Université Libre de Bruxelles
Faculté des Sciences
Service de Physique Théorique et Mathématique

Dynamics at infinity in anti-de Sitter gravity

Thèse présentée en vue de l'obtention
du grade de Docteur en Sciences
(grade légal)

Karin Bautier
Chercheur F.R.I.A.

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Chapter 1

Introduction

The holographic principle assumes that, in any theory of quantum gravity, physics contained within a certain volume should be described on the boundary surrounding this volume by a quantum theory which presents about one degree of freedom per Planck area contained in the boundary surface. In the context of the string theory approach to quantum gravity, a particular set up has been viewed as a manifestation of this principle [1, 2, 3]. It is the conjectured equivalence between some string theory, or its low energy supergravity limit, on anti-de Sitter space-time times a compact space, and some quantum conformal field theory living on the anti-de Sitter boundary [4]. In this framework, the supergravity action turns to be the (logarithm of) the generating functional of correlation functions of the conformal quantum field theory [5, 6].

In this thesis, we show that there are classical degrees of freedom on the conformal boundary of anti-de Sitter and that the asymptotic dynamics of classical gravity with a negative cosmological constant, of which anti-de Sitter is the most symmetric solution, might in fact be controlled by a classical conformal field theory on its boundary. This may be viewed as a classical ancestor for the holographic principle, realizing at the classical level the correspondence between a gravity theory for asymptotically anti-de Sitter space-times and a conformal field theory on their boundary. Classical gravity with a negative cosmological constant will be called anti-de Sitter gravity, in reference to its most symmetric solution.

In three dimensions, the asymptotic symmetries and dynamics of anti-de Sitter gravity have been studied in [7, 8]. The algebra which preserves the boundary conditions for an asymptotically anti-de Sitter space-time in $2 + 1$ dimensions is given by (twice) the Virasoro algebra, which is the conformal algebra in two dimensions. The canonical realization of this algebra presents a central extension, with a central charge proportional to the anti-de Sitter radius and to the inverse of the three-dimensional Newton constant

[7]. The dynamics realizing this symmetry at infinity is described by Liouville theory [8]. The Liouville field constitutes therefore locally the classical degree of freedom of three-dimensional anti-de Sitter gravity.

The non-trivial central charge in the asymptotic symmetry algebra is particular to three dimensions and is related to the enhancement of symmetry on the boundary from the anti-de Sitter isometry group in three dimensions to the infinite-dimensional conformal group in two dimensions. This central charge provides a semi-classical hint into black hole microphysics. Indeed, the degeneracy of states for a conformal field theory with this central charge gives rise to, under appropriate conditions, exactly the Bekenstein-Hawking entropy for the $2+1$ black hole [9]. However, Liouville theory does not satisfy these conditions. The Liouville field alone cannot generate the full black hole entropy and can merely be viewed as a collective coordinate of the underlying microscopic theory [10].

In this thesis, the above properties of three-dimensional anti-de Sitter gravity are extended to the supersymmetric case, which is known to be an important ingredient of black hole entropy computations in string theory and which is a feature of the gravity theories considered in the conjectured duality between string theories on anti-de Sitter space-time and conformal quantum field theories on its boundary.

We next show that classical degrees of freedom on the conformal boundary of anti-de Sitter are also present in higher dimensions. Their effective boundary action has a Weyl anomaly for even dimensions and is conformally invariant for odd ones. These degrees of freedom are encoded in traceless tensor fields appearing in the expansion of the metric near the boundary and generate all the anti-de Sitter Schwarzschild and Kerr black holes. We argue that these fields describe components of the energy-momentum tensor of a boundary theory and that this theory could be given a local expression in terms of local boundary fields. As in the three-dimensional case, the quantization of these fields might fail to provide a microscopic description of black hole entropy. However, in the light of the above correspondence between anti-de Sitter supergravity and a conformal quantum field theory on the boundary, or in any attempt to quantize gravity, it is interesting to better understand the nature of the conformal boundary degrees of freedom of classical gravity with a negative cosmological constant.

The aim of Chapter 2 is to give a pedagogical introduction to the context in which this research takes place. It includes a presentation of the holographic principle for the description of the quantum degrees of freedom of gravity. The geometry of anti-de Sitter space-time is described. The chapter also contains a brief introduction to the equivalence between string and gravity theories on anti-de Sitter space-time and some quantum conformal field theories living on its boundary. Finally, there is a section devoted to anti-de Sitter space-time in three dimensions, reviewing the results about its asymptotic symmetries and dynamics and about their implications, as well as those of the above equivalence, for the entropy of the $2+1$ black hole.

The next two chapters present original results.

In Chapter 3, we extend the known results about the asymptotic symmetries and dynamics of AdS_3 gravity to the supersymmetric case [11, 12]. It is useful to turn to the Chern-Simons formulation of three-dimensional gravity. The boundary conditions for the metric of an asymptotically anti-de Sitter space-time are translated into this formalism. We construct the asymptotic conditions for the Rarita-Schwinger fields, which involve a chiral projection of the spinors at infinity. Together with the boundary conditions on the bosonic fields, these ensure that the asymptotic symmetry algebra is the superconformal algebra. In the canonical realization of this algebra through the Poisson brackets of the generators of the asymptotic symmetries, a central extension appears. The central charge is equal to the one of pure gravity. We then sketch how the asymptotic degrees of freedom are described by super-Liouville theory. We also consider the extended (p,q)-supergravity models, for which the asymptotic symmetry algebras are the superconformal algebras with quadratic non-linearities in the currents.

In Chapter 4, we show that, in all dimensions, classical gravity with a negative cosmological constant possesses boundary degrees of freedom and that these are described by a boundary effective action which is conformally invariant for odd-dimensional boundaries and presents a classical Weyl anomaly for even ones [13, 14]. This last result is obtained by computing the variation of the gravitational action under diffeomorphisms which induce Weyl transformations on the boundary of asymptotically anti-de Sitter space-times. By expanding the Einstein equations near the boundary, we show, for boundaries of dimension two, three and four, that the gravitational boundary degrees of freedom are encoded in a conserved tensor and argue that this tensor might be the energy-momentum tensor of some local boundary fields. The $(2+1)$-dimensional case is analyzed explicitly [13]. Finally, we show how to construct in all dimensions actions which present classically the gravitational Weyl anomaly and illustrate the method in four dimensions.

Our conclusions are summarized in Chapter 5.

Chapter 2

Anti-de Sitter space-time and holography

2.1 The holographic principle

In this section, we review how the semi-classical analysis of Bekenstein [15] and Hawking [16] of the black hole entropy has led 't Hooft and Susskind to advocate that any theory of quantum gravity should satisfy the so-called holographic principle [1, 2].

The entropy of a black hole has been shown by Bekenstein [15] to be proportional to the area of its event horizon:

$$S = \frac{A}{4G}, \tag{2.1}$$

where the exact numerical factor has been derived by Hawking [16] when he discovered that a black hole can emit radiation. Indeed, the Hawking temperature of this radiation as seen by an observer at infinity is given by [16]:

$$T = \frac{1}{8\pi GM}.$$

Writing $\delta M = T\delta S$, one obtains equation (2.1), up to an integration constant which will be assumed to be zero.

The fact that the entropy of a black hole given in equation (2.1) is proportional to its area has been interpreted by Bekenstein as determining an upper bound on the entropy of any region of space-time of volume V enclosed in a surface of size A, in the presence of gravity. This bound says that the entropy contained in that region cannot exceed that of a filling black hole, i.e. cannot be larger the $A/4G$ [17]. Indeed, if the region inside V had an entropy bigger than the one of the filling black hole and an energy just smaller than his, then, by throwing in an infinitesimal amount of matter, such a black hole could be formed and the second law of thermodynamics would be violated.

As an example of a system satisfying the Bekenstein entropy bound, consider a gas at some temperature T inside a region of size V. To be observable from the outside world, the energy of the gas must be less than the mass of a black hole of size V or, equivalently, the radial size of the region V must be larger than its Schwarzschild radius:

$$EG \lesssim V^{\frac{1}{3}}. \tag{2.2}$$

The energy of the gas at temperature T is given by:

$$E \sim VT^4. \tag{2.3}$$

Its entropy is equal to:

$$S \sim VT^3.$$

Using equations (2.2)-(2.3), one obtains the following bound for the entropy:

$$S \lesssim \left(\frac{A}{G}\right)^{\frac{3}{4}},$$

which is stronger that the Bekenstein bound.

In statistical mechanics, the entropy is given by the logarithm of the number of microscopic states corresponding to the same macroscopic state. Therefore, for a system of n binary degrees of freedom corresponding to n spins that can take only two values, n is proportional the maximal entropy of the system. The statistical interpretation of the Bekenstein bound has led 't Hooft [1] to make the assumption that, in any theory of quantum gravity, physics inside a certain region of space-time should be described by degrees of freedom on the surface surrounding that region and that the number of those degrees of freedom is limited roughly to one binary degree of freedom per Planck area in this surface [1, 2]. This has been called the holographic principle.

Notice that the limitation of the number of degrees of freedom by the area of the surface surrounding a certain region of space-time is much stronger than that obtained by putting a cut-off at the Planck scale. Indeed such a cut-off would induce an entropy proportional to the volume of that region, corresponding to about one binary degree of freedom per Planck cell in that region. Now, for sufficiently large volume, this is bigger than the area surrounding it. This radical decrease in the number of degrees of freedom in the presence of gravity is explained by the fact that most of the states of a quantum field theory regulated by a Planckian cut-off would have so much energy that they would collapse into a black hole and could not influence the evolution of the system [1].

In section 2.3, a realization of the holographic principle will be presented in the context of string theory, relating the propagation of string in anti-de Sitter space-time to a conformal quantum field theory on its boundary. The geometry of anti-de Sitter space-time is considered in the next section.

2.2 Anti-de Sitter space-time

Anti-de Sitter is the most symmetric solution of the Einstein equations for gravity with a negative cosmological constant. A review of its geometric properties can be found in [18]. It is a space-time with negative constant curvature. Anti-de Sitter space-time AdS_D in $D = d+1$ dimensions can be constructed as a hyperboloid of radius l embedded in a $(d+2)$-dimensional flat space with two time-coordinates. The metric of the embedding space is:

$$ds^2 = dX_1^2 + \ldots + dX_d^2 - dX_0^2 - dX_{-1}^2. \tag{2.4}$$

The equation of the hyperboloid is the following:

$$X_1^2 + \ldots + X_d^2 - X_0^2 - X_{-1}^2 = -l^2. \tag{2.5}$$

Global coordinates for AdS_D are defined by:

$$\begin{aligned} X_0 &= l\cosh\rho\cos(\tau/l), \\ X_{-1} &= l\cosh\rho\sin(\tau/l), \\ X_i &= l\sinh\rho\,\Omega_i, \qquad i = 1,\ldots,d, \quad \sum_i \Omega_i^2 = 1, \end{aligned} \tag{2.6}$$

where, when $D = 3$, the last line accounts to:

$$X_1 = l\sinh\rho\cos\varphi, \qquad X_2 = l\sinh\rho\sin\varphi.$$

These coordinates yield the following metric for AdS_D:

$$ds^2 = l^2 d\rho^2 - \cosh^2\rho\, d\tau^2 + l^2\sinh^2\rho\, d\Omega_{d-1}^2, \tag{2.7}$$

with:

$$d\Omega_{d-1}^2 = d\varphi_1^2 + \sin^2\varphi_1 d\varphi_2^2 + \ldots + \sin^2\varphi_1 \ldots \sin^2\varphi_{d-2} d\varphi_{d-1}^2.$$

With $0 \le \rho < \infty$ and $0 \le \tau < 2\pi$, they cover the hyperboloid once but there are closed timelike curves. One therefore considers the universal covering space of the hyperboloid unwrapping the time coordinate τ by enlarging its range of variation to $-\infty < \tau < \infty$. It is this space-time which is usually referred to when speaking of anti-de Sitter.

The above coordinates are the ones used in Chapter 3. The metric for AdS_3 is read in equation (2.7) as:

$$ds^2 = l^2 d\rho^2 - \cosh^2\rho\, d\tau^2 + l^2\sinh^2\rho\, d\varphi^2.$$

By the change of radial coordinate $\sinh\rho = r/l$, one obtains the usual metric:

$$ds^2 = -[1 + (r/l)^2]d\tau^2 + [1 + (r/l)^2]^{-1}dr^2 + r^2 d\varphi^2. \tag{2.8}$$

Anti-de Sitter space-time has the property that light rays can travel to spatial infinity in a finite amount of time. Going to $D = 2$ for simplicity, one writes the equation for a lightlike geodesic in AdS_2 as:

$$ds^2 = l^2 d\rho^2 - \cosh^2\rho \, d\tau^2 = 0.$$

The solution of this equation, with $\rho = 0$ at time $\tau = 0$, is given by:

$$\tau/l = \pm \ \arctan\sinh\rho.$$

One sees that the geodesic reaches spatial infinity in a finite amount of time equal to $\tau = l\pi/2$. Reversely, light rays arrive from infinity in a finite time. This shows that anti-de Sitter is not globally hyperbolic i.e. there is no Cauchy surface on which to put initial conditions. In order to make the Cauchy problem well-posed, suitable boundary conditions must be specified at spatial infinity [19].

This particularity of anti-de Sitter shows up clearly in its Penrose diagram. One sets for ρ the following conformal coordinate change:

$$\cosh\rho = \frac{1}{\cos\alpha},$$

that brings conformally (i.e. in a way that light rays are conserved) spatial infinity at a finite value of the radial coordinate, namely at $\alpha = \pi/2$. Forgetting about the angular variables, the metric (2.7) becomes:

$$ds^2 = l^2\cosh^2\rho\,[-d\tau^2/l^2 + d\alpha^2].$$

One sees that AdS_2 is conformal to a vertical band of Minkowski space-time. In this diagram, light rays are at 45 degrees and the above characteristic that they can reach infinity in a time equal to $l\pi/2$ becomes manifest. It is also clear that the boundary at spatial infinity, which is the line $\alpha = \pi/2$, is timelike. These features show the importance of the dynamics at infinity in anti-de Sitter space-time.

Let us now describe the Poincaré coordinates which will be mostly used in Chapter 4. We introduce the coordinates $U = X_{-1} + X_d$ and $V = X_{-1} - X_d$ and solve equation (2.5) for V:

$$V = \frac{1}{U}(l^2 + X_i^2 - X_0^2),$$

where X_i^2 is written for $\sum_{i=1}^{d-1} X_i^2$. Posing, for $U \neq 0$, $x_i = X_i/U$ and $t = X_0/U$, we find the following parametrization of the hyperboloid:

$$\begin{aligned} X_i &= x_i U, \qquad i = 1, \ldots, d-1, \\ X_0 &= tU, \end{aligned}$$

$$X_d = \frac{U}{2}(1 - x_i^2 + t^2) - \frac{l^2}{2U},$$
$$X_{-1} = \frac{U}{2}(1 + x_i^2 - t^2) + \frac{l^2}{2U}.$$

The metric (2.4) becomes in these new variables:

$$ds^2 = l^2 \frac{dU^2}{U^2} + U^2(-dt^2 + dx_1^2 + \ldots + dx_{d-1}^2). \tag{2.9}$$

Notice also that the isometry group of AdS_D, as being a hyperboloid embedded in a $(D+1)$-dimensional space with two times, is $SO(2,d)$ with $D = d+1$. This is precisely the conformal group in d dimensions (for d greater than two). This implies that the holographic theory dual to a gravity theory on anti-de Sitter is likely to be a conformal theory. This relation between symmetry groups in the bulk and on the boundary plays an essential role in the AdS/CFT correspondence presented in the next section. The precise way in which the isometry group of anti-de Sitter generates conformal transformations on its boundary will be presented in section 4.1 of Chapter 4, where it will also be shown how the boundary of anti-de Sitter is identified with compactified Minkowski space-time.

2.3 The AdS/CFT correspondence

The correspondence described in this section between string theory on anti-de Sitter $\mathrm{AdS}_{D=d+1}$ times a compact manifold and a d-dimensional conformal field theory provides a concrete realization of the holographic principle presented in section 2.1 [3]. This duality has been conjectured by Maldacena [4] and a review on the subject can be found in [18] as well as a wide list of references. The correspondence relies on the introduction of D-branes in string theory [20], wich are defined in perturbation theory as surfaces where open strings can end, and on the fact that these surfaces appear in supergravity as the p-branes solitonic solutions which carry Ramond-Ramond charges.

Let us consider an extremal 3-brane solution of ten-dimensional IIB supergravity corresponding to an electric source of charge N for the Ramond-Ramond field. Extremality is translated into the fact that the mass of the solution saturates a lower bound in terms of its charge. This extremality condition is also the BPS condition with respect to ten-dimensional supersymmetry and the extremal solution preserves one half of the supersymmetry. The metric of this 3-brane solution is the following [21]:

$$ds^2 = \frac{1}{\sqrt{1+\frac{l^4}{r^4}}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{1+\frac{l^4}{r^4}}(dr^2 + r^2 d\Omega_5^2), \tag{2.10}$$

where l^4 is proportional to $G_{10}\mu$, with G_{10} the ten-dimensional Newton constant and μ the energy density of the brane. In terms of string parameters, one has $G_{10} = 8\pi^6 g_s^2 l_s^8$, with l_s the string length and g_s the string coupling. The tension of one brane is proportional to $g_s^{-1} l_s^{-4}$. In terms of these parameters, l^4 is then equal to $4\pi g_s l_s^4 N$. There is an horizon at $r = 0$. The dilaton is constant for this solution and the metric can be extended beyond the horizon. The metric (2.10) describes a semi-infinite throat of radius l inserted in flat space. Note that the description of the 3-brane as a supergravity solution is valid as soon as the curvature is small with respect to the string scale i.e. when $l \gg l_s$. The string coupling must also be small in order to suppress string loop corrections: $g_s < 1$. Using $l^4 \sim g_s l_s^4 N$, this is translated into $1 \ll g_s N < N$.

Near the horizon (inside the throat), $r \ll l$ and the 1 can be neglected in the harmonic function appearing in equation (2.10). The metric becomes:

$$ds^2 = \frac{r^2}{l^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + l^2 \frac{dr^2}{r^2} + l^2 d\Omega_5^2,$$

which, comparing with equation (2.9) with $U = r/l$, is the metric of $AdS_5 \times S^5$, with both radii equal to l.

In string perturbation theory, this extremal 3-brane solution can be described in terms of $D3$-branes. These D-branes are surfaces where open strings can end and are also sources for the closed strings. A stack of N $D3$-branes carry the same Ramond-Ramond charge as the corresponding 3-brane solution. The loop expansion parameter of open string perturbation theory is $g_s N$. The D-brane description is therefore valid when $g_s N \ll 1$, which is complementary to the supergravity regime.

We consider a system of N $D3$-branes in IIB string theory. The perturbative excitations of this system are those of the closed strings in empty flat space and of the open strings ending on the D-branes. In the low energy limit $l_s \to 0$, the D-branes decouple from the bulk and the worldvolume theory describing their massless excitations reduces to $(3+1)$-dimensional $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills (SYM) theory. The Yang-Mills coupling is given in terms of the string coupling by $g_{YM}^2 = g_s$.

We now consider IIB string theory in the background described by equation (2.10). Looking at this metric, one sees that the excitations inside the throat (the near horizon region) are strongly redshifted. Therefore the low energy excitations as viewed from infinity are the massless states propagating in flat space and all the string theory excitations inside the throat. Recalling that the near horizon geometry is $AdS_5 \times S^5$, the comparison of this system with the low energy description of the above system of $D3$-branes has led to a conjectured duality between type IIB string theory on $AdS_5 \times S^5$ and $(3+1)$-dimensional $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills [4].

One must take care of the range of validity of the various approximations. Let us go first to the supergravity approximation $1 \ll g_s N < N$. Using $g_{YM}^2 = g_s$, one obtains the weakest form of the conjectured duality, relating IIB supergravity around the $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ SYM theory at large 't Hooft coupling $g_{YM}^2 N$ and in the large N

limit [4]. Stronger versions of the conjecture could also be considered, going beyond the supergravity approximation. Gauge theory at finite 't Hooft coupling but with $N \to \infty$ could then be studied by examining string scale corrections to the supergravity limit. The strongest form of the conjecture would be that $\mathcal{N} = 4$ SYM and string theory on $AdS_5 \times S^5$ are in fact the same for all values of g_s and N [4]. It has been explained in [5, 6] in what precise sense the two dual theories should be identified, suggesting on geometrical grounds that the conformal field theory lives on the boundary of anti-de Sitter and giving a method for computing correlation functions in the gauge theory by supergravity calculations in the anti-de Sitter background. Notice also that it has been shown in [3] that the conformal field theory has about one degree of freedom per Planck area. As one has a duality between a field theory without gravity in $d = 4$ dimensions and a gravity theory in $d+1 = 5$ dimensions, it constitutes a true example of holography.

A similar duality can be elaborated by considering a system of $D1$- and $D5$-branes. The near horizon geometry of the corresponding supergravity solution contains an $AdS_3 \times S^3$ factor, with or without the discrete identifications that lead to a black hole in three dimensions [4, 22]. The low energy field theory on the worldvolume of the D-branes is a $(1+1)$-dimensional conformal field theory. We will come back to this configuration in the next section where we consider the case of AdS_3 in the framework of three-dimensional classical gravity as well as in the context of string theory where the AdS/CFT correspondence takes place. In particular, we examine the entropy of three-dimensional and five-dimensional black holes and how D-branes dynamics provides a microscopic explanation for it.

2.4 AdS_3

We begin this section by recalling the result of Brown and Henneaux [7] according to which any theory of gravity on AdS_3 provides a representation of the conformal algebra in two dimensions with a central charge equal to $c = 3l/2G$, where l is the anti-de Sitter radius and G the three-dimensional Newton constant. This result is reviewed in the next chapter in the Chern-Simons formulation of AdS_3 gravity [23, 24] while extending it to the supersymmetric case. We briefly introduce it here.

The isometry group of anti-de Sitter space-time in $2+1$ dimensions is $SO(2,2)$ and AdS_3 possesses six Killing vectors, three for each $SO(1,2)$ factor of $SO(2,2)$ [25]. Besides anti-de Sitter space-time, the Einstein equations for gravity with a negative cosmological constant $\Lambda = -1/l^2$ present also the BTZ black hole solutions [26]. These black holes can be obtained from anti-de Sitter space-time by appropriate discrete identifications [25]. Their metric is given by:

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\varphi dt + d\varphi)^2, \tag{2.11}$$

with:

$$\begin{aligned} N^2 &= (r/l)^2 - M + (J/2r)^2, \\ N^\varphi &= -J/2r^2. \end{aligned}$$

Notice that AdS corresponds to the same metric with $M = -1$ and $J = 0$. These black holes possess only two Killing vectors [25]. Therefore, acting on them with the AdS Killing vectors generates new configurations, which are used to construct boundary conditions for asymptotically anti-de Sitter space-times [27]. The asymptotic behaviour corresponding to these conditions is the same to leading order as the asymptotic behaviour which is common to the anti-de Sitter and the black holes solutions. The subdominant terms, which correspond to the mass and angular momentum of these solutions, are relaxed in the boundary conditions to arbitrary functions of t and φ. The Killing vectors that preserve these conditions depend now on two arbitrary functions of $u = (t/l) + \varphi$ and $v = (t/l) - \varphi$, enhancing each $so(1,2)$ AdS isometry algebra to the complete set of modes of the Virasoro algebra. These two algebras form the conformal algebra in two dimensions. The canonical realization of this asymptotic symmetry algebra through the Poisson brackets of the symmetry generators presents a central extension whose central charge is equal to $c = 3l/2G$. This non trivial central charge is related to the enhancement of symmetry at the boundary and to the fact that the asymptotic Killing vectors cannot generally be extended in the bulk into exact Killing vectors of any three-dimensional metric [7]. In the case of the BTZ solutions, the zero modes of the Virasoro generators L and $\bar{L}$ are related to the mass and angular momentum of the black hole by the following expressions:

$$\frac{L_0}{k} = \frac{1}{4}\left(M - \frac{J}{l}\right), \qquad \frac{\bar{L}_0}{k} = \frac{1}{4}\left(M + \frac{J}{l}\right), \tag{2.12}$$

with $k = c/6 = l/4G$.

The above conformal symmetry is realized classically in the relation of AdS_3 gravity with Liouville theory [8]. This relation is based on the description of three-dimensional gravity by a Chern-Simons theory [23, 24]. The asymptotic boundary conditions reduce it to a WZW model, with a restriction on the WZW currents that yields the Hamiltonian reduction of the $SO(1,2)$ WZW model to Liouville theory [8]. This reduction is outlined in section 3.5 of the next chapter in the supersymmetric case.

The above result on the asymptotic symmetry of AdS_3 has been derived in classical gravity and is therefore valid as soon as the AdS curvature (or the cosmological constant) is small in Planck units, i.e. as soon as:

$$l \gg G.$$

However, Strominger [9] has shown that the knowledge of the central charge encountered in the asymptotic symmetry algebra allows to compute microscopically the entropy of the three-dimensional black hole. We write the lapse function N appearing in the BTZ metric of equation (2.11) as:

$$N^2 = \frac{1}{r^2}(r^2 - r_+^2)(r^2 - r_-^2),$$

with:

$$r_\pm^2 = \frac{l^2}{2}\left(M \pm \sqrt{M^2 - J^2/l^2}\right).$$

The Bekenstein-Hawking entropy of the black hole is given by:

$$S = \frac{A}{4G} = \frac{\pi r^+}{2G}, \tag{2.13}$$

where A is the area of the black hole horizon.

The asymptotic growth of the degeneracy of states in a conformal field theory is given in terms of the central charge c and of the levels L_0 and $\tilde{L}_0$ of the excited states as:

$$d(c, L_0, \tilde{L}_0) \sim e^{2\pi\sqrt{\frac{cL_0}{6}}} e^{2\pi\sqrt{\frac{c\tilde{L}_0}{6}}},$$

leading to the Cardy formula for the entropy [28]:

$$S = 2\pi\sqrt{\frac{cL_0}{6}} + 2\pi\sqrt{\frac{c\tilde{L}_0}{6}}, \tag{2.14}$$

which is valid for a unitary conformal field theory and for $L_0, \tilde{L}_0 \gg c$. In the case of the BTZ black hole with large M viewed as an excited state of a conformal field theory with central charge given by the one of the asymptotic symmetry algebra $c = 3l/2G$, the values of L_0 and $\tilde{L}_0$ in terms of M and J are given in equation (2.12). They are expressed in terms of r_+ and r_- as:

$$L_0 = \frac{k}{4l^2}(r_+ - r_-)^2, \qquad \tilde{L}_0 = \frac{k}{4l^2}(r_+ + r_-)^2,$$

recalling that $c = 6k$. Inserting these values in equation (2.14) yields a microscopic entropy in exact agreement with the Bekenstein-Hawking result given in equation (2.13) for the BTZ black hole [9].

This statistical derivation of the black hole entropy only needs information on the value of the central charge which is provided by the semi-classical analysis of Brown and Henneaux. It is not clear however which is the conformal quantum field theory that describes the microcopical degrees of freedom of the black hole and to which the Cardy formula applies.

As three-dimensional gravity can be written as a Chern-Simons theory and that this theory is renormalizable [24], it could provide by itself the searched-for theory. However, as mentioned before, the Chern-Simons theory reduces to Liouville theory [8]. Now this theory is not unitary and the effective central charge corresponding to its density of states as given by the Cardy formula of equation (2.14) is $c_{eff} = 1$ [29], rather than $c = 3l/2G$ which is large in the semi-classical limit. This observation led Martinec [10] to propose that the Liouville field is merely a collective coordinate of the underlying conformal field theory, writing:

$$< T_{ij} >_{\text{CFT}} = T_{ij}^{Liouville}.$$

As it will be explained in Chapter 4, the Liouville energy-momentum tensor $T_{ij}^{Liouville}$ is related to the Virasoro generators of the asymptotic symmetry algebra and concentrates all the gravitational classical degrees of freedom. As it is constructed solely from the CFT energy-momentum tensor, gravity cannot distinguish among CFT states of the same energy and charges.

We now turn to the system of $D1$- and $D5$-branes that we mentioned in the previous section in the context of the AdS/CFT correspondence. We will see how D-branes dynamics provides a microscopical derivation of the entropy of string theoretic five-dimensional black holes [30, 31] and how, in the low energy limit used in the previous section, this state counting applies to the three-dimensional BTZ black hole too [22].

Consider in type IIB string theory N_1 coinciding $D1$-branes and N_5 $D5$-branes wrapped on a four-torus T^4 with their non compact direction along the $D1$-branes. As the branes are coinciding in the non compact direction, this corresponds to an extremal 1-brane configuration in the supergravity description. In the near horizon limit of the previous section, its metric becomes that of $AdS_3 \times S^3$, with radius $l^2 = l_{AdS}^2 = l_{S^3}^2 = g_6\sqrt{N_1 N_5} l_s^6$, where g_6 is the six-dimensional string coupling. It is given in terms of the ten-dimensional string coupling g_s by $g_6^2 = g_s^2/v$, with $(2\pi l_s)^4 v$ equal to the volume of the four-torus T^4 [4].

Under the compactification of the remaining non compact direction along the D-branes on a circle of radius R and the addition of momentum N/R in one direction along this circle, the corresponding 1-brane configuration forms a five-dimensional extremal black hole [30]. The addition of momentum in the other direction along the circle leads to a non extremal black hole in the supergravity picture [31], see [32, 33] for a review.

The extremal configuration preserves part of the supersymmetry and corresponds to BPS states. This property has been used to transport a degeneracy counting in the D-brane regime to the supergravity regime in order to provide a microscopical derivation of the five-dimensional black hole entropy [30]. The Bekenstein-Hawking entropy of the extremal black hole is [32]:

$$S = \frac{A}{G} = 2\pi\sqrt{N N_1 N_5}. \tag{2.15}$$

The counting of states in the D-brane picture is described in [30] in its original derivation and in [32] in a slightly different one. The states to be considered are the massless excitations of the strings attached to the D-branes, namely the (1,1), (5,5), (1,5) and (5,1) strings. The momentum N/R implies that the strings are moving along one direction and are left-moving. The states giving rise to the highest degeneracy are the excitations of the (1,5) and (5,1) strings, which correspond to scalars in the matter multiplet of the six-dimensional worldvolume theory. Excitations of the (1,1) and (5,5) strings, which correspond to scalars in the six-dimensional vector multiplet, separate the D-branes and lead therefore to a smaller number of massless states. Moreover, if many (1,5) and (5,1) strings are excited, the (1,1) and (5,5) strings become massive and can be dropped from the state counting. The total number of possible states for the strings corresponds to $4N_1N_5$ bosons and $4N_1N_5$ fermions for each momentum. The state counting is the same as the asymptotic degeneracy formula of a 1+1 conformal field theory of central charge $c = 6N_1N_5$ (where a boson contributes to 1 and a fermion to 1/2) at level N with only left-moving modes. Note that the theory on the brane is a $(1+1)$-dimensional theory as soon as R is large enough. The entropy is therefore given by the Cardy fomula (2.14) and is equal to [30]:

$$S = 2\pi\sqrt{NN_1N_5},$$

which reproduces exactly the Bekenstein-Hawking entropy (2.15) of the five-dimensional extremal black hole. The microscopic entropy in the near extremal case has been computed in [31], by considering the addition of right-moving modes.

