Fluid Queues

Building Upon the Analogy with QBD Processes

Ana da Silva Soares

Thèse présentée en vue de l'obtention
du grade de docteur en sciences

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Brussels, March 2005

Introduction

Markov modulated fluid flow processes are a very popular subject in applied probability, at least since the 1960's, although at that time they were mainly called *storage* or *dam* processes (see, for instance, Loynes [32], Sevastyanov [47]). In this work, we study fluid processes and, starting with the most traditional definition, we shall expand and introduce new models.

A fluid process driven by a Markovian environment, also called fluid queue, can be briefly described as follows. Consider a buffer or reservoir which is filled up with fluid (water, for example) and emptied out. Its content varies linearly with time, and the rate of variation depends on the state of some continuous time Markov process[1] evolving in the background. The fluid queue is thus a two-dimensional Markov process, which we denote by $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$:

- the first component $X(\cdot)$ is continuous and represents the content of the fluid buffer; it is usually called the *level*;
- the second component $\varphi(\cdot)$ is discrete and corresponds to the state of the underlying Markov process; it is usually called the *phase*.

A precise definition of the fluid model will be given later; at this stage, we only provide an example to clarify the type of models that will be studied in this work.

[1] The reader not familiar with Markov processes will find in the Appendix a definition of such processes, as well as some basic properties that will be useful throughout the text.

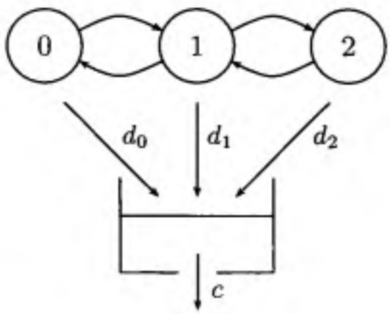


Figure 1: An example of a fluid queue modulated by a Markov process with three states $\{0, 1, 2\}$.

In Figure 1, we illustrate a water reservoir which empties out at a constant rate $c = 1$. The input rate of water into the reservoir is controlled by a continuous time Markov process on the three states $\{0, 1, 2\}$. When the phase process is in state 0, the input rate is $d_0 = 0$, when it is in state 1, the input rate is $d_1 = 1$ and when it is in state 2, the input rate is $d_2 = 2$. Thus, the net rate of variation of the buffer is $r_0 = d_0 - c = -1$ when the phase process is in state 0 and in this case the buffer content decreases at a rate of 1; it is $r_1 = d_1 - c = 0$ when the phase process is in state 1 and in this case the buffer content remains constant; and it is $r_2 = d_2 - c = 1$ and the buffer content increases at a rate of 1 when the phase process is in state 2. If the input rate is equal to -1 for a long time, the buffer may become empty and, in this case, it remains at level zero. On the other hand, if the reservoir is of infinite capacity and if the input rate is equal to $+1$ for a long interval of time, the content increases without stopping. If the buffer is of finite capacity and if the input rate is equal to $+1$ for a sufficiently long period, it may happen that the buffer overflows; in this case, we force the level to remain at its maximum value.

The interest of the applied probability community in fluid models is particularly due to their applicability in telecommunication and computer systems. Among the first to work in this area and to describe computer applications is Kosten [28]; he analyzes the statistical properties of the content of a buffer placed at the input of a central processor handling messages coming from a multitude of terminals. In 1982, Anick, Mitra and Sondhi [4] study a fluid process modelling a data-handling switch with multiple information sources which alternate independently and asynchronously between on and off states. The sojourn times in each of these states are assumed to be random and exponentially distributed. The switch stores in a buffer the information exceeding the maximum transmission rate of the output channel, the buffer being of infinite capacity. The main questions that arise in this context are the following. What is the right buffer size for a predetermined number of sources and quality of service? How does one select the maximum number of sources to be allowed in the system?

In 1988, Mitra [36] considers a continuous time system where fluid is produced by m machines, transferred to a buffer of finite or infinite capacity, and consumed by n other machines. The producing and the consuming machines are allowed to have failures. Such models are well adapted to manufacturing applications, but also to telecommunication systems, for which the machines represent sources and channels, and failures represent service interruptions. The assumption of a finite buffer is essential in manufacturing problems, while in communication systems, buffers may be assumed to be of infinite capacity since the systems considered have, in reality, relatively large buffers compared to the size of the packets, and the overflow probabilities are small.

Elwalid and Mitra [20, 21] in the early 1990's use fluid processes to model an Asynchronous Transfer Mode (ATM) environment. This works well due to the fact that cells are small and have uniform sizes, and that interarrival times between cells are constant for several contiguous cells. Also, the fluid approximation is well suited to circumstances where different time scales co-exist; here, the interarrival time of cells is small with respect to the time between changes in the rate, which is a feature of the ATM environment.

More recently, fluid flow models were used by Van Foreest, Mandjes and Scheinhardt [52, 53] and Mandjes, Mitra and Scheinhardt [33], among others, to model an Internet congestion control protocol, the so-called Transport Control Protocol (TCP). The processes studied in this context are known as *feedback* fluid queues, and differ from the models described above by the fact that the behaviour of the background process changes according to the value of the level of the buffer.

Lately, fluid queues have also appeared to be useful in the analysis of risk processes. One may exploit the relationship between such processes and fluid queues to provide efficient computational algorithms, which allow for the determination of the probability of ruin under different scenarios. These problems will not be studied here for we concentrate our attention on fluid queues proper. We only mention that in Stanford et al. [48], we determine the probability of ruin prior to an Erlangian horizon when the sizes of the claims are phase-type[2] distributed, considering both the Spare-Andersen and the stationary risk models. Also, in Badescu et al. [6], we present the Laplace transform of the time until ruin for a fairly general risk model, again by exploiting the relationship between risk processes and fluid queues.

Various approaches have been developed to study fluid models, mostly to determine the joint distribution of the buffer content and the phase of the background Markov process in the stationary regime:

$$F_j(x) = \lim_{t\to\infty} \mathrm{P}[X(t) \leq x, \varphi(t) = j]$$

for some nonnegative value of the level x and some phase j.

Methods using spectral analysis are probably the most traditional (see, for instance, Anick et al. [4], Kosten [28], Mitra [36], Stern and Elwalid [49], Van Foreest et al. [52, 53]). The equilibrium distribution of the state of the fluid process is described by a set of differential equations, and its solution is expressed in terms of linear combinations of exponentials of the eigenvalues of the system. The limitation of this approach comes from the fact that such eigenvalues are of both signs, and therefore numerical errors, no matter how small, may lead to solutions that are unstable: computed probabilities become negative or grow without bounds.

In spite of the numerical difficulties carried by the spectral approach, very little material exists which discusses these problems in detail. One may find some results in Fiedler and Voos [22], where it is shown how stable the spectral approach might perform provided that some precautions are taken.

Rogers [42] applies results on the Wiener-Hopf factorization of finite Markov chains and shows that the stationary distribution of the fluid buffer content has a matrix-exponential form. In order to compute it, one has to solve a Riccati equation, that is, a matrix equation of the form

$$XCX - AX - XD + B = 0,$$

where A, B, C and D are matrices and X is an unknown matrix. The author considers both cases of finite and infinite buffers, and explores

[2]The definition of a phase-type distribution is given in the Appendix.

algorithmic issues in a subsequent paper with Shi [43]. The conclusion there is that spectral methods are the most efficient.

Asmussen [5] shows that the buffer content has a phase-type stationary distribution. He obtains his results by constructing a new fluid queue, called the *dual fluid queue*, which is a stationary time-reversed version of the original one, and which we shall define later. Asmussen also considers fluid models with Brownian noise. He proposes an algorithmic procedure which is analyzed in Bean, O'Reilly and Taylor [10], and found there to be very efficient under special circumstances.

The state space of the background Markov process is usually assumed to be finite, except, for instance, in Van Doorn and Scheinhardt [50, 51], where the authors analyze fluid queues driven by an infinite-state Birth-and-Death process. Although the state space of the background process is allowed to be infinite, it has to satisfy some other constraints, making the model not completely general either.

Sericola and Tuffin [46] derive a direct approach which leads to simple recursions and to a stable algorithm for the computation of the stationary buffer content in an infinite capacity fluid queue. Sericola [45] generalizes this technique to the finite buffer case.

In her PhD Thesis, Barbot [7] studies fluid queues both in the transient and in the stationary regimes. For the transient case, Barbot derives the joint distribution of the infinite capacity fluid buffer level and the phase of the background Markov process, and studies the busy period of the fluid reservoir. Then, she considers a network of fluid buffers and determines the marginal distributions of the levels of the buffers. For the stationary regime, Barbot obtains the joint distribution of the fluid buffer level and the state of the underlying Markov process in a series form. Then, she considers a network of fluid queues, driven by a common background Markov process, and obtains an expression for the steady state distribution of the level of each reservoir in the case where the fluid buffers are controlled by a unique M/M/1 queue.

More recently, in 2004, Akar and Sohraby [3] study fluid queues with either finite or infinite buffers, using a novel algorithmic approach to solve numerically for the stationary solution of such processes. Their method does not rely on the computation of eigenvalues and eigenvectors, thus avoiding the numerical instability that this may create. They obtain a matrix-exponential form for the stationary distribution in the infinite buffer case, and a modified matrix-exponential form in the finite buffer case. The expressions *matrix-exponential form* and *modified matrix-exponential form* shall become clear throughout this work.

Meanwhile, in 1999, Ramaswami [39] extends the Markov-renewal approach, which he developed for Quasi Birth-and-Death (QBD) processes, to fluid queues.

A QBD process is somewhat similar to a fluid queue, as it is a two-dimensional Markov process of which the first component is called the *level*, and the second one is called the *phase*; one of the main differences is that the level of a QBD process is *discrete*, while the level of a fluid queue is *continuous*. The evolution of the state of a QBD process is such that, when in level n, the process either stays in level n, or it moves to levels $n-1$ or $n+1$. Jumps of more that one level, up or down, are not allowed. A more precise definition of a QBD process will be given later.

The Markov-renewal approach applied to fluid processes lead Ramaswami to obtain a matrix-exponential form for the joint distribution of the buffer content and the phase of the underlying Markov process in the stationary regime. Using the dual version of the given fluid process, he actually shows that this distribution has a phase-type representation. Quite significantly, Ramaswami relates the fluid model to a discrete time, discrete state space QBD process, and this leads to a very efficient computational procedure based on the Logarithmic-Reduction algorithm of Latouche and Ramaswami [29] for QBDs. By efficient, we mean that the algorithm is iterative, easy to implement, numerically stable and it converges quadratically (Bini, Latouche and Meini [11], Guo [25], Meini [34, 35]).

Ramaswami's paper, together with the idea of exploiting the similarity between fluid queues and QBD processes, are the starting point of our work. We further explore the relationship between the two processes and, avoiding the construction of a dual fluid queue, we obtain interesting results concerning simple fluid queues, with either finite or infinite buffers, and also more complex ones, like feedback fluid queues. We also study the necessary and sufficient conditions of independence between the two components of a fluid queue, the level and the phase, in the stationary regime. In some cases, we provide new proofs for known results, on the basis of renewal arguments; other results are new. Our approach reveals a great tractability and is always combined with a very efficient algorithmic procedure. Also, it leads to a unified approach of various fluid models. The results presented in this work have appeared, in large part, in [15, 16, 17, 18].

Building upon the probabilistic interpretation that we have given in [16] to Ramaswami's computational procedure, Ahn and Ramaswami [2] establish a direct connection by stochastic coupling between infinite buffer fluid queues and infinite QBDs. In the finite buffer case, Ahn, Jeon and Ramaswami [1] define a sequence of discrete-customer queues for which the stationary distribution of the work in the system converges to the stationary distribution of the fluid queue.

The Markov-renewal tool is powerful, and many pieces of work on fluid models using this kind of technique have appeared recently. For example, Bean, O'Reilly and Taylor [9] obtain expressions for return probabilities to the same level, Laplace-Stieltjes transforms and moments of the time taken to return to the initial level, excursion probabilities to high or low levels and Laplace-Stieltjes transforms of sojourn times in specified sets. The authors also provide physical interpretations of their results. As another example, Latouche and Takine [30] use the Markov-renewal approach to analyze fluid queues controlled by semi-Markov processes, and give a characterization of the stationary distribution of such systems.

