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Application to $^{16}\text{O}$ and $^{12}\text{C}+\alpha$

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Deducing physical properties of weakly bound states from low-energy scattering data. Application to $^{16}$O and $^{12}$C+$\alpha$.

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Abstract. Possibilities of deducing the asymptotic normalisation constant $C_b$ of the $2^+_1$ subthreshold bound state of $^{16}$O from the low-energy $^{12}$C+$\alpha$ d-wave scattering phase shifts are studied. Within the potential model, it is shown that such a connection exists in principle, either by studying the range of potentials fitting the phase shifts, or by using the effective range expansion. However, present-day data do not seem sufficient to strongly constrain $C_b$, as values ranging from 20 to 180 $\times 10^3$ fm$^{-1/2}$ are obtained, depending on the order of the chosen effective-range expansion.

1. Introduction

Subthreshold bound states, i.e. states with a small binding energy with respect to the lowest scattering channel, constitute a very intense field of research in nuclear physics today, in contexts as varied as radioactive beams, exotic nuclei and nuclear astrophysics. In the simplest case, both the bound and scattering states can be described by a two-nuclei model, with $r$ the radial relative coordinate between these two nuclei, $\mu$ their reduced mass and $Z_1, Z_2$ their atomic numbers. A key feature of such a bound state is then its slowly decreasing radial wave function,

$$ rR_b(r) \sim C_b \exp(-\kappa_b r)/(2\kappa_b r)^{\eta_b}, $$

where $C_b$ is the asymptotic normalisation constant (ANC), $\kappa_b$ is the wave-number modulus, which is related to the binding energy by $E_b = -\hbar^2\kappa_b^2/2\mu$, $\eta_b = -i/a_B\kappa_b$ is the corresponding Sommerfeld parameter and $a_B = \hbar^2/Z_1Z_2e^2\mu$ is the nuclear Bohr radius.

Various methods are available to determine this ANC (radiative cascade, transfer reactions, $\beta$ decay...) [1, 2]. The aim of the present work is to study the possibility to deduce $C_b$ from scattering data, in particular from effective-range expansions (ERE) of the phase shifts belonging to the same partial wave as the considered bound state. The mathematical formalism has been presented in reference [3], both for the charged and neutral cases, together with physical examples like the $^{16}$O+nucleon systems. Here, we focus on a particularly important case, the $2^+_1$ bound state of $^{16}$O, which lies 245 keV below the $^{12}$C+$\alpha$ threshold. This state is believed to have a strong impact on the corresponding radiative capture reaction at energies of astrophysical
interest and its ANC is still subject to strong debate. This state also has a well-known $^{12}$C+$\alpha$ cluster structure, which allows us to describe it with a potential model. We will first revisit the potential models of reference [4], where the connection between bound and scattering states had already been tackled, then use a simple potential model as a test case for the ERE method, and finally directly apply the ERE method to the analysis of the precise experimental data from references [5] and [6]. In the following, we use $\bar{\kappa}$ with a bound state at the physical binding energy $E_b = -245$ keV, with an arbitrary ANC, in order to illustrate the fact that bound-state properties were independent of the scattering properties. It was then argued that to reliably extract this ANC from scattering data, the whole $0^+_2$ rotational band had to be fitted, which led to the value of the ANC $C_b \approx 145(10) \times 10^3$ fm$^{-1/2}$.

It was not realised at that time that among the phase-equivalent family, the nuclear part of some potentials had a more physical behaviour than the others. This is illustrated by figures 1 and 2, where the nuclear part is shown for some of these potentials, among which the ones presented in figure 2 of reference [4]. One notices that the potentials corresponding to an ANC in the range $C_b \approx 190(10) \times 10^3$ fm$^{-1/2}$ decrease faster than the others in the [5-10] fm interval. For smaller ANCs, the potential displays an oscillation at large distance, while for larger ANCs it is always negative but has a slowly-decreasing tail.