Let us return now to the system of $D1$- and $D5$- branes considered before whose near horizon geometry is $AdS_3 \times S^3$. The dual conformal field theory of the AdS/CFT correspondence describing the excitations in the near horizon is in fact the IR fixed point of the field theory living on the $D1$-$D5$ branes. This conformal field theory has been constructed in the original derivation [30] of the extremal black hole entropy, where the $D1$-branes were viewed as instantons in the $U(N_5)$ SYM theory on the worldvolume of the $D5$-branes. Their low energy dynamics is described by a $(1+1)$-dimensional sigma model whose target space is the instanton moduli space, which reduces to the symmetric product $(T^4)^k/S_k$, where $4k$ is the dimension of the instanton moduli space and $k = N_1N_5$. This theory is a superconformal field theory with central charge $c = 6k$, which is the value of the central charge considered when computing the degeneracy of states of the five-dimensional black hole.

Recalling that the radius of the anti-de Sitter appearing in the near horizon satisfies $l^4 = g_s^2N_1N_5l_s^4/v$ and that the three-dimensional Newton constant is given by $G = g_s^2l_s^3/4l^3v$ (using the ten-dimensional Newton constant $G_{10} = 8\pi^6g_s^2l_s^8$ and the volumes of the three-sphere $V_{S^3} = 2\pi^2l^3$ and of the four-torus $V_{T^4} = (2\pi l_s)^4v$), one sees that the central charge $c = 6N_1N_5$ of the dual conformal field theory is equal to the Brown-Henneaux central charge $c = 3l/2G$. Moreover, it has been shown in [22] that the compactification of the direction common to the $D1$- and $D5$-branes together with the momentum along this direction that bring the original 1-brane configuration to a five-dimensional black hole induce near the horizon precisely the discrete identifications that bring the metric of AdS_3 to the BTZ one. Hence the near horizon geometry of the five-dimensional black hole contains a $BTZ \times S^3$ factor [22]. Therefore, the counting of states that explains the entropy of the five-dimensional black hole applies to the BTZ black hole. The three-dimensional black holes are really excited states of the dual conformal field theory and the use of the Cardy formula in [9] to compute their entropy finds here a justification in the extremal and near extremal cases.

Chapter 3

AdS_3 supersymmetric asymptotics

We recall that it has been pointed in [7] that the asymptotic symmetry group of anti-de Sitter gravity in three dimensions is the conformal group in two dimensions with a central charge $c = 3l/2G$. It is a purely asymptotic phenomenon, in the sense that the infinite-dimensional conformal group in two dimensions is not the isometry group of any three-dimensional background geometry. This is one feature that makes the three-dimensional case particularly interesting and which actually allows for a non-trivial central charge in the dynamical realization of the asymptotic symmetry algebra [7]. Moreover, we recall that it was observed in [9] that the degeneracy of states for a conformal field theory with this central charge gives rise to, under appropriate conditions, exactly the Bekenstein-Hawking entropy for the $2+1$ black hole. The central charge appearing in three-dimensional gravity provides therefore a semi-classical contact with the underlying microscopic theory which describes the degrees of freedom of the black hole.

Another feature of three-dimensional gravitational theories is that they have no bulk degrees of freedom, so that the analysis of their asymptotic dynamics is of particular interest. The bosonic case has been studied in [8], where it was shown that the boundary dynamics at infinity is described by Liouville theory up to terms involving the zero modes and the holonomies that were not worked out.

In this chapter, we extend the analysis of [7, 8] to the supersymmetric context, which is known, as mentioned in the previous chapter, to play a central role in black hole physics. We use the Chern-Simons formulation of supergravity [23, 24]. The new non-trivial ingredient to be supplied is the precise asymptotic behaviour of the Rarita-Schwinger fields, which must be compatible with the symmetries. In particular, one must understand how the boundary conditions implement two-dimensional supersymmetry at infinity.

The supersymmetry properties of the three-dimensional black holes were investigated in [34], assuming the existence of asymptotic conditions on the Rarita-Schwinger fields fulfilling the required properties. However, the asymptotic conditions in question were not given. The main object of this chapter is to fill this gap, which appears necessary since otherwise, the discussion of the asymptotic dynamics remains rather formal. We also verify that the given fall-off conditions reduce the theory to super-Liouville. The asymptotic symmetry algebra is shown to be the superconformal algebra with unchanged central charge $c = 3l/2G$. The boundary conditions involve a chiral projection of the spinorial fields on the two-dimensional boundary at infinity. Models with extended supersymmetry are considered, leading to the extended superconformal algebras with quadratic non-linearities in the currents.

The chapter is organized as follows. In section 3.1, we briefly review the Chern-Simons formulation of three-dimensional gravity [23, 24]. We express the metric of the $2+1$ black hole in this formalism. The Chern-Simons formulation is extended to the case of AdS_3 supergravity with the use of supermatrices. Finally, we reproduce the results of [34] concerning the supersymmetry properties of the $2+1$ black hole. In section 3.2, the boundary conditions for the metric of an asymptotically anti-de Sitter space-time of [7] are reexpressed in the connection representation used in the Chern-Simons formalism. Boundary conditions for the Rarita-Schwinger fields are constructed, following the procedure of [27]. These conditions involve a chiral projection of the spinorial fields on the two-dimensional boundary at infinity. In section 3.3, the canonical formulation of the theory is used to derive the generators of the supergauge transformations through the Poisson bracket. These generators are improved by appropriate surface terms related to the global charges of the theory. Section 3.4 is devoted to the analysis of the asymptotic symmetry. The general supergauge parameter preserving the boundary conditions given in section 3.2 is computed. The canonical realization of the asymptotic symmetry algebra is displayed by working out the Poisson brackets of the generators of section 3.3 with supergauge parameter replaced by the above one. The algebra obtained in this way is the superconformal algebra with same central charge as in the bosonic case $c = 3l/2G$. It is sketched in section 3.5 how the boundary conditions on the bosonic and fermionic fields reduce AdS_3 supergravity to super-Liouville theory. Finally, extended supersymmetry models are considered in section 3.6. The last section contains a brief conclusion.

This chapter is mainly based on [11] (see [12] for a summary). The results of section 3.6 can be found in [45].

3.1 Chern-Simons formulation

3.1.1 Pure gravity

Gravity in $2+1$ dimensions can be formulated as a Chern-Simons theory [24]:

$$I_{CS}[A] = \frac{k}{4\pi}\int \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \tag{3.1}$$

where A is a 1-form connection for the isometry group of the ground state of the corresponding gravity theory. As a 1-form and as an element of the algebra of the isometry group, its different components can be defined as follows:

$$A = A_\mu dx^\mu = A^a T_a = A^a{}_\mu T_a dx^\mu,$$

where T_a are the generators of the algebra.

The Chern-Simons formulation takes as variables of general relativity the dreibein e^a_μ and the spin connection $\omega_{\mu ab}$, where the latin indices are tangent-space Lorentz indices and the greak indices are space-time indices. The relations between the metric and the dreibein are given by:

$$\begin{aligned} g_{\mu\nu} &= e^a{}_\mu e^b{}_\nu \eta_{ab}, \\ \eta^{ab} &= e^a{}_\mu e^b{}_\nu g^{\mu\nu}, \end{aligned} \tag{3.2}$$

where η_{ab} is the metric of Minkowski space-time. The spin connection $\omega_{\mu ab}$ is needed to construct the covariant derivative of a spinor ψ, which will be useful when adding supersymmetry:

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \frac{1}{2}\omega_{\mu ab}J^{ab}\psi,$$

where J^{ab} are the generators of the Lorentz algebra $so(1,2)$ in the spinorial representation with indices in the adjoint and $\omega_{\mu ab}$ is therefore antisymmetric in the Lorentz indices. The dreibein satisfies the following constraint:

$$D_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\lambda_{\mu\nu} e^a{}_\lambda = 0. \tag{3.3}$$

This equation implies that the metric is covariantly constant and that the connection is torsion free. The antisymmetric part of equation (3.3) completely determines the spin connection in terms of the dreibein. It can be written more compactly with the use of the 1-forms $e^a = e^a{}_\mu dx^\mu$ and $\omega_{ab} = \omega_{\mu ab}dx^\mu$ as:

$$de^a + \omega^a{}_b \wedge e^b = 0. \tag{3.4}$$

Notice that, in three dimensions, the Lorentz generators J^{ab} can be written more simply as $J_a = \epsilon_{abc}J^{bc}$ with, in our conventions, $\epsilon^{012} = -\epsilon_{012} = 1$. Any element B of the Lorentz algebra $so(1,2)$ can then be decomposed alternatively as:

$$B = B^a J_a \qquad \text{or} \qquad B = \frac{1}{2}B_{ab}J^{ab}.$$

The two different types of components are then related by the following equations:

$$B^a = \frac{1}{2}\epsilon^{abc}B_{bc}, \qquad B_{ab} = -\epsilon_{abc}B^c. \tag{3.5}$$

Those relations clearly apply to the spin connection.

In the absence of a cosmological constant, the ground state of gravity is Minkowski space-time and the relevant symmetry group is $ISO(1,2)$. The Chern-Simons connection is then written as:

$$A = \omega^a J_a + e^a P_a,$$

where the spin connection is the gauge field for the Lorentz transformations and the dreibein is the gauge field for the translations.

When the cosmological constant is negative, the ground state is anti-de Sitter space-time whose isometry group is $SO(2,2)$. The algebra $so(2,2)$ is the same as $iso(1,2)$ apart from the fact that the "translations" do not commute:

$$[J_a, J_b] = \epsilon_{abc}J^c, \qquad [J_a, P_b] = \epsilon_{abc}P^c, \qquad [P_a, P_b] = \epsilon_{abc}J^c.$$

In order to avoid the appearance of the cosmological constant in the algebra, the generators P_a have been redefined taking lP_a as the new generators. The constant l is the anti-de Sitter radius and is related to the cosmological constant Λ through $\Lambda = -1/l^2$. Introducing the new generators $J_a^{(+)}$ and $J_a^{(-)}$ defined by:

$$J_a^{(\pm)} = \frac{1}{2}(J_a \pm P_a),$$

one can show that the algebra of $SO(2,2)$ decomposes into the algebra of $SO(1,2) \otimes SO(1,2)$ as follows:

$$[J_a^{(\pm)}, J_b^{(\pm)}] = \epsilon_{abc}J^{(\pm)c}, \qquad [J_a^{(+)}, J_b^{(-)}] = 0.$$

The $so(2,2)$ connection $A = \omega^a J_a + (e^a/l)P_a$ can then be written as $A = A^{(+)} + A^{(-)}$ with:

$$A_a^{(\pm)} = \omega_a \pm \frac{1}{l}e_a. \tag{3.6}$$

Both $A^{(+)}$ and $A^{(-)}$ are now connections for the algebra $so(1,2)$ in the spinorial representation. A Chern-Simons action can be written for each of them and we will see below that the Einstein-Hilbert action with a negative cosmological constant is equivalent to the difference between these two Chern-Simons actions [23, 24]:

$$I_{EH} \equiv I_{CS}[A^{(+)}] - I_{CS}[A^{(-)}]. \tag{3.7}$$

The generators of each $so(1,2)$ algebra is constructed with the Dirac matrices as $J_a = (1/2)\gamma_a$. The Dirac matrices are taken to be:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

They satisfy the following relations:

$$\gamma_{ab} = \tfrac{1}{2}[\gamma_a, \gamma_b] = \epsilon_{abc}\gamma^c,$$
$$\mathrm{Tr}(\gamma_a\gamma_b) = 2\eta_{ab},$$
$$\mathrm{Tr}(\gamma_a\gamma_b\gamma_c) = 2\epsilon_{abc}.$$

The generators J_{ab} are then equal to $(1/2)\gamma_{ab}$.

Taking the trace explicitly in (3.1), the Chern-Simons action for $SO(1,2)$ becomes:

$$I_{CS}[A] = \frac{k}{4\pi}\int \frac{1}{2}(A_a \wedge dA^a + \frac{1}{3}\epsilon_{abc}A^a \wedge A^b \wedge A^c). \tag{3.8}$$

It is invariant under the gauge transformation δA with gauge parameter the 0-form λ:

$$\delta A = D\lambda,$$

where by definition:

$$D\lambda = d\lambda + [A, \lambda].$$

Since A and λ are elements of $so(1,2)$, this is written in components as:

$$\delta A^a = d\lambda^a + \epsilon^{abc}A_b\lambda_c.$$

The 2-form curvature of the connection A which is associated to the commutator of two D-derivatives is defined as:

$$F = dA + A \wedge A.$$

The equation of motion derived from the action (3.8) is given by the vanishing of the curvature F.

To recover the Einstein-Hilbert action from the Chern-Simons formulation, one needs the expression of the space-time curvature in terms of the spin connection. Computing the commutator of the covariant derivatives of a spinor ψ, one writes:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi = \frac{1}{4}\mathcal{R}_{\mu\nu ab}\gamma^{ab}\psi$$

and obtains:

$$\mathcal{R}_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} + \omega_\mu{}^a{}_c \omega_{\nu cb} - \partial_\nu \omega_{\mu ab} - \omega_\nu{}^a{}_c \omega_{\mu cb}. \tag{3.9}$$

More compactly, the 2-form curvature of the spin connection is defined as:

$$\mathcal{R} = d\omega + \omega \wedge \omega.$$

As an element of $so(1,2)$, its components in the generators J_a are given by:

$$\mathcal{R}^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c. \tag{3.10}$$

Writing $\mathcal{R}_a = (1/2)\mathcal{R}^a{}_{\mu\nu} dx^\mu \wedge dx^\nu$, expressions (3.9) and (3.10) are shown to be equivalent with the use of equation (3.5), that relates both types of components of an element of $so(1,2)$, and of the following formula:

$$\gamma^{lm}\epsilon_{lab}\epsilon_{mcd} = \frac{1}{2}[\gamma_{ab}, \gamma_{cd}].$$

The spin connection curvature contains the same information as the Riemann tensor. Indeed, making use of equation (3.3) that relates the spin connection and the $\Gamma^\lambda_{\mu\nu}$'s, one obtains:

$$e_a{}^\lambda e^b{}_\rho \mathcal{R}_{\mu\nu}{}^a{}_b = \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\lambda_{\rho\mu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\rho\mu} \equiv R^\lambda{}_{\rho\mu\nu}.$$

The scalar curvature is therefore constructed as:

$$R = \mathcal{R}_{\mu\nu ab} e^{a\mu} e^{b\nu}. \tag{3.11}$$

To derive the Einstein-Hilbert action, we write here some additional useful formulae:

$$\begin{aligned} dx^\lambda \wedge dx^\mu \wedge dx^\nu &= \epsilon^{\lambda\mu\nu} d^3x, \\ \epsilon^{\lambda\mu\nu} &= -e e_a{}^\lambda e_b{}^\mu e_c{}^\nu \epsilon^{abc}, \end{aligned} \tag{3.12}$$

where

$$\mathrm{e} = \det(e_{a\mu}) = \sqrt{-g}.$$

Only the terms which are linear and cubic in e survive in the difference between the two Chern-Simons actions $I_{CS}[A^{(+)}]$ and $I_{CS}[A^{(-)}]$. After integrating by parts and taking the trace in the $so(1,2)$ algebra, the linear terms give the following contribution:

$$\frac{k}{2\pi l}\int e_a \wedge \mathcal{R}^a,$$

which, using equations (3.11)-(3.12), becomes:

$$\frac{k}{4\pi l}\int \mathrm{e} R d^3x.$$

The cubic terms in e contribute as:

$$\frac{k}{2\pi l^3}\int \mathrm{e} d^3x.$$

Relating the constant k to the three-dimensional Newton constant G and the anti-de Sitter radius l through $k = l/4G$, the equivalence (3.7) of the Einstein-Hilbert action and of the difference between two Chern-Simons actions for $SO(1,2)$ is then demonstrated:

$$I_{CS}[A^{(+)}] - I_{CS}[A^{(-)}] = \frac{1}{16\pi G}\int \left(\mathrm{e}R + \frac{2\mathrm{e}}{l^2}\right) d^3x \equiv I_{EH},$$

up to surface terms coming from the integration by parts.

3.1.2 The $2 + 1$ black hole

The Einstein equations in $2 + 1$ dimensions with a negative cosmological constant admit black holes solutions discovered by Bañados, Teitelboim and Zanelli (BTZ) [26]. Their metric is given by:

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2(N^\varphi dt + d\varphi)^2, \tag{3.13}$$

with:

$$\begin{aligned} N^2 &= (r/l)^2 - M + (J/2r)^2, \\ N^\varphi &= -J/2r^2. \end{aligned}$$

The constants M and J are, respectively, the mass and angular momentum of the black hole. These solutions can be obtained by making appropriate identifications of the anti-de Sitter metric [25]:

$$ds^2 = -[1 + (r/l)^2]dt^2 + [1 + (r/l)^2]^{-1}dr^2 + r^2 d\varphi^2,$$

which corresponds to (3.13) with $M = -1$ and $J = 0$.

To establish the boundary conditions for asymptotically anti-de Sitter spaces in the Chern-Simons formulation, it will be useful to know the expression of the connections $A^{(+)}$ and $A^{(-)}$ that correspond to the BTZ metric (3.13), at least asymptotically. The dreibein associated to the metric (3.13) satisfies equation (3.2) and is given, up to a Lorentz transformation, by:

$$e^0 = -N dt, \qquad e^1 = \frac{1}{N} dr, \qquad e^2 = rN^\varphi dt + r d\varphi.$$

The related spin connection is computed to satisfy equation (3.4) and is equal to:

$$\omega^0 = -N d\varphi, \qquad \omega^1 = -\frac{N^\varphi}{N} dr, \qquad \omega^2 = \frac{r}{l^2} dt + rN^\varphi d\varphi.$$

For simplicity, we expand the Chern-Simons connection A in the Cartan basis of $so(1,2)$:

$$A = A^+ J_+ + A^- J_- + A^1 J_1,$$

where:

$$J_+ = J_0 + J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = J_2 - J_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

To avoid confusion with the Cartan indices, we rename the connections $A^{(+)}$ and $A^{(-)}$ as A and $\tilde{A}$ respectively. The coordinates $u = (t/l) + \varphi$ and $v = (t/l) - \varphi$ are introduced, giving:

$$A_u = \frac{1}{2}(lA_t + A_\varphi), \qquad A_v = \frac{1}{2}(lA_t - A_\varphi),$$

and the same for $\tilde{A}$. We are now ready to compute the Chern-Simons connections A and $\tilde{A}$ corresponding to the BTZ black hole (3.13), recalling equation (3.6) that relates them to the dreibein and spin connection:

$$A = \omega + \frac{1}{l}e, \qquad \tilde{A} = \omega - \frac{1}{l}e. \tag{3.14}$$

Their non vanishing components are the following:

$$A^1{}_r = \frac{1 - lN^\varphi}{lN}, \quad A^+{}_u = \frac{r}{2l} - \frac{N}{2} + \frac{rN^\varphi}{2}, \quad A^-{}_u = \frac{r}{2l} + \frac{N}{2} + \frac{rN^\varphi}{2}, \tag{3.15}$$

$$\tilde{A}^1{}_r = -\frac{1 + lN^\varphi}{lN}, \quad \tilde{A}^+{}_v = \frac{r}{2l} + \frac{N}{2} - \frac{rN^\varphi}{2}, \quad \tilde{A}^-{}_v = \frac{r}{2l} - \frac{N}{2} - \frac{rN^\varphi}{2}. \tag{3.16}$$

Keeping only the asymptotically leading term in each component, one obtains explicitly in M and J:

$$A_r = b^{-1}\partial_r b, \qquad A_u = b^{-1}\begin{pmatrix} 0 & \frac{1}{4}(M - J/l) \\ 1 & 0 \end{pmatrix} b, \qquad A_v = 0, \tag{3.17}$$

and

$$\tilde{A}_r = b\partial_r b^{-1}, \qquad \tilde{A}_u = 0, \qquad \tilde{A}_v = b\begin{pmatrix} 0 & 1 \\ \frac{1}{4}(M + J/l) & 0 \end{pmatrix} b^{-1}. \tag{3.18}$$

The group element $b(r)$ is equal to:

$$b(r) = \begin{pmatrix} \sqrt{r/l} & 0 \\ 0 & \sqrt{l/r} \end{pmatrix} \tag{3.19}$$

and satisfies $b\gamma_0 b = \gamma_0$.

3.1.3 AdS$_3$ supergravity

The Chern-Simons formulation of gravity can be extended to supergravity theories, by replacing one or both of the Chern-Simons groups by supergroups that still contain $SO(1,2)$ in their bosonic part. In what follows we will mostly treat the case of $OSp(1|2)$ whose bosonic part of the algebra is exactly $sp(2) \equiv so(1,2)$. More general supergroups will be considered in section 3.6 which is devoted to theories with extended supersymmetry. Whether one or the other of the Chern-Simons groups is enlarged to $OSp(1|2)$, the $N = 1$ $(1,0)$- or $(0,1)$-supergravity theory is constructed. The case where both Chern-Simons actions are generalized to $OSp(1|2)$ corresponds to $N = 2$ $(1,1)$-supergravity whose ground state isometry group is $OSp(1|2) \otimes OSp(1|2)$.

The Chern-Simons action for the supergroup $OSp(1|2)$ is:

$$I_{CS}[A,\psi] = \frac{k}{4\pi}\int \left[\mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) + i\bar{\psi} \wedge D\psi\right]. \tag{3.20}$$

The field ψ is a 1-form spinor $\psi = \psi_\mu dx^\mu$ where ψ_μ is a Rarita-Schwinger field. It is real so that $\bar{\psi} = \psi^t \gamma_0$ and D is the derivative constructed with the $so(1,2)$ connection A satisfying:

$$D\psi = d\psi + A \wedge \psi.$$

As before, action (3.20) is invariant under the gauge transformations:

$$\delta_\lambda A = D\lambda = d\lambda + [A, \lambda], \tag{3.21}$$

$$\delta_\lambda \psi = -\lambda\psi, \tag{3.22}$$

where λ is a 0-form gauge parameter that belongs to $so(1,2)$. It possesses also one supersymmetry under:

$$\delta_\rho A_a = i\bar{\rho}\gamma_a\psi, \tag{3.23}$$

$$\delta_\rho \psi = D\rho = d\rho + A\rho, \tag{3.24}$$

where ρ is the parameter of the supersymmetry transformation and is a 0-form spinor.

The $(1,1)$-supergravity action is then obtained as the difference between two $OSp(1|2)$ Chern-Simons actions [23]:

$$I_{CS}[A,\psi] - I_{CS}[\tilde{A},\tilde{\psi}],$$

with A and $\tilde{A}$ still related to the dreibein and the spin connection through (3.14) and $k = l/4G$. Indeed we have:

$$I_{CS}[A,\psi] - I_{CS}[\tilde{A},\tilde{\psi}] = \frac{1}{16\pi G}\int \left(\mathrm{e}R + \frac{2\mathrm{e}}{l^2} + il\epsilon^{\lambda\mu\nu}\bar{\psi}_\lambda \mathcal{D}_\mu \psi_\rho \right.$$
$$\left. -il\epsilon^{\lambda\mu\nu}\bar{\tilde{\psi}}_\lambda \mathcal{D}_\mu \tilde{\psi}_\rho - \frac{i}{2}\mathrm{e}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + \frac{i}{2}\mathrm{e}\bar{\tilde{\psi}}_\mu\gamma^{\mu\nu}\tilde{\psi}_\nu\right) d^3x, \tag{3.25}$$

up to the same surface terms as in the case of pure gravity. The mass terms for the spinors arise because of the dreibein term in the D-derivative. Rescaling ψ and $\tilde{\psi}$ by a factor of $\sqrt{l}$ to remove the parameter l in front of the kinetic terms, one sees that the mass parameter is equal to $1/l$ which is the squareroot of the cosmological constant.

The supersymmetric Chern-Simons action (3.20) can be written more compactly in terms of a superconnection that belongs to the algebra $osp(1|2)$. The generators of this superalgebra are supermatrices. A supermatrix M is a matrix:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

whose Bose-Bose part A and Fermi-Fermi part D have even elements and Bose-Fermi part B and Fermi-Bose part C have odd elements. Its supertrace is defined as:

$$\mathrm{sTr} M = \mathrm{Tr} A - \mathrm{Tr} B.$$

A supertranspose M^{st} is introduced:

$$M^{st} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix},$$

in order to have $(M_1 M_2)^{st} = M_2^{st} M_1^{st}$, noticing that for matrices B_1, B_2 with odd elements, the usual transposition is such that $(B_1 B_2)^t = -B_2^t B_1^t$.

An element $\mathcal{M}$ of the supergroup $OSp(1|2)$ satisfies:

$$\mathcal{M}^{st} H \mathcal{M} = H,$$

with:

$$H = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix},$$

where η is the simplectic form for $Sp(2)$:

$$\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An element M of the $osp(1|2)$ algebra is defined by:

$$M^{st} H + H M = 0.$$

According to this equation, the bosonic generators are those of $sp(2) \oplus so(1) \equiv sp(2)$ which is isomorphic to $so(1,2)$. They are obtained by augmenting the previous $J_a = (1/2)\gamma_a$

with one row and one column of zeros (and will still be denoted by J_a). The Cartan basis is still defined by $J_\pm = J_2 \pm J_0$. The fermionic generators are:

$$e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

They satisfy:

$$\{e,e\} = 2J_+, \qquad \{f,f\} = -2J_-, \qquad \{e,f\} = -2J_1.$$

The other commutators are given by:

$$\begin{aligned} [J_1,e] &= e, & [J_+,e] &= 0, & [J_-,e] &= f, \\ [J_1,f] &= -f, & [J_+,f] &= e, & [J_-,f] &= 0. \end{aligned}$$

One associates to a 2-component spinor ψ the fermionic supermatrix Ψ defined by:

$$\Psi = \frac{1}{\sqrt{2}}(\psi_1 e + \psi_2 f).$$

Let Ψ and Ξ be the supermatrices corresponding to the spinors ψ and ξ respectively. The following equality holds:

$$\mathrm{sTr}(\Psi\Xi) = i\bar{\psi}\xi,$$

provided the product of two fermions in this expression differs by a factor $(-i)$ from the standard Grassmann product fulfilling $(ab)^* = b^* a^*$. This convention will be adopted each time we multiply supermatrices.

We are now ready to rewrite the action (3.20) in the manifest super-Chern-Simons form:

$$I_{CS}[\Gamma] = \frac{k}{4\pi}\int \mathrm{sTr}(\Gamma\wedge d\Gamma + \frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma),$$

with $\Gamma = A + \Psi$ (where A stands for the $so(1,2)$ connection augmented with one row and one column of zeros). The supercurvature is $\mathcal{F} = d\Gamma + \Gamma\wedge\Gamma$ and the equations of motion are just $\mathcal{F} = 0$. The gauge and supersymmetry transformations (3.21)-(3.24) are summarized in the supergauge transformation:

$$\delta\Gamma = d\Lambda + [\Gamma, \Lambda],$$

where Λ is a 0-form element of $osp(1|2)$. The supermatrix formulation simplifies the generalization to other supergroups which will be studied in section 3.6.

3.1.4 Supersymmetry properties of the $2+1$ black hole

To establish the asymptotic conditions for the Rarita-Schwinger fields, the supersymmetry properties of anti-de Sitter space-time and the BTZ black holes will be needed. They have been studied in [34]. The metric of AdS_3 possesses six Killing vectors which make up its isometry group $SO(2,2) = SO(1,2) \otimes SO(1,2)$. Only two of them are compatible with the identifications that lead to the BTZ metric [25]. If regarded as solutions of AdS supergravity with vanishing gravitini, these metrics may possess in addition exact supersymmetries. The "Killing spinors" corresponding to these supersymmetries are, by definition, the spinor parameters of the supersymmetry transformation (3.24) that leave the solution invariant i.e. keep the gravitini to zero. If $(1,1)$-supergravity is considered, the gravitini are the spinors ψ and $\tilde{\psi}$ corresponding to both $OSp(1|2)$ factors and the Killing spinors ρ and $\tilde{\rho}$ are such that:

$$\delta\psi = d\rho + A\rho = 0, \qquad \delta\tilde{\psi} = d\tilde{\rho} + \tilde{A}\tilde{\rho} = 0, \tag{3.26}$$

where A and $\tilde{A}$ are the Chern-Simons connections corresponding to the solution whose supersymmetry properties are studied (see equations (3.15)-(3.16) for their exact expression). One sees that the number of supersymmetries of a bosonic solution depends on the supergravity model which is considered.

The Killing spinors of AdS_3 and of the BTZ black holes were given in [34]. Anti-de Sitter space-time, for which $N^2 = (r/l)^2 + 1$ and $N^\varphi = 0$, possesses four Killing spinors, two for each $OSp(1|2)$ factor. In our conventions, they are given by:

$$\rho = [(N+1)^{\frac{1}{2}} - (N-1)^{\frac{1}{2}}\gamma_1][\cos(u/2) + \sin(u/2)\gamma_0]\alpha,$$
$$\tilde{\rho} = [(N+1)^{\frac{1}{2}} + (N-1)^{\frac{1}{2}}\gamma_1][\cos(v/2) - \sin(v/2)\gamma_0]\tilde{\alpha},$$

where α and $\tilde{\alpha}$ are constant spinors. The extremal black holes for which $|J| = Ml$ possess one Killing spinor. Consider first the case $J = Ml$. The connection A is given by equation (3.15) with $N = r/l - Ml/2r$ and $N^\varphi = -Ml/2r^2$. The corresponding supersymmetry parameter ρ that satisfies equation (3.26) is given by:

$$\rho = B^{-1}(1-\gamma_1)\alpha,$$

where α is a constant spinor and B is the matrix:

$$B = \begin{pmatrix} N^{\frac{1}{2}} & 0 \\ 0 & N^{-\frac{1}{2}} \end{pmatrix}.$$

It is an eigenvector of the radial γ-matrix of eigenvalue -1. There is no parameter $\tilde{\rho}$ that satisfies the Killing spinor equation for $\tilde{A}$. In the case $J = -Ml$, only the equation for $\tilde{\rho}$ has a solution:

$$\tilde{\rho} = B(1+\gamma_1)\tilde{\alpha},$$

with $\tilde{\alpha}$ a constant spinor which is projected on the eigenspace of γ_1 with eigenvalue $+1$. In the limit $M \to 0$, the matrix B reduces to the element b given in equation (3.19) and the connections A and $\tilde{A}$ are equal to those corresponding to the extremal black holes for $J = Ml$ and $J = -Ml$ respectively. Hence the massless black hole possesses two Killing spinors:

$$\rho = b^{-1}(1 - \gamma_1)\alpha, \qquad \tilde{\rho} = b(1 + \gamma_1)\tilde{\alpha}, \tag{3.27}$$

where α and $\tilde{\alpha}$ are constant spinors. In the non extremal case, the connections A and $\tilde{A}$ are asymptotically the same as those of the extremal black holes but for $J = -Ml$ and $J = Ml$ respectively and consequently they admit no Killing spinor. Notice [34] that the Killing spinors of the extremal black holes have the same asymptotic behaviour in r as those of anti-de Sitter space. However they are periodic in φ while those of anti-de Sitter are antiperiodic.