The structure of this thesis is the following. In the first chapter, we focus on an infinite capacity buffer fluid queue, which will be often referred to as the *standard* model. The content of the buffer takes values in $\mathbb{R}^+$ and varies linearly at rate r_i each time the underlying Markov process $\{\varphi(t) : t \in \mathbb{R}^+\}$ is in state i. We first assume without loss of generality that all the rates r_i are equal to $+1$ and -1 only. This considerably simplifies the analysis and we show how to return to the general setting, in Section 1.7 using normalization arguments, then in Section 1.8 using a direct probabilistic approach.

In Section 1.2, we derive the set of differential equations satisfied by the steady-state density vector of the fluid buffer content. Using a Markov-renewal approach and level-crossing type arguments, we derive an expression for the stationary density of the buffer content in Section 1.3. It is expressed in terms of the steady state probability mass vector of being in level zero with a phase corresponding to a negative input rate, and of a matrix which records the expected number of visits to higher levels, starting from level zero, before returning to this initial level. The return probabilities to the initial level turn out to play a crucial role in our analysis. These probabilities are contained in a matrix denoted by Ψ, which is needed to determine the probability mass vector of level zero, as well as the matrix of expected number of visits. We give expressions for all these quantities and show that Ψ is the solution of a Riccati equation for which efficient algorithms exist, as we shall see later. Some of these results may also be found in da Silva Soares and Latouche [16].

In Section 1.9, we determine performance measures for the marginal distribution of the level of a fluid queue with arbitrary input rates.

We briefly present in the last three sections the approaches which have been followed by Ramaswami [39] and Rogers [42] to determine the stationary distribution of the standard fluid queue. We show that these results are in fact the same as ours.

We discuss in Chapter 2 algorithmic issues based on the link between fluid queues and Quasi-Birth-and-Death (QBD) processes. First, we formally define a QBD and recall the matrix-geometric property of its stationary distribution. We also make some comments about the similarity between QBDs and fluid queues.

In [39], Ramaswami presents a computational procedure to solve the Riccati equation for the matrix Ψ of first passage probabilities to the initial level, based on uniformization and on the Logarithmic-Reduction algorithm for QBD processes. We adapt in Section 2.2 this computational method and give it a probabilistic interpretation, which was presented in da Silva Soares and Latouche [16]. The algorithm in [39] is based on the uniformization of two Markov processes related to the fluid queue, the uniformization being performed using the same parameter for the two processes. In Section 2.3, we show that the discretization parameters need not to be the same and we derive a computational procedure in this case. We also show that the determination of the stationary distribution of the fluid buffer content does not depend on the way that the uniformization is carried out.

In Section 2.4, we give a few numerical examples. The computations are performed using both the methods presented in Sections 2.2 and 2.3 leading to the observation that there does not seem to be any real advantages in using one procedure rather than the other. We conclude by confirming this observation by a theoretical analysis of the effect on the number of iterations needed by the Logarithmic-Reduction algorithm produced by varying the values of the two uniformization parameters. These results were already presented in [14].

We describe in Section 2.6 other algorithms for solving the Riccati equation, as well as their probabilistic interpretation in the fluid flow setting. The material of this section is taken from Bean et al. [10]. From now on, we know that computations are entirely feasible and we may concentrate on the mathematical questions that arise in the different problems that we consider.

Having recognized the similarity between standard fluid queues and

QBD processes, we continue in this direction and analyze finite buffer fluid queues along the same lines. The main results of Chapter 3 were already presented in da Silva Soares and Latouche [18].

We start this chapter by defining a QBD process with finitely many levels. We recall the form of its stationary distribution, which is a mixture of two matrix-geometric vectors; it is expressed by means of the stationary probability vectors of the boundary levels and of matrices recording certain expected number of visits, without visiting the boundary levels.

In Section 3.2, we define the finite buffer fluid queue, and we derive in Section 3.3 an expression for the stationary distribution of the buffer content, using the same kind of arguments as those used in Section 1.3 for the infinite buffer case. The expression obtained gives the stationary distribution in terms of the probability mass vectors of the boundary levels and of matrices recording the average number of visits to some level x, without visiting the boundary levels. In Section 3.4, we show how to determine these matrices, which are expressed in terms of matrix exponentials. Then, in Section 3.5, we give two alternative ways of computing the boundary steady state probability mass vectors.

The arguments of these four sections on finite fluid queues require the assumption that the input rates are all equal to $+1$ and -1 only. In Section 3.6, we show that this is without any loss of generality and we give the solution in the general case where the input rates can take any real value.

We determine in Section 3.7 some performance measures for the marginal stationary distribution of the buffer content for the general fluid queue with a finite buffer; we apply these results on a numerical example in Section 3.8.

In Chapter 4, we study a different class of fluid models, in which the behaviour of the underlying phase process may change according to the value of the level of the buffer, and we determine the stationary distribution of the models considered. We start by giving one motivation for studying such systems, which is a model of an Internet congestion control protocol which may be found in Van Foreest et al. [52, 53].

The results of Sections 4.2 and 4.3 were presented in da Silva Soares and Latouche [17]. We analyse there an infinite buffer fluid queue, in which the behaviour of the phase process may change when the buffer is empty, and a finite buffer fluid queue in which the behaviour of the phase process may change when the buffer is either empty or full; in other words, we change the behaviour of the underlying process whenever the level reaches the boundary levels of the buffer. We show that it is quite straightforward to apply in this context the approach which we developed for traditional fluid queues with finite or infinite buffers. To simplify the analysis and the notations, we first assume that the input rates of fluid are equal to $+1$ and -1 only, and we show how to return to the general setting in Section 4.4. Then, in Section 4.5, we illustrate our results by a numerical example.

We then consider, in Sections 4.6 and 4.7, fluid queues of infinite capacity in which the behaviour of the phase process changes when the level reaches certain thresholds. The distinguishing feature between the models of these two sections is that in Section 4.7 we create states at the interior thresholds, which attract the fluid and carry a probability mass, and states which repel the fluid. Again, our renewal-type approach can be perfectly adapted to these situations.

We further exploit in Chapter 5 the similarity between fluid queues and QBD processes. Latouche and Taylor showed in [31] that it is always possible to define the boundary transition probabilities of a QBD in such a way that the level and the phase are independent in steady state. We construct here an infinite capacity fluid queue such that its two components, the level and the phase, are independent in the stationary regime. We do so by modifying in a very particular manner its behaviour at the boundary. The results presented in this chapter may also be found in da Silva Soares and Latouche [15].

We begin by showing in Section 5.2 that, under appropriate assumptions, the level of the fluid queue is asymptotically independent of the phase as the level goes to infinity.

In order to obtain the exact level-phase independence, we need to eliminate the steady state probability mass of level zero, as we explain in Section 5.3. We construct such a fluid queue without steady state probability mass associated to level zero and give the form of its stationary distribution.

In Section 5.4, we give the necessary and sufficient condition to have the level-phase independence in the stationary version, and we construct in Section 5.5 new transition rules for the fluid queue at level zero, which lead to the announced independence.

We conclude by a brief description of some perspectives for future research, opened by this thesis.

The following notations will be used throughout the text. Matrices will be denoted by capital letters, with I standing for the identity matrix, of which the dimension is made clear by the context. Vectors will be denoted by boldface lowercase letters. We will write $\mathbf{0}$ and $\mathbf{1}$ for vectors of zeros and ones, respectively, of the appropriate dimension. Furthermore, the reader will find the main notations used in this text in the corresponding table at the end of the monograph.

1

Fluid Queues with Infinite Buffers

In this chapter, we apply Markov-renewal techniques to derive the stationary density of the buffer content for a fluid queue with an infinite capacity buffer, and show that it has a matrix-exponential form.

We start in Section 1.1 by setting the context and, in Section 1.2, we give the set of differential equations satisfied by the joint density of the level and the phase of the system in both the transient and the stationary regimes. We then assume that the net input rates of fluid into the buffer may only take the values $+1$ and -1; we show later in Sections 1.7 and 1.8 how to return to the general setting.

In Section 1.3, we derive an expression for the stationary density vector of the system, and we study in Section 1.4 one of its main components: the matrix Ψ of first return probabilities to the initial level, which we show is the solution of a Riccati equation. This matrix is needed for the determination of the other ingredients of the stationary density of the system: a matrix of expected number of visits to higher levels, starting from level zero, before returning to the initial level, which is determined in Section 1.5; and the steady state probability mass vector of level zero, which is determined in Section 1.6 by relating it to the steady state probability vector of a censored process, which we call the process of *downward records*. In Section 1.9, we compute some performance measures of the system.

The results presented in Sections 1.3, 1.4, 1.6 and 1.7 have appeared, in large part, in da Silva Soares and Latouche [16, 18].

The last three sections are devoted to the results obtained by Ramaswami in [39] and by Rogers in [42]; we also show how these results relate to ours. Ramaswami obtains a matrix-exponential form for the stationary distribution of a fluid queue, using renewal-type arguments. Then, he constructs the dual fluid queue and shows that the steady state density of the buffer content of a fluid queue with nonzero net input rates has a phase-type representation. We present this result in Section 1.11 as well as an extension for the fluid queue with arbitrary net input rates, obtained without constructing the dual process. The results of Rogers, presented in the last section, are based on the Wiener-Hopf factorization for finite Markov chains.

1.1 Background

We consider a Markov modulated fluid queue, that is, a two-dimensional Markov process $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$, where

- $X(t) \in \mathbb{R}^+$ is called the *level* and represents the content of an infinite capacity fluid buffer at time t,
- and $\varphi(t)$ takes values in some finite set $\mathcal{S}$ and is called the *phase*; it is the state at time t of a Markov process which regulates the evolution of the buffer content.

This regulation is performed as follows: during intervals of time when $\varphi(t)$ is constant and equal to i, the level $X(t)$ varies linearly at the rate r_i, which can take any real value; when $X(t) = 0$ and the rate at time t is negative, the level remains at zero. The evolution of the buffer content can thus be expressed by the following equations:

$$\frac{dX(t)}{dt} = \begin{cases} r_{\varphi(t)}, & \text{if } X(t) > 0, \\ \max(0, r_{\varphi(t)}), & \text{if } X(t) = 0. \end{cases}$$

The underlying Markov process $\{\varphi(t) : t \in \mathbb{R}^+\}$ is assumed to be irreducible and to have a finite state space $\mathcal{S}$. We decompose the set $\mathcal{S}$ into three disjoint subsets $\mathcal{S}_0$, $\mathcal{S}_+$ and $\mathcal{S}_-$, where $\mathcal{S}_0 = \{i \in \mathcal{S} : r_i = 0\}$, $\mathcal{S}_+ = \{i \in \mathcal{S} : r_i > 0\}$ and $\mathcal{S}_- = \{i \in \mathcal{S} : r_i < 0\}$. Roughly speaking, we may say that $\mathcal{S}_0$ is the subset of phases which do not lead to a change of the buffer content, and $\mathcal{S}_+$ and $\mathcal{S}_-$ are the subsets of phases which make the level increase and decrease, respectively. Without any loss of generality, it is assumed throughout the text that $\mathcal{S}_+$ and $\mathcal{S}_-$ are nonempty sets. We denote by s_0, s_+ and s_-, respectively, the cardinalities of the subsets $\mathcal{S}_0$, $\mathcal{S}_+$ and $\mathcal{S}_-$.

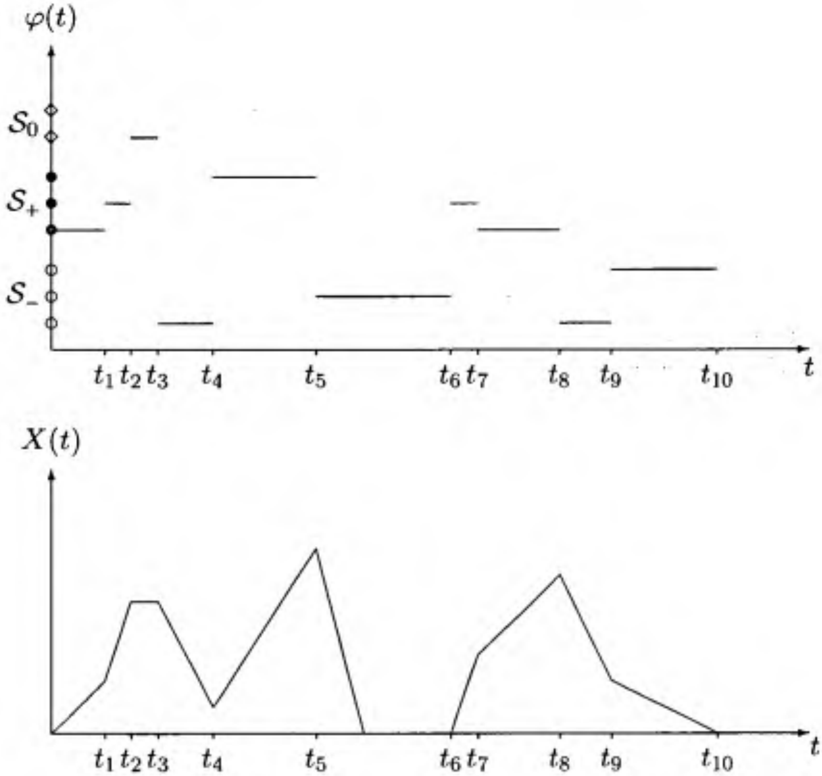


Figure 1.1: Possible evolution of the buffer content. The top graph depicts the evolution of the phase process, whereas the bottom graph depicts the evolution of the level.

