostrovsky, 2008). A second source of endogeneity stems from the possible presence of an agenda setter. A shareholder who is able to
manipulate vote planning for reasons unrelated to ownership (such as political power or informational superiority) could enforce a voting order that would maximize her voting power in a given firm. The interesting question here is how agenda setting affects the distribution of voting power in corporate structures with cross-ownership. Remarkably, cross-ownership-inclusive corporate structures are the only ones where agenda setting can affect shareholders' voting powers.

In sum, this paper provides a general method for measuring the impact of cross-ownership on corporate control. It theorizes that cross-ownership can act as a powerful device for shareholder expropriation and also create management entrenchment. Accordingly, cross-ownership certainly deserves more attention than it has received thus far.
List of Figures

Figure 1. Allianz's ownership structure in 1998

Source: La Porta et al. (1999)
Figure 2. The symmetrical two-firm case with a single controller (Bebchuk et al., 2000)

Figure 3. The symmetrical two-firm case with self-control

Figure 4. The symmetrical two-firm case with two ultimate shareholders


List of Tables

*Table I. Direct Ownership Shares in the Allianz Group (in %)*

This table reports the ownership structure of the Allianz Group in 1998 (Source: La Porta et al., 1999). The structure is made of seven shareholders, including two ultimate ones: Bayerische Vereinsbank (BV) and the Finck Family (FF). Cell $(i,j)$ represents the direct ownership share of shareholder $i$ in firm $j$ (in %).

<table>
<thead>
<tr>
<th>Shareholder $i$</th>
<th>MR</th>
<th>DrB</th>
<th>DeB</th>
<th>BHWB</th>
<th>Allianz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayerische Vereinsbank (BV)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Finck Family (FF)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Münchener Rückversicherung (MR)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>Dresdner Bank (DrB)</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Deutsche Bank (DeB)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Bayerische Hypotheken und Wechsel Bank (BHWB)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Allianz</td>
<td>25</td>
<td>22.5</td>
<td>5</td>
<td>22.6</td>
<td>0</td>
</tr>
</tbody>
</table>
Table II. Voting Powers in Allianz (in %)

This table presents the voting powers in Allianz (in %) produced by our algorithm for two different scenarios. Scenario 1 corresponds to a simulation excluding cross-ownership. It is obtained by zeroing all the shares owned by Allianz as well as the share Dresdner Bank (DrB) holds in Münchener Rückversicherung (MR). Scenario 2 is the actual cross-ownership-inclusive situation reported in Table I. In both scenarios, we assume that the float is neutral.

<table>
<thead>
<tr>
<th>Shareholder</th>
<th>Scenario 1: Simulation excluding cross-ownership</th>
<th>Scenario 2: Real situation including cross-ownership</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayerische Vereinsbank (BV)</td>
<td>6.3</td>
<td>5.0</td>
</tr>
<tr>
<td>Finck Family (FF)</td>
<td>6.3</td>
<td>5.0</td>
</tr>
<tr>
<td>Münchener Rückversicherung (MR)</td>
<td>81.3</td>
<td>35.9</td>
</tr>
<tr>
<td>Dresdner Bank (DrB)</td>
<td>18.8</td>
<td>7.7</td>
</tr>
<tr>
<td>Deutsche Bank (DeB)</td>
<td>18.8</td>
<td>7.8</td>
</tr>
<tr>
<td>Bayrische Hypotheken- und Wechsel-Bank (BHWB)</td>
<td>18.8</td>
<td>7.7</td>
</tr>
<tr>
<td>Allianz</td>
<td>0</td>
<td>59.4</td>
</tr>
</tbody>
</table>
Appendix A: Cross-Holding versus Cross-Ownership

Faccio and Lang (2002, p. 373) illustrate the notion of “cross-holding” borrowed from Claessens et al. (2000) by the graph in Figure A1, where firm A is an ultimate shareholder. According to the terminology presented in Section 3, this graph is acyclic and the corresponding corporate structure is a pyramid. This is because the ownership links in Figure A1 are top-down only. In particular, firm B has no shareholding in the graph. Creating a cycle requires reverting at least one vertex in Figure A1. This is illustrated by Figure A2, where vertex A-B has been reverted. Note that in Figure A2 there is no ultimate shareholder anymore. Comparing the cross-holding case in Figure A1 to the cross-ownership case in Figure A2 emphasizes that cross-holding and cross-ownership refer to quite different situations.