3.2 Boundary conditions

The ground state of $(1,1)$-supergravity is anti-de Sitter space-time which is the solution with the larger isometry group, namely $OSp(1|2) \otimes OSp(1|2)$ whose bosonic part is $SO(2,2)$. The boundary conditions to be imposed on the fields at infinity should be invariant under the anti-de Sitter supergroup since otherwise a symmetry transformation would map an allowed configuration onto a non-allowed one. Moreover they should include the asymptotically anti-de Sitter solutions of physical interest, such as the BTZ black holes. These two requirements prescribe [27] the following procedure to construct the boundary conditions: one starts with the known physical metrics that should be included in the theory - here, the BTZ solutions - and acts on them with the anti-de Sitter supergroup. As the black holes metrics possess less Killing vectors and spinors than the anti-de Sitter one, this will generate new asymptotic behaviour. The remarkable feature is that the resulting class of allowed asymptotic fields admits a much larger symmetry group.

The boundary conditions at infinity on the bosonic fields for an asymptotically anti-de Sitter space have been given in [7] in the metric representation and they were reexpressed in the connection representation in [8]. The asymptotic form of the Chern-Simons connection A associated to the BTZ solutions (with AdS corresponding to $M = -1$ and $J = 0$) was given in equation (3.17). The gauge transformation (3.21) that preserves exactly the asymptotically leading term of these solutions is characterized by the gauge parameter λ that fulfills to leading order:

$$\lambda = b^{-1}\eta(u)b,$$

with all $so(1,2)$ Cartan components of η expressed in terms of η^- as:

$$\begin{aligned}\eta^+ &= \frac{1}{4}(M-J/l)\eta^- - \frac{1}{2}(\eta^-)'',\\ \eta^1 &= -(\eta^-)',\end{aligned}$$

and η^- constrained by the following equation:

$$(M-J/l)(\eta^-)' - (\eta^-)''' = 0. \tag{3.28}$$

Only in the case of anti-de Sitter space has this equation a non trivial solution:

$$\eta^- = A\cos u + B\sin u + C.$$

In the case of the BTZ black holes, where $M - J/l$ is positive, η^- has to be constant. The parameter $\tilde{\lambda}$ that preserves the asymptotic form of the connection $\tilde{A}$ given in equation (3.18) satisfies:

$$\begin{aligned}\tilde{\lambda} &= b\tilde{\eta}(v)b^{-1},\\ \tilde{\eta}^- &= \frac{1}{4}(M-J/l)\tilde{\eta}^+ - \frac{1}{2}(\tilde{\eta}^+)'',\\ \tilde{\eta}^1 &= (\tilde{\eta}^+)',\end{aligned}$$

with η^+ equally constrained by equation (3.28). This agrees with the number of Killing vectors of these solutions [25]: six for anti-de Sitter space and two for the BTZ black holes.

Acting on the black holes solutions with the AdS gauge parameter, one generates new terms in the components A_u and $\tilde{A}_v$ which behave then asymptotically as:

$$A_u = b^{-1}\begin{pmatrix} 0 & L/k \\ 1 & 0 \end{pmatrix} b, \qquad \tilde{A}_v = b\begin{pmatrix} 0 & 1 \\ \tilde{L}/k & 0 \end{pmatrix} b^{-1}, \tag{3.29}$$

where L (resp. $\tilde{L}$) is a linear combination of $\cos u$ and $\sin u$ (resp. v). This motivates to adopt (3.29) with L and $\tilde{L}$ arbitrary functions of t and φ as boundary conditions for the connections, supplemented by the conditions for A_r, A_v, $\tilde{A}_r$ and $\tilde{A}_u$ that are just those given in equations (3.17)-(3.18). Weakening the boundary conditions by allowing arbitrary L and $\tilde{L}$ is shown to be necessary to have at least $SO(2,2)$ as asymptotic symmetry group by acting twice with an AdS transformation. The vanishing of A_v and $\tilde{A}_u$ is related to the fact that the "Killing gauge parameters" λ and $\tilde{\lambda}$ of anti-de Sitter do not depend on v and u respectively. We recall that for constant and positive values of L and $\tilde{L}$, the boundary conditions (3.29) represent a black hole, with $M = (2/k)(L+\tilde{L})$ and $J = (2l/k)(\tilde{L} - L)$. Anti-de Sitter space corresponds to $L/k = \tilde{L}/k = -1/4$.

To construct appropriate boundary conditions for the fermionic fields, one follows the same procedure [27]: one performs on a general black hole solution with no gravitini a supersymmetry transformation (3.24) whose supersymmetry parameter is a Killing spinor of a solution that presents more supersymmetries, namely anti-de Sitter space or the massless black hole. For example, acting on an extremal black hole solution with the massless black hole Killing spinors (3.27), one generates gravitini that typically behave as:

$$\begin{aligned} \psi_r &\sim r^{-\frac{5}{2}}(1-\gamma_1)\chi_r, & \tilde{\psi}_r &\sim r^{-\frac{5}{2}}(1+\gamma_1)\tilde{\chi}_r, \\ \psi_u &\sim r^{-\frac{1}{2}}(1+\gamma_1)\chi, & \tilde{\psi}_v &\sim r^{-\frac{1}{2}}(1-\gamma_1)\tilde{\chi}, \end{aligned} \tag{3.30}$$

with χ_r, χ and $\tilde{\chi}_r$, $\tilde{\chi}$ constant spinors. The action of the AdS Killing spinors would make them depend on u and v respectively. Again the components ψ_v and $\tilde{\psi}_u$ remain zero. The preceeding computation suggests to adopt (3.30) as boundary conditions for the Rarita-Schwinger fields with χ_r, χ, $\tilde{\chi}_r$, $\tilde{\chi}$ allowed to be arbitrary functions of t and φ. We write them:

$$\begin{aligned} \psi_r = \psi_v &= 0, & \tilde{\psi}_r = \tilde{\psi}_u &= 0, \\ \psi_u &= b^{-1}\begin{pmatrix} Q/k \\ 0 \end{pmatrix}, & \tilde{\psi}_v &= b\begin{pmatrix} 0 \\ \tilde{Q}/k \end{pmatrix}, \end{aligned} \tag{3.31}$$

asymptotically, with Q and $\tilde{Q}$ arbitrary functions of t and φ. Apart from an irrelevant replacement of γ_1 by $-\gamma_1$ (due to conventions) in equation (3.30), these boundary conditions differ from those of [27] in two respects. First they involve a slower rate of decrease at infinity (one less power of r). This was to be expected since we are one dimension lower and also holds for the bosonic fields [7]. Second ψ_v and $\tilde{\psi}_u$ vanish, which is consistent with the fact that the AdS Killing spinors of $(1,0)$- and $(0,1)$-supergravity depend only on u and v respectively. The boundary conditions are otherwise indentical and in particular, they crucially involve a projection onto the eigenspaces of the radial γ-matrix, which makes the induced spinors chiral in two dimensions (recall that γ_1 appears as the "γ_5"-matrix on the surface at infinity). This chirality condition is translated in equation (3.31) by the fact that one of the components of the spinors are zero. Notice also that the spinors of both $OSp(1|2)$ factors are of opposite chirality.

The boundary conditions for the bosonic connections (3.29) and for the Rarita-Schwinger fields (3.31) combine to impose the following asymptotic behaviour on the Chern-Simons superconnections written in terms of supermatrices as:

$$\begin{aligned} \Gamma_u &= b^{-1}\begin{pmatrix} 0 & L/k & Q/\sqrt{2}k \\ 1 & 0 & 0 \\ 0 & Q/\sqrt{2}k & 0 \end{pmatrix} b, & \tilde{\Gamma}_v &= b\begin{pmatrix} 0 & 1 & 0 \\ \tilde{L}/k & 0 & \tilde{Q}/\sqrt{2}k \\ -\tilde{Q}/\sqrt{2}k & 0 & 0 \end{pmatrix} b^{-1}, \\ \Gamma_r = b^{-1}\partial_r b, & \quad \Gamma_v = 0, & \tilde{\Gamma}_r = b\partial_r b^{-1}, & \quad \tilde{\Gamma}_u = 0, \end{aligned} \tag{3.32}$$

where b is now the 3×3 supermatrix obtained by completing the above b by adding 0 in the fermionic positions and 1 in position $(3,3)$. The advantage of the connection representation is that one can completely eliminate the r-dependence through the gauge transformation generated by the similarity transformation b, which acting on Γ and $\tilde{\Gamma}$ gives:

$$\bar{\Gamma} = b\partial_i b^{-1} + b\Gamma_i b^{-1}, \qquad \bar{\tilde{\Gamma}} = b^{-1}\partial_i b + b^{-1}\tilde{\Gamma}_i b.$$

After this gauge transformation has been performed, all the asymptotically relevant components of the fields occur at the same order $\mathcal{O}(1)$.

3.3 Hamiltonian formalism

Before computing the symmetries that preserve the above boundary conditions, we study the canonical formulation of the theory. We will see that the constraints derived from action (3.20) generate the supergauge transformations (3.21)-(3.24) through the Poisson bracket and that a projective representation of the asymptotic symmetry algebra is provided by the bracket of the canonical generators themselves. This canonical realization of the asymptotic symmetry algebra is displayed in the next section where the asymptotic behaviour of the supergauge parameter of the asymptotic symmetry is computed.

Assuming that the topology of the three-dimensional manifold M is $\Sigma \times \Re$, the action (3.20) is recast in Hamiltonian form, where time is set apart from the other variables, as:

$$I = \int \left[-\frac{k\epsilon^{ij}}{4\pi} \left(\frac{\eta_{ab}}{2} A_i^a \dot{A}_j^b + i\bar{\psi}_i \dot{\psi}_j \right) - A_0^a \mathcal{G}_a - \bar{\psi}_0 \mathcal{S} \right] dt d^2x, \qquad (3.33)$$

where the constraints $\mathcal{G}_a$ and $\mathcal{S}$ are given by:

$$\mathcal{G}_a \equiv -\frac{k}{8\pi}\epsilon^{ij}\eta_{ab}(F_{ij}^b - i\bar{\psi}_i\gamma^b\psi_j) = 0, \qquad \mathcal{S} \equiv -\frac{ik}{2\pi}\epsilon^{ij}D_i\psi_j = 0.$$

In the Hamiltonian formalism [35], phase space is endowed with a Poisson structure through which the constraints generate the gauge transformations. This structure is completely captured by a rank-two, contravariant, antisymmetric tensor σ^{ij}. In the case of a first-order action in the time derivative of the fields:

$$I[y] = \int (a_i \dot{y}^i - h) dt,$$

the function $a_i(y)$ constitutes a one-form potential for the closed two-form σ_{ij}:

$$\sigma_{ij} = \frac{\partial a_j}{\partial y^i} - \frac{\partial a_i}{\partial y^j}.$$

This two-form is called the symplectic form and its inverse σ^{ij} completely characterizes the Poisson bracket of the fields:

$$[y^i, y^j] = \sigma^{ij}.$$

This construction is adapted to the case of fermionic variables as follows. Starting from the first-order action:

$$I[z] = \int (\dot{z}^A a_A - H)dt,$$

the symplectic two-form is defined as:

$$\sigma_{AB} = -\frac{\partial^R a_A}{\partial z^B} - (-)^{(\varepsilon_A+1)(\varepsilon_B+1)}\frac{\partial^R a_B}{\partial z^A},$$

where ε_A is the Grassmann parity of a_A. The Poisson bracket of the fields is then given by:

$$\{z^A, z^B\} = \sigma^{AB}.$$

The Poisson brackets derived from action (3.33) by this method are:

$$[A_i^a, A_j^b] = \frac{4\pi}{k}\eta^{ab}\epsilon_{ij}, \qquad \{\psi_i^\alpha, \psi_j^\beta\} = \frac{2\pi i}{k}\epsilon_{ij}(\gamma_0)^{\alpha\beta}. \tag{3.34}$$

Recalling the basic non vanishing Poisson brackets between the canonical coordinates q^i, θ^α and momenta p_i, π_α:

$$[q^i, p_j] = \delta^i_j, \qquad \{\theta^\alpha, \pi_\alpha\} = -\delta^\alpha_\beta, \tag{3.35}$$

and writing down the kinetic term of action (3.33) in canonical form:

$$\dot{q}^i p_i + \dot{\theta}^\alpha \pi_\alpha = \frac{k}{4\pi}\eta_{ab}\dot{A}_r^a A_\varphi^b + \frac{ik}{2\pi}\dot{\psi}_r^\alpha(\gamma_0)^{\alpha\beta}\psi_\varphi^\beta,$$

one reads that A_r^a and $(k/4\pi)\eta_{ab}A_\varphi^b$ are conjugated, as well as ψ_r^α and $(ik/2\pi)(\gamma_0)^{\alpha\beta}\psi_\varphi^\beta$. Using (3.35), the Poisson brackets given in (3.34) are recovered.

The canonical generator of the supergauge transformation (3.21)-(3.24) is $G(\lambda)+S(\rho)$ with:

$$G(\lambda) = \int_\Sigma \lambda^a \mathcal{G}_a d^2x, \qquad S(\rho) = \int_\Sigma \bar{\rho} S d^2x, \tag{3.36}$$

up to surface terms that will be considered later. Indeed the following Poisson brackets are satisfied (forgetting for the moment about boundary terms):

$$\begin{aligned} [A^a, G(\lambda)] &= D\lambda^a = \delta_\lambda A^a, & [\psi, G(\lambda)] &= -\lambda\psi = \delta_\lambda\psi, \\ [A^a, S(\rho)] &= i\bar{\rho}\gamma^a\psi = \delta_\rho A^a, & \{\psi, S(\rho)\} &= D\rho = \delta_\rho\psi. \end{aligned} \tag{3.37}$$

The gauge (super)transformations (3.21)-(3.24) form an algebra, namely $osp(1|2)$. Indeed the commutator of two transformations is again a gauge transformation:

$$[\delta_{\lambda_1}, \delta_{\lambda_2}] = \delta_{\lambda_{(\lambda_1,\lambda_2)}}, \qquad [\delta_\lambda, \delta_\rho] = \delta_{\rho_{(\lambda,\rho)}}, \qquad [\delta_{\rho_1}, \delta_{\rho_2}] = \delta_{\lambda_{(\rho_1,\rho_2)}}, \tag{3.38}$$

whose gauge parameters are the following:

$$\lambda_{(\lambda_1,\lambda_2)} = [\lambda_1, \lambda_2], \qquad \rho_{(\lambda,\rho)} = \lambda\rho, \qquad \lambda^a_{(\rho_1,\rho_2)} = -i\bar{\rho}_1\gamma^a\rho_2. \tag{3.39}$$

Expressing the action on the fields of both sides of equations (3.38) by the Poisson brackets (3.37) and making use of the Jacobi identity, one obtains, up to functions that do not depend on the canonical variables and neglecting boundary terms as before:

$$\begin{aligned} [G(\lambda_1), G(\lambda_2)] &= -G([\lambda_1, \lambda_2]), \\ [G(\lambda), S(\rho)] &= -S(\lambda\rho), \\ \{S(\rho_1), S(\rho_2)\} &= -G(-i\bar{\rho}_1\gamma^a\rho_2). \end{aligned} \tag{3.40}$$

The minus sign that appears in these equations is a matter of convention related to the fact that the commutator $[\delta_1, \delta_2]$ acts on the fields from the left while the generators sit in the right position in the Poisson brackets (3.37). Apart from that, these equations express that the generators satisfy the $OSp(1|2)$ algebra in the Poisson brackets.

In the next section, the supergauge parameter (λ, ρ) that preserves the boundary conditions on the fields given in equation (3.32) will be computed. To each infinitesimal asymptotic symmetry parameter (λ, ρ) is associated a functional $G(\lambda) + S(\rho)$ that generates the corresponding transformation of the canonical variables and it is generally taken for granted that the Poisson bracket algebra (3.40) of the generators with (λ, ρ) equal to the asymptotic symmetry parameter is just isomorphic to the Lie superalgebra of the infinitesimal asymptotic symmetries. However the Poisson brackets (3.37) make sense only if G and S have well defined functional derivatives. To guarantee this property, they must be improved by appropriate surface terms B and F:

$$G(\lambda) = \int_\Sigma \lambda^a \mathcal{G}_a d^2x + B, \qquad S(\rho) = \int_\Sigma \bar{\rho} \mathcal{S} d^2x + F. \tag{3.41}$$

It has been established in [36] that these surface terms are related to the global charges of the theory. They must be chosen in such a way that their variation cancels the surface terms appearing in the variation of the "volume piece" of the generators so that the improved generators (3.41) contain no surface term in their variation. The method to construct them consists then in looking at the surface terms coming from the variation of the generators (3.36) and to rewrite them as the total variation of surface integrals. The charges B and F are then, within a constant, the negative of these surface integrals.

By means of this procedure, one obtains:

$$B = \frac{k}{8\pi}\int_{\partial\Sigma} \eta_{ab}\lambda^a A^b_\varphi d\varphi, \qquad F = \frac{ik}{2\pi}\int_{\partial\Sigma} \bar{\rho}\psi_\varphi d\varphi. \tag{3.42}$$

up to constants. It remains to be checked whether these charges are well defined i.e. if, taking into account the asymptotic behaviour of the fields and of the parameters of the asymptotic symmetries, they are finite. This will be done in the next section. It was proved in [37] that the Poisson bracket of two well defined generators is again a well defined generator. The Poisson brackets (3.40) are then also satisfied for the improved generators (3.41) still up to constants, when the boundary terms showing up in the variations are taken into account. Nevertheless, the fact that the improved generators (3.41) are only defined up to arbitrary constants implies that the canonical generator associated to an infinitesimal asymptotic symmetry is not unique. The Poisson bracket of the generators of two given symmetries can then differ by a constant from the generator associated with the commutator of these symmetries, having for example:

$$[G(\lambda_1), G(\lambda_2)] = -G([\lambda_1, \lambda_2]) + K(\lambda_1, \lambda_2).$$

The constant $K(\lambda_1, \lambda_2)$ commutes with everything and constitutes a central term in the Poisson bracket algebra of the generators. Because of this central extension, the canonical generators only yield what is called a projective representation of the asymptotic symmetry group. In our case, this central extension will be shown to be non trivial in the sense that it cannot be absorbed in a redefinition of the charges. This is related to the fact that the asymptotic symmetry group turns out to be much larger than the exact symmetry group of any background configuration.

3.4 Asymptotic symmetry

The most general supergauge transformations that preserve the boundary conditions (3.32) for the Chern-Simons superconnection Γ are characterized by gauge parameters (λ, ρ) that must fulfill, to leading order:

$$\lambda(r,u) = b^{-1}\eta(u)b, \qquad \rho(r,u) = b^{-1}\varepsilon(u), \tag{3.43}$$

with:

$$\begin{aligned} \eta^+ &= \frac{\eta^- L}{k} - \frac{1}{2}(\eta^-)'' - \frac{i\epsilon Q}{2k}, \\ \eta^1 &= -(\eta^-)', \\ \varepsilon &= \begin{pmatrix} -\epsilon' + \eta^- Q/k \\ \epsilon \end{pmatrix}, \end{aligned} \tag{3.44}$$

where $'$ denotes derivative with respect to the argument. As usual the algebra element η has been expanded in the Cartan basis $\eta = \eta^1 J_1 + \eta^+ J_+ + \eta^- J_-$. Equations (3.44) imply that the full residual symmetry can be expressed in terms of two functions of the light-like coordinate u, one bosonic η^- and one fermionic ϵ. Any three-dimensional gauge transformation whose parameters fulfill (3.43)-(3.44) keeps the asymptotic behaviour of the fields (3.32) intact and is called an asymptotic symmetry. It only changes the (t,φ)-functions L and Q as follows:

$$\delta L = (\eta^- L)' + (\eta^-)' L - \frac{k}{2}(\eta^-)''' - i\epsilon' Q - \frac{i}{2}(\epsilon Q)', \tag{3.45}$$

$$\delta Q = \epsilon L - k\epsilon'' + (\eta^- Q)' + \frac{1}{2}(\eta^-)' Q. \tag{3.46}$$

As we will see, these functions enter the surfaces terms of the improved generators (3.42) and their transformation law is then useful to compute the representation of the asymptotic symmetry algebra through the Poisson brackets of the generators. The asymptotic form of the gauge parameter $(\tilde{\lambda}, \tilde{\rho})$ of the transformations that preserve the boundary conditions on $\tilde{\Gamma}$ can be computed in the same way, but from now on, we consider only one $OSp(1|2)$ copy, the other copy being treated similarly.

Among the asymptotic symmetries, two gauge transformations that tend to the same η^- and ϵ at infinity should be identified because they differ by a gauge transformation for which $\eta^- = \epsilon = 0$ and which is called a pure gauge transformation. As it will be clear below, the pure gauge transformations have no associated charge and their generators vanish weakly. They then produce effects which are not to be considered as physically meaningful and as they form an ideal, it is legitimate to quotientize the asymptotic symmetry by them [36, 38]. The asymptotic symmetry superalgebra is therefore defined as the resulting quotient algebra.

The commutator of two gauge transformations fulfilling the above asymptotic conditions is again a gauge transformation that satisfies the same conditions. Using equation (3.39), one computes the relation between the asymptotic parameters (η^-, ϵ) of the transformation corresponding to the commutator and the asymptotic parameters of the transformations whose commutator is taken. One obtains:

$$\begin{aligned} \eta^-_{(\eta_1^-, \eta_2^-)} &= (\eta_1^-)'\eta_2^- - \eta_1^-(\eta_2^-)', \\ \epsilon_{(\eta^-, \epsilon)} &= -\eta^-\epsilon' + \frac{1}{2}(\eta^-)'\epsilon, \\ \eta^-_{(\epsilon_1, \epsilon_2)} &= -i\epsilon_1\epsilon_2. \end{aligned} \tag{3.47}$$

These correspond exactly to the graded commutation rules of the super-Virasoro algebra. This infinite-dimensional algebra contains $osp(1|2)$ as a subalgebra when the fermions are antiperiodic (Fourier modes 0 and ± 1 of η^- and modes $\pm\frac{1}{2}$ of ϵ). Note, in particular, that the parameter of the Lie algebra commutator $[\delta_{\lambda_1}, \delta_{\lambda_2}]$ of two bosonic transformations restricted by (3.43)-(3.44) reduces at infinity to the Lie bracket of the residual functions η_1^- and η_2^- viewed as vector fields on the circle. Together with the super-Virasoro algebra of the asymptotic symmetry of the other Chern-Simons $OSp(1|2)$ copy, the asymptotic symmetry group defined above is isomorphic to the superconformal group in two dimensions, which has the particularity to be infinite-dimensional and consequently much larger than the anti-de Sitter supergroup. This is different than in higher dimensions where the anti-de Sitter group is isomorphic to the conformal group on the boundary, i.e. in one dimension lower. This particular feature of $2+1$ dimensions will be shown below to prescribe a non trivial central extension in the canonical representation of the asymptotic symmetry algebra.

We now turn to the discussion of the canonical realization of the asymptotic symmetry algebra. As explained in the previous section, the bracket of the canonical generators of the asymptotic symmetries provides a projective representation of the asymptotic symmetry algebra. To determine the central extension of the algebra corresponding to (3.47), one must first work out the complete form of the generators (3.41). This is now possible since the asymptotic form of both the fields and the symmetry transformations has been obtained. Replacing in equation (3.42) the fields and symmetry parameters by their asymptotic value, the obtained charges are:

$$B = \frac{1}{2\pi}\int_{\partial\Sigma} \eta^- L d\varphi, \qquad F = \frac{-i}{2\pi}\int_{\partial\Sigma} \epsilon Q d\varphi, \tag{3.48}$$

i.e. the surface terms of the generators are precisely L and Q (up to numerical factors). The constants in the charges have been adjusted so that these vanish for the zero mass black hole, which has $L = 0$. The surface terms (3.48) are of course equal to the surface terms that one would obtain through a more orthodox "non-Chern-Simons-based" approach (see [27] for the four-dimensional treatment). In particular, the bosonic piece B is equal to the charge (4.11) of [7] written in terms of the metric.

We are now ready to compute the Poisson brackets of the improved generators (3.41) which provide, as explained in the previous section, a projective representation of the asymptotic symmetry algebra. The generators of the asymptotic symmetries are just the ones given in equation (3.41) with the fields and gauge parameters fulfilling the conditions (3.32) and (3.43)-(3.44) respectively. Therefore they satisfy the Poisson brackets given in (3.40) up to central terms, according to the same arguments that apply to any gauge transformations algebra and that were exposed in the previous section. It only remains to determine the possible central extensions to these brackets. This can be achieved by a direct calculation of the brackets of the improved generators of the asymptotic symmetries or by using the general argument given in [7]. This last method relies on the fact that the Poisson bracket $[G(\lambda_1), G(\lambda_2)]$, for example, can be interpreted as the change in the improved generator $G(\lambda_1)$ under the transformation generated by $G(\lambda_2)$, so that:

$$\delta_{\lambda_2} G(\lambda_1) = [G(\lambda_1), G(\lambda_2)].$$

On the other hand, since the generators form a central extension of the asymptotic symmetry algebra, one can write:

$$\delta_{\lambda_2} G(\lambda_1) = -G([\lambda_1, \lambda_2]) + K(\lambda_1, \lambda_2).$$

The central charge $K(\lambda_1, \lambda_2)$ is a surface term which is independent of the canonical variables and can therefore be evaluated on any background solution. It is most easily evaluated on the massless black hole solution. Indeed, the improved generator vanishes when evaluated on this solution because its volume piece is proportional to the constraint which is zero on the equations of motion while the constant in the surface term B has been chosen so that it vanishes for this solution. In particular $G([\lambda_1, \lambda_2)]) = 0$ and the central charge $K(\lambda_1, \lambda_2)$ reduces to the value of the generator $G(\lambda_1)$ for the solution obtained by the deformation of the massless black hole by the transformation generated by λ_2. It is non zero for a general λ_2 that satisfies the asymptotic conditions (3.43)-(3.44) because, in that case, λ_2 does not parametrize an exact symmetry of the massless black hole. By the above reasoning, the different terms coming from the variation of $G(\lambda_1)$ will combine to give only surface terms that are independent of the canonical variables. These surface terms arise from the variation of the surface terms B and correspond to the variables-independent terms that appear in the transformation law of L given in equation (3.45). Keeping the relevant term, one obtains on the massless black hole solution:

$$\delta_{\lambda_2} G(\lambda_1) = K(\lambda_1, \lambda_2) = -\frac{k}{4\pi} \int_{\partial\Sigma} \eta_1^- (\eta_2^-)''' d\varphi. \tag{3.49}$$

Proceeding in the same way for the other brackets, one finds that the Poisson brackets of the improved generators, whose gauge parameters and fields satisfy the boundary conditions (3.43)-(3.44) and (3.32), are:

$$\begin{aligned} [G(\lambda_1), G(\lambda_2)] &= -G([\lambda_1, \lambda_2]) - \frac{k}{4\pi} \int_{\partial\Sigma} \eta_1^- (\eta_2^-)''' d\varphi, \\ [G(\lambda), S(\rho)] &= -S(\lambda\rho), \\ \{S(\rho_1), S(\rho_2)\} &= -G(-i\bar{\rho}_1 \gamma^a \rho_2) + \frac{ik}{2\pi} \int_{\partial\Sigma} \epsilon_1 (\epsilon_2)'' d\varphi. \end{aligned} \tag{3.50}$$

As the gauge parameters appearing in this algebra satisfy the boundary conditions (3.43)-(3.44), their asymptotic form is expressed in terms of residual functions η^- and ϵ. The exact relation between the functions corresponding to the gauge parameters present on the left and right hand sides of these equations were given in (3.47). As the asymptotic symmetry is not an exact symmetry of any background, the central extensions are non trivial i.e. they cannot be absorbed by a redefinition of the charges B and F. Indeed there exists no background, on which they could be defined as to vanish, that is preserved by the asymptotic symmetry and the deformed solution would always have non vanishing charge, i.e. equation (3.49) would be non zero for any reference background. Let us also notice that if one of the symmetry transformation in the brackets (3.50) is a pure gauge transformation, the associated central charge vanishes. Refering to our above reasoning to compute the central terms, this is related to the fact that the charges associated to such a transformation vanish for all admissible fields configurations.

For the moment, the above Poisson bracket algebra involves both the pure gauge symmetries and the non pure ones [36, 38]. The pure gauge symmetries have weakly vanishing Poisson brackets with all the other generators, i.e. form also an ideal in the Poisson sense. It follows that the asymptotic symmetries are "first class" and well defined in the reduced phase space which is obtained by quotientizing the pure gauge symmetries and where the constraints are zero. Using standard terminology, they are "observables". The Poisson bracket of these observables in the reduced phase space is the same as their Poisson bracket in the original phase space (see e.g. [35]). The asymptotic symmetry algebra is therefore realized in the space of physical observables where it is generated by B and F (because the volume term of the generators is proportional to the constraints which are zero in the reduced phase space). According to this, one will derive the explicit form of the super-Virasoro algebra with central extension in two different ways.

One way is to rewrite the Poisson bracket algebra (3.50) with the constraints put to zero and the gauge parameters at infinity explicitly expressed in terms of the functions η^- and ϵ according to equation (3.47). One obtains:

$$\begin{aligned}
[B[\eta_1^-], B[\eta_2^-]] &= B[\eta_1^-(\eta_2^-)' - (\eta_1^-)'\eta_2^-] - \frac{k}{4\pi}\int_{\partial\Sigma} \eta_1^-(\eta_2^-)''' d\varphi, \\
[B[\eta^-], F[\epsilon]] &= F[\eta^-\epsilon' - \frac{1}{2}(\eta^-)'\epsilon], \\
\{F[\epsilon_1], F[\epsilon_2]\} &= B[i\epsilon_1\epsilon_2] + \frac{ik}{2\pi}\int_{\partial\Sigma} \epsilon_1(\epsilon_2)'' d\varphi,
\end{aligned}$$

where $B[\eta^-]$ and $F[\epsilon]$ are given as in (3.48) by:

$$B[\eta^-] = \frac{1}{2\pi}\int_{\partial\Sigma} \eta^- L d\varphi, \qquad F[\epsilon] = \frac{-i}{2\pi}\int_{\partial\Sigma} \epsilon Q d\varphi.$$

It follows from the above algebra that the Fourier components $L_n = B[e^{inu}]$ and $Q_n = F[e^{inu}]$ of the functions L and Q satisfy the super-Virasoro algebra in its familiar form (using quantum-mechanical notation and rescaling Q by $\sqrt{2}$):

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{k}{2}n^3\delta_{m+n,0},$$

$$\begin{aligned} [L_m, Q_n] &= (n - \frac{1}{2}m)Q_{m+n}, \\ \{Q_m, Q_n\} &= 2L_{m+n} - 2km^2\delta_{m+n,0}. \end{aligned} \tag{3.51}$$

Another way to derive this algebra is to notice that, as the canonical generators in the reduced phase space are just the functions L and Q, their Poisson brackets could have been read directly in the expression of their transformation law that was given in equations (3.45)-(3.46). Indeed, writing $\delta L = [L, B + F]$ and $\delta Q = [Q, B] + \{Q, F\}$, one extracts the following Poisson brackets:

$$\begin{aligned} [L(\sigma), L(\sigma')] &= 2\pi(L(\sigma) + L(\sigma'))\delta'(\sigma - \sigma') - \pi k\delta'''(\sigma - \sigma'), \\ [L(\sigma), Q(\sigma')] &= 2\pi(Q(\sigma) + \frac{1}{2}Q(\sigma'))\delta'(\sigma - \sigma'), \\ \{Q(\sigma), Q(\sigma')\} &= 2\pi i L(\sigma)\delta(\sigma - \sigma') - 2\pi i k\delta''(\sigma - \sigma'). \end{aligned} \tag{3.52}$$

After a renormalization taking $L/2\pi$ and $\sqrt{2}Q/2\pi$ as generators and a Fourier transformation, the algebra (3.51) is recovered.