Such a fluid process will often be referred to throughout the text as the *standard* fluid queue.

We illustrate one possible trajectory for the evolution of the fluid buffer content in Figure 1.1, which is to be interpreted as follows. The piecewise constant curve on the top graph shows how the phase evolves in time, and the piecewise linear curve on the bottom graph refers to the evolution of the fluid level with respect to time. In the interval $(0, t_1)$, the fluid level increases continuously since the phase belongs to $\mathcal{S}_+$. At time t_1, there is a phase transition, but the new phase is still in $\mathcal{S}_+$ and therefore the level continues to build up, but at a different rate. At time t_2, the phase changes to some phase in $\mathcal{S}_0$, and the level remains constant until time instant t_3, when the phase changes to $\mathcal{S}_-$ and the level starts to decrease, etc. Observe that in the interval (t_5, t_6), the phase is in

$\mathcal{S}_-$, therefore the level decreases and eventually reaches level zero. We maintain the level at zero until the background process switches to some phase in $\mathcal{S}_+$, at which time the level builds up again.

The infinitesimal transition generator of $\{\varphi(t)\}$ is denoted by T. Its stationary probability row vector is denoted by $\boldsymbol{\xi}$, is such that

$$\xi_i = \lim_{t\to\infty} \mathrm{P}[\varphi(t) = i|\varphi(0) = j]$$

for all $i, j \in \mathcal{S}$ and it is the unique solution of the system

$$\begin{cases} \boldsymbol{\xi} T &= \mathbf{0} \\ \boldsymbol{\xi}\mathbf{1} &= 1. \end{cases}$$

1.2 Differential Equations

This section is devoted to a classical result in the literature about fluid flow models, giving a set of partial differential equations satisfied by the joint density of the level and the phase at time t. The proof provided here is inspired from the elegant one that can be found in the PhD Thesis of Barbot [7, Section 1.3].

For j in $\mathcal{S}$ and $x \geq 0$, define the joint distribution of the level and the phase at time t by

$$F_j(x;t) = \mathrm{P}[X(t) \leq x, \varphi(t) = j],$$

and its density by

$$f_j(x;t) = \frac{\partial}{\partial x} F_j(x;t)$$

for $x > 0$, $j \in \mathcal{S}$, with $f_j(0;t) = \lim_{x\to 0^+} f_j(x;t)$ being defined by continuity. The stationary density vector $\boldsymbol{\pi}(x) = (\pi_i(x) : i \in \mathcal{S})$ of the fluid buffer content is obtained by taking the limit as t goes to infinity of the density function:

$$\pi_i(x) = \lim_{t\to\infty} f_i(x;t).$$

It exists if and only if the *mean stationary drift* of fluid into the buffer is negative, that is, if and only if $\boldsymbol{\xi}\boldsymbol{r} < 0$, where $\boldsymbol{r}$ is the column vector with components r_i for $i \in \mathcal{S}$.

The following theorem gives the set of partial differential equations satisfied by the joint density of the level and the phase at time t of a fluid queue.

Theorem 1.2.1 *For all $j \in \mathcal{S}$ and for $x > 0$, the density functions $f_j(x;t)$ are a solution of the system of partial differential equations*

$$\frac{\partial}{\partial t} f_j(x;t) = \sum_{i \in \mathcal{S}} f_i(x;t) T_{ij} - r_j \frac{\partial}{\partial x} f_j(x;t). \tag{1.1}$$

Proof Take $j \in \mathcal{S}$, $x \geq 0$ and $t > 0$. Define

$$\begin{aligned} G_j(x;t) &= \mathrm{P}[X(t) > x, \varphi(t) = j] \\ &= \mathrm{P}[\varphi(t) = j] - F_j(x;t). \end{aligned} \tag{1.2}$$

We first show that the function $G_j(x;t)$ is a solution of the differential equation (1.1).

Define, for $h > 0$, $N_{t,t+h}$ as the number of state changes of the Markov process $\{\varphi(t)\}$ in the interval $[t, t+h]$. We clearly have that

$$\mathrm{P}[N_{t,t+h} \geq 2 | \varphi(t) = j] = o(h) \tag{1.3}$$

where the notation $o(h)$ has the usual meaning $\lim_{h \to 0} o(h)/h = 0$.

Since there is a finite number of phases in $\mathcal{S}$, there exists a real value m_j such that

$$m_j = \sup_{i \in \mathcal{S}} |r_i|.$$

From (1.3), we can write

$$\begin{aligned} &G_j(x + m_j h; t + h) \\ &= \mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 0] \\ &\quad + \mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 1] + o(h). \end{aligned} \tag{1.4}$$

The first term on the right-hand side of (1.4) yields

$$\begin{aligned} &\mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 0] \\ &\quad = \mathrm{P}[\max(0, X(t) + r_j h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 0] \end{aligned}$$

by observing that, since the process $\{\varphi(t)\}$ stays in the phase j during the whole interval $[t, t+h]$, the level $X(t+h)$ reached at the end of this interval is $X(t) + r_j h$ if this quantity is positive, and zero otherwise. We may also write

$$\begin{aligned} &\mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 0] \\ &= \mathrm{P}[X(t) > x + (m_j - r_j)h, \varphi(t) = j, N_{t,t+h} = 0] \\ &= \mathrm{P}[N_{t,t+h} = 0 | X(t) > x + (m_j - r_j)h, \varphi(t) = j] G_j(x + (m_j - r_j)h; t) \\ &= \mathrm{P}[N_{t,t+h} = 0 | \varphi(t) = j] G_j(x + (m_j - r_j)h; t). \end{aligned}$$

This last equality follows from the fact that the number of events for the background Markov process is conditionally independent of $\{X(t)\}$ given the phase at time t. The probability that $\{\varphi(t)\}$ stays in phase j for an interval of time of length at least equal to h is $e^{T_{jj}h}$, and we therefore obtain

$$\begin{aligned} &\mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 0] \\ &\qquad = (1 + T_{jj}h) G_j(x + (m_j - r_j)h; t) + o(h). \end{aligned}$$

The second term on the right-hand side of (1.4) verifies the inequality

$$\begin{aligned} &\mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 1] \\ &\qquad \leq \mathrm{P}[X(t) > x, \varphi(t+h) = j, N_{t,t+h} = 1]. \end{aligned}$$

Indeed, if $\varphi(t+h) = j$ and if there is only one state change of $\{\varphi(t)\}$ in the interval $[t, t+h]$, then the net input rate of fluid into the buffer during this interval cannot be bigger than m_j, thus

$$\begin{aligned} &\mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 1] \\ &\qquad \leq \mathrm{P}[X(t) > x, \varphi(t+h) = j, N_{t,t+h} = 1]. \end{aligned}$$

Therefore, we may write

$$\begin{aligned} &\mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 1] \\ &\qquad \leq \sum_{\substack{i \in \mathcal{S} \\ i \neq j}} \mathrm{P}[\varphi(t+h) = j, N_{t,t+h} = 1 | \varphi(t) = i] G_i(x; t) \\ &\qquad = \sum_{\substack{i \in \mathcal{S} \\ i \neq j}} T_{ij} h G_i(x; t) + o(h). \end{aligned}$$

On the other hand, m_j being positive,

$$\begin{aligned} &\mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 1] \\ &\qquad \geq \mathrm{P}[X(t) > x + 2m_j h, \varphi(t+h) = j, N_{t,t+h} = 1] \\ &\qquad = \sum_{\substack{i \in \mathcal{S} \\ i \neq j}} T_{ij} h G_i(x + 2m_j h; t) + o(h). \end{aligned}$$

Thus, since $G_j(\cdot; t)$ is right-continuous, we deduce that

$$\lim_{h \to 0^+} \frac{1}{h} \mathrm{P}[X(t+h) > x + m_j h, \varphi(t+h) = j, N_{t,t+h} = 1] = \sum_{\substack{i \in \mathcal{S} \\ i \neq j}} T_{ij} G_i(x; t).$$

Rewriting (1.4), one finds

$$\begin{aligned} & G_j(x+m_jh;t+h) \\ &= (1+T_{jj}h)G_j(x+(m_j-r_j)h;t) \\ &\quad +\mathrm{P}[X(t+h)>x+m_jh, \varphi(t+h)=j, N_{t,t+h}=1]+o(h). \end{aligned}$$

Subtracting $G_j(x+m_jh;t)$ on both sides, dividing by h and taking the right limit as h goes to zero, we obtain

$$\begin{aligned} \lim_{h\to 0^+} \frac{1}{h}(G_j(x+m_jh;t+h) - G_j(x+m_jh;t)) & \\ = \lim_{h\to 0^+} \frac{1}{h}(G_j(x+(m_j-r_j)h;t) - G_j(x+m_jh;t)) & \\ + G_j(x;t)T_{jj} + \sum_{\substack{i\in\mathcal{S}\\ i\neq j}} G_i(x;t)T_{ij}. & \end{aligned}$$

It follows that

$$\frac{\partial}{\partial t}G_j(x;t) = \sum_{i\in\mathcal{S}} G_i(x;t)T_{ij} - r_j\frac{\partial}{\partial x}G_j(x;t).$$

Replacing $G_j(x;t)$ by its expression (1.2), we may write

$$\begin{aligned} \frac{\partial}{\partial t}\mathrm{P}[\varphi(t)=j] - \frac{\partial}{\partial t}F_j(x;t) &= \sum_{i\in\mathcal{S}} \mathrm{P}[\varphi(t)=j]T_{ij} - \sum_{i\in\mathcal{S}} F_i(x;t)T_{ij} \\ & \quad + r_j\frac{\partial}{\partial x}F_j(x;t), \end{aligned}$$

which yields

$$\frac{\partial}{\partial t}F_j(x;t) = \sum_{i\in\mathcal{S}} F_i(x;t)T_{ij} - r_j\frac{\partial}{\partial x}F_j(x;t) \tag{1.5}$$

because

$$\frac{\partial}{\partial t}\mathrm{P}[\varphi(t)=j] = \sum_{i\in\mathcal{S}} \mathrm{P}[\varphi(t)=j]T_{ij}.$$

Taking the partial derivative with respect to x on both sides of (1.5) and using the fact that $\partial/\partial x F_j(x;t) = f_j(x;t)$, the result follows. □

Upon letting t go to infinity in (1.1), we obtain the following immediate corollary.

Corollary 1.2.2 *For $x > 0$ and for $j \in \mathcal{S}$, the stationary density functions $\pi_j(x)$ solve the following system of differential equations:*

$$-r_j \frac{d}{dx}\pi_j(x) + \sum_{i \in \mathcal{S}} \pi_i(x) T_{ij} = 0. \tag{1.6}$$

□

Remark 1.2.3 The Markov-renewal approach followed in this work is justified by the fact that finding the numerical solution of (1.6) by spectral methods is an ill-behaved problem. Indeed, rewrite (1.6) in matrix form as

$$\frac{d}{dx}\boldsymbol{\pi}(x) = \boldsymbol{\pi}(x) T C^{-1}$$

where $C = \mathrm{diag}(\boldsymbol{r})$ is a diagonal matrix such that $C_{ii} = r_i$ for all i in $\mathcal{S}$; it is assumed here that $r_i \neq 0$ for all i in $\mathcal{S}$. The corresponding eigenvalue problem is $\lambda \boldsymbol{u} = \boldsymbol{u} T C^{-1}$, where λ and $\boldsymbol{u}$ denote an eigenvalue and the corresponding left eigenvector of the matrix TC^{-1}. If the stability condition $\boldsymbol{\xi r} < 0$ is satisfied, independent solutions of (1.6) are of the form $\sum_i c_i x^{k_i} e^{\lambda_i x} \boldsymbol{u}_i$ (see Barbot [7], for instance), where the λ_i's have a strictly negative real part. The eigenvalues of the system being computed with finite precision, approximation errors may appear, leading to solutions that are numerically unstable and to results that do not have a physical interpretation. As we shall see later, our solution is numerically stable, because only eigenvalues with a non positive real part are present.