This confirms the general argument of reference [7] that, for potentials decreasing faster than $\exp(-2\kappa_b r)$ (in the absence of Coulomb term), the ANC of a bound state with binding energy $E_b = -\hbar^2/2\mu$ is uniquely determined by the scattering matrix for the corresponding partial wave. In fact, all nuclear parts of the one-bound-state phase-equivalent potentials constructed in reference [4] decrease slowly, i.e. as $\exp(-2\kappa_b r)/r^{2|\eta_b|}$, at very large distances (see figure 2 for $r \geq 25$ fm), a drawback which should be eliminated before considering the quoted ANC values as final.

![Figure 1](image1.png)  
**Figure 1.** Nuclear part of the $^{12}$C+$\alpha$ d-wave inversion potentials [4], as a function of the relative distance, with a bound state at the physical energy but with different ANCs.

![Figure 2](image2.png)  
**Figure 2.** Same as figure 1 but in logarithmic scale.
3. Effective-range expansion

Let us now recall the essential formulae required for the direct extraction of the ANC from the low-energy phase shifts, without reference to the potential model [3]. Let us define the effective-range function for the charged case (for the neutral case, see reference [3]),

\[ K_I(k^2) = \frac{2w_l(\eta^2)}{l^2a_B^{2l+1}} \left( \frac{\pi \cot \delta_l(k) - i}{e^{2\pi \eta} - 1} + h(\eta^2) \right), \]

where \( k \) and \( E = \hbar^2k^2/2\mu \) are the complex wave number and energy, \( \eta = 1/a_Bk \) is the Sommerfeld parameter, \( \delta_l \) is the Coulomb-modified short-range phase shift, \( w_l \) and \( h \) are functions defined in reference [3]. Function \( K_I \) is analytic at the origin of the energy complex plane, and hence can be expanded as

\[ K_I(k^2) = -\frac{1}{a_l} + \frac{r_l}{2}k^2 - P_l r_l^2 k^4 + Q_l k^6 + R_l k^8 + S_l k^{10} + O(k^{12}), \]

where \( a_l, r_l \) and \( P_l \) are the scattering length, effective range and shape parameter, respectively.

For real energies, this function is real. For scattering states, i.e. positive real energies, the imaginary part of equation (1) vanishes and \( K_I \) is directly related to the Coulomb-modified short-range phase shift \( \delta_l \). For a bound state of positive imaginary wave number \( k = i\kappa_b \), one has \( \cot \delta_l(i\kappa_b) = i \). Hence, a bound-state energy \( E_b \) must satisfy the relation\(^1\)

\[ K_I(-\kappa_b^2) = \frac{2w_l(\eta_b^2)}{l^2a_B^{2l+1}} h(\eta_b^2). \]

Moreover, the bound state ANC is given by [8]\(^1\)

\[ |C_b| = \kappa_b^l \frac{\Gamma(l + 1 + |\eta_b|)}{l!} \left[ -\left( -\frac{dK_l}{dk^2} + \frac{2}{l^2a_B^{2l+1}} \frac{d(w_l h)}{dk^2} \right) \right]_{k^2 = -\kappa_b^2}^{\frac{1}{2}}. \]  

4. Test on a \(^{12}\)C+α potential model

Let us now apply relations (2) and (3) to the \(^{12}\)C+α d-wave case. As in reference [3], we choose a nuclear Gaussian potential \(-112.3319\exp(-r^2/2.8^2) \) MeV, where \( r \) is the internucleus distance in fm, and a Coulomb potential \( 12e^2\text{erf}(r/2.5)/r \), where \( \text{erf} \) is the error function. This potential has a d-wave bound state at \( E_b = -245.0 \) keV. Its effective-range function and phase shift have been calculated numerically and are shown in figures 3 and 4, respectively (“exact” curves), for energies corresponding to the available experimental data (see next section). The condition on the bound-state energy (equation (2)) is satisfied: it corresponds to the effective-range function going through the “bound-state” point displayed in figure 3. As for the bound-state ANC, it has also been calculated numerically and has the value \( C_b = 138.4 \times 10^3 \text{ fm}^{-1/2} \) for this potential.