In our conventions, the central term in the algebra of the L_n's in equation (3.51) is proportional to $c/12$. The central charge is then equal to $c = 6k = 3l/2G$, which is the same as the one arising from the analysis of the asymptotic symmetry algebra in the purely bosonic case [7]. It is related to the facts that the surface term at infinity in the Virasoro generator involves only the (bosonic) gravitational variables, i.e. the dreibein and the spin connection, and that the asymptotic boundary conditions that dictate the approach of these variables to anti-de Sitter remain the same. Any theory with these features will have the same central charge in the commutator involving the L_n's. This central charge is non trivial in the sense that it cannot be absorbed in a redefinition of the generators. In fact, as mentioned before, the bosonic part of the asymptotic symmetry group is isomorphic to the conformal group in two dimensions. The latter has the particularity, unlike in higher dimensions, to be infinite-dimensional and hence much larger than the anti-de Sitter group. Anti-de Sitter space being the solution with maximal symmetry, there exists no background that has the asymptotic symmetry group as exact symmetry group. Hence, by the reasoning presented above while computing the central extension of the algebra, the central charge cannot be absorbed in a redefinition of the generators [7]. The situation is different in four dimensions [27] where the asymptotic symmetry group is the conformal group in three dimensions, which is isomorphic to the symmetry group of anti-de Sitter spacetime in four dimensions. Adapting the constants in the surface terms of the generators so as to make them vanish on this solution, the canonical representation of the asymptotic symmetry algebra yields no central extension. We recall that it was observed in [9] that the degeneracy of states for a conformal field theory with the above central charge give rise to, under appropriate conditions, exactly the Bekenstein-Hawking entropy for the BTZ black hole (see also [39]).

The super-Virasoro algebra (3.51) has the form of the Ramond graded extension of the Virasoro algebra: it is adapted to the case of periodic fermions and the moding of the Fourier modes of the fermionic generators is integer. The central charge vanishes for the sub-algebra generated by (L_0, Q_0) which, together with the corresponding two modes of the other $OSp(1|2)$ copy, constitutes the exact Killing vectors and spinors of the massless black hole. According to this, configurations with periodic spinor fields will be referred as belonging to the Ramond sector whose ground state appears to be the zero mass black hole [34]. The anti-de Sitter background has $M = -1$, i.e. $L_0 = -c/24$. Shifting L_0 by $c/24$ so that the charge B vanishes on the anti-de Sitter solution, the algebra (3.51) takes the form of the Neveu-Schwarz extension of the Virasoro algebra. It is adapted to antiperiodic fermions and the index on Q is half-integer moded. In that case, the central charge vanishes for the sub-algebra generated by $(L_{\pm 1}, L_0, Q_{\pm\frac{1}{2}})$ which are true $OSp(1|2)$ symmetries of the anti-de Sitter background, recalling that the four Killing spinors of anti-de Sitter are indeed antiperiodic [34]. Accordingly, anti-de Sitter space appears as the ground state of the Neveu-Schwarz sector, made up by the solutions with anti-periodic spinor fields [34]. Taking into account both $OSp(1|2)$ copies, one sees that the anti-de Sitter group $SO(2,2)$ constitutes a subgroup of the asymptotic symmetry group but not an invariant one. There is therefore no obvious way to restrict the asymptotic symmetries to just the anti-de Sitter supergroup [7], i.e. to strengthen the boundary conditions (3.32) so as to have exactly $OSp(1|2) \otimes OSp(1|2)$ as asymptotic symmetry group.

3.5 Dynamics at infinity

The emergence of the super-conformal algebra at infinity with a non-vanishing central charge can be understood at the dynamical level, in the light of Polyakov's discovery of the hidden $SL(2,R)$ symmetry of induced two-dimensional gravity, which is described by Liouville theory in the conformal gauge [40, 41, 42]. We recall that the $SL(2,R)$ algebra is isomorphic to the $SO(1,2)$ one. The argument runs as follows [8]. As shown by [43], the $SO(1,2)$ Chern-Simons theory under the boundary condition $A_v = 0$ induces the chiral Wess-Zumino-Witten (WZW) model at the boundary. The corresponding Kac-Moody currents are just the φ-components of the connection. Combining the two chiral WZW models of opposite chiralities obtained from each $SO(1,2)$ factor, one finds a non-chiral $SL(2,R)$ WZW theory (modulo zero modes and holonomies not discussed here because they affect neither the asymptotic symmetry nor the central charge). The constraints on the Kac-Moody currents arising from the boundary conditions on A_u, which together with the above condition constitutes anti-de Sitter asymptotics, lead then to Liouville theory.

In a similar way, the further constraint that the component of the Kac-Moody current along the fermionic generator f vanishes (see (3.31)) turns out to be precisely the constraint that reduces the WZW theory based on the supergroup $OSp(1|2)$ to chiral two-dimensional induced supergravity [41, 44]. Altough the $OSp(1|2)$-WZW theory is not superconformal, the resulting theory is. What happens is that the other component (along e) of the Kac-Moody supercurrent is transmuted into the super-Virasoro generator since its transformation law becomes (3.46) once the gauge parameters are restricted by (3.43)-(3.44). From the WZW point of view, supersymmetry on the worldsheet arises therefore in a non trivial way. Bringing the other $OSp(1|2)$ factor leads to the non chiral two-dimensional induced $(1,1)$-supergravity, which is described, in the conformal gauge, by super-Liouville theory. It has been explicitly checked, using the Gauss decomposition and following the same lines as in [8], that the three-dimensional supergravity action (3.25) yields the super-Liouville action (up to zero modes and holonomies that we have not explicitly worked out).

The two steps leading from Chern-Simons theory to WZW theory and from the latter to super-Liouville theory can be implemented kinematically while computing the asymptotic symmetry algebra. Indeed, if we had imposed only the chirality condition $\Gamma_v = 0$ as is done in the Chern-Simons $\to$ chiral WZW reduction, we would have obtained a current Kac-Moody algebra rather than the super-Virasoro algebra. The key point leading to the super-Virasoro algebra is the presence of extra boundary conditions in (3.32) which transmute the residual gauge field components functions L and Q into super-Virasoro charges [40, 41, 42]. From the point of view of the chiral bosonic WZW theory, this transmutation can be seen explicitly as follows. Let affine $SL(2,R)_k$ be generated by $J^{\pm}$, J^1. Impose $J^- = 1$ and $J^1 = 0$ (see (3.32)). These are second class constraints because their Poisson bracket is an invertible matrix. It follows that $J^+ = L$ satisfies, in the Dirac bracket, the Virasoro algebra with $c = 6k$.

3.6 Extended supersymmetry

Thanks to the Chern-Simons formulation, the results of this chapter can be adapted to the case of extended supersymmetry quite straightforwardly. Indeed, the difference of two Chern-Simons theories based on supergroups which contain $SO(1,2)$ in their even part possibly describes AdS_3 supergravity theories. The relevant supergroups have their fermionic generators transforming as $SO(1,2)$ spinors [45] and were listed in [46]. As explained before, any theory with these properties must have twice the Virasoro algebra in its asymptotic symmetry algebra with a central charge equal to $c = 3l/2G$.

In particular, the (p,q)-supergravity theories of [23] can be formulated as a Chern-Simons theory based on the anti-de Sitter supergroup $OSp(p|2) \otimes OSp(q|2)$. The $SO(2,2) \equiv SO(1,2) \otimes SO(1,2)$ part of the Chern-Simons connections Γ and $\tilde{\Gamma}$ describes the gravitational variables (the dreibein and the spin connection). In addition, each $OSp(N|2)$ connection contains N gravitini ψ^i and $O(N)$ gauge fields B^{ij}. The $OSp(N|2)$ generators are constructed as in the $OSp(1|2)$ case presented in section 3.2 and the superconnection Γ takes the following form:

$$\begin{pmatrix} A^1/2 & A^+ & \psi_1^j/\sqrt{2} \\ A^- & -A^1/2 & \psi_2^j/\sqrt{2} \\ -\psi_2^i/\sqrt{2} & \psi_1^i/\sqrt{2} & B^{ij} \end{pmatrix},$$

where B^{ij} is antisymmetric and $i, j = 1, \dots, N$. Taking into account the similar form of the superconnection $\tilde{\Gamma}$ and with A and $\tilde{A}$ related to the dreibein and spin connection as usual, one has [23], up to surface terms:

$$\begin{aligned} I_{CS}[\Gamma] - I_{CS}[\tilde{\Gamma}] &= \frac{1}{16\pi G}\int \Big\{ eR + \frac{2e}{l^2} + il\epsilon^{\lambda\mu\nu}\bar{\psi}^i_\lambda \mathcal{D}_\mu \psi^i_\rho - il\epsilon^{\lambda\mu\nu}\bar{\tilde{\psi}}^i_\lambda \mathcal{D}_\mu \tilde{\psi}^i_\rho \\ &\quad -\frac{i}{2}e\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^i_\nu + \frac{i}{2}e\bar{\tilde{\psi}}^i_\mu\gamma^{\mu\nu}\tilde{\psi}^i_\nu - l\epsilon^{\lambda\mu\nu}\left(B^{ij}_\lambda\partial_\mu B^{ji}_\nu + \frac{2}{3}B^{ij}_\lambda B^{jk}_\mu B^{ki}_\nu\right) \\ &\quad +l\epsilon^{\lambda\mu\nu}\left(\tilde{B}^{ij}_\lambda\partial_\mu \tilde{B}^{ji}_\nu + \frac{2}{3}\tilde{B}^{ij}_\lambda \tilde{B}^{jk}_\mu \tilde{B}^{ki}_\nu\right)\Big\} d^3x, \end{aligned}$$

which is the action for AdS$_3$ (p,q)-supergravity. From now, we take $p = q = N$ and consider one $OSp(N|2)$ factor only.

The dreibein, spin connection and spinor fields obey the same boundary conditions (3.32) as in the $N = 1$ case, while the $SO(N)$-currents fulfill $B^{ij}_r = B^{ij}_v = 0$. This is translated into the generalized asymptotic conditions:

$$\begin{aligned} \Gamma_u &= b^{-1}\begin{pmatrix} 0 & L/k & Q^j\sqrt{2}k \\ 1 & 0 & 0 \\ 0 & Q^i\sqrt{2}k & T_{ij}/k \end{pmatrix} b, \\ \Gamma_r &= b^{-1}\partial_r b, \qquad \Gamma_v = 0, \end{aligned} \tag{3.53}$$

to leading order, where L, Q^i and T^{ij} are arbitrary functions of t and φ. The most general supergauge transformations that preserve these boundary conditions are characterized by gauge parameters $(\lambda, \rho^i, \beta^{ij})$ that must fulfill to leading order:

$$\lambda(r,u) = b^{-1}\eta(u)b, \qquad \rho^i(r,u) = b^{-1}\varepsilon^i(u), \qquad \beta^{ij}(r,u) = \beta^{ij}(u), \tag{3.54}$$

with:

$$\begin{aligned} \eta^+ &= \frac{\eta^- L}{k} - \frac{1}{2}(\eta^-)'' - \frac{i\epsilon^i Q^i}{2k}, \\ \eta^1 &= -(\eta^-)', \\ \varepsilon^i &= \begin{pmatrix} -(\epsilon^i)' + \eta^- Q^i/k - \epsilon^j T^{ij}/k \\ \epsilon^i \end{pmatrix}, \end{aligned} \tag{3.55}$$

implying that the full residual symmetry can be expressed in terms of arbitrary functions η^-, ϵ^i and β^{ij} of the light-like coordinate u.

The generators of the asymptotic symmetry transformations are constructed as in section 3.4 and their surface terms are equal to:

$$B = \frac{1}{2\pi}\int \eta^- L d\varphi, \qquad F = \frac{-i}{2\pi}\int \epsilon^i Q^i d\varphi, \qquad D = \frac{1}{2\pi}\int \beta^{ij}T^{ij}d\varphi, \tag{3.56}$$

where the latter is the surface term of the generator of the $O(N)$ gauge transformations. As before the charges (3.56) are proportional to the functions L, Q^i and T^{ij} which appear in the boundary conditions (3.53) and transform under (3.54)-(3.55) as follows:

$$\begin{aligned}
\delta L &= (\eta^- L)' + (\eta^-)'L - \frac{k}{2}(\eta^-)''' - \frac{i}{2}(\epsilon^i Q^i)' - i(\epsilon^i)'Q^i + \frac{i}{k}\epsilon^i Q^j T^{ij}, &(3.57)\\
\delta Q^i &= \epsilon^i L - k(\epsilon^i)'' + \frac{1}{k}\epsilon^k T^{ij}T^{kj} - (\epsilon^j T^{ij})' - (\epsilon^j)'T^{ij} &\\
&\quad +(\eta^- Q^i)' + \frac{1}{2}(\eta^-)Q^i + \frac{1}{k}\eta^- T^{ij}Q^j - \beta^{ij}Q^j, &(3.58)\\
\delta T^{ij} &= \beta^{ik}T^{jk} - \beta^{jk}T^{ik} + k(\beta^{ij})' - \frac{i}{2}(\epsilon^i Q^j - \epsilon^j Q^i). &(3.59)
\end{aligned}$$

As in section 3.4, we can compute the Poisson brackets between the functions L, Q^i and T^{ij} by writing their variations (3.57)-(3.59) through their brackets with the canonical generators in the reduced phase space i.e. with the surface terms (3.56). As they stand, they do not provide a representation of (half) a superconformal algebra because of the vanishing of the Poisson bracket of L and T^{ij} and of the unwanted term proportional to $Q^j T^{ij}$ that appears in (3.57).

This can be solved by defining a new Virasoro generator $\bar{L}$ which presents the same central extension characterized by $c = 6k$. To this purpose, one notices by looking at equation (3.59) that the T^{ij}'s form a Kac-Moody algebra. Now it happens that a Virasoro generator with no central charge can be formed out of Kac-Moody generators by what is called a Sugawara construction. In the abelian case, this works as follows. Starting from a Kac-Moody algebra with central charge α:

$$[T(\sigma), T(\sigma')] = \alpha\delta'(\sigma - \sigma'),$$

the generator $T^2/2\alpha$ can be constructed, which satisfies a Virasoro algebra with no central term:

$$[\frac{1}{2\alpha}T^2(\sigma), \frac{1}{2\alpha}T^2(\sigma')] = \frac{1}{2\alpha}(T^2(\sigma) + T^2(\sigma'))\delta'(\sigma - \sigma').$$

This is extended to the non-abelian case in the following way. Starting from the Kac-Moody algebra:

$$[T^a(\sigma), T^b(\sigma')] = f^{abc}T^c(\sigma)\delta(\sigma - \sigma') + \alpha g^{ab}\delta'(\sigma - \sigma'),$$

the Sugawara construction gives:

$$[\frac{1}{2\alpha}g^{ab}T^a(\sigma)T^b(\sigma), \frac{1}{2\alpha}g^{cd}T^c(\sigma')T^d(\sigma')] = \frac{1}{2\alpha}g^{ab}(T^a(\sigma)T^b(\sigma) + T^a(\sigma')T^b(\sigma'))\delta'(\sigma - \sigma').$$

In our case, defining $\bar{L}$ as:

$$\bar{L} = L + \frac{1}{2k}T^{ij}T^{ij},$$

one preserves the central charge $c = 6k$ of the Virasoro subalgebra and generates the right Poisson bracket between $\bar{L}$ and T^{ij}. Moreover one gets rid of the unwanted term in (3.57). Indeed one has:

$$\delta\bar{L} = (\eta^- L)' + (\eta^-)'L - \frac{k}{2}(\eta^-)''' - \frac{i}{2}(\epsilon^i Q^i)' - i(\epsilon^i)'Q^i + (\beta^{ij})'T^{ij} + \frac{2}{k}\beta^{ik}T^{ij}T^{jk}.$$

Computing the Poisson brackets of the generators $\bar{L}$, Q^i and T^{ij}, one comes upon the extended superconformal algebras described in [47]. These algebras close quadratically in the $SO(N)$-currents T^{ij}, except for $N = 2$ and $N = 4$ (with boundary conditions breaking $SO(4)$ to one of its $SU(2)$ subgroups) for which one recovers the linear algebras of [48]. The non-linear extension appearing in these asymptotic symmetry algebras is due to the quadratic term in T^{ij} in equation (3.58) which is totally absorbed in the redefinition of the Virasoro generator L only in the case $N = 2$ and disappears when $N = 4$ through the breaking of one $SU(2)$. By a redefinition of the spinorials fields, the $N = 4$ asymptotic symmetry superalgebra is shown to be the same as the one arising from the asymptotic analysis of $SU(1,1|2)$ AdS$_3$ supergravity. The hamiltonian reduction of the WZW models corresponding to the supergroups studied in this section has been analyzed in [49] and a detailed derivation of the super-Liouville theory at infinity through the Gauss decomposition has been presented in [45].

3.7 Conclusion

We have shown in this chapter that the anti-de Sitter boundary conditions in $(1,1)$ three-dimensional supergravity lead to an asymptotic symmetry algebra which is twice the super-Virasoro algebra with a central charge equal to $3l/2G$. The precise boundary conditions given on the spinors involve a chiral projection on the boundary and legitimate the assumptions of [34]. The theory which realizes this symmetry on the boundary at infinity has been shown to be super-Liouville theory.

The appearance of the Virasoro algebra as the boundary symmetry algebra of anti-de Sitter space-time is purely kinematical in the sense that the only ingredients that enter the derivation of both the asymptotic symmetry algebra and its central charge are, on the one hand, the asymptotic boundary conditions that dictate the approach to anti-de Sitter and, on the other hand, the fact that the surface terms at infinity in the Virasoro generators involve only the (bosonic) gravitational variables, i.e. the dreibein and the spin connection. Any theory with these features will have the same central charge in the commutator involving two L_n's. It is, for instance, the case of all the supergravity models with extended supersymmetry, whose asymptotic symmetries and dynamics have been studied in [45]. Actually, this result applies to any gravity theory which admits an anti-de Sitter solution. In particular, the Virasoro algebra appears in the space-time symmetry algebra arising from string propagation on AdS_3 times a compact space [50]. The asymptotic central charge received a microscopic derivation in the study of the system of $D1$- and $D5$-branes [30], which is dual to the above string theory in view of the AdS/CFT correspondence conjectured by Maldacena [4].

Chapter 4

Anti-de Sitter gravity and classical boundary degrees of freedom

The conjectured equivalence, in the string theory approach to quantum gravity, between supergravity in D-dimensional anti-de Sitter space-time and some quantum conformal field theory living on its d-dimensional boundary has been briefly described in Chapter 2 [4, 5, 6]. It has been viewed as a manifestation of the holographic principle for quantum degrees of freedom [1, 2, 3]. In this chapter, we show that there are classical degrees of freedom on the conformal boundary of AdS and we discuss their relation with the Fefferman-Graham description of gravity with a negative cosmological constant. These degrees of freedom generate all AdS Schwarzschild and Kerr black holes. We verify that the boundary action that constitutes the finite part of the gravitational action is conformal invariant for odd-dimensional space-time boundaries and presents the well known Weyl anomaly for even ones [33]. We derive here this anomaly for general asymptotically anti-de Sitter space-times from local transformations of the boundary, avoiding ambiguities arising from global transformations. This Weyl anomaly was interpreted in [33], through the above equivalence, as the anomaly of the corresponding conformal quantum field theory. We show here that it could be viewed as the classical anomaly of a d-dimensional theory which would describe the boundary degrees of freedom of anti-de Sitter gravity. These considerations are applied explicitly to the case of AdS_3 gravity, whose asymptotic dynamics is known to be described by Liouville theory [8]. In higher dimensions, we construct an action exhibiting the same particular behaviour under Weyl transformations.

The chapter is organized as follows. In section 4.1, the boundary of AdS is defined and it is shown how bulk diffeomorphisms generate Weyl transformations in d dimensions on the boundary of asymptotically AdS space-times. In section 4.2, by analysing the expansion of the Einstein equations near the boundary, it is shown how classical boundary degrees of freedom are encoded in the coefficient of order d of the Fefferman-Graham expansion of the metric and how these degrees of freedom generate all AdS Schwarzschild and Kerr black holes. In section 4.3, the variation of the gravitational action under diffeomorphisms is computed, showing that its finite part is conformally invariant when d is odd and presents a Weyl anomaly when d is even. In section 4.4, we show, when $d = 2$, 3 and 4, that the dynamical equations for the degrees freedom hidden in the Fefferman-Graham coefficient are summarized into the conservation of a rank-two tensor on the boundary. This tensor is shown to have the same trace and conformal properties as the energy-momentum tensor of any d-dimensional action which has the same classical Weyl anomaly as the finite part of the gravitational action. Section 4.5 is devoted to the explicit treatment of the above considerations in the case of AdS_3 gravity, showing that the Fefferman-Graham coefficient describes, on the equations of motion, the components of the Liouville energy-momentum tensor. The asymptotic symmetry algebra presented in the previous chapter is recovered. In section 4.6, we propose a method to construct, in all dimensions, an action with the same classical Weyl anomaly as that computed in section 4.3. The last section contains a general discussion on the above results, examining if the description of the boundary degrees of freedom of AdS_3 gravity by a local field on the boundary could be extended to the cases of higher-dimensional boundaries.

This chapter describes the results of [14] (see also [13]), although in a more detailed and extended version, particularly in sections 4.2 and 4.4, where some issues about the logarithmic term present in the expansion of the metric in the case of an even-dimensional boundary are clarified. Section 4.5 constitutes an extended version of [13]. Finally, section 4.6 contains results which were developed in collaboration with F. Englert.

4.1 Bulk diffeomorphisms and Weyl transformations on the boundary

This section is devoted to the study of some geometrical properties of anti-de Sitter space-time which induce the particular behaviour of its boundary under conformal transformations. We will see how the boundary of anti-de Sitter space-time can be identified with compactified Minkowski space-time and how this boundary is mapped into itself by the transformations of the $AdS_{D=d+1}$ isometry group $O(2,d)$ [6]. This group acts on the boundary as the conformal group in d dimensions. A coordinate system will be introduced in which the anti-de Sitter metric is shown to be singular at spatial infinity. This property will lead to the definition of a conformal class of boundary metrics subject to a Weyl equivalence relation. In this coordinate system, we will see how bulk diffeomorphisms generate transformations belonging to the extended conformal group on the boundary, which contains the reparametrization group and the Weyl group as subgroups [51].

4.1.1 Anti-de Sitter boundary

We recall from Chapter 2 that anti-de Sitter space-time AdS_D in $D = d + 1$ dimensions can be constructed as a hyperboloid of radius l embedded in a $(d + 2)$-dimensional flat space with two time-coordinates. The metric of the embedding space is:

$$ds^2 = dX_1^2 + \ldots + dX_d^2 - dX_0^2 - dX_{-1}^2. \tag{4.1}$$

The equation satisfied by the points of the hyperboloid is the following:

$$X_1^2 + \ldots + X_d^2 - X_0^2 - X_{-1}^2 = -l^2. \tag{4.2}$$

Clearly, its isometry group is $O(2, d)$. This space-time contains closed timelike curves, so one replaces it by its universal cover.

We will see now how Minkowski space-time appears at the boundary of anti-de Sitter. Sending all the coordinates to infinity while preserving equation (4.2), one may take as boundary of the hyperboloid the solution of equation (4.2) with $l = 0$:

$$X_1^2 + \ldots + X_d^2 - X_0^2 - X_{-1}^2 = 0, \tag{4.3}$$

and with the coordinates subject to an overall scaling equivalence under a non zero constant factor. Equation (4.3) expresses the fact that the hyperboloid meets the asymptotic cone at the boundary. Together with the overall scaling equivalence relation, this equation describes the so-called projective cone.

In fact, the projective cone taken as the boundary of anti-de Sitter space-time is the compactified Minkowski space-time. Indeed, this can be shown [6] by introducing the coordinates $U = X_{-1} + X_d$ and $x_i = X_i/U$ for $U \neq 0$. Solving equation (4.2) for $V = X_{-1} - X_d$, the induced metric on the hyperboloid is:

$$ds^2 = l^2 \frac{dU^2}{U^2} + U^2(dx_1^2 + \ldots + dx_{d-1}^2 - dx_0^2). \tag{4.4}$$

Going to the boundary, or equivalently setting $l = 0$ together with the scaling equivalence as in equation (4.3), the metric of the asymptotic projective cone is then given by:

$$ds^2 = U^2(dx_1^2 + \ldots + dx_{d-1}^2 - dx_0^2), \tag{4.5}$$

which can be reduced to the Minkowskian metric by using the scaling relation to set $U = 1$. The compactification of Minkowski space-time is obtained by adding the points at infinity corresponding to $U = 0$.

The isometry group $O(2,d)$ of anti-de Sitter also preserves the cone given by equation (4.3) and therefore induces a transformation of the space corresponding to (4.1) that maps compactified Minkowski space-time to itself. These isometries leave the metric (4.5) invariant and are realized in compactified Minkowski space-time by the conformal group in d dimensions. It is straightforward to check that the rotations $O(1,d-1)$ leave U invariant, while the dilatations, the $2d$ translations and special conformal transformations require a compensating transformation on U. For $d=2$, the invariance of (4.5) under the conformal transformations is larger than $O(2,2)$ and is generated by two copies of the Virasoro algebra. Let us note that the compactification of Minkowski space-time by the addition of the points at infinity is essential to perform conformal transformations. Indeed a conformal transformation can map an ordinary point to infinity and is therefore not defined on Minkowski space-time itself.

Spaces which, at spacelike infinity, can be described locally by the metric given in equation (4.5) will be called asymptotically anti-de Sitter space-times. This definition has however to be made precise by specifying the limiting procedure. To this effect, it is useful to enlarge the set of boundary geometries to the class of the d-dimensional space-times described by metrics which differ from (4.5) by a Weyl transformation and to introduce a Weyl equivalence relation between them. Up to reparametrizations in d dimensions, this conformal class of boundary metrics is then invariant under the extended conformal group, which contains as subgroups the reparametrization group and the Weyl group [51].

We now introduce a particular coordinate system in which the Weyl equivalence relation among the boundary metrics shows up clearly. It will be shown in the next subsection how, in this coordinate system, the extended conformal group can be generated by bulk diffeomorphisms. Writing $U^2 = y^{-1}$, the metric (4.4) takes the more general Fefferman-Graham form [52]:

$$ds^2 = g_{yy}dy^2 + g_{ij}dx^i dx^j = \frac{l^2 dy^2}{4y^2} + \frac{1}{y}\bar{g}_{ij}(x,y)dx^i dx^j, \tag{4.6}$$

where here the d-dimensional metric $\bar{g}_{ij}(x,y)$ is simply the y-independent Minkowskian metric η_{ij}. The boundary is defined by the limit $y \to 0$. Clearly, the induced metric on the boundary $g_{ij}(x, y=0)$ is singular and it is because of this feature that the boundary metric of anti-de Sitter space-time is identified to the Minkowskian metric only up to a Weyl rescaling. Indeed, this Weyl scaling equivalence can be understood by introducing a so-called defining function $f(x,y)$, taken to be positive and with a first order zero on the boundary $y=0$. The boundary metric corresponding to (4.6) is then defined as the value on the boundary of the product $f(x,y)g_{ij}(x,y)$. As the defining function $f(x,y)$ is arbitrary, only a conformal equivalence class of boundary metrics can be defined. The Minkowskian metric $\bar{g}_{ij}(x, y=0)$ is the boundary metric corresponding to the particular choice of defining function $f(x,y) \equiv y$ and constitutes only a favoured representative of the conformal equivalence class of boundary metrics.

From here, let us assume that $\tilde{g}_{ij}(x,y)$ in equation (4.6) can be any metric which tends, when $y \to 0$, to a regular metric $\tilde{g}_{(0)ij}(x)$. It will be shown in the next section how, by theorems due to Fefferman and Graham, this generalized form of the metric (4.6) is shared by the solutions of the Einstein equations with a negative cosmological constant. As mentioned above, the metric $\tilde{g}_{(0)ij}(x)$ constitutes only a representative of the conformal equivalence class of boundary metrics but it will be referred to as the boundary metric.

4.1.2 Diffeomorphisms and Weyl transformations

We will now see how Weyl transformations on the boundary can be generated by bulk diffeomorphisms, having in mind that a Weyl transformation of $\tilde{g}_{(0)ij}(x)$ preserves the conformal equivalence class of boundary metrics. Moreover the diffeomorphisms considered will be shown to be isomorphic to the extended conformal group, for which we recall that the reparametrization group and the Weyl group are subgroups [51]. Let us consider the diffeomorphisms which keep the $(d+1)$-dimensional metric in the Fefferman-Graham form (4.6). They are characterized by the vanishing of the Lie derivatives of g_{yy} and g_{yi}. They are given by [13, 53]:

$$
\begin{aligned}
\delta y &= -2\sigma(x)y, && (4.7)\\
\delta x^i &= \frac{l^2}{2}\int_0^y \tilde{g}^{ij}(x,y')dy'\ \partial_j\sigma(x) + \chi^i(x), && (4.8)
\end{aligned}
$$

with $\sigma(x)$ and $\chi^i(x)$ arbitrary. By computing the Lie derivative of $g_{ij}(x,y)$, it is easily shown that it induces on $\tilde{g}_{(0)ij}(x)$, up to a reparametrization generated by $\chi^i(x)$, the following Weyl transformation:

$$\delta_W \tilde{g}_{(0)ij}(x) = 2\sigma(x)\tilde{g}_{(0)ij}(x). \tag{4.9}$$

Hence, these diffeomorphisms act on the boundary metric $\tilde{g}_{(0)ij}$ as the extended conformal group.