1.3 Stationary Density

We assume that all the net input rates of fluid into the buffer are different from zero and that they are all equal to $+1$ or -1. That is, we assume that $\mathcal{S}_0$ is empty and that $\mathcal{S}$ is partitioned into $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$, where $\mathcal{S}_+ = \{i \in \mathcal{S} : r_i = +1\}$ and $\mathcal{S}_- = \{i \in \mathcal{S} : r_i = -1\}$. This assumption is without loss of generality, for we show in Sections 1.7 and 1.8 how to obtain the distribution in the general setting, once one has the distribution for this simplified process.

We depict in Figure 1.2 one possible evolution of the buffer content in this context. This figure, as well as all the figures throughout this chapter, is to be interpreted as follows. We assume that there are four phases in all, with $\mathcal{S}_+ = \{1, 2\}$ and $\mathcal{S}_- = \{3, 4\}$. The graph is drawn

Figure 1.2: Possible evolution of the buffer content when the net input rates are equal to 1 or -1.

with a thin line when the phase is 1 or 3, with a thick line otherwise. We see that the process is in phase 1 at time 0, then it jumps to phase 2, then to phase 3, at which time the level begins to decrease, then to phase 2 again, at which time the level starts to build up again, etc. At the time when the level becomes zero with a phase in $\mathcal{S}_-$, then it remains at zero until the background process switches to some phase in $\mathcal{S}_+$.

We partition matrices and vectors in a manner conformant to the decomposition of $\mathcal{S}$. Thus,

$$T = \begin{bmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{bmatrix},$$

that is, T_{++} contains the components T_{ij} for $i, j \in \mathcal{S}_+$, T_{+-} contains the components T_{ij} for $i \in \mathcal{S}_+$ and $j \in \mathcal{S}_-$, and so on. Similarly, we write $\boldsymbol{\xi} = (\boldsymbol{\xi}_+, \boldsymbol{\xi}_-)$.

In this setting, the stability condition can be determined by the sign of the difference $\boldsymbol{\xi}_+\mathbf{1} - \boldsymbol{\xi}_-\mathbf{1}$, which we call the *mean stationary drift* of the fluid queue and denote by μ. If $\mu < 0$, the stationary density of the buffer content exists and the fluid queue is positive recurrent. It is null recurrent if $\mu = 0$ and transient if $\mu > 0$.

We now introduce a matrix Ψ which plays a key role in what follows. For i in $\mathcal{S}_+$ and j in $\mathcal{S}_-$, we denote by Ψ_{ij} the probability that, starting from $(0, i)$ at time 0, the fluid queue returns to the level zero in a finite amount of time and does so in phase j. More precisely, if we define $\theta = \inf\{t > 0 : X(t) = 0\}$ as the first return time to level zero,

$$\Psi_{ij} = \mathrm{P}[\theta < \infty \text{ and } \varphi(\theta) = j | X(0) = 0, \varphi(0) = i].$$

If the fluid queue is recurrent, this return time is finite almost surely and thus we have that $\Psi\mathbf{1} = \mathbf{1}$.

The transition rates and the net input rates are independent of the value x of the level, for $x > 0$; this property will be referred to throughout the text as the *spatial homogeneity* of the process. It implies that, for any $x \geq 0$, Ψ_{ij} is also the probability that, starting from (x, i) at time 0, the fluid queue returns to the level x, in phase j, in a finite amount of time.

The fluid process has a stationary density for strictly positive values of the buffer content, and a stationary probability mass vector for level zero. Denote by $\boldsymbol{p} = (\boldsymbol{p}_+, \boldsymbol{p}_-)$ the steady state probability mass vector of the empty buffer:

$$p_i = \lim_{t\to\infty} \mathrm{P}[X(t) = 0, \varphi(t) = i],$$

for each i in $\mathcal{S}$. Since the fluid queue instantaneously leaves the level zero if the phase is in $\mathcal{S}_+$, the sub-vector $\boldsymbol{p}_+ = \{p_i : i \in \mathcal{S}_+\}$ is equal to $\mathbf{0}$, and therefore $\boldsymbol{p} = (\mathbf{0}, \boldsymbol{p}_-)$.

We give in the next theorem an expression for the stationary density vector $\boldsymbol{\pi}(x)$ of the buffer content. It is expressed in terms of the steady state probability mass vector of the boundary level of the buffer and of a matrix recording certain expected number of visits. For readers familiar with QBD processes, the similarity between the stationary distributions of the two families of processes should now become apparent; this will be discussed later. We define $\varphi(\tau - 0) = \lim_{t\to\tau, t<\tau} \varphi(t)$.

Theorem 1.3.1 *If $\mu < 0$, then the stationary density of the infinite buffer fluid queue is given by*

$$\boldsymbol{\pi}(x) = \boldsymbol{p}_- T_{-+} N(x) \tag{1.7}$$

for $x > 0$, where, for $i \in \mathcal{S}_+$ and $j \in \mathcal{S}$, $[N(x)]_{ij}$ is the expected number of crossings of (x, j), starting from $(0, i)$, before the first return to the level zero.

Proof Assume that $X(0) = 0$ and take $j \in \mathcal{S}$. By a decomposition based on the last visit to level zero before time t, we find that $(X(t), \varphi(t)) = (x, j)$ in one of two cases:

- either there exist some time $\tau < t$ and $i \in \mathcal{S}_-$ such that $X(t-\tau) = 0$ and $\varphi((t - \tau) - 0) = i$, at time $t - \tau$ the phase changes from $\mathcal{S}_-$ to $\mathcal{S}_+$ and the fluid starts building up, and in the interval $(t - \tau, t)$, it continuously remains above level zero (see Figure 1.3),

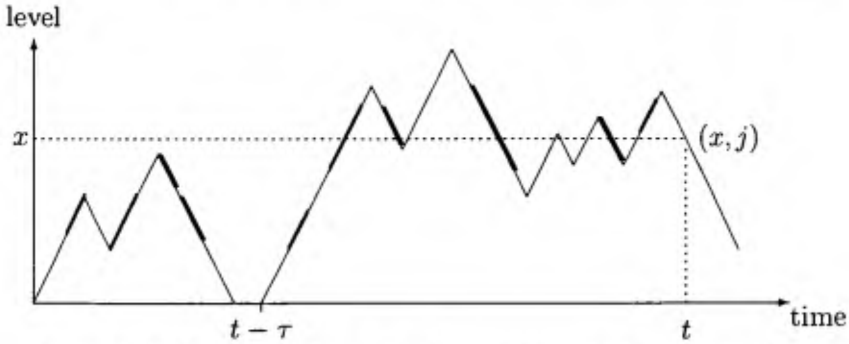


Figure 1.3: The last visit to level zero occurs at time $t-\tau > 0$.

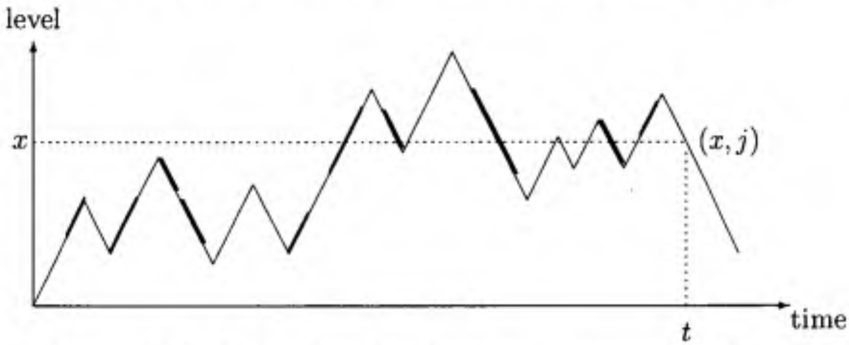


Figure 1.4: The last visit to level zero occurs at time 0.

- or in the interval $[0, t)$, the fluid level continuously remains above level zero, which can only happen if $\varphi(0)$ is in $\mathcal{S}_+$ (see Figure 1.4).

Hence, we have

$$\begin{aligned} f_j(x;t) &= \sum_{\substack{i \in \mathcal{S}_- \\ k \in \mathcal{S}_+}} \int_0^t F_i(0; t-\tau)(T_{-+})_{ik}[\phi(x;\tau)]_{kj} d\tau \\ &\quad + \sum_{i \in \mathcal{S}_+} \mathrm{P}[\varphi(0) = i][\phi(x;t)]_{ij}, \end{aligned} \tag{1.8}$$

where $[\phi(x;t)]_{kj} = \partial / \partial x \, F^{(0)}_{kj}(x;t)$, with $F^{(0)}_{kj}(x;t)$ being the conditional probability, given that the process starts in $(0, k)$, that it remains above level zero in the interval $(0, t)$ and that it is at a level at most equal to x, in phase $j \in \mathcal{S}$ at time t.

The integral $[N(x)]_{kj} = \int_0^\infty [\phi(x;\tau)]_{kj} d\tau$ is then the expected number of crossings of level x in phase j, starting from $(0, k)$, before the first return to level zero. Since the process is ergodic, the expected return time to level zero is finite, so that $[N(x)]_{kj} < \infty$ for all k and j.

This implies, on the one hand, that $\lim_{t\to\infty} \phi(x;t) = 0$, so that the second term in the right-hand side of (1.8) converges to zero as $t \to \infty$; on the other hand, since $F_i(0;t)$ and $(T_{-+})_{ik}$ are bounded and $\lim_{t\to\infty} F_i(0;t) = p_i$ exists, we have by the dominated convergence theorem

$$\lim_{t\to\infty} \int_0^t F_i(0;t-\tau)(T_{-+})_{ik}[\phi(x;\tau)]_{kj} d\tau = [\lim_{t\to\infty} F_i(0;t)](T_{-+})_{ik}[N(x)]_{kj}.$$

Replacing this in (1.8) and using matrix notations, we obtain (1.7). □

We are now interested in showing that the quantity $N(x)$ has a matrix-exponential form. We write $N(x) = [N_{++}(x),\ N_{+-}(x)]$.

Theorem 1.3.2 *For $x > 0$, the matrix $N(x)$ giving the expected number of crossings of level x, starting from level zero, before returning to level zero, is given by*

$$N(x) = e^{Kx}[I,\ \Psi] \tag{1.9}$$

for some matrix K, which will be given later.

Remark 1.3.3 We are not concerned here with the determination of the matrix K itself, but mainly on showing that the expected number of visits $N(x)$ has a matrix-exponential form. A precise expression of K will be given in Section 1.5.

Proof We first show that

$$N_{++}(x+y) = N_{++}(x)N_{++}(y). \tag{1.10}$$

Take $x,\ y > 0$ arbitrary but fixed, and define $[V(x+y)]_{ij}$, for $i, j \in \mathcal{S}_+$, as the number of visits to $(x+y, j)$, starting from $(0, i)$, before the first return to the level zero. We thus have

$$\mathrm{E}[V(x+y)] = N_{++}(x+y).$$

Also, define $[W(x+y|x)]_{ij}$, for $i, j \in \mathcal{S}_+$, as the number of visits to $(x+y, j)$, starting from (x, i), before the first return to the level x. By the spatial homogeneity of the process, $[W(x+y|x)]_{ij}$ has the same distribution as the number of visits to (y, j), starting from $(0, i)$, before the first return to the level 0, and therefore

$$\mathrm{E}[W(x+y|x)] = \mathrm{E}[V(y)] = N_{++}(y).$$

Starting from level zero and in order to cross level $x + y$ from below, the process has first to cross level x in a phase of $\mathcal{S}_+$. We organize the visits to $(x + y, \mathcal{S}_+)$, starting from level zero, into several groups: the visits which occur after the first passage through $(x, \mathcal{S}_+)$ but before the second, those which occur after the second passage through $(x, \mathcal{S}_+)$ but before the third, and so on. By the strong Markov property, we find that

$$\begin{aligned} \mathrm{E}\{[V(x+y)]_{ij}\} &= \sum_{k\in\mathcal{S}_+} \mathrm{E}\{[V(x)]_{ik}\}\,\mathrm{E}\{[W(x+y|x)]_{kj}\} \\ &= \sum_{k\in\mathcal{S}_+} [N_{++}(x)]_{ik}[N_{++}(y)]_{kj}, \end{aligned}$$

which, written in matrix form, yields (1.10).