Figure A1. The cross-holding example of Faccio and Lang (2002)
Figure A2: A cross-ownership example
Appendix B: Additional Examples

Let us illustrate with two examples how top managers in subsidiaries can seize voting power through cross-ownership. The first example in Figure B1 involves own shares, the simplest form of cross-ownership. Firm C has two ultimate shareholders, firms A and B, each holding 49% of the shares. The remaining 2% are own shares controlled by the top management of firm C. Following the majority voting rule, any coalition of two players among A, B, and C makes the decision. With only 2% of the voting shares, the top management of firm C has the same voting power as any ultimate shareholder. Put formally, the voting game has four equilibria:

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Omega_1)</td>
<td>({0,1}^2 \rightarrow {0,1}^3 : (0,0) \rightarrow (0,0,0),(1,1) \rightarrow (1,1,1),(0,1) \rightarrow (0,1,0),(1,0) \rightarrow (1,0,1))</td>
</tr>
<tr>
<td>(\Omega_2)</td>
<td>({0,1}^2 \rightarrow {0,1}^3 : (0,0) \rightarrow (0,0,0),(1,1) \rightarrow (1,1,1),(0,1) \rightarrow (0,1,1),(1,0) \rightarrow (1,0,0))</td>
</tr>
<tr>
<td>(\Omega_3)</td>
<td>({0,1}^2 \rightarrow {0,1}^3 : (0,0) \rightarrow (0,0,0),(1,1) \rightarrow (1,1,1),(0,1) \rightarrow (0,1,0),(1,0) \rightarrow (1,0,0))</td>
</tr>
<tr>
<td>(\Omega_4)</td>
<td>({0,1}^2 \rightarrow {0,1}^3 : (0,0) \rightarrow (0,0,0),(1,1) \rightarrow (1,1,1),(0,1) \rightarrow (0,1,1),(1,0) \rightarrow (1,0,1))</td>
</tr>
</tbody>
</table>

The four equilibria share the following characteristics: \((0,0) \rightarrow (0,0,0),(1,1) \rightarrow (1,1,1)\). This means that firms A and B together dictate their common will to firm C. The differences between the four equilibria only concern situations were firms A and B disagree. In equilibrium \(\Omega_1\), the management of C coalesces with firm A. Indeed, we have \((0,1) \rightarrow (0,1,0)\) and \((1,0) \rightarrow (1,0,1)\), which implies that firm C aligns on firm A’s vote. In equilibrium \(\Omega_2\), the management of C coalesces with firm B. In equilibria \(\Omega_3\) and \(\Omega_4\), the management of firm C has no fixed coalition.
Figure B1. Voting power of top managers: An example with own shares

![Diagram of voting power with own shares](image)

Figure B2. Voting power of top managers: An example with cross-ownership

![Diagram of voting power with cross-ownership](image)

The second example in Figure B2 emphasizes that, unlike pyramids, there may be cross-ownership-inclusive corporate structures without any ultimate shareholders. In Figure B2, firm A has three 100%-controlled subsidiaries: firms B, C, and D. These subsidiaries hold shares in their parent firm A. As a result, the graph has no ultimate shareholders and the notion of equilibrium in Equation (3) loses significance. Still, it is obvious from Figure B2 that there are at least two
admissible voting states. In the first, firm A dictates its will to its three subsidiaries. In the second, the three subsidiaries form a coalition and control their parent firm A. At this stage, we will not elaborate further on this example. Our point is simply to stress that cross-ownership deeply challenges the intuition that full control is secured by a majority voting share. Moreover, this example shows that the definition of equilibrium needs to be generalized.
Appendix C: Proof of Theorem 1