Under the assumption that the metric $\tilde{g}_{ij}(x,y)$ can be expanded in a power series in y:

$$\tilde{g}_{ij}(x,y) = \sum_n \tilde{g}_{(2n)ij}(x)y^n,$$

the expression of the Lie derivative of g_{ij} under the diffeomorphism (4.7)-(4.8) prescribes a certain transformation law for each term in the expansion. The transformation of $\tilde{g}_{ij}(x,y)$ is given by:

$$\delta_W \tilde{g}_{ij}(x,y) = 2\sigma(x)\tilde{g}_{ij}(x,y) - y2\sigma(x)\partial_y\tilde{g}_{ij}(x,y) + \partial_k\tilde{g}_{ij}(x,y)\delta x^k(x,y)$$

$$+\tilde{g}_{ik}(x,y)\partial_j\delta x^k(x,y)+\tilde{g}_{jk}(x,y)\partial_i\delta x^k(x,y). \tag{4.10}$$

The first terms of its expansion transform as [13, 53]:

$$\begin{aligned}
\delta_W\tilde{g}_{(2)ij}(x) &= l^2 D_i\partial_j\sigma(x),\\
\delta_W\tilde{g}_{(4)ij}(x) &= -2\sigma(x)\tilde{g}_{(4)ij}(x)+\frac{l^2}{4}\tilde{g}_{(2)}{}^k{}_j D_i\partial_k\sigma(x)+\frac{l^2}{4}\tilde{g}_{(2)}{}^k{}_i D_j\partial_k\sigma(x)\\
&\quad+\frac{l^2}{4}\left(2D^k\tilde{g}_{(2)ij}(x)-D_i\tilde{g}_{(2)}{}^k{}_j+D_j\tilde{g}_{(2)}{}^k{}_i\right)\partial_k\sigma(x),
\end{aligned} \tag{4.11}$$

where indices are raised with the inverse of the boundary metric $\tilde{g}_{(0)ij}(x)$ and D_i is the covariant derivative in the same metric. These are the transformation laws prescribed for the coefficients in the expansion of $\tilde{g}_{ij}$, while the Weyl transformation (4.9) is performed on the boundary metric. We will see in the next section that the hypothesis that the metric $\tilde{g}_{ij}$ has a power series expansion in a neighbourhood of $y = 0$ is correct for the solutions of the Einstein equations. However, for d even and greater than two, a logarithmic term must be added in the expansion in y of $\tilde{g}_{ij}$. We will show in section 4.4 how this term contributes to the Weyl transformation of $\tilde{g}_{(d)}$ and how, according to this contribution, the Weyl transformation of $\tilde{g}_{(4)}$ given in equation (4.11) is slightly modified when $d=4$. Some of the coefficients $\tilde{g}_{(2n)}$'s will be shown to be expressible in terms of the metric $\tilde{g}_{(0)}$ and in that case, by a simple scaling argument, $\tilde{g}_{(2n)}$ contains $2n$ derivatives with respect to the x variables, hence this notation.

4.2 The Fefferman-Graham ambiguity : classical boundary degrees of freedom

In the previous section, while constructing the d-dimensional metric that describes the boundary geometry of a space-time characterized by the $(d+1)$-dimensional metric given in equation (4.6), we have seen that only a conformal equivalence class of boundary metrics can be defined, for which $\tilde{g}_{(0)ij}(x)$ constitutes a distinctive representative. In particular, the boundary of anti-de Sitter space-time is described by the class of conformally flat metrics.

Now we will see that, given a general boundary metric $\tilde{g}_{(0)ij}(x)$, it is possible to reconstruct bulk metrics of the type of equation (4.6) which satisfy in a neighbourhood of $y=0$ the Einstein equations for pure gravity with a negative cosmological constant $\Lambda = -d(d-1)/2l^2$. However, this recontruction is not unique and yields, in the case of conformally flat $\tilde{g}_{(0)ij}(x)$, distinct solutions in addition to the anti-de Sitter one. In particular, this variety of solutions will be shown to include the AdS Schwarzschild and Kerr black holes solutions. We will study in this section how the Einstein equations constrain the metric $\tilde{g}_{ij}(x,y)$ in equation (4.6). We will see that the equations for $\tilde{g}_{ij}(x,y)$, when expanded in a power series expansion in y, reduce to algebraic equations for the coefficients of the expansion, apart from the traceless part of one of them which has to satisfy a differential equation. This coefficient will be shown to describe the classical degrees of freedom on the conformal boundary of AdS. The specific form of the dynamical equations satisfied by them will be examined in section 4.4.

4.2.1 Expansion of the Einstein equations near the boundary

The Einstein equations for pure gravity with a negative cosmological constant are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0,$$

with $\Lambda = -d(d-1)/2l^2$. Solving for R by taking the trace of this tensor, one obtains:

$$R_{\mu\nu} + \frac{d}{l^2}g_{\mu\nu} = 0.$$

In the metric of equation (4.6), these equations provide the following equations for the metric $\tilde{g}_{ij}(x,y)$ [55, 56]:

$$y[2\tilde{g}'' - 2\tilde{g}'\tilde{g}^{-1}\tilde{g}' + \mathrm{Tr}(\tilde{g}^{-1}\tilde{g}')\tilde{g}'] - l^2\mathrm{Ricci}(\tilde{g}) - (d-2)\tilde{g}' - \mathrm{Tr}(\tilde{g}^{-1}\tilde{g}')\tilde{g} = 0, \quad (4.12)$$

$$D_i h^i{}_j - \partial_j h^i{}_i = 0, \quad (4.13)$$

$$\mathrm{Tr}(\tilde{g}^{-1}\tilde{g}'') - \tfrac{1}{2}\mathrm{Tr}(\tilde{g}^{-1}\tilde{g}'\tilde{g}^{-1}\tilde{g}') = 0, \quad (4.14)$$

where $'$ denotes partial derivative with respect to y and:

$$h^i{}_j = \tilde{g}^{ik}\tilde{g}'_{kj}.$$

Multiplying equation (4.12) on the left by $\tilde{g}^{-1}$, it is expressed in terms of $h^i{}_j$ as:

$$l^2\tilde{R}^i{}_j + (d-2)h^i{}_j + h^k{}_k\delta^i{}_j - y(2\partial_y h^i{}_j + h^k{}_k h^i{}_j) = 0.$$

The y-lapse constraint, which will be useful in the next section:

$$G^y{}_y + \Lambda = -\frac{1}{2}(R^{ij}{}_{ij} - 2\Lambda) = 0, \quad (4.15)$$

is a combination of the trace of equation (4.12) and of equation (4.14) and takes the form:

$$l^2\tilde{R} + 2(d-1)h^i{}_i + y(h^i{}_j h^j{}_i - h^i{}_i h^j{}_j) = 0.$$

The y-shift constraints are equivalent to equation (4.13). Equations (4.12)-(4.14) can be solved iteratively by expanding $\tilde{g}_{ij}(x,y)$ in a power series expansion in y and one will see how there exist different bulk solutions $\tilde{g}_{ij}(x,y)$ corresponding to the same $\tilde{g}_{(0)ij}(x)$.

Equation (4.12) prescribes the general form of the expansion of $\tilde{g}_{ij}(x,y)$ and provides algebraic recurrence relations for most of its coefficients [52, 57]. The generic term in the expansion is:

$$\tilde{g}_{(2n)}(x)y^n,$$

where n is an integer. When d is odd, there is an additional term:

$$\tilde{g}^t_{(d)}(x)y^{d/2},$$

where the coefficient $\tilde{g}^t_{(d)}$ is traceless with respect to $\tilde{g}_{(0)}$, namely $\mathrm{Tr}(\tilde{g}^{-1}_{(0)}\tilde{g}^t_{(d)}) = 0$. When d is even and greater than two, there is, in addition of the $\tilde{g}_{(d)}(x)y^{d/2}$ term, a logarithmic term:

$$\tilde{k}_{(d)}(x)y^{d/2}\ln y,$$

where $\tilde{k}_{(d)}$ is traceless with respect to $\tilde{g}_{(0)}$. The coefficients $\tilde{g}_{(2n)}(x)$ are obtained algebraically in terms of $\tilde{g}_{(0)}(x)$ up to $n=(d-1)/2$ for d odd and $n=d/2-1$ for d even. For d odd, the traceless coefficients $\tilde{g}^t_{(d)}(x)$ is algebraically undetermined. For d even, while the coefficient $\tilde{k}_{(d)}(x)$ is still determined by $\tilde{g}_{(0)}(x)$, the traceless part $\tilde{g}^t_{(d)}(x)$ of $\tilde{g}_{(d)}(x)$ is algebraically undetermined. The higher order coefficients are all algebraically expressed in terms of $\tilde{g}_{(0)}(x)$ and of the undetermined traceless coefficient $\tilde{g}^t_{(d)}(x)$.

The two lowest order algebraic expressions for the coefficients $\tilde{g}_{(2n)}$ have been computed explicitly from the expansion of equation (4.12), using the expansion of the Ricci tensor computed from equation (A.4) in Appendix A. They are given by:

$$\begin{aligned}
\tilde{g}_{(2)ij} &= -\frac{l^2}{d-2}\left[\tilde{R}_{ij} - \frac{1}{2(d-1)}\tilde{g}_{(0)ij}\tilde{R}\right], \\
\tilde{g}_{(4)ij} &= \frac{l^4}{d-4}\left\{\frac{1}{4(d-2)}\left[\frac{1}{d-2}\tilde{g}_{(0)ij}\tilde{R}^{kl}\tilde{R}_{kl} - \frac{3d}{4(d-1)^2(d-2)}\tilde{g}_{(0)ij}\tilde{R}^2\right.\right. \\
&\quad -\frac{4}{d-2}\tilde{R}^k{}_i\tilde{R}_{kj} + \frac{4}{(d-1)(d-2)}\tilde{R}\tilde{R}_{ij} + D_kD_i\tilde{R}^k{}_j + D_kD_j\tilde{R}^k{}_i \\
&\quad \left.\left.\frac{1}{2(d-1)}\tilde{g}_{(0)ij}\Box\tilde{R} - \Box\tilde{R}_{ij} - \frac{d}{2(d-1)}D_i\partial_j\tilde{R}\right] - \frac{d-8}{2l^4}\tilde{k}_{(4)ij}\right\} \qquad (4.16)
\end{aligned}$$

with non zero $\tilde{k}_{(4)ij}$ when $d=4$ only. These expressions without the logarithmic coefficient $\tilde{k}_{(4)ij}$ have been given in [53]. The curvature tensors appearing in these expressions and in what follows are the ones for the metric $\tilde{g}_{(0)}$, D_i is its covariant derivative and indices are raised with the same metric. The indeterminacy of (the traceless part of) $\tilde{g}_{(d)}$ appears clearly in the above equation for $d=2$, as the coefficient of $1/(d-2)$ vanishes identically. When $d=4$, $\tilde{g}_{(4)ij}$ would be singular in the absence of the $\tilde{k}_{(4)ij}$ term, hence the necessity to introduce the logarithmic term in the expansion of $\tilde{g}_{ij}(x,y)$ in order to have a regular expansion [54]. The equation for $\tilde{g}_{(4)ij}$ is non singular only if the algebraic expression of $\tilde{k}_{(4)ij}$ in terms of the boundary metric $\tilde{g}_{(0)ij}$ cancels the numerator of the right-hand side of equation (4.16). This condition enforces the following expression for $\tilde{k}_{(4)ij}$ in terms of the boundary metric $\tilde{g}_{(0)ij}$:

$$\begin{aligned} \tilde{k}_{(4)ij} &= -\frac{l^4}{16}\left[\frac{1}{2}\tilde{g}_{(0)ij}\left(\tilde{R}^{kl}\tilde{R}_{kl}-\frac{1}{3}\tilde{R}^2\right)-2\tilde{R}^k{}_i\tilde{R}_{kj}+\frac{2}{3}\tilde{R}\tilde{R}_{ij}\right. \\ &\quad \left.+D_kD_i\tilde{R}^k{}_j+D_kD_j\tilde{R}^k{}_i+\frac{1}{6}\tilde{g}_{(0)ij}\Box\tilde{R}-\Box\tilde{R}_{ij}-\frac{2}{3}D_i\partial_j\tilde{R}\right]. \end{aligned} \tag{4.17}$$

This expression could have been derived directly by considering the expansion of equation (4.12). The equation for (the traceless part of) $\tilde{g}_{(4)ij}$ is then algebraically undetermined when $d=4$.

The traces with respect to $\tilde{g}_{(0)}$ of the expressions given in equation (4.16) are algebraically determined in terms of the metric $\tilde{g}_{(0)}$. Indeed, recalling the tracelessness of $\tilde{k}_{(d)}$, these traces are given by:

$$\begin{aligned} \mathrm{Tr}(\tilde{g}_{(0)}^{-1}\tilde{g}_{(2)}) &= -\frac{l^2}{2(d-1)}\tilde{R}, \\ \mathrm{Tr}(\tilde{g}_{(0)}^{-1}\tilde{g}_{(4)}) &= \frac{l^4}{4(d-2)^2}\left[\tilde{R}^{ij}\tilde{R}_{ij}-\frac{3d-4}{4(d-1)^2}\tilde{R}^2\right], \end{aligned} \tag{4.18}$$

which are well defined also when $d=2$ and 4 respectively.

The indeterminacy appearing at the order $y^{d/2}$ in the expansion of $\tilde{g}_{ij}(x,y)$ corresponds to degrees of freedom not encoded in the boundary metric. We shall refer to these indeterminacies as the Fefferman-Graham ambiguity. A solution in a neighbourhood of $y=0$ with boundary metric $\tilde{g}_{(0)}$ is then determined by the specification of the traceless fields $\tilde{g}^t_{(d)}$: although algebraically undetermined, these fields must satisfy the differential equations obtained by expanding in y equation (4.13). The precise form of these equations will be investigated in section 4.4.

Notice that the expressions for $\tilde{g}_{(2)}$ and $\tilde{g}_{(4)}$ without the logarithmic coefficient $\tilde{k}_{(4)ij}$ given in equation (4.16) have been obtained in [53] without the use of the equations of motion: upon the hypothesis that the coefficients in the expansion of $\tilde{g}_{ij}(x,y)$ are all expressible in terms of $\tilde{g}_{(0)ij}(x)$, their exact expression has been computed by integrating their transformation law (4.11) under the Weyl transformation (4.9) of $\tilde{g}_{(0)}$. This computation indicates that geometry alone prescribes the algebraic expressions for $\tilde{g}_{(2n)}$ as well as the indeterminacy at the order $y^{d/2}$ and the necessity to introduce the logarithmic term $\tilde{k}_{(d)}$ when d is even and greater than two [54]. The transformation law of $\tilde{g}_{(d)}$ predicts the existence of degrees of freedom not encoded in the metric on the boundary since it cannot be reproduced by any well defined local function of $\tilde{g}_{(0)}$. This indicates that the existence of degrees of freedom is not restricted to the use of the Einstein equations. Now, giving up locality, the Weyl transformation rule of $\tilde{g}_{(d)}$ corrected by the contribution of $\tilde{k}_{(d)}$ can be reproduced by a unique and well-defined function of $\tilde{g}_{(0)}$. Indeed, it has been shown in [54], where a cohomological problem corresponding to the Weyl transformation properties of the Fefferman-Graham coefficients has been studied, that the cohomology of the Weyl transformation of $\tilde{g}_{(d)}$ admits a non-trivial solution corresponding to a non-local expression of $\tilde{g}_{(0)}$. A non-ambiguous non-local expression for $\tilde{g}_{(d)}$ can then be written. However, we will see in section 4.6 that this expression can be related to a local one by the introduction of a further degree of freedom, which restores the above ambiguity.

Boundary conditions can be constructed by putting the coefficients of the expansion of $\tilde{g}_{ij}(x,y)$ that can be expressed algebraically in terms of $\tilde{g}_{(0)}$ to their value in terms of $\tilde{g}_{(0)}$. We therefore define as asymptotically anti-de Sitter space-time $\mathcal{G}$ any space-time which can be parametrized in a neighbourhood of $y=0$ by a Fefferman-Graham metric (4.6) and such that $\tilde{g}_{(0)}$ is conformally flat and that the following boundary fields:

$$\begin{array}{ll} \tilde{g}_{(2)}(x), \tilde{g}_{(4)}(x), \ldots, \tilde{g}_{(d-1)}(x) & d \text{ odd}, \\ \tilde{g}_{(2)}(x), \tilde{g}_{(4)}(x), \ldots, \tilde{g}_{(d-2)}(x), \tilde{k}_{(d)}(x), \mathrm{Tr}(\tilde{g}_{(0)}^{-1}\tilde{g}_{(d)}(x) & d \text{ even}, \end{array} \tag{4.19}$$

are expressed algebraically in terms of $\tilde{g}_{(0)}(x)$ in accordance with equation (4.12). As mentioned above, the work of reference [53] shows that these boundary conditions are in fact a direct consequence of the transformation of $\tilde{g}_{ij}(x,y)$ given in equation (4.10) under the diffeomorphism (4.7) and (4.8).

4.2.2 The Fefferman-Graham ambiguity and AdS black holes

We now see how the degrees of freedom encoded in the fields $\tilde{g}^t_{(d)}$ generate all AdS Schwarzschild and Kerr black holes in all dimensions $D = d+1$. The metric of AdS Schwarzschild black holes of mass M is:

$$ds^2 = -\left(1 - \frac{\lambda}{r^{d-2}} + \frac{r^2}{l^2}\right) dt^2 + \left(1 - \frac{\lambda}{r^{d-2}} + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 \Omega^2_{d-1}, \tag{4.20}$$

where $\lambda = \nu_D G_D M$. G_D is the gravitational constant in D dimensions and ν_D is a D-dependent numerical coefficient. When $r \to \infty$, the relation between r and the variable y in the metric (4.6) is $dr/r \to -dy/2y$ or equivalently by a suitable choice of the integration constant $r/l \to y^{-1/2}$. Hence the leading order in y of the mass term in the coefficient of dt^2 is of order $y^{d/2}$. As the mass term has no counterpart in the pure AdS_{d+1} geometry for which $\tilde{g}_{ij}(x,y) \equiv \tilde{g}_{(0)ij}(x)$, its leading contribution to the expansion of $\tilde{g}_{ij}(x,y)$ is the traceless quantity $\tilde{g}^t_{(d)}(x)$ which is here a constant independent of x.

The tracelessness of the mass term can be verified explicitly. The relation between r and y is defined by:

$$\left(1 - \frac{\lambda}{r^{d-2}} + \frac{r^2}{l^2}\right)^{-\frac{1}{2}} dr = -\frac{l dy}{2y}.$$

Setting, for $d > 2$, $2r/l = \xi - 1/\xi$, one obtains, up to order ξ^{-d}:

$$\ln\xi - \frac{2^{d-1}}{l^{d-2}\lambda^2 d}\xi^{-d} = -\frac{1}{2}\ln\frac{y}{4}.$$

Expressing this equality to the same order in terms of r, one gets:

$$\frac{r}{l} = \frac{1}{y^{\frac{1}{2}}}\left(1 + \frac{1}{4}y + \frac{\lambda^2}{2l^{d-2}\lambda^2 d}y^{\frac{d}{2}}\right),$$

from which one easily verifies that $\mathrm{Tr}(\bar{g}_{(0)}^{-1}\bar{g}_{(d)})$ vanishes. For $d = 2$, the exact relation between r and y is obtained along similar lines by the change of variable $2r/l = \xi + 1/\xi$ and the vanishing of $\mathrm{Tr}(\bar{g}_{(0)}^{-1}\bar{g}_{(2)})$ follows.

These conclusions can be extended to Kerr black holes. The dependence on r and hence the leading order in y of the mass term in equation (4.20) follows from dimensional considerations: $G_D M$ has dimension $D - 3 = d - 2$ and its contribution to g_{tt} cannot depend on l. Similarly the angular momentum J will give a term proportional to $G_D J$ in $g_{t\varphi}$ of the same dimensionality. This is the leading dependence on the angular momentum in the Kerr metric and is again encoded in the traceless field $\bar{g}^t_{(d)}(x)$.

4.3 Weyl anomaly of the gravitational action

In this section, we compute the variation of the gravitational action with a negative cosmological constant under diffeomorphisms that do not vanish on the boundary. After removing divergent counterterms, the remaining part of the action is shown to be finite when the boundary is taken to spatial infinity. The divergent terms are shown to be independent of the Fefferman-Graham fields $\bar{g}^t_{(d)}$ introduced in the previous section and the "renormalized" action constitutes an effective action for these boundary degrees of freedom. Computing the variation of the gravitational action for the particular diffeomorphisms which induce Weyl transformations on the boundary, it is shown that the effective action presents the well known classical Weyl anomaly when the dimension of the boundary is even and is Weyl invariant when it is odd [55]. This Weyl anomaly is related to the presence of a logarithmically divergent term in the original action when d is even [6].

4.3.1 Variation of the gravitational action under bulk diffeomorphisms

The Einstein-Hilbert action in $D = d + 1$ dimensions with a negative cosmological constant $\Lambda = -d(d-1)/2l^2$ and with suitable boundary term is:

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}\left(R + \frac{d(d-1)}{l^2}\right) d^{d+1}x + I, \tag{4.21}$$

$$I = -\frac{1}{8\pi G}\int_{\mathcal{S}}\sqrt{-g^{(d)}}(K-c)d^dx, \tag{4.22}$$

where $\mathcal{S}$ is a $y=\bar{y}$ d-dimensional boundary with topology $S_{d-1}\times R$. The surface term I is introduced as usual to render the action stationary for solutions of the Einstein equations with fixed fields on the boundary $\mathcal{S}$ [58]. The notation $g^{(d)}$ is for the determinant of the d-dimensional metric g_{ij} of equation (4.6) and $K = K^i{}_i$ where K_{ij} is the extrinsic curvature tensor in the same metric. The constant c is introduced for convenience and will be fixed later. As the variation of S is sensitive only to diffeomorphisms that do no vanish on the boundary $\mathcal{S}$, we perform a diffeomorphim in D dimensions defined by an infinitesimal displacement field $\xi^\mu(x,y)$ which vanishes outside a neighbourhood of the surface $y=0$ which contains the surface $\mathcal{S}$.

To compute the corresponding variation of the action (4.21), we first write the surface term (4.22) in covariant form by introducing the D-vectors n_μ normal to the surface element $d\Sigma_\mu$ on $\mathcal{S}$. By embedding the displaced surface $\mathcal{S}$ under the diffeomorphism in a family of surfaces characterized by infinitesimal displacements $\alpha\xi^\mu$ where α varies from 0 to 1, one defines a field n_μ normal to the surfaces $\alpha =$ constant. The extrinsic curvature scalar can be written as a D-dimensional scalar:

$$K \equiv -n^\mu{}_{;\mu}.$$

The surface element $d\Sigma_\mu$ is written as:

$$d\Sigma_\mu = \frac{1}{d!}\epsilon_{\mu\lambda_1\ldots\lambda_d}dx^{\lambda_1}\wedge\ldots\wedge dx^{\lambda_d},$$

and one has:

$$\sqrt{-g^{(d)}}d^dx = \sqrt{-g}n^\mu d\Sigma_\mu.$$

The surface term (4.22) becomes:

$$I = \frac{1}{8\pi G}\int_{\mathcal{S}}\sqrt{-g}(n^\lambda{}_{;\lambda}+c)n^\mu d\Sigma_\mu.$$

It is then written as a volume integral of a D-divergence, which gives:

$$I = \frac{1}{8\pi G}\int_{\mathcal{M}}\sqrt{-g}[(n^\lambda{}_{;\lambda}+c)n^\mu]_{;\mu}d^{d+1}x,$$

such that the variation of the action is obtained by taking the Lie derivative of integrands of volume terms only. We get:

$$\delta_\xi S = \frac{1}{16\pi G}\int_{\mathcal{S}}\sqrt{-g}\left\{R+\frac{d(d-1)}{l^2}+2[(n^\lambda{}_{;\lambda}+c)n^\mu]_{;\mu}\right\}\xi^\nu d\Sigma_\nu. \tag{4.23}$$

In this expression, we substitute the following identity:

$$(n^\lambda{}_{;\lambda}n^\mu)_{;\mu} = n^\lambda{}_{;\lambda}n^\mu{}_{;\mu} - n^\lambda{}_{;\mu}n^\mu{}_{;\lambda} - R_{\lambda\mu}n^\mu n^\lambda + (n^\mu n^\lambda{}_{;\mu})_{;\lambda},$$

where it is easily checked that:

$$n^\lambda{}_{;\lambda}n^\mu{}_{;\mu} = K^i{}_i K^j{}_j, \qquad n^\lambda{}_{;\mu}n^\mu{}_{;\lambda} = K^i{}_j K^j{}_i, \qquad R_{\lambda\mu}n^\mu n^\lambda = R^{yi}{}_{yi},$$

and the last term gives no contribution on $\mathcal{S}$. Using the Gauss-Codazzi equation:

$$R^i{}_{jkl} = {}^{(d)}R^i{}_{jkl} - (K^i{}_k K_{jl} - K^i{}_l K_{jk}), \tag{4.24}$$

where ${}^{(d)}R^i{}_{jkl}$ is the curvature tensor in the same metric g_{ij} as before, one finds the following relation between the curvature scalars in D and d dimensions:

$$R = R^{(d)} + 2R^{yi}{}_{yi} - (K^i{}_i K^j{}_j - K^i{}_j K^j{}_i).$$

Using the identity $\sqrt{-g}\xi^\nu d\Sigma_\nu = \sqrt{-g^{(d)}}\xi d^d x$ with $\xi = \xi^\mu n_\mu$ and the above ones, equation (4.23) becomes:

$$\delta_\xi S = \frac{1}{16\pi G}\int_{\mathcal{S}} \sqrt{-g^{(d)}}\left(R^{(d)} + \frac{d(d-1)}{l^2} + K^i{}_i K^j{}_j - K^i{}_j K^j{}_i - 2cK\right)\xi d^d x.$$

Inserting the Gauss-Codazzi equation (4.24) into the y-lapse constraint (4.15):

$$R^{ij}{}_{ij} + \frac{d(d-1)}{l^2} \equiv R^{(d)} + \frac{d(d+1)}{l^2} - (K^i{}_i K^j{}_j - K^i{}_j K^j{}_i) = 0,$$

we get:

$$\delta_\xi S = \frac{1}{8\pi G}\int_{\mathcal{S}} \sqrt{-g^{(d)}}\left(R^{(d)} + \frac{d(d-1)}{l^2} - cK\right)\xi d^d x. \tag{4.25}$$

Writing the identity (4.24) in terms of the metric $\tilde{g}_{ij}$ of equation (4.6), one obtains:

$$K^i{}_j = \frac{y}{l}h^i{}_j - \frac{1}{l}\delta^i{}_j,$$

recalling that $h^i{}_j = \tilde{g}^{ik}\partial_y \tilde{g}_{kj}$. Taking $c = -(d-1)/l$ and using:

$$\sqrt{-g^{(d)}} = \sqrt{-\tilde{g}}y^{-d/2}, \qquad R^{(d)} = y\tilde{R}, \qquad \xi = -\frac{l}{2y}\delta y,$$

equation (4.25) can be written:

$$\delta_\xi S = -\frac{l}{16\pi G}\int_{\mathcal{S}} \sqrt{-\tilde{g}}\left(\tilde{R} + \frac{d-1}{l^2}h^i{}_i\right)\frac{\delta\bar{y}}{\bar{y}^{d/2}}d^d x. \tag{4.26}$$

This equation is similar to the equation obtained in reference [56] for the variation of the Einstein-Hilbert action but there are two noticeable differences. First, our result is valid for arbitrary local variations $\delta\bar{y}$ around the surface $y = \bar{y}$. Second, as shown below, in the limit $\bar{y} \to 0$, equation (4.26) does not require the evaluation of the action (4.21) on a solution of the Einstein equations but only on an arbitrary asymptotically anti-de Sitter space-time $\mathcal{G}$.

4.3.2 Expansion of the gravitational action and classical Weyl anomaly of its finite part

We study the behaviour in $\bar{y}$ of equation (4.26) by expanding the coefficient of $\delta\bar{y}$ in power series of $\bar{y}$. It is divergent when $\bar{y} \to 0$ and takes the form:

$$\begin{aligned} \delta_\xi S &= \int \left[\frac{A_{d/2}}{\bar{y}^{d/2}} + \ldots + \frac{A_{3/2}}{\bar{y}^{3/2}} + \mathcal{O}\left(\frac{1}{\bar{y}^{1/2}}\right)\right] \delta\bar{y}\; d^d x, \qquad d \text{ odd}, \\ \delta_\xi S &= \int \left[\frac{A_{d/2}}{\bar{y}^{d/2}} + \ldots + \frac{A_1}{\bar{y}} + \mathcal{O}(1)\right] \delta\bar{y}\; d^d x, \qquad d \text{ even}, \end{aligned} \tag{4.27}$$

where the integrals are carried out on the surface $y = \bar{y}$. The higher order terms that are not explicitly written in these expressions derive from terms in the expansion of the action that vanish when $\bar{y}$ goes to zero and need not be considered here. The terms retained in equation (4.27) arise from order less than $\bar{y}^{d/2}$ in the expansion of $\sqrt{-\bar{g}}$, $\bar{R}$ and $h^i{}_i$, and potentially from terms in $h^i{}_i$ (which contains a $\bar{y}$-derivative) of order $\bar{y}^{d/2}$ or $\bar{y}^{d/2} \ln \bar{y}$ when d is even in the expansion of the metric $\bar{g}_{ij}$. However, due to tracelessness, there is no contribution containing $\bar{g}^t_{(d)}$. There is neither contribution from the logarithmic term when d is even for the same reason. The divergent terms that multiply $\delta\bar{y}$ in equation (4.27) depend then only on the fields appearing in the boundary conditions given in equation (4.19). For an asymptotically anti-de Sitter space-time $\mathcal{G}$, these fields are defined as algebraic expressions of $\bar{g}_{(0)}$ for which the y-lapse constraint (4.15) used in the derivation of (4.26) is identically satisfied. Thus, as announced, when $\bar{y}$ tends to zero, the computed variation (4.26) is valid for all asymptotically anti-de Sitter space-times $\mathcal{G}$. The divergent terms in equation (4.27) have no dynamical content.

Notice that, in the case of a boundary $\mathcal{S}$ staying at $y = \bar{y}$ with $\bar{y}$ a non vanishing constant, the variation of the action (4.21) under diffeomorphism arises from the variation δy normal to the boundary surface. Indeed equation (4.26) vanishes if δy is zero on the surface $y = \bar{y}$, i.e. if the normal part of the displacement field $\xi^\mu(x, y)$ defining the diffeomorphism vanishes on the boundary surface $\mathcal{S}$.

We denote by $S(\mathcal{G})$ the action S evaluated on a space $\mathcal{G}$. The variation given in equation (4.27) which is valid for such a space can be integrated. Its integration with respect to the functional variation $\delta\bar{y}(x)$ gives:

$$\begin{aligned} S(\mathcal{G}) &= \int \left[\frac{B_{d/2-1}}{\bar{y}^{d/2-1}} + \ldots + \frac{B_{1/2}}{\bar{y}^{1/2}} + B_c(\mathcal{G}) + \mathcal{O}\left(\bar{y}^{1/2}\right)\right] d^d x, \quad d \text{ odd}, \\ S(\mathcal{G}) &= \int \left[\frac{B_{d/2-1}}{\bar{y}^{d/2-1}} + \ldots + \frac{B_1}{\bar{y}} + B_0 \ln \bar{y} + B_c(\mathcal{G}) + \mathcal{O}(\bar{y})\right] d^d x, \quad d \text{ even}. \end{aligned} \tag{4.28}$$

The higher order terms that are not explicitly written vanish in the limit $\bar{y} \to 0$. Comparing these equations with equation (4.27) for a local variation $\delta\bar{y}$, one obtains the following identifications among the coefficients:

$$\begin{aligned} B_n &= -\frac{1}{n}A_{n+1}, \qquad \text{for} \qquad n = 1, \ldots, d/2-1, \\ B_0 &= A_1, \end{aligned} \tag{4.29}$$

where the coefficients A_n are given by algebraic expressions of the metric $\tilde{g}_{(0)}$ deduced from the expansion of equation (4.26). The $B_c(\mathcal{G})$ term is independent of $\bar{y}$ and finite. It corresponds to the arbitrary integration "constant" with respect to $\bar{y}$. From equation (4.29), one learns that the divergent terms in equation (4.28) are surface integrals of local functions of $\tilde{g}_{(0)}$ only. The fields $\tilde{g}^t_{(d)}$ enter only $B_c(\mathcal{G})$ which contains therefore all the dynamics. After substracting the divergent terms and taking the limit $\bar{y} \to 0$, one may view $\int B_c(\mathcal{G})d^dx$ as an effective boundary action for these dynamical degrees of freedom and writes $\int B_c(\mathcal{G})d^dx = S_{fin}(\mathcal{G})$. Its behaviour under Weyl transformations will be analyzed in what follows, using the particular diffeomorphisms of section 4.1.