Equation (1.10) shows that $N_{++}(x)$ must be a matrix-exponential, thus, since $N_{++}(0) = I$, we write that

$$N_{++}(x) = e^{Kx}$$

for some matrix K.

One proves in a similar manner that

$$N_{+-}(x+y) = N_{++}(x)N_{+-}(y). \tag{1.11}$$

Taking the right limit as y goes to zero in (1.11), one obtains that

$$N_{+-}(x) = N_{++}(x) \lim_{y\to 0^+} N_{+-}(y),$$

which gives

$$N_{+-}(x) = e^{Kx}\Psi,$$

since $\lim_{y\to 0^+} N_{+-}(y)$ is the expected number of visits to a state in $(0, \mathcal{S}_-)$, starting from $(0, \mathcal{S}_+)$, before the first return to the level zero, and is therefore equal to Ψ. This concludes the proof. □

Theorem 1.3.1 and Theorem 1.3.2 together lead to the following straightforward corollary.

Corollary 1.3.4 *If $\mu < 0$, then the stationary density of the infinite buffer fluid queue is given by*

$$\boldsymbol{\pi}(x) = \boldsymbol{p}_{-}T_{-+}e^{Kx}[I,\ \Psi] \tag{1.12}$$

for $x > 0$.

□

In order to actually compute $\boldsymbol{\pi}(x)$, we need to know $\boldsymbol{p}_{-}$, K and Ψ. In fact, we shall see later that the first two quantities are expressed in terms of the latter, thus our main purpose now is to concentrate on the matrix Ψ of first return probabilities to the initial level.

1.4 First Return Probabilities

We first introduce the process $\{D(t) : t \in \mathbb{R}^+\}$, which we call the process of *downwards records*. It is a censored process obtained by restricting $\{\varphi(t)\}$ to those epochs of time during which the phase is in $\mathcal{S}_{-}$ and the level reaches temporary record low levels.

More precisely, let $\{(Y(t), \varphi(t)) : t \in \mathbb{R}^+\}$ be the unconstrained random walk defined by

$$Y(t) = \int_0^t r_{\varphi(s)} ds. \tag{1.13}$$

Clearly, $Y(t)$ takes both positive and negative values and we may think of it as describing the evolution of a fluid queue with a bottomless buffer. The process of downward records $\{D(t)\}$ corresponds to the phase process observed only during those intervals of time in which $Y(t) = \min_{0 \le u \le t} Y(u)$.

We illustrate it in Figure 1.5, which we now explain. Take t_0 arbitrary, such that $\varphi(t_0)$ is in $\mathcal{S}_{-}$. Define the sequence $\{y_k, d_k, t_k : k \in \mathbb{N}\}$ as follows: $y_0 = Y(t_0)$, $d_k = \inf\{t > t_k : \varphi(t) \in \mathcal{S}_{+}\}$, $y_{k+1} = Y(d_k)$, and $t_{k+1} = \inf\{t > d_k : Y(t) = y_{k+1}, \varphi(t) \in \mathcal{S}_{-}\}$. During the intervals (t_k, d_k), the fluid is steadily decreasing; at time d_k, a temporary record low level y_{k+1} is reached and the fluid begins to build up; during the interval (d_k, t_{k+1}), the fluid goes up and down until time t_{k+1} when it reaches its previous record y_{k+1}. The process $\{D(t)\}$ is obtained if one excises the intervals (d_k, t_{k+1}) and only keeps track of the phase during the intervals (t_k, d_k). This is the reason why in Figure 1.5 we only indicate the phase changes during the intervals (t_k, d_k). We project the

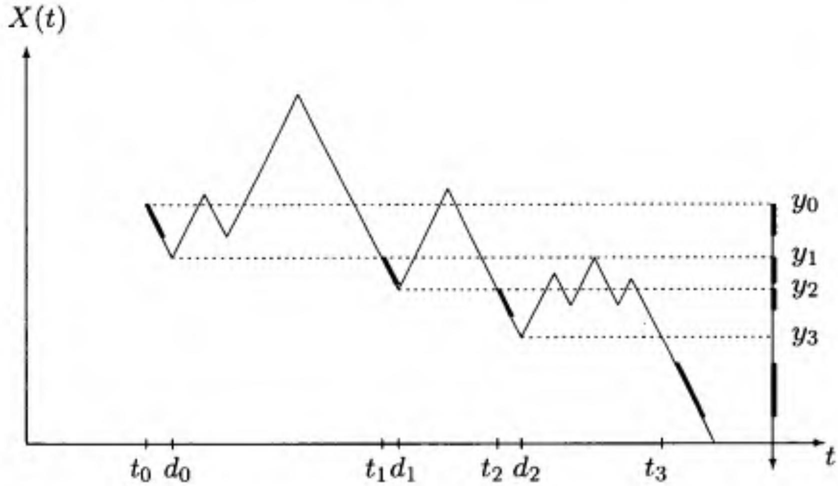


Figure 1.5: Illustration of the process of downward records.

phases on the vertical line on the right, marked with an arrow to indicate the direction of the flow.

Theorem 1.4.1 *The matrix*

$$U = T_{--} + T_{-+}\Psi \tag{1.14}$$

is the infinitesimal transition generator of the process of downward records $\{D(t) : t \in \mathbb{R}^+\}$.

Proof Take $i, j \in \mathcal{S}_-$ and assume that $D(t) = i$ for some t. Over an interval of length h, the phase may directly move from i to j, with probability $T_{ij}h + o(h)$, or it may move to some phase $k \in \mathcal{S}_+$, with probability $T_{ik}h+o(h)$, at which time the level starts to increase. Assume that this happens at some level $x \geq 0$. Later, the fluid queue returns to the level x in phase j, with probability Ψ_{kj}. Summing over all k, we find that U is given by (1.14). □

Remark 1.4.2 An immediate consequence of Theorem 1.4.1 is the following. Since U is the generator of the process of downward records, we have that $(e^{Uh})_{ij}$, for $i, j \in \mathcal{S}_-$, is the probability that, starting from (y, i), for any y, the process reaches level $y - h$ in finite time, and that

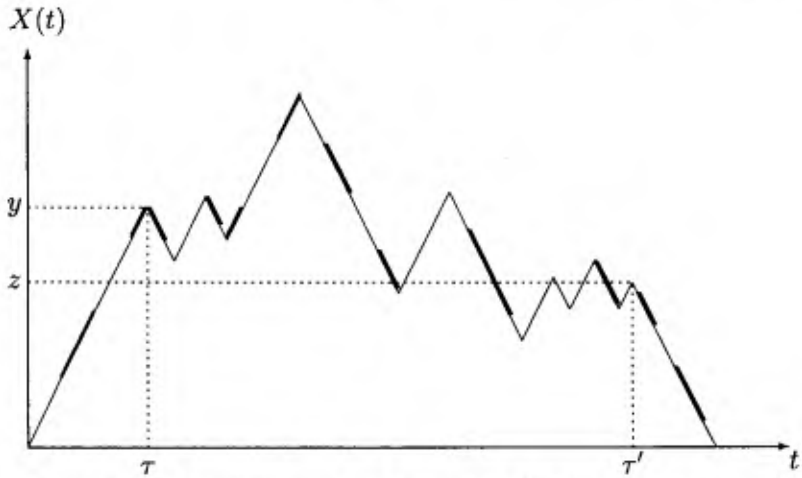


Figure 1.6: Conditioning on the end of the first slope upward or on the beginning of the last downturn.

$(y-h, j)$ is the first state visited in level $y-h$. Denote by $[G_{--}(x)]_{ij}$, for $i, j \in \mathcal{S}_-$, the first passage probability from state (x, j) to state $(0, i)$:

$$[G_{--}(x)]_{ij} = \mathrm{P}[\theta < \infty \text{ and } \varphi(\theta) = j | X(0) = x, \varphi(0) = i],$$

with θ being the first epoch when the fluid level becomes zero, already defined in the preceding section. It follows that $G_{--}(x) = e^{Ux}$. This will be important in the sequel.

If the fluid queue is recurrent, the bottomless process $\{Y(t)\}$ drifts to $-\infty$. In this case, e^{Ux} is a stochastic matrix and U is singular. If, on the contrary, the fluid queue is transient, the process $\{Y(t)\}$ drifts to $+\infty$, the matrix e^{Ux} is sub-stochastic and U is nonsingular.

We derive in the next theorem an expression for the matrix Ψ, obtained by a Markov-renewal type argument which consists on conditioning on the first transition from a phase in $\mathcal{S}_+$ to a phase in $\mathcal{S}_-$.

Theorem 1.4.3 *The matrix Ψ of first return probabilities to the initial level is given by*

$$\Psi = \int_0^\infty e^{T_{++}y} T_{+-} e^{Uy} dy. \tag{1.15}$$

Proof Assume that $X(0) = 0$. Starting in $(0, \mathcal{S}_+)$ at time 0, the fluid queue returns to the level zero at a time which is positive and finite if and only if the following event takes place (see Figure 1.6 for an illustration): there exist a time τ and a level y such that

- $\varphi(t)$ is in $\mathcal{S}_+$ for $0 < t < \tau$, with probabilities given by the elements of the matrix $e^{T_{++}\tau}$,
- $X(\tau) = y$,
- at time τ, the phase changes from $\mathcal{S}_+$ to $\mathcal{S}_-$, with probabilities given by the elements of $T_{+-}d\tau$, and
- the queue returns to the level zero in a finite time afterwards; this event has probability $G_{--}(y)$ by definition of this matrix.

Since $r_i = +1$ for all $i \in \mathcal{S}_+$, necessarily τ and y are equal, and

$$\Psi = \int_0^\infty e^{T_{++}y} T_{+-} G_{--}(y) dy.$$

By Remark 1.4.2, $G_{--}(y)$ is equal to e^{Uy}, and the proof is completed. □

As a consequence of this theorem, we find that the matrix Ψ is the solution of a nonsymmetric algebraic Riccati equation, that is an equation of the form

$$XCX - AX - XD + B = 0,$$

where A, B, C, D and X are real matrices of sizes $m \times m$, $m \times n$, $n \times m$, $n \times n$ and $m \times n$, respectively.

Corollary 1.4.4 *The matrix Ψ is the solution of the Riccati equation*

$$\Psi T_{-+} \Psi + T_{++} \Psi + \Psi T_{--} + T_{+-} = 0. \tag{1.16}$$

Proof The proof is purely algebraic. Pre-multiplying both sides of (1.15) by T_{++} and integrating by parts, one obtains

$$\begin{aligned} T_{++}\Psi &= \int_0^\infty T_{++} e^{T_{++}y} T_{+-} e^{Uy} dy \\ &= \left[e^{T_{++}y} T_{+-} e^{Uy}\right]_0^\infty - \int_0^\infty e^{T_{++}y} T_{+-} e^{Uy} U dy \\ &= -T_{+-} - \Psi U. \end{aligned}$$

Indeed, $\lim_{y\to\infty} e^{T_{++}y} = 0$ because T_{++} is the infinitesimal generator of a transient Markov process, and e^{Uy} is finite by the interpretation of e^{Uy} given in Remark 1.4.2.

Recalling that $U = T_{--} + T_{-+}\Psi$, we obtain the announced result. □

Ramaswami introduced in [39] a very efficient algorithm for the computation of Ψ, based on QBD processes and for which we gave a probabilistic interpretation in [16]. Furthermore, several computational procedures are developed in Guo [24] to solve Riccati equations, and in Bean et al. [10] specifically for (1.15). A discussion of the algorithm of Ramaswami and its probabilistic interpretation, and of some other algorithmic issues, is postponed to Chapter 2.

To complete this section on first return probabilities, we derive another expression for the matrix Ψ, which was already given in Ramaswami [39]. We shall give in Section 1.10 the approach used by Ramaswami to obtain this expression; our proof is similar to that of Theorem 1.4.3, but here we condition on the beginning of the last downturn, that is, on the last transition from a phase in $\mathcal{S}_+$ to a phase in $\mathcal{S}_-$, instead of on the first.