Let us now compare these exact results with truncated effective-range expansions. In contrast with reference [3], we also use here 4th- and 5th-order effective range expansions; in addition, we also test the use of precise values for the product \( w_l h \), instead of truncating it to the same order as \( K_I \). Finally, to mimic the analysis of experimental data as much as possible, we assume here that the scattering length is unknown and determine it from the binding energy, through equation (2); this allows a comparison with the exact value obtained numerically, \( a_2 = 58.91 \times 10^3 \text{ fm}^5 \). For the chosen potential, the numerical method of reference [9] provides the effective-range-expansion parameters \( r_2 = 0.1580 \text{ fm}^{-3} \) and \( P_2 = -65.96 \text{ fm}^8 \). The next-order parameters are

\(^1\) This relation only holds for potentials decreasing faster than \( \exp(-2\kappa_b r) \), i.e. for either a sufficiently weak binding energy or a sufficiently fast decreasing potential [7, 3].
estimated by fitting the phase shifts, starting from low energies; this gives $Q_2 = 0.1411(2) \text{ fm}$, $R_2 = -0.00986(500) \text{ fm}^3$ and $S_2 = -0.03675(100) \text{ fm}^3$, where the values are obtained using an automatic minimisation program, but the error bars are reevaluated by hand through a graphical procedure.

The truncated effective-range functions corresponding to these parameters are displayed in figure 3. By construction, they all satisfy the bound-state-energy condition. For third order and above, the fit seems very satisfactory on the whole energy range. However, figure 4 shows that the phase shift is a more sensitive quantity than the effective-range function: a fifth-order fit is necessary to get a precise fit above 2 MeV, the region where experimental data are available.

Let us now check the accuracies of equations (2) and (3) to estimate the scattering length and ANC from these parameters. The obtained results are displayed in table 1, both for exact and for truncated values of $w_2 h$. As the first term of this expansion is linear in energy, it is not possible to perform a consistent expansion of zeroth order. Precise values of both $a_2$ and $C_b$ are obtained for the third-order expansion and higher. For lower orders, it is crucial to use a consistent order for the expansions of $w_2 h$ and $K_2$. There is an overall large sensitivity of both the scattering length and the ANC to the precise values of the parameters; this is due to the fact that they result from the difference of two very close terms in equations (2) and (3). The rather precise values obtained from the first and second orders in the case of consistent order expansions is rather remarkable but we will see below that it does not hold in general.

5. Analysis of $^{12}\text{C}+\alpha$ experimental data
Let us now proceed to the analysis of the experimental $d$-wave phase shifts, obtained in reference [6] from the analysis of the experiment of reference [5]. The effective-range function corresponding to these phase shifts and the phase shifts themselves are represented in figures 5 and 6 respectively. The narrow resonance at 2.7 MeV has been removed with an R-matrix formula in order to get smooth data to fit on a wide enough region. The higher-energy resonance has not been removed, as it is wider and its removal is more model dependent (we defer its study to a future work, see the conclusion).

The effective-range function is then fitted by truncated polynomial expansions on the [1.95-3.1] MeV interval, which corresponds to 92 experimental points. As in the previous section, we impose the fit to satisfy the bound-state energy condition, which corresponds to it going through...
Table 1. Scattering length and asymptotic normalisation constant of the $d$-wave $^{12}$C+$\alpha$ potential (“exact” line), together with their estimates from truncated effective-range expansions. Columns 2 and 3, respectively 4 and 5, correspond to the exact, respectively consistently truncated, $w_2 h$ product.