Let us recall that there are two ways to compute the quantity $\delta_\xi S$ given in equation (4.27) which is the variation of the action $S(\mathcal{G})$ given in equation (4.28) for a local variation $\delta\bar{y}$ of the boundary surface $y = \bar{y}$. One way is to compare the action with boundary at $y = \bar{y}$ with the action with boundary at $y = \bar{y} + \delta\bar{y}$ (where in our case $\bar{y}$ is constant whether $\delta\bar{y}$ depends on $\bar{y}$ and on the x-variables that parametrize the surface $y = \bar{y}$). This is how the integration of equation (4.27) has been carried out to find equation (4.28). The other way of computing the variation is to keep fixed the parameters space on which the integrals appearing in the action are performed i.e. the boundary surface remains described by the equation $y = \bar{y}$. In that case, it is the fields on which depend the integrands that are varied according to their Lie derivative under the diffeomorphism whose normal part is characterized by $\xi^y = \delta y$. Of course, these two methods give the same result and, for the particular diffeomorphism of section 4.1, this provides a way to compute the Weyl variation of $S_{fin}(\mathcal{G})$.

Hence, we now apply this to the diffeomorphism described by equations (4.7) and (4.8) to obtain information on the behaviour of the surface integrals $\int B_n d^dx$ and $S_{fin}(\mathcal{G})$ under Weyl transformations. Indeed, we have shown in section 4.1 that the Lie derivative of the metric $g_{ij}(x,y)$ for the diffeomorphism (4.7) and (4.8):

$$\begin{aligned} \delta y &= -2\sigma(x)y, \\ \delta x^i &= \frac{l^2}{2}\int_0^y \tilde{g}^{ij}(x,y')dy'\ \partial_j\sigma(x) + \chi^i(x), \end{aligned}$$

induces on $\tilde{g}_{(0)}$ the Weyl transformation given in equation (4.9). The transformation rule of $\tilde{g}_{(d)}$ which enters the integrand $B_c(\mathcal{G})$ of the finite term is prescribed by the expansion of the Lie derivative of $g_{ij}(x,y)$ and is given in equation (4.11) for $d = 2$ and 4, to be corrected by adding the contribution of the $\tilde{k}_{(4)}$ term. The variation of $S(\mathcal{G})$ induced by these Weyl transformation rules is then equal to its variation under $\delta\bar{y} = -2\sigma(x)\bar{y}$. Computing the Lie derivatives of the coefficients B_n and B_c in equation (4.28) for the diffeomorphism (4.7) and (4.8) and comparing with equation (4.27) with $\delta\bar{y}$ replaced by $-2\sigma(x)\bar{y}$, one finds, with the use of equation (4.29):

$$\delta_W \int B_n \, d^dx = 2n \int B_n\sigma \, d^dx, \tag{4.30}$$

$$\delta_W \int B_0 \, d^dx = 0, \tag{4.31}$$

$$\delta_W S_{fin}(\mathcal{G}) = \begin{cases} 0 & d \text{ odd} \\ -2\int B_0\sigma \, d^dx & d \text{ even.} \end{cases} \tag{4.32}$$

One sees in equation (4.32) that the Weyl variation of the $\bar{y}$-independent term $S_{fin}(\mathcal{G})$ of $S(\mathcal{G})$ is proportional to the coefficient of the logarithmic term in $S(\mathcal{G})$ and $S_{fin}(\mathcal{G})$ presents therefore a classical Weyl anomaly when d is even. This is due to the fact that, for the particular variation $\delta\bar{y} = -2\sigma(x)\bar{y}$, the variation of $\ln\bar{y}$ is independent of $\bar{y}$ and is therefore finite in the limit $\bar{y} \to 0$. Notice that if we integrate equation (4.27) for the Weyl transformations induced by the Lie derivatives, we find equation (4.28) up to Weyl invariant terms, i.e. up to the logarithmic term when d is even and up to $S_{fin}(\mathcal{G})$ when d is odd, according to equations (4.31) and (4.32).

We now analyse equations (4.30)-(4.32) in more details.

The Weyl transformation (4.30) of the coefficients of negative powers of $\bar{y}$ contributes to the variation given in equation (4.27) by the same quantity as when they are varied with respect to $\bar{y}$ by $\delta\bar{y} = -2\sigma\bar{y}$. Removing these divergences will then have no effect on the equality between the Weyl variation of the remaining terms (the logarithmically divergent term if present and the finite term) and their variation under $\delta\bar{y} = -2\sigma(x)\bar{y}$. As a check of equation (4.30), we give here the variation of the B_1-term in four dimensions. Using equations (4.27) and (4.29), we see that B_1 is equal to minus the zero order coefficient of the expansion of the coefficient of $\delta\bar{y}/\bar{y}^2$ in equation (4.26), i.e.:

$$B_1 = \frac{l}{32\pi G}\sqrt{-\bar{g}_{(0)}}\bar{R},$$

where here and in what follows $\bar{R}$ is written for the curvature in the metric $\bar{g}_{(0)ij}$. The variation of B_1 under the Weyl transformation (4.9) is the following:

$$\delta_W B_1 = \frac{l}{16\pi G}\sqrt{-\bar{g}_{(0)}}(\bar{R} - 3\Box\sigma),$$

which is equal to $2B_1\sigma$ up to a total derivative as stated in equation (4.30).

Equation (4.32) expresses that the boundary effective action $S_{fin}(\mathcal{G})$ is Weyl invariant when d is odd and presents, when d is even, the well known Weyl anomaly $\mathcal{A}_d$ [6, 55]

defined through:

$$\delta_W S_{fin}(\mathcal{G}) = \int \sqrt{-\tilde{g}_{(0)}} \mathcal{A}_d \; \sigma \; d^d x.$$

It is equal to:

$$\sqrt{-\tilde{g}_{(0)}} \mathcal{A}_d = -2B_0, \tag{4.33}$$

where B_0, according to equations (4.27) and (4.29), is the coefficient of $\delta\bar{y}/\bar{y}$ in the expansion of the integrand of equation (4.26). Let us emphasize again that the Weyl anomaly of $S_{fin}(\mathcal{G})$ is related to the presence of the logarithmic term in (4.28) for d even [6]. Indeed, after the subtraction of the terms in negative powers of $\bar{y}$, the $\delta\bar{y}$-variation of the sum of the finite term $S_{fin}(\mathcal{G})$ and of the logarithmic term (if present) is still equal to their Weyl variation. It is independent of $\bar{y}$ and the Weyl variation of $S_{fin}(\mathcal{G})$ is thus equal to the $\delta\bar{y}$-variation of the logarithmic term (or to zero when this term is absent). The Weyl anomaly of the "renormalized" action $S_{fin}(\mathcal{G})$ can therefore be interpreted as the variation of the logarithmically divergent term under the variation $\delta\bar{y}$ around the boundary surface $y = \bar{y}$, i.e. the $\bar{y}$-independent term in the expansion of the action $S(\mathcal{G})$ is sensitive to the way in which the limit $\bar{y} \to 0$ is taken.

The absence of logarithmic divergence in the variation of the gravitational action given in equation (4.27) also shows that the coefficient of the logarithmic term in the action (which is proportional to the Weyl anomaly) is Weyl invariant. Indeed, putting together equations (4.31) and (4.33) gives the following condition on the Weyl anomaly:

$$\delta_W \int \sqrt{-\tilde{g}_{(0)}} \mathcal{A}_d \; d^d x = 0. \tag{4.34}$$

This relation reproduces the general classification of trace anomalies in any even dimensions [59, 60]. Indeed, there are two ways of solving equation (4.34): $\mathcal{A}_d$ is either a total derivative or a Weyl invariant up to a total derivative. In the first case, $\mathcal{A}_d$ is proportional to a topological invariant and the only available parity-even candidate is the Euler density. It is referred to as the type A anomaly. The case where $\mathcal{A}_d$ is Weyl invariant is called a type B anomaly and can be constructed out of products of contractions of the Weyl tensor and their covariant derivatives. There is also a third type of anomaly which is called trivial in the sense that it can be removed by local counterterms depending only on the boundary metric [60].

4.3.3 Weyl anomaly in two and four dimensions

The precise form of the Weyl anomaly of $S_{fin}(\mathcal{G})$, depending only on the boundary metric, is computed, according to equation (4.33), by looking at the coefficient of σ in the finite part of the variation $\delta_\xi S$ given in equation (4.26) for the particular $\delta\bar{y} = -2\sigma\bar{y}$.

When $d = 2$, this gives [61]:

$$\mathcal{A}_2 = \frac{3l}{2G}\frac{\tilde{R}}{24\pi}, \tag{4.35}$$

which corresponds to the type A anomaly. This value of the anomaly corresponds to the central charge $c = 3l/2G$ of the asymptotic symmetry algebra discovered by Brown and Henneaux [7] and reviewed in Chapter 3. The precise link between the Weyl anomaly and the central charge of the Virasoro algebra will be examined in section 4.5 where the $d = 2$ case will be studied in detail.

In the $d = 4$ case, the Weyl anomaly of $S_{fin}(\mathcal{G})$ derived from equation (4.26) is the following:

$$\mathcal{A}_4 = \frac{l^3}{8\pi G}\left(\frac{1}{8}\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{24}\tilde{R}^2\right). \tag{4.36}$$

This expression can be decomposed into the sum of a type A anomaly $\mathcal{A}_4^A$ and a type B one $\mathcal{A}_4^B$ as follows [55]:

$$\begin{aligned}
\mathcal{A}_4^A &= -\frac{l^3}{8\pi G}\frac{1}{16}\left(\tilde{R}^{ijkl}\tilde{R}_{ijkl} - 4\tilde{R}^{ij}\tilde{R}_{ij} + \tilde{R}^2\right), \\
\mathcal{A}_4^B &= \frac{l^3}{8\pi G}\frac{1}{16}\left(\tilde{R}^{ijkl}\tilde{R}_{ijkl} - 2\tilde{R}^{ij}\tilde{R}_{ij} + \frac{1}{3}\tilde{R}^2\right),
\end{aligned}$$

where $\mathcal{A}_4^A$ is the Euler density in four dimensions and $\mathcal{A}_4^B$ is the Weyl tensor squared.

We have shown in this section that the finite part $S_{fin}(\mathcal{G})$ of the action (4.21) for asymptotically anti-de Sitter space-times $\mathcal{G}$ contains all the dynamics encoded in $\tilde{g}^t_{(d)}$. It is Weyl invariant when d is odd and presents the well known Weyl anomaly when d is even. Notice that even if the anomaly is a local expression of $\tilde{g}_{(0)}$, $S_{fin}(\mathcal{G})$ may be non-local. However we will see in the next section, by analyzing the dynamical equations for $\tilde{g}^t_{(d)}$ in $d = 2$, 3 and 4, that these fields could be related to the energy-momentum tensor of some boundary local fields Φ.

4.4 Conserved tensor on the boundary

4.4.1 Dynamical equations for the boundary degrees of freedom

We now examine the form of the equations of motion for $\tilde{g}_{(d)}$, whose traceless part $\tilde{g}^t_{(d)}$ encodes the degrees of freedom on the boundary of asymptotically anti-de Sitter space-times. These equations arise from the expansion in y of the Einstein equations (4.13), which correspond to the y-shift constraints and are given by:

$$D_i h^i{}_j - \partial_j h^i{}_i = 0,$$

where we recall that:

$$h^i{}_j = \tilde{g}^{ik}\partial_y \tilde{g}_{kj}.$$

The algebraic recurrence relations coming from equation (4.12) give already information on the form of the expansion of $D_i h^i{}_j - \partial_j h^i{}_i$ in a power series in y. Indeed all terms of order less than $y^{d/2-1}$ must vanish identically because they contain only coefficients $\tilde{g}_{(2n)}(x)$ which are determined algebraically in terms of $\tilde{g}_{(0)}(x)$. Consequently the term in $y^{d/2-1}$, which gives the dynamical equation for $\tilde{g}_{(d)}$, has the property to reduce to an identity for all higher dimensional boundaries when its explicit dependence on the dimension is taken into account. For example, the order zero in y of equation (4.13), with the use of equation (4.18) on the trace of $\tilde{g}_{(2)}$, gives:

$$D_i \left[\tilde{g}_{(2)}{}^i{}_j + \frac{l^2}{2(d-1)} \delta^i{}_j \tilde{R} \right] = 0. \tag{4.37}$$

This provides when $d = 2$ the equations of motion for $\tilde{g}_{(2)}$, while for $d > 2$, using the expression of $\tilde{g}_{(2)}$ in terms of $\tilde{g}_{(0)}$ given in equation (4.16), it reduces to the Bianchi identity:

$$-\frac{l^2}{d-2} D_i \left(\tilde{R}^i{}_j - \frac{1}{2}\delta^i{}_j \tilde{R} \right) \equiv 0.$$

Notice that the structure of the equations of motion has been studied in [54] in a cohomological approach.

In general, the term in $y^{d/2-1}$ in the expansion of equation (4.13) takes the form:

$$D_i \tilde{g}_{(d)}{}^i{}_j + \psi_j[\tilde{g}_{(0)}] = 0. \tag{4.38}$$

Apart from the terms coming from the power series expansion of $\tilde{g}^{ik}(x, y)$ and from the coefficient $\tilde{k}_{(d)}(x)$ of the logarithmic term when d is even and greater than two, the quantities $\psi_i[\tilde{g}_{(0)}]$ arise from the expansion of the Christoffel symbols in the covariant derivative with respect to the metric $\tilde{g}_{ij}(x, y)$ in (4.13). These quantities contain then only terms determined in terms of derivatives of $\tilde{g}_{(0)}(x)$, hence expressible in terms of curvature terms of the boundary metric. They depend explicitly on d. We show for the cases $d = 2$, 3 and 4 that they can be brought into a covariant derivative of the form $D_i \xi_{(d)}{}^i{}_j$. This characteristic allows to express the dynamical equations for $\tilde{g}_{(d)}$ given in equation (4.38) as the conservation of a rank-two tensor.

When $d = 2$, the equations of motion for $\tilde{g}_{(2)}$ have been given in equation (4.37) and satisfy this property. For $d = 3$, we take the derivative of equation (4.13) with respect to $y^{1/2}$. The Christoffel symbols are not affected at this order and taking into account the vanishing of the trace $\tilde{g}_{(3)}{}^i{}_i$, we simply get:

$$D_i \tilde{g}_{(3)}{}^i{}_j = 0. \tag{4.39}$$

For $d=4$, we take into account the first two order terms coming from the expansion of the Christoffel symbols in the covariant derivative and obtain:

$$D_i \left\{ \tilde{g}_{(4)}{}^i{}_j + \frac{l^4}{16} \left[\frac{1}{2}\delta^i{}_j \left(\tilde{R}^{kl}\tilde{R}_{kl} - \frac{5}{9}\tilde{R}^2 \right) + \tilde{R}\tilde{R}^i{}_j - 2\tilde{R}^{ik}\tilde{R}_{kj} \right] \right\} = 0. \tag{4.40}$$

The terms of order y in the expansion of equation (4.13) prescribes the addition of $(1/2)D_i\tilde{k}_{(4)}{}^i{}_j$ in this equation but this term cancels identically in agreement with the terms in $y \ln y$ of the same equation and with the fact that $\tilde{k}_{(d)}$ is an algebraic expression of $\tilde{g}_{(0)}$.

The form of equations (4.37), (4.39) and (4.40) can be summarized in:

$$D_i \left(\tilde{g}_{(d)}{}^i{}_j + \xi_{(d)}{}^i{}_j \right) = 0, \tag{4.41}$$

where $\xi_{(d)ij}$ is constructed out of curvature tensors of $\tilde{g}_{(0)ij}$. From the conservation law (4.41), we may define a conserved tensor:

$$\mathcal{T}_{(d)ij} = \alpha_d \left(\tilde{g}_{(d)ij} + \xi_{(d)ij} \right), \tag{4.42}$$

where α_d is a numerical coefficient. The traces of those tensors are equal to:

$$\begin{aligned} \mathcal{T}_{(2)}{}^i{}_i &= \alpha_2 \frac{l^2}{2}\tilde{R}, \\ \mathcal{T}_{(3)}{}^i{}_i &= 0, \\ \mathcal{T}_{(4)}{}^i{}_i &= \alpha_4 \frac{l^4}{16}\left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2 \right), \end{aligned} \tag{4.43}$$

and depend only on the boundary metric $\tilde{g}_{(0)ij}$. Provided the constants α_2 and α_4 are set to the following values:

$$\alpha_2 = \frac{1}{8\pi G l}, \qquad \alpha_4 = \frac{1}{4\pi G l},$$

these traces reproduce the gravitational Weyl anomalies $\mathcal{A}_d$ given in equations (4.35) and (4.36) for $d=2$ and 4 and its vanishing for $d=3$.

Note that any traceless and conserved quantity could be added to the right-hand side of equation (4.42) without changing the properties of the tensor $\mathcal{T}_{(d)ij}$. In particular and as mentioned above, this is the case of the logarithmic coefficient $\tilde{k}_{(4)ij}$ in $d=4$.

Remark that in flat space the equations of motion given in equation (4.38) already express for all d the conservation of the quantity $\tilde{g}_{(d)ij}$ because then $\psi_i[\tilde{g}_{(0)}]$ vanishes. The extension of this conservation equation to the case of general boundary metric $\tilde{g}_{(0)}$ relies on the precise form of $\psi_i[\tilde{g}_{(0)}]$ and does not involve the dynamical fields $\tilde{g}^t_{(d)}(x)$.

A conserved tensor of the type of equation (4.42) could then possibly be constructed in higher dimensions (see [62] for all d odd and for $d = 6$).

These results suggest that, for all d, the degrees of freedom hidden in the Fefferman-Graham ambiguity $\tilde{g}^t_{(d)}(x)$ can be expressed in terms of a conserved tensor and that this tensor is the energy-momentum tensor of some local boundary fields. Its trace would, on the equations of motion, be equal to the gravitational anomaly $\mathcal{A}_d$.

Indeed we recall that the above properties of the tensor $\mathcal{T}_{(d)ij}$, which were proven for $d = 2$, 3 and 4, are those of the energy-momentum tensor $T_{(d)ij}$ of any action whose classical Weyl anomaly is $\mathcal{A}_d$. Consider such a d-dimensional action $S[\tilde{g}_{(0)ij}, \Phi]$ of some boundary fields Φ in the background boundary metric $\tilde{g}_{(0)ij}$. Its energy-momentum tensor $T_{(d)ij}$ is defined by:

$$\frac{1}{2}\sqrt{-\tilde{g}_{(0)}}T^{ij}_{(d)} = \frac{\delta S[\tilde{g}_{(0)ij}, \Phi]}{\delta \tilde{g}_{(0)ij}}. \tag{4.44}$$

The equality between the trace of the energy-momentum tensor on the equations of motion and the Weyl anomaly of the action under the Weyl transformation (4.9) is expressed in the following relation:

$$\begin{aligned} \frac{\delta S[\tilde{g}_{(0)ij}, \Phi]}{\delta \tilde{g}_{(0)ij}}\delta_W \tilde{g}_{(0)ij} + \frac{\delta S[\tilde{g}_{(0)ij}, \Phi]}{\delta \Phi}\delta_W \Phi &= \sqrt{-\tilde{g}_{(0)}}T_{(d)}{}^i{}_i\, \sigma + \frac{\delta S[\tilde{g}_{(0)ij}, \Phi]}{\delta \Phi}\delta_W \Phi \\ &= \begin{cases} \sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d\, \sigma & d \text{ even} \\ 0 & d \text{ odd}, \end{cases} \end{aligned} \tag{4.45}$$

where $\delta_W \Phi$ denotes the variation of the fields Φ under the Weyl transformation (4.9).

We emphasize that, on the equations of motion, $T_{(d)ij}$ and $\mathcal{T}_{(d)ij}$ have the same trace (at least, in the cases of $d = 2$, 3 and 4 where it was computed) and are both conserved. The conservation of $T_{(d)ij}$ proceeds from the invariance of $S[\tilde{g}_{(0)ij}, \Phi]$ under d-dimensional reparametrization, while that of $\mathcal{T}_{(d)ij}$ is imposed by the Einstein equations through equation (4.38) at least for $d = 2$, 3 and 4. Notice that the conservation of $T_{(d)ij}$, as well as the value of its trace, do not rely on the precise form of the equations of motion of the fields Φ.

4.4.2 Weyl transformation of the boundary conserved tensor

We now show that the tensor $\mathcal{T}_{(d)ij}$ of equation (4.42) and the energy-momentum tensor $T_{(d)ij}$ not only obey the same conservation law and have the same trace. They also vary in the same way under Weyl transformations, at least when $d = 2$ and 4.

The Weyl transformation of the d-dimensional energy-momentum tensor (4.44) is completely determined by the Weyl anomaly $\mathcal{A}_d$ of the action from which it is derived.

Indeed, applying δ_W to equation (4.44) and commuting, in its right-hand side, δ_W and the functional derivative with respect to $\tilde{g}^{ij}_{(0)}$ gives, with the use of equation (4.45):

$$\delta_W T_{(d)ij} = -(d-2)\sigma T_{(d)ij} - 2\frac{1}{\sqrt{-\tilde{g}_{(0)}}}\frac{\delta}{\delta \tilde{g}^{ij}_{(0)}}\left(\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d\sigma\right), \qquad (4.46)$$

under the assumption that the transformation rule $\delta_W\Phi$ for the fields Φ on which depends the action $S[\tilde{g}_{(0)ij}, \Phi]$ is independent of $\tilde{g}_{(0)ij}$ and its derivatives. In the right-hand term of equation (4.46), the first term arises from the contribution of the determinant of the metric and from the partial derivative of $\delta_W\tilde{g}_{(0)ij}$ with respect to $\tilde{g}^{ij}_{(0)}$. The second term comes from the exact commutation of δ_W and the functional derivative with respect to $\tilde{g}^{ij}_{(0)}$. One sees that in the absence of the Weyl anomaly, which is the case of odd-dimensional theories, the energy-momentum tensor transforms homogeneously according to its conformal weight which is equal to $-(d-2)$. This homogeneous term arises from the Weyl transformation of $\sqrt{-\tilde{g}_{(0)}}$ and from the partial derivative of $\delta_W\tilde{g}_{(0)ij}$ with respect to $\tilde{g}^{ij}_{(0)}$.

In two dimensions, the value of the gravitational Weyl anomaly is given in equation (4.35). It is equal to:

$$\mathcal{A}_2 = \alpha_2\frac{l^2}{2}\tilde{R},$$

with $\alpha_2 = 1/8\pi Gl$. The conformal weight of $T_{(2)ij}$ is zero and the anomalous part of the transformation (4.46) gives (see equation (A.6) in Appendix A) [62]:

$$\delta_W T_{(2)ij} = \alpha_2 l^2\left(D_i\partial_j\sigma - \tilde{g}_{(0)ij}\Box\sigma\right). \qquad (4.47)$$

In four dimensions, the Weyl anomaly to be considered is given by equation (4.36) as:

$$\mathcal{A}_4 = \alpha_4\frac{l^4}{16}\left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2\right),$$

with $\alpha_4 = 1/4\pi Gl$. The transformation rule of the energy-momentum tensor dictated by equation (4.46) is then (see equation (A.7) in Appendix A) [62]:

$$\begin{aligned}\delta_W T_{(4)ij} \;=\;& -2\sigma T_{(4)ij} + \alpha_4\frac{l^4}{8}\left\{\left[\frac{1}{2}\tilde{g}_{(0)ij}\left(\tilde{R}^{kl}\tilde{R}_{kl} - \frac{1}{3}\tilde{R}^2\right) - 2\tilde{R}^k{}_i\tilde{R}_{kj} + \frac{2}{3}\tilde{R}\tilde{R}_{ij}\right]\sigma\right.\\ &+D_kD_i\left(\tilde{R}^k{}_j\sigma\right) + D_kD_j\left(\tilde{R}^k{}_i\sigma\right) - \tilde{g}_{(0)ij}D_kD_l\left(\tilde{R}^{kl}\sigma\right)\\ &\left.-\Box\left(\tilde{R}_{ij}\sigma\right) - \frac{2}{3}D_i\partial_j\left(\tilde{R}\sigma\right) + \frac{2}{3}\tilde{g}_{(0)ij}\Box\left(\tilde{R}\sigma\right)\right\}. \qquad (4.48)\end{aligned}$$

We now compare the above expressions for $d = 2$ and 4 with the transformation of $\mathcal{T}_{(d)ij}$ generated by the variation of $\tilde{g}_{(0)ij}$ and $\tilde{g}_{(d)ij}$ prescribed by the Lie derivative of

$\tilde{g}_{ij}(x,y)$ given in equation (4.10), for which $\tilde{g}_{(0)ij}$ undergoes the Weyl transformation (4.9).

When $d=2$, the expression of $\mathcal{T}_{(2)ij}$ is read from equations (4.37) and (4.42) as:

$$\mathcal{T}_{(2)ij} = \alpha_2 \left(\tilde{g}_{(2)ij} + \frac{l^2}{2} \tilde{g}_{(0)ij} \tilde{R} \right),$$

with $\alpha_2 = 1/8\pi Gl$. It is easily checked that under the Weyl transformation of $\tilde{g}_{(0)ij}$ given in equation (4.9) and for the transformation rule of $\tilde{g}_{(2)ij}$ given in equation (4.11), the transformation of $\mathcal{T}_{(2)ij}$ reproduces exactly the transformation of the two-dimensional energy-momentum tensor given in equation (4.47).

When $d=4$, the matching of the variation of $\mathcal{T}_{(4)ij}$ under the diffeomorphism given in equations (4.7) and (4.8) and the Weyl transformation of the four-dimensional energy-momentum tensor given in equation (4.48) is a little more subtle. Indeed, looking at the order $y^{d/2}$ of the expansion of equation (4.10), one sees that the logarithmic term $\tilde{k}_{(d)ij} y^{d/2} \ln y$, which appears in the expansion of $\tilde{g}_{ij}(x,y)$ for d even and greater than two, contributes to the variation $\delta_W \tilde{g}_{(d)ij}$ by a term of the form [54]:

$$-2\sigma \tilde{k}_{(d)ij}. \tag{4.49}$$

In the particular case of $d=4$, the remaining contribution to $\delta_W \tilde{g}_{(4)ij}$ coming from the coefficient $\tilde{g}_{(4)ij}$ was given in equation (4.11). The expression of $\mathcal{T}_{(4)ij}$ is read from equations (4.40) and (4.42) as:

$$\mathcal{T}_{(4)ij} = \alpha_4 \left\{ \tilde{g}_{(4)ij} + \frac{l^4}{16} \left[\frac{1}{2} \tilde{g}_{(0)ij} \left(\tilde{R}^{kl} \tilde{R}_{kl} - \frac{5}{9} \tilde{R}^2 \right) + \tilde{R} \tilde{R}_{ij} - 2 \tilde{R}_{ik} \tilde{R}^k{}_j \right] \right\},$$

with $\alpha_4 = 1/4\pi Gl$. Replacing in equation (4.11) $\tilde{g}_{(2)ij}$ by its expression (4.16) in terms of $\tilde{g}_{(0)ij}$ and using the transformation rule of $\tilde{g}_{(4)ij}$ given in equation (4.11) and corrected by the contribution of $\tilde{k}_{(4)ij}$ as in equation (4.49), one obtains for the variation of $\mathcal{T}_{(4)ij}$ under the diffeomorphism that induces on $\tilde{g}_{(0)ij}$ the Weyl transformation (4.9) the following result:

$$\begin{aligned}
\delta_W \mathcal{T}_{(4)ij} &= -2\sigma \mathcal{T}_{(4)ij} + \alpha_4 \frac{l^4}{8} \Big[D_i \left(\tilde{R}^k{}_j \partial_k \sigma \right) + D_j \left(\tilde{R}^k{}_i \partial_k \sigma \right) - \tilde{g}_{(0)ij} \tilde{R}^{kl} D_k \partial_l \sigma \\
&\quad -2 D^k \tilde{R}_{ij} \partial_k \sigma - \tilde{R}_{ij} \Box \sigma - \frac{1}{6} \partial_i R \partial_j \sigma - \frac{1}{6} \partial_j R \partial_i \sigma - \frac{2}{3} D_i \partial_j \sigma \\
&\quad \frac{2}{3} \tilde{g}_{(0)ij} \tilde{R} \Box \sigma + \frac{1}{3} \tilde{g}_{(0)ij} \partial^k \tilde{R} \partial_k \sigma \Big] - 2\sigma \alpha_4 \tilde{k}_{(4)ij}.
\end{aligned}$$

Replacing $\tilde{k}_{(4)ij}$ by its value given in equation (4.17), this variation reproduces exactly the Weyl transformation (4.48) of the four-dimensional energy-momentum tensor derived from any action whose Weyl anomaly is equal to $\mathcal{A}_4$ and for which the Weyl transformation rules of the field Φ on which it depends are independent of the background metric $\tilde{g}_{(0)ij}$.

Let us mention the fact that the coefficient $\tilde{k}_{(d)}$ of the logarithmic term can be expressed in terms of the Weyl anomaly [54], at least for the cases where we computed it explicitly, namely when $d = 2$, 3 and 4. We write:

$$\tilde{k}_{(d)ij} = \frac{1}{\alpha_d} \frac{1}{\sqrt{-\tilde{g}_{(0)}}} \frac{\delta}{\delta \tilde{g}^{ij}_{(0)}} \left(\sqrt{-\tilde{g}_{(0)}} \mathcal{A}_d \right), \tag{4.50}$$

such that, according to equation (4.49), the contribution of this term to the variation of $\mathcal{T}_{(d)ij}$ is given by:

$$-2\sigma \frac{1}{\sqrt{-\tilde{g}_{(0)}}} \frac{\delta}{\delta \tilde{g}^{ij}_{(0)}} \left(\sqrt{-\tilde{g}_{(0)}} \mathcal{A}_d \right).$$

Through equation (4.50), the absence of logarithmic term in the expansion of $\tilde{g}_{ij}(x, y)$ in the odd-dimensional case appears to be related to the cancellation of the Weyl anomaly in that case. Notice that only the type B part of the Weyl anomaly $\mathcal{A}$ contributes to $\tilde{k}_{(d)ij}$ because the type A anomaly being a total derivative has vanishing Euler-Lagrange derivatives [54]. This explains the absence of logarithmic term in the expansion of $\tilde{g}_{ij}(x, y)$ in two dimensions where the Weyl anomaly is only of type A. The rightness of equation (4.50) is verified when $d = 4$ by comparing equation (4.17) of section 4.2 and equation (A.8) of Appendix A.