Theorem 1.4.5 *The matrix Ψ of first return probabilities to the initial level is given by*

$$\Psi = \int_0^\infty e^{Kz} T_{+-} e^{T_{--}z} dz, \tag{1.17}$$

where e^{Kz} is defined in Theorem 1.3.2.

Proof Assume that $X(0) = 0$. Starting from $(0, \mathcal{S}_+)$ at time 0, the queue returns to the level zero in a finite time if and only if the following event holds (see Figure 1.6 for an illustration): there exist a time τ' and a level z such that

- $X(h) > 0$ for $0 < h < \tau'$,
- $X(\tau') = z$ and $\varphi(\tau' - 0)$ is in $\mathcal{S}_+$,
- at time τ' the phase changes to $\mathcal{S}_-$,
- after time τ', the phase remains in $\mathcal{S}_-$ at least until the level returns to zero.

Since $r_i = -1$ for all i in $\mathcal{S}_-$, it takes exactly z units of time for the fluid queue to become empty if it starts in $(z, \mathcal{S}_-)$ and continuously remains in $\mathcal{S}_-$. Therefore, we find that

$$\Psi = \int_0^\infty \int_0^\infty \phi(z; \tau') T_{+-} e^{T_{--}z} dz d\tau',$$

where, for $i \in \mathcal{S}_+$ and $j \in \mathcal{S}$, $[\phi(z;\tau')]_{ij} = \partial/\partial z\, F_{ij}^{(0)}(z;\tau')$ with $F_{ij}^{(0)}(z;\tau')$ being the conditional probability, given that the process starts in $(0,i)$, that it remains above level zero in the interval $(0,\tau')$ and that it is at a level at most equal to z, in phase $j \in \mathcal{S}$ at time τ'.

The integral $\int_0^\infty \phi(z;\tau')d\tau'$ is the expected number of crossings of level z, starting from level zero, and avoiding level zero, and is therefore equal to e^{Kz} by Theorem 1.3.2. □

1.5 Expected Number of Crossings

As shown in Section 1.3, the average number of visits $N(x)$ to any positive level x, given that the fluid process starts in level zero, before returning to this initial level, has the matrix-exponential form $N(x) = [e^{Kx}, e^{Kx}\Psi]$ for some matrix K, and for Ψ given in Section 1.4. Following the same lines as in [39], we now determine the matrix K. For this purpose, we need a preliminary lemma stating the Kolmogorov differential equations for the taboo process avoiding level zero. Recall the definition of $[\phi(x;t)]_{ij}$ given in the proof of Theorem 1.3.1, and denote by $\phi_{++}(x,t)$ and $\phi_{+-}(x,t)$ respectively the matrices containing the components $[\phi(x;t)]_{ij}$ for $i,j \in \mathcal{S}_+$ and for $i \in \mathcal{S}_+$, $j \in \mathcal{S}_-$.

Lemma 1.5.1 *For $x > 0$, we have*

$$\begin{aligned} \frac{\partial}{\partial t}\phi_{++}(x,t) &= \phi_{++}(x,t)T_{++} + \phi_{+-}(x,t)T_{-+} - \frac{\partial}{\partial x}\phi_{++}(x,t), \quad (1.18) \\ \frac{\partial}{\partial t}\phi_{+-}(x,t) &= \phi_{++}(x,t)T_{+-} + \phi_{+-}(x,t)T_{--} + \frac{\partial}{\partial x}\phi_{+-}(x,t). \end{aligned}$$

□

In fact, we only need (1.18), but we provide the two equations here for the sake of completeness. We omit the proof of this lemma since it is similar to that of Theorem 1.2.1.

Theorem 1.5.2 *The matrix K is given by*

$$K = T_{++} + \Psi T_{-+}. \tag{1.19}$$

Proof Taking the integral from 0 to $+\infty$ in both sides of (1.18), we obtain

$$N_{++}(x)T_{++} - \frac{\partial}{\partial x}N_{++}(x) + N_{+-}(x)T_{-+} = 0,$$

since for x fixed and strictly positive, $\phi_{++}(x,t)$ goes to zero both as $t \to 0$ and as $t \to \infty$. Substituting $N_{++}(x)$ and $N_{+-}(x)$ by (1.9), we deduce that

$$e^{Kx}T_{++} - \frac{\partial}{\partial x}e^{Kx} + e^{Kx}\Psi T_{-+} = 0.$$

Since $\partial/\partial x e^{Kx} = Ke^{Kx}$, taking the limit as x goes to zero in the above equation leads to (1.19). □

The following result states some spectral properties of the matrix K in the case where the drift of the fluid queue is negative. We only consider this case for the time being because it is in this context that we work in this chapter. We shall later give the corresponding spectral properties for the matrix K for other values of the drift.

Theorem 1.5.3 *If $\mu < 0$, then all the eigenvalues of K have a strictly negative real part. Therefore,*

$$\lim_{x\to\infty} e^{Kx} = 0$$

and K is a nonsingular matrix.

Proof By the existence of the stationary distribution (1.12), we know that we can integrate $\boldsymbol{\pi}(x)\mathbf{1}$ with respect to x over $(0,+\infty)$ and obtain something which is bounded by one. Therefore, $\int_0^\infty e^{Kx}dx$ must be finite, implying that all the eigenvalues of K have a strictly negative real part. This directly leads to the conclusion that e^{Kx} goes to zero as x goes to infinity and that K is nonsingular. □

In the sequel, we will often need to integrate e^{Kx} with respect to x over the interval $(0,+\infty)$. This is the object of the next corollary.

Corollary 1.5.4 *If $\mu < 0$, then*

$$\int_0^\infty e^{Kx}dx = -K^{-1}.$$

Proof First, let us write

$$e^{Kx} = I + \int_0^x Ke^{Ku}du; \tag{1.20}$$

to prove this, it suffices to differentiate both sides of the equation and to verify that both sides are equal at $x = 0$.

By Theorem 1.5.3, we know that K is nonsingular. We can thus pre-multiply (1.20) by K^{-1} and find

$$K^{-1}e^{Kx} = K^{-1} + \int_0^x e^{Ku}\,du. \tag{1.21}$$

Taking the limit as x goes to infinity in (1.21), one obtains the announced result since $\lim_{x\to\infty} e^{Kx} = 0$ by Theorem 1.5.3. □

1.6 Boundary Probability Vector

To actually compute the stationary density of the buffer content (1.12), we need to determine the vector $\boldsymbol{p}_-$ of the steady state probability that the process is in the level zero with a phase in $\mathcal{S}_-$. For this purpose, we consider the censored process when the fluid queue is at level zero, which in fact has the same characteristics as the process of downward records $\{D(t)\}$ introduced in Section 1.4, and we show that the vector $\boldsymbol{p}_-$ is the equilibrium probability vector of this process.

Theorem 1.6.1 *If $\mu < 0$, the vector $\mathbf{p}_-$ is the unique solution of the system*

$$\begin{aligned} \boldsymbol{p}_- U &= \mathbf{0} & (1.22)\\ \boldsymbol{p}_-(\mathbf{1} - 2T_{-+}K^{-1}\mathbf{1}) &= 1, & (1.23)\end{aligned}$$

where U is given by (1.14).

Proof We introduce the restriction $\{\chi(\cdot)\}$ of the process $\{\varphi(\cdot)\}$ observed during the intervals of time when the fluid level is zero. In order to do this, we define the sequences of epochs $\{a_n\}$, when the buffer becomes empty, and $\{d_n\}$, when it starts filling up:

$$a_0 = \inf\{t \geq 0 : X(t) = 0\},$$
$$d_n = \inf\{t > a_n : X(t) > 0\},$$

and

$$a_{n+1} = \inf\{t > d_n : X(t) = 0\},$$

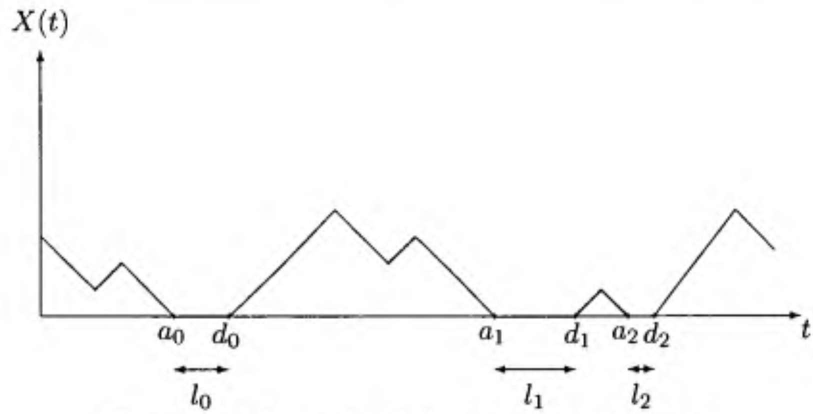


Figure 1.7: Intervals of time spent at level zero.

for $n \geq 0$. Next, we introduce the lengths $l_n = d_n - a_n$ of the intervals (a_n, d_n) spent at level zero (see the example depicted in Figure 1.7). The cumulative lengths L_n of these intervals are $L_n = \sum_{0 \leq i \leq n} l_i$, $n \geq 0$. We define $\nu(t)$ for $t > 0$ by $L_{\nu(t)-1} \leq t < L_{\nu(t)}$, taking $L_{-1} = 0$. Finally, we define

$$\chi(t) = \varphi(a_{\nu(t)} + t - L_{\nu(t)-1}).$$

This means that $\chi(t)$ evolves in the interval (L_{n-1}, L_n) exactly like $\varphi(t)$ in the interval (a_n, d_n) (see Freedman [23] for further details about this construction). The vector $\boldsymbol{p}_-$ is proportional to the steady state probability vector of the process $\{\chi(t)\}$ and we need to determine its infinitesimal generator.

A moment of reflection shows that it is $T_{--} + T_{-+}\Psi = U$; this can be obtained by arguing as in the proof of Theorem 1.4.1.

To prove the normalizing equation, we integrate $\boldsymbol{\pi}(x)\mathbf{1}$ over $(0, \infty)$, and add it to $\boldsymbol{p}_-\mathbf{1}$. This must be equal to 1 and, by (1.12), we have

$$\boldsymbol{p}_-\mathbf{1} + \int_0^\infty \boldsymbol{p}_- T_{-+} e^{Kx} [I, \ \Psi] \mathbf{1} dx = 1.$$

Using Corollary 1.5.4 and the fact that $\Psi\mathbf{1} = \mathbf{1}$, we readily obtain (1.23), which completes the proof. □

1.7 General Fluid Input Rates

As announced previously, we show in this section that we do not lose generality in restricting our analysis to fluid queues with net input rates equal to $+1$ or -1 only.

Define the fluid queue $\{(\tilde{X}(t), \tilde{\varphi}(t)) : t \in \mathbb{R}^+\}$ for which the net input rates $\tilde{r}_i$ may take any real value, including zero, and partition the set of phases into the subsets $\mathcal{S}_0$, $\mathcal{S}_+$ and $\mathcal{S}_-$, with $\tilde{r}_i = 0$ for i in $\mathcal{S}_0$, $\tilde{r}_i > 0$ for i in $\mathcal{S}_+$ and $\tilde{r}_i < 0$ for i in $\mathcal{S}_-$. Denote by $\tilde{T}$ the generator of the underlying phase process, with

$$\tilde{T} = \begin{bmatrix} \tilde{T}_{00} & \tilde{T}_{0+} & \tilde{T}_{0-} \\ \tilde{T}_{+0} & \tilde{T}_{++} & \tilde{T}_{+-} \\ \tilde{T}_{-0} & \tilde{T}_{-+} & \tilde{T}_{--} \end{bmatrix},$$

and denote by $\tilde{\boldsymbol{\pi}}(x)$ and $\tilde{\boldsymbol{p}}$ respectively the stationary density of the buffer content and the probability mass vector of the empty buffer for this process. As before, $\tilde{\boldsymbol{p}}_+ = \boldsymbol{0}$ and the density vector is a solution of the system

$$\frac{d}{dx}\tilde{\boldsymbol{\pi}}(x)C = \tilde{\boldsymbol{\pi}}(x)\tilde{T}, \tag{1.24}$$

with $C = \operatorname{diag}(\tilde{r}_i : i \in \mathcal{S})$.