Let $X^0$ be an initial voting state. We build a finite voting sequence leading from $X^0$ to an admissible voting state. We choose the voting order in the following way. First, we place the $L$ firms that, according to Equation (5), change their vote from 0 to 1. Namely, the first firm to vote, $w_1$, is defined by the following conditions:

a) If $\forall i \in F: x_i^0 = 0 \Rightarrow g_i(X^0) = 0$, then $L = 0$ and $X^L = X^0$

b) Otherwise, $\exists w_1 \in F$ such that: $x_{w_1}^0 = 0$ and $g_{w_1}(X^0) = 1$

and Equation (5) gives voting state $X^1$.

Similarly, the second firm to vote, $w_2$, is defined by:

a) If $\forall i \in F: x_i^1 = 0 \Rightarrow g_i(X^1) = 0$, then $L = 0$ and $X^L = X^1$

b) Otherwise, $\exists w_2 \in F$ such that: $x_{w_2}^1 = 0$ and $g_{w_2}(X^1) = 1$

and Equation (5) gives voting state $X^2$.

And so on, until the following condition is met:

$$\forall i \in F: x_i^k = 0 \Rightarrow g_i(X^k) = 0.$$  \hspace{1cm} (C1)

Condition (B1) is reached in a finite number of steps, $L \leq n - s$, because the graph is finite. As a consequence, after $L$ steps, the first part of the voting sequence $(X^1, X^2, ..., X^L)$ is such that:

$$\forall j \in \{1, 2, ..., L\}, x_{w_j}^{j-1} = 0 \text{ and } x_{w_j}^j = g_{w_j}(X^{j-1}) = 1$$  \hspace{1cm} (C2)

and:

$$\forall i \in F: x_i^L = 0 \Rightarrow x_i^L = g_i(X^L)$$  \hspace{1cm} (C3)
Next, we place in the sequence all the firms that change their vote from 1 to 0. Firm $w_{L+1}$ is defined by:

a) If $\forall i \in F : x_i^L = 1 \Rightarrow g_i(X^L) = 1$, and $X^{L+M} = X^L$.

b) Otherwise, $\exists w_{L+1}$ such that: $x_{w_{L+1}}^L = 1$ and $g_{w_{L+1}}(X^L) = 0$

and Equation (7) gives voting state $X^{L+1}$.

And so on, until the following condition is met:

$\forall i \in F : x_i^{L+k} = 1 \Rightarrow g_i(X^{L+k}) = 1$ (C4)

Condition (B4) is reached in a finite number of steps, say $M \leq n - s$. The associated voting subsequence, $(X^{L+1}, X^{L+2}, ..., X^{L+M})$ is such that:

$\forall j \in \{L + 1, L + 2, ..., L + M\} : x_{w_j}^{j-1} = 1$ and $x_{w_j}^j = g_{w_j}(X^{j-1}) = 0$ (C5)

and: $\forall i \in F : x_i^{L+M} = 1 \Rightarrow x_i^{L+M} = g_i(X^{L+M})$ (C6)

Last, we prove that the final voting state $X^{L+M}$ is admissible. From Equation (B6), we have:

If $x_i^{L+M} = 1$, then $x_i^{L+M} = g_i(X^{L+M})$.

Alternatively, if $x_i^{L+M} = 0$, then either $x_i^L = 0$ and Equation (B3) yields $x_i^L = g_i(X^L) = 0$, or $\exists m \in \{1, 2, ..., M\}$ such that $x_i^{L+m-1} = 1$ and $x_i^{L+m} = g_i(X^{L+m-1}) = 0$. In both cases, $\exists m \in \{0, 1, ..., M\}$ such that $g_i(X^{L+m}) = 0$. Condition (B5) then implies that: $g_i(X^{L+m}) = 0 \Rightarrow g_i(X^{L+M}) = 0 = x_i^{L+M}$.

Finally, we obtain $x_i^{L+M} = g_i(X^{L+M})$, which implies that $X^{L+M}$ is an admissible voting state. Moreover, the length of the complete voting sequence is: $L + M \leq 2(n - s)$.