<table>
<thead>
<tr>
<th>ERE order</th>
<th>$a_2$ ($10^3$ fm$^5$)</th>
<th>$C_b$ ($10^3$ fm$^{-1/2}$)</th>
<th>$a_2$ ($10^3$ fm$^5$)</th>
<th>$C_b$ ($10^3$ fm$^{-1/2}$)</th>
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<td>/</td>
<td>/</td>
</tr>
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<tr>
<td>exact</td>
<td>58.91</td>
<td>138.4</td>
<td>58.91</td>
<td>138.4</td>
</tr>
</tbody>
</table>

Figure 5. Effective-range function extracted from the $^{12}$C+$\alpha$ $d$-wave experimental phase shifts, as a function of center-of-mass energy, together with its polynomial fits satisfying the bound-state energy condition.

Figure 6. Phase shifts corresponding to the effective-range functions of figure 5 for positive energies. Experimental data and error bars are from reference [6], except for the removal of the 2.7 MeV narrow resonance.

the “bound-state” point in figure 5. Since a curvature is clearly seen on the experimental data, we only use 2nd-order fits and higher, using the parameters of the potential model of the previous section as starting values. For comparison with these previous results, we test fits up to the 5th order. Satisfactory fits are obtained for all orders, with no improvement above the third-order, as seen on the $\chi^2$ values of the effective-range-function fits given in table 2.

In contrast with table 1, no strong difference is observed between the values of $a_2$ and $C_b$ obtained with the exact or truncated values of $w_2 h$. Moreover, neither the second- nor the third-order expansion provides values of the ANC compatible with other estimates, though they precisely fit the phase shifts. Once again, one observes a strong sensitivity of both the scattering length and the ANC to the parameters, which probably implies that the ERE method is not able to extract a precise value of the ANC from the present-day experimental phase shifts.
Table 2. Scattering length and asymptotic normalisation constant of the $d$-wave $^{12}$C+$\alpha$ system, as estimated from truncated effective-range expansions fitted to experiment. Column 1 is the order of the expansions; column 2 is the total $\chi^2$ on the 92 fitted points; columns 3 and 4, respectively 5 and 6, correspond to exact, respectively consistently truncated, values of $w_2h$.

<table>
<thead>
<tr>
<th>order</th>
<th>ERF $\chi^2$</th>
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<th>$C_b$ ($10^3$ fm$^{-1/2}$)</th>
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<th>$C_b$ ($10^3$ fm$^{-1/2}$)</th>
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</table>

6. Conclusions and perspectives

The above examples illustrate the fact that the analytic continuation of an effective-range expansion to a weakly-bound-state energy can in principle be used to deduce the value of this state’s ANC from the low-energy scattering phase shifts. This result holds for a single partial wave and confirms a general property of the potential model.

For the $^{12}$C+$\alpha$ $d$-wave case, the method seems however unable to strongly constrain the ANC from the present-day experimental phase shifts, as several effective-range expansions, all compatible with the data, provide very different ANCs. Several points would however deserve a further study. First, resonances could probably be directly introduced as poles of the effective-range function, allowing one to fit the data on a wider energy interval without requiring an R-matrix based resonance removal. The Taylor effective-range expansion would then be replaced by a Padé one, a possibility that could also be useful for the fit of the background phase shifts themselves. Second, a truncated expansion of $w_2h$ could also be used in the extraction of the effective range function from the experimental phase shifts; it would be worth testing whether making the order of this expansion consistent with that of the effective-range expansion does improve the stability of the ANC extraction.

Revisiting the inversion of the same phase shifts would also be interesting, as a possibility of constraining the ANC within the potential model also exists. The link between the effective-range-expansion and scattering-inversion approaches should also be clarified.

Finally, the same analysis should be performed for the $^{12}$C+$\alpha$ $p$-wave subthreshold bound state, for which no potential model exists, but for which the extracted ANC could be compared with that obtained by other methods [1, 2].

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