We denote by $\mathcal{S}_\bullet = \mathcal{S}_+ \cup \mathcal{S}_-$ the subset of phases for which the input rate is different from zero. We partition the infinitesimal generator $\tilde{T}$, the stationary vectors $\tilde{\boldsymbol{\pi}}(x)$ and $\tilde{\boldsymbol{p}}$ in the corresponding manner

$$\tilde{T} = \begin{bmatrix} \tilde{T}_{00} & \tilde{T}_{0\bullet} \\ \tilde{T}_{\bullet 0} & \tilde{T}_{\bullet\bullet} \end{bmatrix},$$

$$\tilde{\boldsymbol{\pi}}(x) = (\tilde{\boldsymbol{\pi}}_0(x), \tilde{\boldsymbol{\pi}}_\bullet(x)) \quad \text{and} \quad \tilde{\boldsymbol{p}} = (\tilde{\boldsymbol{p}}_0, \tilde{\boldsymbol{p}}_\bullet).$$

We separate the phases in $\mathcal{S}_0$ from the others and consider the censored process $\{(\bar{X}(t), \bar{\varphi}(t))\}$ which is the fluid queue observed only during those intervals of time when the phase is in $\mathcal{S}_\bullet$. Using the above notations, it is easy to verify that the system (1.24) is equivalent to

$$\tilde{\boldsymbol{\pi}}_0(x) = \tilde{\boldsymbol{\pi}}_\bullet(x)\tilde{T}_{\bullet 0}(-\tilde{T}_{00})^{-1}, \tag{1.25}$$

$$\frac{d}{dx}\tilde{\boldsymbol{\pi}}_\bullet(x)C_\bullet = \tilde{\boldsymbol{\pi}}_\bullet(x)\bar{T}, \tag{1.26}$$

where $C_\bullet = \operatorname{diag}(\tilde{r}_i : i \in \mathcal{S}_\bullet)$, and

$$\begin{aligned} \bar{T} &= \tilde{T}_{\bullet\bullet} + \tilde{T}_{\bullet 0}(-\tilde{T}_{00})^{-1}\tilde{T}_{0\bullet} \\ &= \begin{bmatrix} \tilde{T}_{++} & \tilde{T}_{+-} \\ \tilde{T}_{-+} & \tilde{T}_{--} \end{bmatrix} + \begin{bmatrix} \tilde{T}_{+0} \\ \tilde{T}_{-0} \end{bmatrix} (-\tilde{T}_{00})^{-1} \begin{bmatrix} \tilde{T}_{0+} & \tilde{T}_{0-} \end{bmatrix} \end{aligned}$$

is the infinitesimal generator of the censored phase process $\{\bar{\varphi}(t)\}$. Moreover, one shows by conditioning on the last visit to the states in $(0, \mathcal{S}_-)$ that

$$\tilde{\boldsymbol{p}}_0 = \tilde{\boldsymbol{p}}_- \tilde{T}_{-0}(-\tilde{T}_{00})^{-1}.$$

To obtain this, it suffices to mimic the proof of Theorem 1.3.1.

With $\bar{\boldsymbol{\pi}}(x)$ and $\bar{\boldsymbol{p}}$ denoting the stationary density and probability vectors of the fluid queue $\{(\bar{X}(t), \bar{\varphi}(t))\}$, we find that they are proportional to $\tilde{\boldsymbol{\pi}}_\bullet(x)$ and $\tilde{\boldsymbol{p}}_\bullet$, and eventually we obtain

$$(\tilde{\boldsymbol{\pi}}_+(x), \tilde{\boldsymbol{\pi}}_-(x)) = \tilde{\alpha}(\bar{\boldsymbol{\pi}}_+(x), \bar{\boldsymbol{\pi}}_-(x))$$

and

$$\tilde{\boldsymbol{p}}_\bullet = \tilde{\alpha}\bar{\boldsymbol{p}}.$$

The factor $\tilde{\alpha}$ is determined in the following manner. Since

$$\bar{\boldsymbol{p}}\mathbf{1} + \int_0^\infty (\bar{\boldsymbol{\pi}}_+(x), \bar{\boldsymbol{\pi}}_-(x))dx\mathbf{1} = 1,$$

it follows that

$$\tilde{\boldsymbol{p}}_\bullet\mathbf{1} + \int_0^\infty (\tilde{\boldsymbol{\pi}}_+(x), \tilde{\boldsymbol{\pi}}_-(x))dx\mathbf{1} = \tilde{\alpha}.$$

To compute the left-hand side of the above expression, it suffices to note that $\tilde{\boldsymbol{p}}_+ + \int_0^\infty \tilde{\boldsymbol{\pi}}_+(x)dx = \tilde{\boldsymbol{\xi}}_+$ and $\tilde{\boldsymbol{p}}_- + \int_0^\infty \tilde{\boldsymbol{\pi}}_-(x)dx = \tilde{\boldsymbol{\xi}}_-$ where $\tilde{\boldsymbol{\xi}}$ is the stationary probability vector of $\tilde{T}$, that is, $\tilde{\boldsymbol{\xi}}\tilde{T} = \mathbf{0}$ and $\tilde{\boldsymbol{\xi}}\mathbf{1} = 1$. It readily follows that $\tilde{\alpha} = \tilde{\boldsymbol{\xi}}_+\mathbf{1} + \tilde{\boldsymbol{\xi}}_-\mathbf{1}$.

We have thus reduced the initial problem to that of finding the stationary characteristics of the fluid queue $\{(\bar{X}(t), \bar{\varphi}(t))\}$ for which $\mathcal{S}_0$ is empty.

Denote by $|M|$ the matrix obtained from a matrix M by taking the absolute values of all its entries.

Define $T = |C_\bullet|^{-1}\bar{T}$, $r_i = 1$ for i in $\mathcal{S}_+$ and $r_i = -1$ for i in $\mathcal{S}_-$. Note that T is still the generator of a Markov process, since it has non-negative off-diagonal entries and its row sums are equal to zero. This transformation reflects a change of time scale and net input rates: for each phase i, the length of the sojourn intervals in i are multiplied by $|\tilde{r}_i|$ and the net input rates are divided by $|\tilde{r}_i|$, the fluid level changing by the same amount overall. The stationary density vector for this fluid queue is proportional to $\bar{\boldsymbol{\pi}}(x)|C_\bullet|$. To see this, it suffices to rewrite (1.26) in the equivalent form

$$\frac{d}{dx}\bar{\boldsymbol{\pi}}_\bullet(x)|C_\bullet| \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} = \bar{\boldsymbol{\pi}}_\bullet(x)|C_\bullet||C_\bullet|^{-1}\bar{T}$$

where I_+ and I_- respectively denote identity matrices of orders s_+ and s_-.

Denoting by $\{(X(t), \varphi(t))\}$ the fluid queue with generator T and net input rates equal to $+1$ and -1, and by $\boldsymbol{\pi}(x)$ and $\boldsymbol{p}$ its stationary characteristics, one has

$$(\bar{\boldsymbol{\pi}}_+(x), \bar{\boldsymbol{\pi}}_-(x)) = \bar{\alpha}(\boldsymbol{\pi}_+(x), \boldsymbol{\pi}_-(x))|C_\bullet|^{-1},$$

and

$$\bar{\boldsymbol{p}} = \bar{\alpha}\boldsymbol{p}|C_\bullet|^{-1}. \tag{1.27}$$

To determine $\bar{\alpha}$, one uses the facts that

$$\int_0^\infty (\bar{\boldsymbol{\pi}}_+(x), \bar{\boldsymbol{\pi}}_-(x))\mathbf{1}\, dx + \bar{\boldsymbol{p}}\mathbf{1} = 1,$$

and that

$$\int_0^\infty \boldsymbol{\pi}_+(x) dx = \boldsymbol{\xi}_+ \quad \text{and} \quad \int_0^\infty \boldsymbol{\pi}_-(x) dx + \boldsymbol{p}_- = \boldsymbol{\xi}_-,$$

where $\boldsymbol{\xi}$ is the stationary probability vector of T, that is, $\boldsymbol{\xi}$ solves the system $\boldsymbol{\xi}T = \mathbf{0}$ and $\boldsymbol{\xi}\mathbf{1} = 1$. Thus, we infer that

$$\bar{\alpha} \int_0^\infty (\boldsymbol{\pi}_+(x), \boldsymbol{\pi}_-(x))|C_\bullet|^{-1}\mathbf{1}\, dx + \bar{\alpha}\boldsymbol{p}|C_\bullet|^{-1}\mathbf{1} = 1,$$

which leads to $\bar{\alpha} = (\boldsymbol{\xi}|C_\bullet|^{-1}\mathbf{1})^{-1}$.

Retracing our steps back, we see that one immediately obtains the stationary distribution of the original fluid queue $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$, once one has determined that of the simpler process $\{(X(t), \varphi(t))\}$. Furthermore, one shows that $\bar{\boldsymbol{\xi}} = \tilde{\alpha}^{-1}(\tilde{\boldsymbol{\xi}}_+, \tilde{\boldsymbol{\xi}}_-)$, where $\bar{\boldsymbol{\xi}}$ is the stationary probability vector of $\bar{T}$, and that $\boldsymbol{\xi} = (\bar{\boldsymbol{\xi}}|C_\bullet|\mathbf{1})^{-1}\bar{\boldsymbol{\xi}}|C_\bullet|$ which leads to $\boldsymbol{\xi}|C_\bullet|^{-1}\mathbf{1} = (\bar{\boldsymbol{\xi}}|C_\bullet|\mathbf{1})^{-1}$. We can now compute the product γ of the scaling factors $\tilde{\alpha}$ and $\bar{\alpha}$:

$$\gamma = \tilde{\alpha}\bar{\boldsymbol{\xi}}|C_\bullet|\mathbf{1} = (\tilde{\boldsymbol{\xi}}_+, \tilde{\boldsymbol{\xi}}_-)|C_\bullet|\mathbf{1} = \tilde{\boldsymbol{\xi}}|C|\mathbf{1}, \tag{1.28}$$

the last equality following from the fact that $\tilde{\boldsymbol{\xi}}_0|C_0| = \mathbf{0}$.

In summary, the stationary marginal density of the fluid level is given by

$$\begin{aligned} \tilde{\boldsymbol{\pi}}(x) &= (\tilde{\boldsymbol{\pi}}_0(x), \tilde{\boldsymbol{\pi}}_+(x), \tilde{\boldsymbol{\pi}}_-(x)) \\ &= \gamma \boldsymbol{p}_- T_{-+} e^{Kx}[(C_+^{-1}\tilde{T}_{+0} + \Psi|C_-|^{-1}\tilde{T}_{-0})(-\tilde{T}_{00})^{-1},\ C_+^{-1},\ \Psi|C_-|^{-1}] \end{aligned}$$

and the stationary probability mass vector of level zero is

$$(\tilde{\boldsymbol{p}}_-, \tilde{\boldsymbol{p}}_0) = \gamma \boldsymbol{p}_-[|C_-|^{-1},\ |C_-|^{-1}\tilde{T}_{-0}(-\tilde{T}_{00})^{-1}].$$

The following expressions will be useful in the sequel: the stationary density $\mu(x) = \tilde{\boldsymbol{\pi}}(x)\mathbf{1}$ is

$$\begin{aligned} \mu(x) &= \gamma\{\boldsymbol{\pi}_+(x)\boldsymbol{w}_+ + \boldsymbol{\pi}_-(x)\boldsymbol{w}_-\} \\ &= \gamma \boldsymbol{p}_- T_{-+} e^{Kx}(\boldsymbol{w}_+ + \Psi \boldsymbol{w}_-), \end{aligned} \tag{1.29}$$

with $\boldsymbol{\pi}(x)$ given in Corollary 1.3.4, and with

$$\boldsymbol{w}_+ = C_+^{-1}\{\mathbf{1} + \tilde{T}_{+0}(-\tilde{T}_{00})^{-1}\mathbf{1}\} \tag{1.30}$$

and

$$\boldsymbol{w}_- = |C_-|^{-1}\{\mathbf{1} + \tilde{T}_{-0}(-\tilde{T}_{00})^{-1}\mathbf{1}\}, \tag{1.31}$$

and the probability mass at level zero is

$$\tilde{m}_0 = \tilde{\boldsymbol{p}}\mathbf{1} = \gamma \boldsymbol{p}_- \boldsymbol{w}_-. \tag{1.32}$$

The vector $\boldsymbol{p}_-$ is given in Theorem 1.6.1.