QED
Appendix D: Proof of Theorem 2

Once the Markov chain defined by Equation (5) has reached an admissible voting state, $X^*$, it stays there forever:

$$\Pr[\tilde{X}^k = X^* | \tilde{X}^{k-1} = X^*] = 1$$

This implies that all admissible voting states of the Markov chain in Equation (5) are absorbing states (Fu and Lou 2003). Moreover, Theorem 1 ensures that, for any initial voting state, there exists an accessible, and hence absorbing, admissible voting state. The Markov chain is therefore absorbing. Finally, in an absorbing Markov chain, the probability that the process is absorbed in a finite number of steps is equal to one (Grinstead and Snell 1997, p. 417).

QED.
Appendix E: Proof of Theorem 3

In a pyramid, the equilibrium exists and is unique. Moreover, the sources have no predecessors and will thus never change their initial votes in $X^0_S$. As shown by Gambarelli and Owen (1994), the votes of the sources fully determine the votes of the other firms. Therefore, $X^0_F$ is irrelevant to the computation of the $Z_{ij}$’s and can be arbitrarily chosen.

If firm $i$ is a source ($i \in S$), then its vote is fixed to either 1 or 0, and we have:

$$Z_{ij} = \frac{1}{2^{n-1}} \left[ \sum_{X_S^0 \in \{0,1\}^n : x^0_i = 1} x^*_j \left( X^0_S \right) - \sum_{X^0_S \in \{0,1\}^n : x^0_i = 0} x^*_j \left( X^0_S \right) \right]$$

Alternatively, if firm $i$ is not a source ($i \in F$), then $E(\tilde{x}^*_j | x^0_i = 1) = E(\tilde{x}^*_j | x^0_i = 0)$, meaning that firm $i$ has no influence on the votes of any firm. It follows that:

$$Z_{ij} = 0.$$
Appendix F: Proof of Theorem 4\textsuperscript{16}

From Equation (4), we have:

\[ C_j = \frac{1}{2^n} \sum_{x^j \in \{0,1\}^n} (x^j_0 \cdot \Pr[\tilde{x}_j^* = 1] + (1 - x^j_0) \cdot \Pr[\tilde{x}_j^* = 0]) \]

Then, simple derivations yield:

\[
2C_j - 1 = \frac{1}{2^{n-1}} \left( \sum_{x^j \in \{0,1\}^n} (x^j_0 \cdot \Pr[\tilde{x}_j^* = 1] + (1 - x^j_0) \cdot \Pr[\tilde{x}_j^* = 0]) - 1 \right)
\]

\[
= \frac{1}{2^{n-1}} \left( \sum_{x^j \in \{0,1\}^n \setminus x^j_0 = 1} \Pr[\tilde{x}_j^* = 1] + \sum_{x^j \in \{0,1\}^n \setminus x^j_0 = 0} \Pr[\tilde{x}_j^* = 0] - 1 \right)
\]

\[
= \frac{1}{2^{n-1}} \left( \sum_{x^j \in \{0,1\}^n \setminus x^j_0 = 1} \Pr[\tilde{x}_j^* = 1] - \sum_{x^j \in \{0,1\}^n \setminus x^j_0 = 0} \Pr[\tilde{x}_j^* = 1] \right)
\]

\[
= \frac{1}{2^{n-1}} \left( \sum_{x^j \in \{0,1\}^n \setminus x^j_0 = 1} E(\tilde{x}_j) - \sum_{x^j \in \{0,1\}^n \setminus x^j_0 = 0} E(\tilde{x}_j) \right)
\]

\[
= \frac{1}{2^{n-1}} \sum_{x^j \in \{0,1\}^n} \left( E(\tilde{x}_j = 1 | x^j_0 = 1) - E(\tilde{x}_j = 1 | x^j_0 = 0) \right)
\]

\[ = Z_j \]

QED

\textsuperscript{16} This proof is inspired by Levy (2011)
Appendix G: Algorithm

We extend Levy’s (2011) algorithm to corporate structures with cross-ownership. Using coincidence matrices makes it possible to derive all control powers in the corporate structure from a single Monte-Carlo simulation. Moreover, Levy (2011) considers several modelizations of the float. Here, we restrict ourselves to a single modelization and assume that the float is split, with equal parts voting 0 and 1. This split is assumed for the sake of simplicity, but our algorithm could easily be adapted to other situations for the float.