One should notice that any traceless and conserved tensor which transforms homogeneously with conformal weight $-(d-2)$ under a Weyl transformation could be added to $\mathcal{T}_{(d)ij}$ without changing its above properties. As mentioned before, this is the case of $\tilde{k}_{(d)ij}$, whose transformation rule is derived from equation (4.10) and is indeed given by [54]:

$$\delta_W \tilde{k}_{(d)ij} = -(d-2)\sigma \tilde{k}_{(d)ij}.$$

Looking at equation (4.46), one sees that it is also true for the energy-momentum tensor derived from any Weyl invariant action.

We recall here that it has been shown in [54] that the cohomological problem set up from the Weyl transformations properties of $\mathcal{T}_{(d)ij}$, while corrected by the contribution of the logarithmic coefficient $\tilde{k}_{(d)ij}$, has a non-trivial solution which depends on the boundary metric $\tilde{g}_{(0)ij}$ only. However this solution is obtained [54] by computing the energy-momentum tensor of the non-local effective action generating the Weyl anomaly [60]. We will see in section 4.6 how this non-local action can be rendered local by introducing a further field ϕ.

4.5 The $d = 2$ case

In this section, we recall and apply the results of the previous sections to the case of two-dimensional boundaries. For AdS_3 gravity, there is the well known result of Brown and Henneaux [7] according to which the asymptotic symmetry algebra is given by twice the Virasoro algebra with a central extension whose central charge is equal to $c = 3l/2G$. This result has been reviewed in Chapter 3 in the Chern-Simons formalism while giving its supersymmetric extension. The reduction of AdS_3 gravity to Liouville theory for asymptotically anti-de Sitter spacetimes has been shown in [8] in the case of flat boundary metrics. That reduction has been outlined in section 3.5 of the previous chapter, generalizing it to the reduction of AdS_3 supergravity to super-Liouville (see also [45]). The reduction in the case of curved boundary metrics has been treated in [63, 64]. Parts of these results will be recovered here, in the case of a general boundary metric which, being two-dimensional, is conformally flat.

4.5.1 Weyl anomaly and boundary degrees of freedom of AdS_3 gravity

Integrating the variation of the gravitational action under diffeomorphism, we found in section 4.3 that this action is, when $d = 2$, the sum of an action which is finite in the limit of a boundary staying at $\bar{y} = 0$ and of a logarithmically divergent term. Putting together equations (4.28) and (4.33), we write:

$$S(\mathcal{G}) = -\frac{1}{2}\int \sqrt{-\bar{g}_{(0)}}\mathcal{A}_2 \ln \bar{y}\; d^2x + S_{fin}(\mathcal{G}), \tag{4.51}$$

where, according to equation (4.35):

$$\mathcal{A}_2 = \frac{\alpha_2 l^2}{2}\bar{R}, \tag{4.52}$$

with $\alpha_2 = 1/8\pi G l$. We have shown that the diffeomorphism:

$$\begin{aligned} \delta y &= -2\sigma(x)y, \\ \delta x^i &= \frac{l^2}{2}\partial_j\sigma(x)\int_0^y \bar{g}^{ij}(x,y')dy', \end{aligned} \tag{4.53}$$

which was given in equations (4.7) and (4.8), induces the following Weyl transformation on the boundary:

$$\delta_W \bar{g}_{(0)ij}(x) = 2\sigma(x)\bar{g}_{(0)ij}(x).$$

According to this and because the logarithmically divergent term is Weyl invariant (or equivalently because the variation of $S(\mathcal{G})$ contains only finite terms), the quantity $\mathcal{A}_2$

which appears in the coefficient of the logarithmic term in $S(\mathcal{G})$ is precisely the Weyl anomaly of the finite boundary action $S_{fin}(\mathcal{G})$.

We now recall the results of sections 4.2 and 4.4 concerning the asymptotic solutions of the Einstein equations for AdS_3 gravity and some of their geometrical properties. We have seen in section 4.2 that geometry predicts, when $d = 2$, an indeterminacy at the order y of the expansion of the metric $\tilde{g}_{ij}(x, y)$ appearing in the Fefferman-Graham form of the metric, which was given in equation (4.6). Indeed the equation for $\tilde{g}_{(2)ij}$ obtained by integrating the Lie derivative of the $(2 + 1)$-dimensional metric under the diffeomorphism (4.53) has its traceless part undetermined, due to a factor of $(d-2)$ in the denominator while the numerator is a vanishing geometrical identity in two dimensions [53]. Notice that this is different than in higher even dimensions, where the contribution of the coefficient $\tilde{k}_{(d)}$ of the logarithmic term appearing in the expansion of $\tilde{g}_{ij}(x, y)$ is needed to remove a singularity and to obtain an indeterminacy. The coefficient $\tilde{k}_{(d)}$ has been shown to be equal to the Euler-Lagrange derivative of the Weyl anomaly with respect to the boundary metric $\tilde{g}_{(0)ij}$ and its absence in two dimensions is due to the fact that, in that case, the Weyl anomaly $\mathcal{A}_2$ is of type A only and has therefore vanishing Euler-Lagrange derivatives [54].

In section 4.2, we have seen how the indeterminacy for $\tilde{g}_{(2)}$ is found in the expansion of the Einstein equations, which provide algebraic recurrence relations for all the coefficients of the expansion of $\tilde{g}_{ij}(x, y)$ in terms of the lower order ones, besides for the traceless part of $\tilde{g}_{(2)}$ which remains indeterminate. This indeterminacy has been referred to as the Fefferman-Graham ambiguity and the traceless fields $\tilde{g}^t_{(2)}$ carry the gravitational boundary degrees of freedom not encoded in the boundary metric $\tilde{g}_{(0)}$. However, to provide a solution of the Einstein equations, $\tilde{g}_{(2)}$ must still satisfy a differential equation which concentrates all the dynamics of AdS_3 gravity. This equation can be expressed in the form of the conservation of a tensor $\mathcal{T}_{(2)ij}$ related to $\tilde{g}_{(2)ij}$ by the following expression (see equations (4.37) and (4.42) in section 4.4):

$$\mathcal{T}_{(2)ij} = \alpha_2 \left(\tilde{g}_{(2)ij} + \frac{l^2}{2} \tilde{g}_{(0)ij} \tilde{R} \right). \tag{4.54}$$

The trace of this tensor is equal to the Weyl anomaly $\mathcal{A}_2$ of $S_{fin}(\mathcal{G})$ given in equation (4.52), provided α_2 is fixed as before to $\alpha_2 = 1/8\pi Gl$. Its Weyl transformation rule is deduced from the Lie derivative of the coefficient $\tilde{g}_{(2)ij}$ under the diffeomorphism (4.53) and has been shown in the previous section to be given by:

$$\delta_W \mathcal{T}_{(2)ij} = \alpha_2 l^2 (D_i \partial_j \sigma - \tilde{g}_{(0)ij} \Box \sigma). \tag{4.55}$$

We recall from the previous section that the above properties of the tensor $\mathcal{T}_{(2)ij}$ (its conservation law, the value of its trace and the way it varies under Weyl transformations) are those of the energy-momentum tensor $T_{(2)ij}$ of any two-dimensional action with same classical Weyl anomaly $\mathcal{A}_2$ as $S_{fin}(\mathcal{G})$.

4.5.2 Gravitational degrees of freedom and Liouville theory

We will see now how the Liouville action $S_{(L)}$ constitutes such an action, i.e. it presents a classical Weyl anomaly equal to $\mathcal{A}_2$ and its energy-momentum tensor $T_{(L)ij}$ has therefore its trace equal to $\mathcal{A}_2$ on the equations of motion and has the same Weyl transformation rule as that presented in equation (4.55). Moreover it is conserved by covariance of the Liouville action. We will show later how the degrees of freedom carried by the Liouville field are sufficient to describe the asymptotic solutions of the Einstein equations.

The Liouville action is the following:

$$S_{(L)} = -\frac{\alpha_2 l^2}{2} \int \sqrt{-\tilde{g}_{(0)}} \left(\partial^i \phi \partial_i \phi + R\phi + \lambda e^{2\phi}\right) d^2x, \tag{4.56}$$

and its energy-momentum tensor is derived as:

$$T_{(L)ij} = \alpha_2 l^2 \left[\partial_i \phi \partial_j \phi - D_i \partial_j \phi + \tilde{g}_{(0)ij} \left(\Box\phi - \frac{1}{2}\partial^k \phi \partial_k \phi - \frac{1}{2}\lambda e^{2\phi}\right)\right]. \tag{4.57}$$

It is easily shown that under the Weyl transformation:

$$\begin{aligned} \delta_W \tilde{g}_{(0)ij} &= 2\sigma \tilde{g}_{(0)ij}, \\ \delta_W \phi &= -\sigma, \end{aligned}$$

the Weyl anomaly of $S_{(L)}$ is equal to that of $S_{fin}(\mathcal{G})$ and the transformation rule of $T_{(L)ij}$ is that given in equation (4.55), when α_2 has the same value as before, namely $\alpha_2 = 1/8\pi Gl$. Moreover, the trace of $T_{(L)ij}$ is equal to this anomaly when the equation of motion of ϕ is satisfied. The constant λ in front of the Weyl invariant potential term is arbitrary because it does not contribute to the Weyl anomaly. Moreover any Weyl invariant matter action could be added to the Liouville action without changing its conformal properties.

We will now show that, on the equations of motion, we can identify the tensor $\mathcal{T}_{(2)ij}$ of equation (4.54) which carries the gravitational degrees of freedom encoded in $\tilde{g}^t_{(2)}$ with the Liouville energy-momentum tensor $T_{(L)ij}$. We write:

$$\mathcal{T}_{(2)ij} = T_{(L)ij}, \tag{4.58}$$

with the expression of $T_{(L)ij}$ in terms of ϕ and $\tilde{g}_{(0)ij}$ given in equation (4.57) and taking into account the equations of motion. This equation provides, on the equations of motion, an expression of the Fefferman-Graham coefficient $\tilde{g}_{(2)ij}$ in terms of the Liouville field ϕ and of the boundary metric $\tilde{g}_{(0)ij}$. We will show that, for the solutions of the equations of motion, this equation is integrable, giving rise to an expression of the Liouville field ϕ in terms of the gravitational field $\tilde{g}_{(2)ij}$. This provides a bijection between the asymptotic solutions of the Einstein equations, which are translated into the conservation of the tensor $\mathcal{T}_{(2)ij}$, and the energy-momentum tensors constructed out of the solutions of the Liouville equation and shows that the suggestion mentioned in the previous section, according to which the degrees of freedom hidden in the Fefferman-Graham ambiguity $\tilde{g}^t_{(2)}$ can be expressed in terms of the energy-momentum tensor of some boundary fields, is verified when $d = 2$. This bijection is translated, through equation (4.58), into the fact that, for any value of the tensor $\mathcal{T}_{(2)ij}$ constrained only by its conservation law, there exists a solution ϕ of the Liouville equation of motion for which it is equal to the value of its energy-momentum $T_{(L)ij}$.

We begin to show that it is sufficient to check this for a flat boundary metric $\tilde{g}_{(0)}$. In two dimensions, a general curved metric is conformally flat and the metric $\tilde{g}_{(0)}$ can locally always be put in the following form:

$$\tilde{g}_{(0)uv} = -\frac{1}{2}e^{2\rho}dudv, \qquad \tilde{g}_{(0)uu} = \tilde{g}_{(0)vv} = 0, \tag{4.59}$$

with $u = (t/l) + \varphi$ and $v = (t/l) - \varphi$. In this coordinate system, the fact that the trace of an energy-momentum tensor $T_{(2)ij}$ is equal to the Weyl anomaly $\mathcal{A}_2$, given in equation (4.52) is expressed as:

$$T_{(2)uv} = -\alpha_2 l^2 \partial_u \partial_v \rho.$$

In that case, the conservation of $T_{(2)ij}$ can be written as [65]:

$$\partial_v t_u(u,v) = 0, \qquad \partial_u t_v(u,v) = 0, \tag{4.60}$$

where t_u and t_v are related to the components $T_{(2)uu}$ and $T_{(2)vv}$ of the energy-momentum tensor by the following expressions:

$$\begin{aligned} T_{(2)uu} &= \alpha_2 l^2 \left[\partial_u^2 \rho - (\partial_u \rho)^2\right] + t_u, \\ T_{(2)vv} &= \alpha_2 l^2 \left[\partial_v^2 \rho - (\partial_v \rho)^2\right] + t_v. \end{aligned} \tag{4.61}$$

Computing the functions t_u and t_v for the particular case of the Liouville energy-momentum tensor given in equation (4.57), one finds:

$$t_{(L)u} = \alpha_2 l^2 \left[\left(\partial_u \overline{\phi}\right)^2 - \partial_u^2 \overline{\phi}\right], \qquad t_{(L)v} = \alpha_2 l^2 \left[\left(\partial_v \overline{\phi}\right)^2 - \partial_v^2 \overline{\phi}\right], \tag{4.62}$$

where $\overline{\phi} = \phi + \rho$. We recognize the components $T_{(L)uu}$ and $T_{(L)vv}$ of equation (4.57) for the field $\overline{\phi}$ with flat $\tilde{g}_{(0)ij}$, corresponding to vanishing ρ in equation (4.59). Moreover equation (4.60), which expresses the conservation of the tensor $T_{(L)ij}$ in the case of a general curved metric $\tilde{g}_{(0)ij}$, is the conservation equation of the energy-momentum tensor $T_{(L)ij}$ computed for the field $\overline{\phi}$ in flat space. Therefore, it is sufficient to demonstrate the integrability on the equations of motion of equation (4.57) in flat space in order to show that it is valid for any curved boundary metric $\tilde{g}_{(0)ij}$.

We are now ready to study the integrability of equation (4.57) with respect to ϕ. The equation of motion derived from the Liouville action (4.56) is the following:

$$-2\Box\phi + \tilde{R} + 2\lambda e^{2\phi} = 0. \tag{4.63}$$

With $\tilde{g}_{(0)}$ given by equation (4.59), it can be written:

$$4\partial_u\partial_v\overline{\phi} + \lambda e^{2\overline{\phi}} = 0.$$

A general solution of this equation is given by [66]:

$$e^{2\overline{\phi}} = -\frac{4f'g'}{\lambda(1-fg)^2}, \tag{4.64}$$

where f and g are arbitrary functions of u and v respectively and $'$ denotes derivative with respect to the argument. Computing $t_{(L)u}$ and $t_{(L)v}$ given in equation (4.62) or equivalently the components $T_{(L)uu}$ and $T_{(L)vv}$ as given in equation (4.57) with flat $\tilde{g}_{(0)}$ and ϕ replaced by $\overline{\phi}$, one finds:

$$t_{(L)u} = -\alpha_2 l^2 \mathcal{D}^s_u f, \qquad t_{(L)v} = -\alpha_2 l^2 \mathcal{D}^s_v g, \tag{4.65}$$

where $\mathcal{D}^s_u f$ (similarly for g) denotes the Schwarzian derivative of f with respect to its argument u:

$$\mathcal{D}^s_u f = \frac{1}{2}\frac{f'''}{f'} - \frac{3}{4}\left(\frac{f''}{f'}\right)^2.$$

The integrability of equation (4.57), which, taking into account the Liouville equation of motion, is equivalent to equation (4.65), is then translated into the integrability of the following equation for $f(x)$:

$$F(x) = \mathcal{D}^s_x f(x).$$

This equation has a solution which is given by $f = w_1/w_2$, with w_i $(i = 1, 2)$ solving the following differential equation:

$$w_i'' + F w_i = 0.$$

Given $t_{(L)u}$ and $t_{(L)v}$, related through equations (4.54), (4.58) and (4.61) to a gravitational solution specified by $\tilde{g}_{(2)ij}$, it is therefore possible to solve equation (4.65) for f and g and to determine the corresponding Liouville solution through equation (4.64). $\mathcal{T}_{(d)ij}$ is then equal to the energy-momentum of this Liouville solution. Hence, at the level of the equations of motion, we have shown locally that all the degrees of freedom encoded in the Fefferman-Graham traceless field $\tilde{g}^t_{(2)}$ are described by the Liouville field on an arbitrary curved two-dimensional background and the Liouville action is locally equivalent to $S_{fin}(\mathcal{G})$. This equivalence is consistent with the generalization to curved boundaries [64] of the reduction of AdS_3 gravity to Liouville theory demonstrated in [8].

4.5.3 Back to the asymptotic symmetry algebra

We now turn to the analysis of the Brown-Henneaux asymptotic symmetry algebra [7] which is given, as reviewed in Chapter 3, by twice the Virasoro algebra with central charge $c = 3l/2G$. Looking at the gravitational side, we first identify in the formalism of this chapter the Brown-Henneaux Virasoro generators and specify their transformation under the diffeomorphism that reproduces the asymptotic symmetry transformation. For simplicity, we consider the flat boundary metric $\tilde{g}_{(0)}$ given in equation (4.59) with $\rho = 0$. In this case $\alpha_2 \tilde{g}_{(2)ij}$ is just equal to $\mathcal{T}_{(2)ij}$. Similarly to what has been shown for $T_{(L)ij}$ in the previous subsection, the generalization to curved boundaries can be done by considering the functions t_u and t_v whose relations with $\mathcal{T}_{(2)uu}$ and $\mathcal{T}_{(2)vv}$ are given in equation (4.61).

We begin by looking at the BTZ black hole solution [26]. We express the coefficients $\tilde{g}_{(2)ij}$ in terms of the mass M and the angular momentum J of the black hole or equivalently in terms of the functions L and $\bar{L}$ introduced in Chapter 3. These functions appear in the boundary conditions satisfied by an asymptotically anti-de Sitter space-time. In the case of the BTZ solutions, they are constant and equal to:

$$\frac{L}{k} = \frac{1}{4}\left(M - \frac{J}{l}\right), \qquad \frac{\bar{L}}{k} = \frac{1}{4}\left(M + \frac{J}{l}\right),$$

with $k = l/4G$. We recall that the BTZ metric is given by:

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\varphi dt + d\varphi)^2,$$

with:

$$\begin{aligned} N^2 &= (r/l)^2 - M + (J/2r)^2, \\ N^\varphi &= -J/2r^2. \end{aligned}$$

Through the following change of radial variable:

$$r^2 = \frac{1}{y} + \frac{l^2 L}{k} + \frac{l^2 \bar{L}}{k} + y \frac{l^4 L \bar{L}}{k^2},$$

and $u = (t/l) + \varphi$, $v = (t/l) - \varphi$, the BTZ metric takes the Fefferman-Graham form [67]:

$$ds^2 = \frac{l^2 dy^2}{4y^2} - \frac{1}{y} du dv + \frac{l^2 L}{k} du^2 + \frac{l^2 \bar{L}}{k} dv^2 - y \frac{l^4 L \bar{L}}{k^2} du dv,$$

whose expression has the particularity to stop at the order y. We extract from this expression the following identification between the components of $\mathcal{T}_{(2)ij}$ and the functions L and $\bar{L}$:

$$\mathcal{T}_{(2)uu} = \frac{L}{2\pi}, \qquad \mathcal{T}_{(2)vv} = \frac{\bar{L}}{2\pi}.$$

Enlarging this identification to the arbitrary functions $L(t,\varphi)$ and $\tilde{L}(t,\varphi)$ appearing in the boundary conditions (see [68]), one recognizes $\mathcal{T}_{(2)uu}$ and $\mathcal{T}_{(2)vv}$ as the Brown-Henneaux generators whose algebra was given in equation (3.52) in Chapter 3.

We are now able to describe the $(2+1)$-diffeomorphism which induce on L and $\tilde{L}$ the transformation that corresponds to the asymptotic symmetry transformation presented in Chapter 3 and leads to the Virasoro algebra. We consider the diffeomorphism given in equations (4.7)-(4.8). Let us notice that in the particular case where $\Box\sigma = 0$, the Weyl transformation:

$$\delta_W \bar{g}_{(0)ij} = 2\sigma \bar{g}_{(0)ij},$$

does not change the curvature of the boundary and can be compensated by a two-dimensional conformal reparametrization generated by the functions $\chi^i(x)$ appearing in equation (4.8) with [69]:

$$\sigma(u,v) = -\frac{1}{2}\left(\frac{d\chi^u}{du}(u) + \frac{d\chi^v}{dv}(v)\right).$$

Let us write δ_W the variation under the diffeomorphism (4.7)-(4.8) with $\chi^i = 0$ and δ_χ the variation under the diffeomorphism (4.7)-(4.8) with $\sigma = 0$. Taking:

$$\chi^u = \eta(u), \qquad \chi^v = \bar{\eta}(v), \qquad \sigma = -\frac{1}{2}(\eta' + \bar{\eta}'), \tag{4.66}$$

with $'$ denoting derivative with respect to the argument, we have:

$$\begin{aligned} \delta_W \bar{g}_{(2)uu} &= -\frac{l^2}{2}\eta''', & \delta_W \bar{g}_{(2)vv} &= -\frac{l^2}{2}\bar{\eta}''', \\ \delta_\chi \bar{g}_{(2)uu} &= (\eta \bar{g}_{(2)uu})' + \eta' \bar{g}_{(2)uu}, & \delta_\chi \bar{g}_{(2)vv} &= (\bar{\eta} \bar{g}_{(2)vv})' + \bar{\eta}' \bar{g}_{(2)vv}. \end{aligned}$$

Consequently, the diffeomorphism characterized by equation (4.66) does not change the boundary metric $\bar{g}_{(0)}$ and induces on $\mathcal{T}_{(2)uu}$ and $\mathcal{T}_{(2)vv}$ the following transformation:

$$\begin{aligned} (\delta_W + \delta_\chi)\mathcal{T}_{(2)uu} &= (\eta \mathcal{T}_{(2)uu})' + \eta' \mathcal{T}_{(2)uu} - \frac{\alpha_2 l^2}{2}\eta''', \\ (\delta_W + \delta_\chi)\mathcal{T}_{(2)vv} &= (\bar{\eta} \mathcal{T}_{(2)vv})' + \bar{\eta}' \mathcal{T}_{(2)vv} - \frac{\alpha_2 l^2}{2}\bar{\eta}'''. \end{aligned} \tag{4.67}$$

These transformation rules are the same as the bosonic part of the one given in equation (3.45) for $L/2\pi$, indicating that $\mathcal{T}_{(2)uu}$ and $\mathcal{T}_{(2)vv}$ form two Virasoro algebras with central charge $c = 3l/2G$.

It is clear from our derivation that the non central part of the Virasoro algebras is just generated by the longitudinal two-dimensional conformal reparametrization decomposed as usual into two one-dimensional reparametrizations. The central term arises from the

δy-part of the $(2+1)$-diffeomorphism i.e. from the deformations normal to the surfaces $y =$ constant.

We now look at the emergence of the Virasoro algebra on the Liouville side, considering only the algebra that corresponds to the coordinate u (the coordinate v is treated similarly). We rederive the Virasoro algebra, using the identification of the Brown-Henneaux generator $L/2\pi$ with the Liouville energy-momentum tensor component $T_{(L)uu}$ through equation (4.58) and computing the Poisson bracket of $T_{(L)uu}$ with itself. The Liouville action (4.56) in canonical form, with flat $\tilde{g}_{(0)}$ and potential term set to zero for simplicity, is given by:

$$S_{(L)} = \frac{\alpha_2 l^2}{2} \int \left[l^2 \dot{\phi}^2 - (\phi')^2 \right] dt d\varphi,$$

where $\dot{\phi}$ denotes $\partial\phi/\partial t$ and ϕ' denotes $\partial\phi/\partial\varphi$. The momentum p conjugated to ϕ derivated from this action is $p = \alpha_2 l^3 \dot{\phi}$. The component $T_{(L)uu}$ of the Liouville energy-momentum tensor (which is equal to $t_{(L)u}$ when ϕ is replaced by $\overline{\phi} = \phi + \rho$ in the case of curved $\tilde{g}_{(0)}$) is expressed in terms of ϕ and p as:

$$T_{(L)uu} = \frac{\alpha_2 l^2}{4} \left[\left(\frac{p}{\alpha_2 l^2} + \phi' \right)^2 - 2 \left(\frac{p}{\alpha_2 l^2} + \phi' \right)' \right],$$

with the use of the equation of motion $l^2 \ddot{\phi} = \phi''$. Using the canonical Poisson bracket:

$$[\phi(\sigma), p(\sigma')] = \delta(\sigma - \sigma'),$$

one obtains:

$$[T_{(L)uu}(\sigma), T_{(L)uu}(\sigma')] = \left(T_{(L)uu}(\sigma) + T_{(L)uu}(\sigma') \right) \delta'(\sigma - \sigma') - \frac{\alpha_2 l^2}{2} \delta'''(\sigma - \sigma'),$$

which reproduces the Virasoro algebra with the Brown-Henneaux central charge $c = 24\pi\alpha_2 l^2/2 = 3l/2G$.

The generator $t_{(L)u}$ adapted to the case of curved boundaries allows to understand how the central extension in the Virasoro algebra is related to the Weyl anomaly, which is zero when the boundary is flat. Indeed, $t_{(L)u}$ differs from the tensor $T_{(L)uu}$ by a function of the conformal factor $e^{2\rho}$ of the boundary metric (see equation (4.61). Therefore, under a conformal reparametrization $u = u(\tilde{u})$, the transformation law of $t_{(L)u}$ differs from a tensorial one. The function ρ appearing in equation (4.59) transforms according to:

$$\tilde{\rho} = \rho + \frac{1}{2} \ln u',$$

where $u' \equiv du/d\tilde{u}$. From this equation and the tensorial transformation properties of $T_{(L)uu}$, one gets [65]:

$$t_{(L)\tilde{u}}(\tilde{u}) = u'^2 t_{(L)u}(u) - \alpha_2 l^2 \left[\frac{1}{2} \frac{u'''}{u'} - \frac{3}{4} \left(\frac{u''}{u'} \right)^2 \right], \tag{4.68}$$

where we recognize the Schwarzian derivative of $u(\tilde{u})$. For an infinitesimal transformation $u = \tilde{u} + \eta$, one obtains the following transformation for $t_{(L)u}$:

$$\delta t_{(L)u} = (\eta t_{(L)u})' + \eta' t_{(L)u} - \frac{\alpha_2 l^2}{2}\eta''',$$

recovering the transformation of the Virasoro generator as in equation (4.67). We see that the central term occurs because of the non tensorial transformation properties of $t_{(L)u}$. This deviation of $t_{(L)u}$ from a tensor is related through energy-momentum conservation to the value of the Weyl anomaly (see equation (4.61)).

Similarly, the transformation property of the field $\overline{\phi} = \phi + \rho$ under the conformal reparametrization $u = u(\tilde{u})$ differs from that of a scalar. It is given by:

$$\tilde{\overline{\phi}}(\tilde{u}) = \overline{\phi}(u) + \frac{1}{2}\ln u',$$

which gives, for the infinitesimal transformation $u = \tilde{u} + \eta$:

$$\delta\overline{\phi} = \eta\overline{\phi}' + \frac{1}{2}\eta'.$$

It is easily checked that it is the same transformation rule as the one of ϕ which is generated by its Poisson bracket with $\int \eta T_{(L)uu} d\varphi$. Moreover its non tensorial part is equal to $\delta_W \phi = -\sigma$ for $\sigma = -(1/2)\eta'$ as in equation (4.66) (with $\bar{\eta} = 0$).

Let us notice that the non tensorial transformation of $t_{(L)u}(u)$ in equation (4.68) under the conformal reparametrization $u = u(\tilde{u})$ is encoded in the fact that $t_{(L)u}(u)$ itself is proportional to the Schwarzian derivative of the function f that is used to construct a Liouville solution as shown in equation (4.65). Indeed the Schwarzian derivative has the following behaviour under composition:

$$\mathcal{D}^s_{\tilde{u}} f(u(\tilde{u})) = u'^2 \mathcal{D}^s_u f + \mathcal{D}^s_{\tilde{u}} u,$$

giving rise to the Schwarzian derivative of $u(\tilde{u})$ in equation (4.68).

4.6 Higher dimensions

In this section, we give a method to construct a d-dimensional action which presents the classical Weyl anomaly computed in section 4.3 and we apply it explicitly to the four-dimensional case. This construction involves a generalization of the Liouville action to higher even dimensions relying on the particular structure that relates its equation of motion, the form of the action and the Weyl anomaly. We can consider the Liouville action with the potential term put to zero as our analysis of the gravitational action gives information only on the anomalous behaviour of its finite part $S_{fin}(\mathcal{G})$ under Weyl transformation and does not constrain the addition of Weyl invariant terms. Looking at equation (4.56), one sees that the second term of the Liouville lagrangian can be expressed in terms of the Weyl anomaly as:

$$-\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_2\phi.$$

The variation of this term for $\delta_W\phi = -\sigma$ produces the anomaly while its variation induced by the Weyl transformation of the metric $\delta_W\tilde{g}_{(0)ij} = 2\sigma\tilde{g}_{(0)ij}$ compensates exactly the Weyl variation of the first term. Another feature of Liouville theory is that its equation of motion given in equation (4.63) is Weyl invariant and can be expressed more compactly in terms of the metric $\bar{g}_{ij} \equiv \tilde{g}_{(0)ij}e^{2\phi}$ as $\bar{R} + 2\lambda = 0$. This equation, when $\lambda = 0$, is just the expression of the vanishing of the anomaly $\mathcal{A}_2$ when computed for the metric $\bar{g}_{ij}$.

These considerations suggest a receipt to build generalizations of the Liouville action to higher even-dimensional cases. We begin by constructing a d-dimensional equation of motion for a field ϕ from the expression of the gravitational Weyl anomaly computed in section 4.3. We then show that the action integrated from this equation presents this classical Weyl anomaly.

Let us write:

$$\sqrt{-\bar{g}}\mathcal{A}_d(\bar{g}_{ij}) = 0, \tag{4.69}$$

with $\mathcal{A}_d(\bar{g}_{ij})$ the Weyl anomaly computed for the metric $\bar{g}_{ij} \equiv \tilde{g}_{(0)ij}e^{2\phi}$ in d dimensions. The left hand side of this equation is by construction Weyl invariant if the Weyl transformation of the field ϕ is $\delta_W\phi = -\sigma$. Due to this property, the Euler-Lagrange derivative with respect to ϕ of the Weyl transformation of the lagrangian $\mathcal{L}_d$ from which the equation of motion (4.69) is derived vanishes. Indeed, because the Weyl transformation rules $\delta_W\tilde{g}_{(0)ij}$ and $\delta_W\phi$ do not depend on the field ϕ, the Euler-Lagrange derivative with respect to ϕ and the variation under Weyl transformation do commute. Therefore, doing this commutation in the following expression of the Weyl invariance of the equation of motion derived from $\mathcal{L}_d$:

$$\delta_W\frac{\delta\mathcal{L}_d}{\delta\phi} = 0,$$

one obtains:

$$\frac{\delta}{\delta\phi}(\delta_W\mathcal{L}_d) = 0.$$

Solving this equation implies that the Weyl transformation of the lagrangian $\delta_W\mathcal{L}_d$ is independent of the field ϕ and its derivatives up to a total derivative. Recalling that the classical Weyl anomaly A_d associated to the action S_d corresponding to this lagrangian is defined through:

$$\delta_W S_d = \int \delta_W\mathcal{L}_d\ d^dx = \int \sqrt{-\tilde{g}_{(0)}}A_d\ \sigma\ d^dx,$$

one sees that the possible ϕ-dependent total derivative in $\delta_W \mathcal{L}_d$ does not contribute to the anomaly A_d. The Weyl anomaly A_d of the action S_d corresponding to the equation of motion given in equation (4.69) is then independent of ϕ and its derivatives, allowing to compute the Weyl variation of S_d with ϕ and its derivatives set to zero, as follows:

$$\delta_W S_d = \delta_W S_d|_{\phi\equiv 0}.$$

We now use this feature to show that A_d is in fact equal to the original $\mathcal{A}_d$ from which the equation of motion (4.69) has been constructed. The left hand side of equation (4.69) can be written:

$$\sqrt{-\bar{g}}\mathcal{A}_d(\bar{g}_{ij}) = \sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d(\tilde{g}_{(0)ij}) + \mathcal{E}_d^{\phi}(\tilde{g}_{(0)ij}, \phi),$$

where $\mathcal{A}_d(\bar{g}_{ij})$ and $\mathcal{A}_d(\tilde{g}_{(0)ij})$ are the gravitational Weyl anomalies of section 4.3 computed for the metrics $\bar{g}_{ij} \equiv \tilde{g}_{(0)ij}e^{2\phi}$ and $\tilde{g}_{(0)ij}$ respectively and $\mathcal{E}_d^{\phi}(\tilde{g}_{(0)ij}, \phi)$ is an expression depending at least linearly on ϕ or its derivatives. The lagrangian from which equation (4.69) is derived has then the following form:

$$\mathcal{L}_d = a\left[\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d(\tilde{g}_{(0)ij})\phi + \mathcal{L}_d^{\phi}(\tilde{g}_{(0)ij}, \phi)\right], \tag{4.70}$$

with a a constant which will be fixed later and $\mathcal{L}_d^{\phi}(\tilde{g}_{(0)ij}, \phi)$ an at least quadratic expression in ϕ and its derivatives, which comes from the integration of $\mathcal{E}_d^{\phi}(\tilde{g}_{(0)ij}, \phi)$. The Weyl transformation of this lagrangian under $\delta_W \tilde{g}_{(0)ij} = 2\sigma\tilde{g}_{(0)ij}$ and $\delta_W \phi = -\sigma$ gives:

$$\delta_W \mathcal{L}_d = a\left[-\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d(\tilde{g}_{(0)ij})\sigma + \mathcal{F}_d^{\phi}(\tilde{g}_{(0)ij}, \phi, \sigma)\right], \tag{4.71}$$

where $\mathcal{F}_d^{\phi}(\tilde{g}_{(0)ij}, \phi, \sigma)$ comes from the Weyl variation of $\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d(\tilde{g}_{(0)ij})$ and $\mathcal{L}_d^{\phi}(\tilde{g}_{(0)ij}, \phi)$ in equation (4.70) and depends at least linearly on ϕ or its derivatives. Now, using the above demonstrated property that $\delta_W \mathcal{L}_d$ is independent of ϕ and its derivatives up to a total derivative, we compute the Weyl anomaly of the action associated to equation (4.69) by putting ϕ and its derivatives to zero in equation (4.71). In this case, $\mathcal{F}_d^{\phi}(\tilde{g}_{(0)ij}, \phi, \sigma)$ vanishes and one has:

$$\delta_W S_d = \delta_W S_d|_{\phi\equiv 0} = -a\int \sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d(\tilde{g}_{(0)ij})\ \sigma\ d^d x.$$

Setting $a = -1$, we have shown that the Weyl anomaly of the lagrangian (4.70) obtained by integrating the equation of motion (4.69) is precisely the one we start with to build this equation, namely the gravitational Weyl anomaly computed in section 4.3. While the variation under $\delta_W \phi = -\sigma$ of the first term of the lagrangian given in equation (4.70) provides the Weyl anomaly, the Weyl transformation of the remaining terms written as $\mathcal{L}_d^{\phi}(\tilde{g}_{(0)ij}, \phi)$ compensates the variation of the first term under $\delta_W \tilde{g}_{(0)ij} = 2\sigma\tilde{g}_{(0)ij}$.

In the particular case of $d = 4$, we recall the value of the Weyl anomaly given in equation (4.36):

$$\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_4 = \alpha_4 \frac{l^4}{16}\sqrt{-\tilde{g}_{(0)}}\left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2\right), \tag{4.72}$$

with $\alpha_4 = 1/4\pi Gl$. Computing this anomaly with the curvature tensors equal to those of the metric $\bar{g}_{ij}$ and expressing it in terms of the metric $\tilde{g}_{(0)ij}$ and of the field ϕ, one obtains the following equation of motion for ϕ (see equation (A.3) in Appendix A):

$$\alpha_4 \frac{l^4}{16}\sqrt{-\tilde{g}_{(0)}}\left[\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2 - 4\left(\tilde{R}^{ij} - \frac{1}{2}\tilde{g}^{ij}_{(0)}\tilde{R}\right)D_i\partial_j\phi + 4\tilde{R}^{ij}\partial_i\phi\partial_j\phi\right.$$
$$\left. -4\Box\phi\Box\phi + 4D^i\partial^j\phi D_i\partial_j\phi - 4\Box\phi\partial^i\phi\partial_i\phi - 8D^i\partial^j\phi\partial_i\phi\partial_j\phi\right] = 0.$$

The lagrangian which can be obtained by integrating this equation of motion is constructed by starting with a term of the form $-\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_d\phi$. As in the Liouville case, the variation of this term induced by $\delta_W\phi = -\sigma$ gives the anomaly while the other terms in the lagrangian must compensate its variation for the Weyl transformation of the background metric $\tilde{g}_{(0)ij}$. In four dimensions, one finds the following action [70]:

$$\begin{aligned} S_4 &= -\alpha_4 \frac{l^4}{16}\int\sqrt{-\tilde{g}_{(0)}}\left[\left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2\right)\phi\right. \\ &\left. +2\left(\tilde{R}^{ij} - \frac{1}{2}\tilde{g}^{ij}_{(0)}\tilde{R}\right)\partial_i\phi\partial_j\phi + 2\Box\phi\partial^i\phi\partial_i\phi + \partial^i\phi\partial_i\phi\partial^j\phi\partial_j\phi\right] d^4x, \end{aligned} \tag{4.73}$$

which has the particularity to be quartic in the derivatives of ϕ. It is easily checked that the equation of motion derived from this action is in fact the above one.

A related action for the conformal factor ϕ is described in [70, 71, 72], which is quadratic in the derivatives of ϕ. Its Weyl anomaly differs from the one given in equation (4.72) by a term proportional to $\Box\tilde{R}$, which constitutes a trivial type of anomaly in the sense that in can be removed by the addition of a local counterterm depending on the metric $\tilde{g}_{(0)ij}$ only. This action is the following [70, 71, 72]:

$$S_\phi = -\alpha_4 \frac{l^4}{16}\int\sqrt{\tilde{g}_{(0)}}\left[\left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2 + \frac{1}{3}\Box\tilde{R}\right)\phi - \phi\Delta_4\phi\right], \tag{4.74}$$

where coefficients have been adapted to the comparison with equation (4.73) and Δ_4 is the Weyl covariant fourth order operator acting on ϕ [70]:

$$\begin{aligned} \Delta_4 &\equiv \Box^2 + 2D_i\left(\tilde{R}^{ij} - \frac{1}{3}\tilde{g}^{ij}_{(0)}\tilde{R}\right)\partial_j \\ &= \Box^2 + 2\tilde{R}^{ij}D_i\partial_j - \frac{2}{3}\tilde{R}\Box + \frac{1}{3}\left(\partial^i\tilde{R}\right)\partial_i. \end{aligned}$$

The Weyl anomaly associated to the above action is given by:

$$\delta_W S_\phi = \alpha_4 \frac{l^4}{16} \int \sqrt{-\tilde{g}_{(0)}} \left(\tilde{R}^{ij} \tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2 + \frac{1}{3}\Box\tilde{R} \right) \sigma \; d^4x, \tag{4.75}$$

which, as mentioned before, differs from the variation of the action S_4 given in equation (4.73) by a trivial type term.

The difference between actions (4.73) and (4.74) contains the cubic and quartic terms of action (4.73) and is equal to:

$$\begin{aligned} S_{R^2} = S_4 - S_\phi = -\alpha_4 \frac{l^4}{16} \int \sqrt{-\tilde{g}_{(0)}} \bigg[-\frac{1}{3}\left(\Box\tilde{R}\right)\phi \\ -\frac{1}{3}\tilde{R}\partial^i\phi\partial_i\phi + \phi\Box^2\phi + 2\Box\phi\partial^i\phi\partial_i\phi + \partial^i\phi\partial_i\phi\partial^j\phi\partial_j\phi \bigg] \; d^4x. \end{aligned} \tag{4.76}$$

This action can be written [70, 71, 72] as the finite variation of a local expression in the boundary metric under the Weyl transformation sending $\tilde{g}_{(0)ij}$ to $\bar{g}_{ij} \equiv \tilde{g}_{(0)ij}e^{2\phi}$. Indeed, using equation (A.1) in Appendix A, one has:

$$\begin{aligned} \frac{1}{36}\sqrt{-\bar{g}}\bar{R}^2 &= \frac{1}{36}\sqrt{-\tilde{g}_{(0)}}\tilde{R}^2 + \sqrt{-\tilde{g}_{(0)}} \big(\Box\phi\Box\phi + \partial^i\phi\partial_i\phi\partial^j\phi\partial_j\phi \\ &\quad -\frac{1}{3}\tilde{R}\Box\phi - \frac{1}{3}\tilde{R}\partial^i\phi\partial_i\phi + 2\Box\phi\partial^i\phi\partial_i\phi \big). \end{aligned}$$

Equation (4.76) is then written as follows:

$$S_{R^2} = -\alpha_4 \frac{l^4}{16}\frac{1}{36} \int \left(\sqrt{-\bar{g}}\bar{R}^2 - \sqrt{-\tilde{g}_{(0)}}\tilde{R}^2 \right) \; d^4x. \tag{4.77}$$

This property implies that the anomaly associated to this action is of the trivial type. Indeed, any function of $\bar{g}_{ij}$ is Weyl invariant under $\delta_W \tilde{g}_{(0)ij} = 2\sigma\tilde{g}_{(0)ij}$ and $\delta_W\phi = -\sigma$ and we have, using equation (A.5) in Appendix A:

$$\delta_W S_{R^2} = \alpha_4 \frac{l^4}{16}\frac{1}{36}\delta_W \int \sqrt{-\tilde{g}_{(0)}}\tilde{R}^2 \; d^4x = \alpha_4 \frac{l^4}{16} \int \sqrt{-\tilde{g}_{(0)}} \left(-\frac{1}{3}\Box\tilde{R} \right) \sigma \; d^4x,$$

where the Weyl anomaly is induced by the Weyl variation of a local expression in the metric $\tilde{g}_{(0)ij}$.

Action S_ϕ given in equation (4.74) and whose Weyl anomaly given in equation (4.75) is equal to:

$$\sqrt{-\tilde{g}_{(0)}}\mathcal{A}_\phi = \alpha_4 \frac{l^4}{16} \left(\tilde{R}^{ij}\tilde{R}_{ij} - \frac{1}{3}\tilde{R}^2 + \frac{1}{3}\Box\tilde{R} \right),$$

and contains non trivial contributions, can also be written as the finite Weyl variation of a functional of the boundary metric. However, in constradistinction with the case of S_{R^2}, this functional is a non-local action [70, 71, 72]. It is the four-dimensional analogue to the Polyakov action [73], which is the non-local functional whose finite Weyl variation is equal to the Liouville action in two dimensions. Notice that the presence of the S_{R^2} terms in action S_ϕ, and hence their trivial contribution proportional to $\Box\tilde{R}$ in the corresponding Weyl anomaly, is needed to have a quadratic action [70, 72]. Moreover, this trivial part of the anomaly is necessary to define the operator Δ_4 which appears in the expression of the action for the conformal factor ϕ as the finite Weyl variation of a non-local functional of the boundary metric [70]. The quantization of the action S_ϕ given in equation (4.74) has been considered in [74].

Non-local effective actions for Weyl anomalies have been presented in [75] for all even dimensions. A way of obtaining them by dimensional regularization has been given in [72] where their local expression in terms of the conformal factor field ϕ has been constructed. The energy-momentum tensors derived from these non-local actions are the non-trivial solutions of the cohomological problem set up in [54] for the Weyl transformation properties of the Fefferman-Graham coefficients $\tilde{g}_{(d)}$, providing non-ambiguous, well-defined, albeit non-local expressions of the boundary metric $\tilde{g}_{(0)}$ for these coefficients [54].

Of course, any Weyl invariant action can be added to action S_d (given in equation (4.73) for $d = 4$) without changing its anomaly. In particular, one can add the usual Weyl invariant scalar action in d dimensions:

$$S = -\frac{1}{2}\int\sqrt{-\tilde{g}_{(0)}}\left(\partial^i\varphi\partial_i\varphi + \frac{d-2}{4(d-1)}\tilde{R}\varphi^2 + \lambda\varphi^{\frac{2d}{d-2}}\right)d^dx.$$

This action is indeed invariant under the Weyl transformation:

$$\delta_W\tilde{g}_{(0)ij} = 2\sigma\tilde{g}_{(0)ij}, \qquad \delta_W\varphi = -\frac{d-2}{2}\sigma\varphi,$$

in agreement with the conformal weight of a scalar field in d dimensions which is equal to $-(d-2)/2$. As the Weyl transformation rule of the higher-dimensional generalization of the Liouville field ϕ is $\delta_W\phi = -\sigma$, it is related to the scalar field φ by the following equation:

$$\phi = \frac{2}{d-2}\ln\varphi.$$

This scalar action reproduces the vanishing of the Weyl anomaly computed from the gravitational action in the case of odd-dimensional boundaries.

We have given in this section a way to construct an even d-dimensional local action S_d which can be viewed as a generalization of the Liouville action in the sense that it exhibits the classical Weyl anomaly $\mathcal{A}_d$ found in the variation of the gravitational action under a particular diffeomorphism. The vanishing of this anomaly when d is odd is appropriately reproduced by the usual Weyl invariant scalar action. However it is not clear that the energy-momentum tensor $T_{(d)ij}$ derived from these actions can describe, on the equation of motion of the field ϕ, all the asymptotic solutions of the Einstein equations encoded in the conserved tensor $\mathcal{T}_{(d)ij}$, as it is the case when $d=2$. This issue will be discussed in the next section.

Concerning the fact that, for even d, the finite part $S_{fin}(\mathcal{G})$ of the gravitational action, the action S_d and the non-local functional from which it can be constructed have all the same classical Weyl anomaly $\mathcal{A}_d$, let us mention the work of reference [76] where the variation of the gravitational action under the finite form [77] of the diffeomorphism (4.7)-(4.8) is computed for $d=2$ and 4. As for its infinitesimal version, this diffeomorphism induces a Weyl transformation of the boundary metric $\tilde{g}_{(0)ij} \to e^{2\sigma}\tilde{g}_{(0)ij}$, where σ is now a finite parameter [77]. It is shown in [76] that the finite part of the variation of the gravitational action under this diffeomorphism leads, when $d=4$, to minus the action $S_4[\tilde{g}_{(0)ij},\sigma]$ given in equation (4.73), with ϕ replaced by σ which parametrizes the finite diffeomorphism (and the corresponding Weyl transformation). By the same reasoning as the one presented in section 4.3 in the case of infinitesimal transformations, this variation can be interpreted as the finite Weyl transformation of $S_{fin}(\mathcal{G})$. Now $S_4[\tilde{g}_{(0)ij},\sigma]$ is precisely equal to minus the variation of the action $S_4[\tilde{g}_{(0)ij},\phi]$ itself under the finite Weyl transformations $\tilde{g}_{(0)ij} \to e^{2\sigma}\tilde{g}_{(0)ij}$ and $\phi \to \phi - \sigma$. It is also the finite Weyl variation of the non-local action from which $S_4[\tilde{g}_{(0)ij},\phi]$ can be constructed. Indeed, recalling from equation (4.76) that $S_4 = S_\phi + S_{R^2}$, we can write:

$$S_4[\tilde{g}_{(0)ij},\phi] = S_g[e^{2\phi}\tilde{g}_{(0)ij}] - S_g[\tilde{g}_{(0)ij}],$$

where $S_g[\tilde{g}_{(0)ij}]$ contains the non-local functional of the boundary metric mentioned previously [70, 71, 72] whose finite Weyl variation generates S_ϕ, plus the local piece generating S_{R^2} as in equation (4.77). Therefore, under the finite Weyl transformations $\tilde{g}_{(0)ij} \to e^{2\sigma}\tilde{g}_{(0)ij}$ and $\phi \to \phi - \sigma$, one has:

$$\Delta_W S_4[\tilde{g}_{(0)ij},\phi] = -\Delta_W S_g[\tilde{g}_{(0)ij}] = -S_4[\tilde{g}_{(0)ij},\sigma].$$

Consequently, the finite Weyl transformation of $S_{fin}(\mathcal{G})$ computed in [76] is the same as that of the action $S_4[\tilde{g}_{(0)ij},\phi]$ given in equation (4.73), as well as that of minus its ϕ-independent non-local associate $S_g[\tilde{g}_{(0)ij}]$.

4.7 Discussion

In this chapter, we have recovered the classical Weyl anomaly $\mathcal{A}_d$ of the finite part $S_{fin}(\mathcal{G})$ of the gravitational action with a negative cosmological constant in the case of even-dimensional boundaries and its vanishing in the case of odd-dimensional ones [55]. The value of the anomaly has been computed by acting with particular diffeomorphisms which generate Weyl transformations on the boundary. This anomaly originates from the presence, for d even, of a logarithmically divergent term in the gravitational action [6].

Classical boundary degrees of freedom have been identified by considering the Fefferman-Graham expansion of the metric near the boundary [52, 57]. These degrees of freedom are encoded in the traceless part $\tilde{g}^t_{(d)}$ of the coefficient $\tilde{g}_{(d)}$ of this expansion which, in contradistinction with the lower order coefficients, is not determined algebraically in terms of the boundary metric $\tilde{g}_{(0)}$ by the Einstein equations. This indeterminacy has been referred to as the Fefferman-Graham ambiguity. It has been shown to generate all the anti-de Sitter Schwarzschild and Kerr black holes.

By analyzing the form of the dynamical equations for $\tilde{g}_{(d)}$, which are induced by the expansion of the Einstein equations, we have shown, for $d = 2$, 3 and 4, that this coefficient defines a conserved tensor $\mathcal{T}_{(d)ij}$ on the boundary (see [62] for $d = 6$ and all d odd). Moreover, the trace of this tensor has been shown to reproduce the value of the Weyl anomaly of $S_{fin}(\mathcal{G})$. These conservation and trace properties are exactly those of the energy-momentum tensor of any d-dimensional theory which presents the same classical Weyl anomaly as $S_{fin}(\mathcal{G})$. These two tensors have also been shown to vary in the same way under Weyl transformations [62]. However the above features do not allow to answer the following two questions: (1) is $\mathcal{T}_{(d)ij}$ the energy-momentum tensor derived from $S_{fin}(\mathcal{G})$ and (2) can it be expressed as the energy-momentum tensor of some local boundary fields?

Concerning the first question, we recall that, as shown in section 4.3, the divergent terms that are removed from the gravitational action in order to obtain $S_{fin}(\mathcal{G})$ do not depend on the dynamical fields $\tilde{g}^t_{(d)}$. The remaining piece $S_{fin}(\mathcal{G})$ contains therefore all the dynamics in the limit where the boundary is sent to spatial infinity and should describe the boundary degrees of freedom encoded in $\tilde{g}^t_{(d)}$.

A boundary energy-momentum tensor for asymptotically anti-de Sitter space-times has been constructed in [78]. Its definition was inspired by the energy-momentum tensor of Brown and York [79] and it has been made finite by removing local counterterms from the gravitational action. These counterterms are the same as those used to construct $S_{fin}(\mathcal{G})$ [62]. This tensor has been explicitly computed in [62], for all odd d and even d up to six, by working out its expression in terms of the coefficients of the Fefferman-Graham expansion of the metric: it has been shown to be equal to $\mathcal{T}_{(d)ij}$ and is indeed the energy-momentum tensor derived from the finite part $S_{fin}(\mathcal{G})$ of the gravitational action [62]. Thus, at least for these cases, the answer to the first question is yes. We stress that this result implies that the dynamical part of the Einstein equations, which is translated into the conservation of $\mathcal{T}_{(d)ij}$, derives from the invariance of $S_{fin}(\mathcal{G})$ under reparametrizations in d dimensions.

We now turn to the question of the existence of local boundary fields whose energy-momentum tensor could provide a description of $\mathcal{T}_{(d)ij}$. We recall that the trace of this tensor, as well as its conservation law and its transformation rule under Weyl transformations, are characteristic of the energy-momentum tensor derived from any covariant d-dimensional action with Weyl anomaly equal to $\mathcal{A}_d$ (which is zero when d is odd).

Inspired by the reduction of three-dimensional AdS gravity to Liouville theory [8], we have shown in section 4.5 that the expression of $\mathcal{T}_{(2)ij}$ as the energy-momentum tensor of the Liouville field ϕ is integrable on the equations of motion. This provides a local boundary description of any Einstein solution in terms of a solution ϕ of the Liouville equation of motion. Therefore, the Liouville action describes the degrees of freedom encoded in $\tilde{g}^t_{(2)}$.

We have given in section 4.6 a way to construct, in all even dimensions d, a classical Weyl anomalous action. This action has the same classical Weyl anomaly as $S_{fin}(\mathcal{G})$ and depends on a field ϕ which is the generalization of the Liouville field in higher dimensions. We have also recalled the form of the Weyl invariant scalar action for all d and the scalar field has been expressed in terms of ϕ.

Besides the fact that the energy-momentum tensor derived from these actions has the same conformal and trace properties as $\mathcal{T}_{(d)ij}$, we have not proven, for $d \neq 2$, that the field ϕ is sufficient to describe all the relevant degrees of freedom encoded in $\tilde{g}^t_{(d)}$. Hence the second question remains open.

Chapter 5

Conclusion

We have shown that the asymptotic central charge of AdS_3 gravity, which is related to the number of microscopic degrees of freedom describing the $2+1$ black hole, is the same whether the classical theory considered is purely bosonic or supersymmetric. This central charge provides therefore no information about the supersymmetry of the theory.

We have shown next that in pure gravity with a negative cosmological constant, there are classical degrees of freedom on the d-dimensional conformal boundary of anti-de Sitter in all dimensions. Their boundary effective action has a Weyl anomaly for even dimensions and is conformally invariant for odd ones. These degrees of freedom are encoded in the traceless part $\tilde{g}^t_{(d)}$ of the coefficient $\tilde{g}_{(d)}$ of the Fefferman-Graham expansion of the metric near the boundary and generate all the AdS Schwarzschild and Kerr black holes. The analysis of the Einstein equations suggests that this coefficient can be expressed in terms of the energy-momentum tensor of some local boundary fields.

One has given a way to formulate an even-dimensional action which has the same classical Weyl anomaly as the one computed from the gravitational action and depends on a generalized Liouville field ϕ. Together with the usual conformal invariant scalar field theory which exists in all dimensions, it may provide the energy-momentum tensor describing the Einstein solutions on the boundary. This is true locally for $d=2$. In higher dimensions, it remains to be seen if a scalar field, or the related generalized Liouville field in even dimensions, is sufficient to describe all the gravitational solutions. This question deserves further examination, in which the global aspects of this description should be taken into account.

At the microscopic level however, this boundary field might fail to describe the degrees of freedom responsible for the black hole entropy. In $2+1$ dimensions for instance, even though the central charge corresponding to the Weyl anomaly permits to compute the entropy of the BTZ black hole, the quantization of the Liouville field on the boundary cannot account for it. In higher dimensions, the asymptotic gravitational degrees of freedom, whether or not expressible in terms of ϕ, are encoded in $\tilde{g}^t_{(d)}$. As for $d = 2$, these fields do not contain enough degrees of freedom to generate the black hole entropy. This failure casts doubts on the existence of a theory of quantum gravity based on the degrees of freedom of pure classical gravity only.

Appendix A

Weyl transformations of curvature tensors

In this appendix, we give the transformation properties of curvature tensors under Weyl transformations and some Euler-Lagrange derivatives of expressions containing these tensors. These results are useful to derive some equations of Chapter 4.

The computed transformations are induced by the Weyl variation of the metric g_{ij}, which appears in the curvature tensors through the Christoffel symbols Γ^i_{jk}'s as follows:

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\right), \\
R^i{}_{jkl} &= \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}, \\
R_{ij} &= R^k{}_{ikj} = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{kj} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{li}\Gamma^l_{kj}.
\end{aligned}$$

The Weyl variations under the finite Weyl transformation that sends the metric g_{ij} to $\bar{g}_{ij} = e^{2\phi}g_{ij}$ are read from the following expressions, which relate functions computed for the metric $\bar{g}_{ij}$ in terms of the metric g_{ij}:

$$\begin{aligned}
\sqrt{-\bar{g}} &= e^{d\phi}\sqrt{-g}, \\
\bar{\Gamma}^i_{jk} &= \Gamma^i_{jk} + \delta^i{}_j\partial_k\phi + \delta^i{}_k\partial_j\phi - g_{jk}\partial^i\phi, \\
\bar{R}_{ij} &= R_{ij} - (d-2)D_i\partial_j\phi - g_{ij}\Box\phi + (d-2)\partial_i\phi\partial_j\phi - (d-2)g_{ij}\partial^k\phi\partial_k\phi, \\
\bar{R} &= e^{-2\phi}\left[R - 2(d-1)\Box\phi - (d-1)(d-2)\partial^i\phi\partial_i\phi\right].
\end{aligned} \tag{A.1}$$

We work out the finite variation of expressions which contain the above quantities and are proportional to the Weyl anomalies in two and four dimensions, computed in Chapter 4. In two dimensions, one needs:

$$\sqrt{-\bar{g}}\bar{R} = \sqrt{-g}\left(R - 2\Box\phi\right). \tag{A.2}$$

In four dimensions, the Weyl transformation of the anomaly is given by:

$$\sqrt{-\bar{g}}\left(\bar{R}^{ij}\bar{R}_{ij}-\frac{1}{3}\bar{R}^2\right)=\sqrt{-g}\left[R^{ij}R_{ij}-\frac{1}{3}R^2-4\left(R^{ij}-\frac{1}{2}g^{ij}R\right)D_i\partial_j\phi\right.$$
$$\left.+4R^{ij}\partial_i\phi\partial_j\phi-4\Box\phi\Box\phi+4D^i\partial^j\phi D_i\partial_j\phi-4\Box\phi\partial^i\phi\partial_i\phi-8D^i\partial^j\phi\partial_i\phi\partial_j\phi\right] \quad (A.3)$$

We give hereafter some infinitesimal variations which are useful to compute Euler-Lagrange derivatives and infinitesimal Weyl transformations:

$$\begin{aligned}
\delta\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left(D_j\delta g_{kl}+D_k\delta g_{jl}-D_l\delta g_{jk}\right),\\
\delta R_{ij} &= \frac{1}{2}\left(D^kD_i\delta g_{kj}+D^kD_j\delta g_{ki}-\Box\delta g_{ij}-g^{kl}D_iD_j\delta g_{kl}\right),\\
\delta R &= -R^{ij}\delta g_{ij}+D^iD^j\delta g_{ij}-g^{ij}\Box\delta g_{ij}.
\end{aligned} \quad (A.4)$$

We are now able to compute the following infinitesimal Weyl transformations under $\delta_W g_{ij}=2\sigma g_{ij}$, in accordance with equations (A.1) for ϕ equal to an infinitesimal parameter σ:

$$\begin{aligned}
\delta_W\sqrt{-g} &= d\sigma\sqrt{-g},\\
\delta_W\Gamma^i_{jk} &= \delta^i{}_j\partial_k\sigma+\delta^i{}_k\partial_j\sigma-g_{jk}\partial^i\sigma,\\
\delta_W R_{ij} &= -(d-2)D_i\partial_j\sigma-g_{ij}\Box\sigma,\\
\delta_W R &= -2\sigma R-2(d-1)\Box\sigma.
\end{aligned} \quad (A.5)$$

From equation (A.4), we compute the Euler-Lagrange derivatives of expressions which are related to the Weyl anomalies in two and four dimensions. These formulas are needed to compute in section 4.4 of Chapter 4 the Weyl transformations of the energy-momentum tensors derived from the anomalous actions. In two dimensions, we have:

$$\frac{\delta}{\delta g^{ij}}\left(\sqrt{-g}R\phi\right)=\sqrt{-g}\left(D_i\partial_j\phi-g_{ij}\Box\phi\right). \quad (A.6)$$

In four dimensions, the expression of interest is:

$$\begin{aligned}
&\frac{\delta}{\delta g^{ij}}\left[\sqrt{-g}\left(R^{ij}R_{ij}-\frac{1}{3}R^2\right)\phi\right]=-\sqrt{-g}\left\{\left[\frac{1}{2}g_{ij}\left(R^{kl}R_{kl}-\frac{1}{3}R^2\right)\right.\right.\\
&\quad\left.-2R^k{}_iR_{kj}+\frac{2}{3}RR_{ij}\right]\phi+D_kD_i\left(R^k{}_j\phi\right)+D_kD_j\left(R^k{}_i\phi\right)-g_{ij}D_kD_l\left(R^{kl}\phi\right)\\
&\quad\left.-\Box\left(R_{ij}\phi\right)-\frac{2}{3}D_i\partial_j\left(R\phi\right)+\frac{2}{3}g_{ij}\Box\left(R\phi\right)\right\}\\
&=-\sqrt{-g}\left\{\left[\frac{1}{2}g_{ij}\left(R^{kl}R_{kl}-\frac{1}{3}R^2\right)-2R^k{}_iR_{kj}+\frac{2}{3}RR_{ij}+D_kD_iR^k{}_j+D_kD_jR^k{}_i\right.\right.
\end{aligned}$$

$$-\Box R_{ij} - \frac{2}{3}D_i\partial_j R + \frac{1}{6}g_{ij}\Box R\Big]\,\phi + D_i\left(R^k{}_j\partial_k\phi\right) + D_j\left(R^k{}_i\partial_k\phi\right)$$
$$-2D^k R_{ij}\partial_k\phi - R^{ij}\Box\phi - \frac{1}{6}\partial_i R\partial_j\phi - \frac{1}{6}\partial_j R\partial_i\phi - \frac{2}{3}RD_i\partial_j\phi$$
$$+g_{ij}\left(-R^{kl}D_k\partial_l\phi + \frac{1}{3}\partial^k R\partial_k\phi + \frac{2}{3}R\Box\phi\right)\Big\}, \tag{A.7}$$

where the coefficient of ϕ corresponds to:

$$\frac{\delta}{\delta g^{ij}}\left[\sqrt{-g}\left(R^{ij}R_{ij} - \tfrac{1}{3}R^2\right)\right] = -\sqrt{-g}\left[\tfrac{1}{2}g_{ij}\left(R^{kl}R_{kl} - \tfrac{1}{3}R^2\right) - 2R^k{}_iR_{kj} + \tfrac{2}{3}RR_{ij}\right.$$
$$\left.+D_kD_iR^k{}_j + D_kD_jR^k{}_i - \Box R_{ij} - \tfrac{2}{3}D_i\partial_j R + \tfrac{1}{6}g_{ij}\Box R\right]. \tag{A.8}$$

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