The new algorithm simulates the Markov stochastic voting process and stops as soon as an admissible voting state is reached. The two steps of the algorithm are driven by the theoretical framework presented in Sections 3 and 4. First, for a given initial voting state, we determine the limit admissible voting states through simulations. Second, we proxy the coincidence matrix by replacing probabilities with the corresponding simulated frequencies.

Step 1: Simulations

For each initial voting state, \( X^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \{0,1\}^n \), we generate \( M \) random voting paths. In each simulation, the voting path is driven by integers taken randomly in set \( \{s+1, s+2, \ldots, n\} \).

In the first step, we randomly pick \( i_i \in \{s+1, s+2, \ldots, n\} \), which designates the first voting firm. The first simulated voting state is \( X^1 = (x_1^1, x_2^1, \ldots, x_n^1) \), where:

\[
    x_i^1 = \begin{cases} 
    g_i(X^0) & \text{if } i = i_i \\
    x_i^0 & \text{otherwise}
    \end{cases}
\]
In the \((k+1)^{th}\) step, \(i_{k+1} \in \{s+1, s+2, \ldots, n\}\) defines voting state \(X^{k+1} = (x_1^{k+1}, x_2^{k+1}, \ldots, x_n^{k+1})\) where:

\[
x_i^{k+1} = \begin{cases} 
  g_i(X^k) & \text{if } i = i_k \\
  x_i^k & \text{otherwise}
\end{cases}
\]

The \(m^{th}\) simulated voting path stops after \(k\) steps if \(\forall j \in \{s+1, s+2, \ldots, n\}\) we have \(x_j^k = g_j(X^k)\). Then, we write: \(X^k = X^*_m\).

**Step 2: Computation of control powers**

For each \(X^0 \in \{0,1\}^n\) and each corresponding admissible voting state \(X^*_m, m \in \{1, \ldots, M\}\), we compute matrix \(S(X^0, m) = (s_{ij}(X^0, m))_{i \in V, j \in F}\) such that:

\[
s_{ij}(X^0, m) = \begin{cases} 
  1 & \text{if } x_j^* = x_i^0 \\
  0 & \text{otherwise}
\end{cases}
\]

Entry \(s_{ij}(X^0, m)\) represents the number of times that, in the \(m^{th}\) simulated path starting from \(X^0\), firm \(j\) ultimately votes in the same way that firm \(i\) voted initially in \(X^0\).

Matrix \(S(X^0, .)\) is computed \(M\) times, even when some simulated paths lead to the same admissible voting state. This is how the algorithm acknowledges the possibility that multiple admissible voting states may have different probabilities. Indeed, the frequency of an admissible voting state indicates how probable this particular state is when the stochastic voting process starts from \(X^0\).
The coincidence matrix $C$ is then obtained by averaging all the $s_{ij} \left( X^0, m \right)$'s:

$$ C_{ij} = \frac{1}{2^nM} \sum_{X^0 \in \{0,1\}^n} \sum_{m=1}^{M} s_{ij} \left( X^0, m \right) $$

This generates the simulated counterpart of the generalized coincidence matrix. According to Theorem 4, the voting powers are then proxied by:

$$ Z_{ij} = 2C_{ij} - 1 $$

This algorithm requires $M 2^n$ simulated samples. Each sample is characterized by an initial voting state and a voting order. Therefore, when the number of firms (especially non-source firms) is large, the algorithm can become highly time-consuming. In this case, it may be better to work with a random sample of initial states $X^0$ as proposed by Crama and Leruth (2007). This, in turn, would make the algorithm less precise.
References