1.8 General Fluid Input Rates by a Probabilistic Approach

In the preceding section, we obtained the stationary distribution of a fluid queue with arbitrary real valued net input rates, by relating it to the stationary distribution of a fluid queue with net input rates equal to $+1$ or -1, using normalization arguments. Our goal here is to show that the same results may be reached by a probabilistic approach [8].

We consider a fluid queue $\{(\bar{X}(t), \bar{\varphi}(t)) : t \in \mathbb{R}^+\}$ with net input rates which can take any real value except zero, and with generator $\bar{T}$. The reason for choosing to analyze such a system instead of a more general one, is that we have seen in (1.25) how to obtain the stationary density for the phases in $\mathcal{S}_0$ once one has the stationary density for the phases in $\mathcal{S}_\bullet = \mathcal{S}_+ \cup \mathcal{S}_-$. We assume that the mean drift μ in equilibrium is negative, so that the stationary density exists. The next theorem gives an expression for the steady state density vector $\bar{\pi}_\bullet(x)$ of the buffer content for the phases in $\mathcal{S}_\bullet$; its proof is based on probabilistic arguments.

The following quantities will be used in the proof of the theorem. We define the matrix Ψ so that, for $i \in \mathcal{S}_+$ and $j \in \mathcal{S}_-$, Ψ_{ij} is the probability that, starting from level zero in phase i, the fluid queue $\{(\bar{X}(t), \bar{\varphi}(t))\}$ returns to level zero in a finite amount of time, and does so in phase j. We denote by e^{Kx} the square matrix of order s_+ which records the expected number of crossings of level x by the process $\{(\bar{X}(t), \bar{\varphi}(t))\}$, starting from level zero, before returning to this initial level. Observe that we use the same notations as for the fluid queue with net input rates equal to $+1$ or -1. The reason for doing so is that these quantities are the same for the two processes: the fact that we shrink or expand the lengths of the intervals of time spent in the different phases does not change the average number of times that the process crosses any given level x, nor the probability that it returns to the initial level in a finite time.

Theorem 1.8.1 *If $\mu < 0$, then the stationary density of the buffer content of the process $\{(\bar{X}(t), \bar{\varphi}(t))\}$ is given by*

$$\bar{\boldsymbol{\pi}}_\bullet(x) = \bar{\boldsymbol{p}}_- \bar{T}_{-+} e^{Kx} [C_+^{-1},\ \Psi |C_-|^{-1}]. \tag{1.33}$$

The steady state probability mass vector $\bar{\boldsymbol{p}}_-$ of the states $(0, \mathcal{S}_-)$ is the unique solution of the system

$$\begin{aligned} \bar{\boldsymbol{p}}_- \bar{U} &= \mathbf{0} \qquad (1.34) \\ \bar{\boldsymbol{p}}_- (\mathbf{1} - \bar{T}_{-+} K^{-1} [C_+^{-1},\ \Psi |C_-|^{-1}] \mathbf{1}) &= 1 \end{aligned}$$

where $\bar{U} = \bar{T}_{--} + \bar{T}_{-+}\Psi$.

Proof Assume that $\bar{X}(0) = 0$ and take $i, j \in \mathcal{S}$. Define

$$\bar{G}_{ij}(x;t) = \mathrm{P}_{(0,i)}[\bar{X}(t) > x, \bar{\varphi}(t) = j],$$

where, for any event A, $\mathrm{P}_{(0,i)}[A] = \mathrm{P}[A | \bar{X}(0) = 0, \bar{\varphi}(0) = i]$. Following the same argumentation as in the proof of Theorem 1.3.1, we condition on the last visit to level zero, and obtain that

$$\begin{aligned} \bar{G}_{ij}(x;t) = \sum_{\substack{k \in \mathcal{S}_- \\ l \in \mathcal{S}_+}} \int_0^t & \mathrm{P}_{(0,i)}[\bar{X}(t-u) = 0, \bar{\varphi}(t-u) = k] \\ & \mathrm{P}_{(0,i)}[\bar{\varphi}(t-u+du) = l | \bar{\varphi}(t-u) = k] \\ & \mathrm{P}_{(0,i)}[\bar{X}(t) > x, \bar{\varphi}(t) = j, \bar{X}(s) > 0\ \forall s \in (t-u, t] | \bar{\varphi}(t-u) = l] \\ & + \mathrm{P}_{(0,i)}[\bar{X}(t) > x, \bar{\varphi}(t) = j, \bar{X}(s) > 0\ \forall s \in (0, t]] \end{aligned}$$

Define

$$\bar{F}_{ij}(x;t) = \mathrm{P}[\bar{X}(t) \le x, \bar{\varphi}(t) = j | \bar{X}(0) = 0, \bar{\varphi}(0) = i].$$

Using the strong Markov property and the fact that $\{\bar{\varphi}(t)\}$ has infinitesimal generator $\bar{T}$, we obtain

$$\begin{aligned} \bar{G}_{ij}(x;t) &= \sum_{\substack{k\in\mathcal{S}_- \\ l\in\mathcal{S}_+}} \int_0^t \bar{F}_{ik}(0;t-u)\bar{T}_{kl}\,du \\ &\qquad \mathrm{P}_{(0,l)}[\bar{X}(u) > x, \bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]] \\ &\quad + \mathrm{P}_{(0,i)}[\bar{X}(t) > x, \bar{\varphi}(t) = j, \bar{X}(s) > 0 \ \forall s \in (0,t]]. \end{aligned} \tag{1.35}$$

Furthermore, we have that

$$\begin{aligned} \bar{G}_{ij}(x;t) &= \mathrm{P}_{(0,i)}[\bar{\varphi}(t) = j] - \bar{F}_{ij}(x;t) \\ &= \sum_{\substack{k\in\mathcal{S}_- \\ l\in\mathcal{S}_+}} \int_0^t \bar{F}_{ik}(0;t-u)\bar{T}_{kl}\,du \\ &\qquad \mathrm{P}_{(0,l)}[\bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]] \\ &\quad + \mathrm{P}_{(0,i)}[\bar{\varphi}(t) = j, \bar{X}(s) > 0 \ \forall s \in (0,t]] - \bar{F}_{ij}(x;t), \end{aligned}$$

again by conditioning on the last visit to level zero. Replacing this in (1.35), we obtain

$$\begin{aligned} &\bar{F}_{ij}(x;t) \\ &= \sum_{\substack{k\in\mathcal{S}_- \\ l\in\mathcal{S}_+}} \int_0^t \bar{F}_{ik}(0;t-u)\bar{T}_{kl}\,du \mathrm{P}_{(0,l)}[\bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]] \\ &\quad - \sum_{\substack{k\in\mathcal{S}_- \\ l\in\mathcal{S}_+}} \int_0^t \bar{F}_{ik}(0;t-u)\bar{T}_{kl}\,du \\ &\qquad\quad \mathrm{P}_{(0,l)}[\bar{X}(u) > x, \bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]] \\ &\quad + \mathrm{P}_{(0,i)}[\bar{\varphi}(t) = j, \bar{X}(s) > 0 \ \forall s \in (0,t]] \\ &\quad - \mathrm{P}_{(0,i)}[\bar{X}(t) > x, \bar{\varphi}(t) = j, \bar{X}(s) > 0 \ \forall s \in (0,t]]. \end{aligned}$$

Taking the limit as t goes to infinity, we see that the last two terms vanish because the drift is negative and the fluid process is recurrent.

Defining $\bar{F}_j(x) = \lim_{t\to\infty} \mathrm{P}[\bar{X}(t) \le x, \bar{\varphi}(t) = j]$, we may therefore write that

$$\bar{F}_j(x) = \sum_{\substack{k\in\mathcal{S}_-\\ l\in\mathcal{S}_+}} \bar{p}_k \bar{T}_{kl} \int_0^\infty \mathrm{P}_{(0,l)}[\bar{X}(u) \le x, \bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]]\, du \tag{1.36}$$

where, for $k \in \mathcal{S}_-$, $\bar{p}_k = \bar{F}_k(0)$ is the steady state probability mass of state $(0,k)$.

Since we are interested in the stationary density of the buffer content, we take the derivative of (1.36) with respect to x and find

$$\begin{aligned}
\bar{\pi}_j(x) & \\
&= \sum_{\substack{k\in\mathcal{S}_-\\ l\in\mathcal{S}_+}} \bar{p}_k \bar{T}_{kl} \int_0^\infty \frac{\partial}{\partial x} \mathrm{P}_{(0,l)}[\bar{X}(u) \le x, \bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]] du \\
&= \sum_{\substack{k\in\mathcal{S}_-\\ l\in\mathcal{S}_+}} \bar{p}_k \bar{T}_{kl} \int_0^\infty \mathrm{P}_{(0,l)}[\bar{X}(u) \in (x, x+du), \bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]] \\
&= \sum_{\substack{k\in\mathcal{S}_-\\ l\in\mathcal{S}_+}} \bar{p}_k \bar{T}_{kl} \int_0^\infty \mathrm{P}_{(0,l)}[x \in \bar{X}(u, u+\frac{du}{|r_j|}), \bar{\varphi}(u) = j, \bar{X}(s) > 0 \ \forall s \in (0,u]]
\end{aligned}$$

since, in order for $\bar{X}(u)$ to be in $(x, x+du)$, we must have that $\bar{X}(\cdot)$ crosses level x in the interval of time $(u, u + du/|r_j|)$; this is expressed by the notation $x \in \bar{X}(u, u + du/|r_j|)$. This interval of time is determined by taking into account the fact that, since $\bar{\varphi}(u) = j$, the level varies linearly at a rate $|r_j|$ (see Figure 1.8). We therefore obtain that

$$\bar{\pi}_j(x) = \sum_{\substack{k\in\mathcal{S}_-\\ l\in\mathcal{S}_+}} \bar{p}_k \bar{T}_{kl} \int_0^\infty [\phi(x;u)]_{lj}\, du \frac{1}{|r_j|} \tag{1.37}$$

where $[\phi(x;u)]_{lj}$ is the conditional density of state (x,j), given that the process starts from $(0,l)$, and avoiding level zero. The integral $\int_0^\infty [\phi(x;u)]_{lj} du$ is then the expected number of visits to level x in phase j, starting from level zero in phase l, and without returning to level zero. From what we have seen before, we know that this quantity is $e^{Kx}[I,\ \Psi]$. Writing (1.37) in matrix notation yields

$$\bar{\boldsymbol{\pi}}_\bullet(x) = \bar{\boldsymbol{p}}_- \bar{T}_{-+} e^{Kx} [I,\ \Psi] |C_\bullet|^{-1}, \tag{1.38}$$

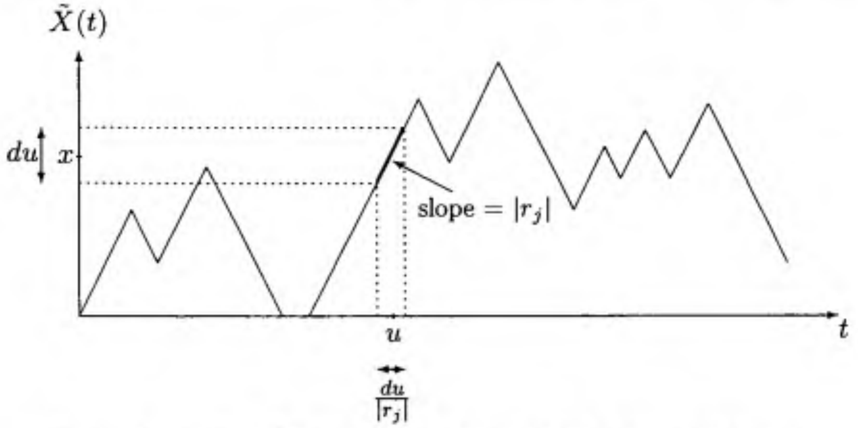


Figure 1.8: For $\bar{X}(u)$ to be in $(x, x+du)$ when $\bar{\varphi}(u) = j$, the process $\bar{X}(\cdot)$ must cross level x in the interval of time $(u, u + du/|r_j|)$.

which proves our first claim.

To show that $\bar{\boldsymbol{p}}_-\bar{U} = \mathbf{0}$, we might mimic the proof of Theorem 1.6.1 and show that $\bar{\boldsymbol{p}}_-$ is the steady state probability vector of the censored process in $(0, \mathcal{S}_-)$, which has generator $\bar{U}$. Instead of doing so, we provide a different approach, which is purely algebraic.

Integrating $\bar{\boldsymbol{\pi}}_\bullet(x)$ with respect to x from zero to infinity, we obtain the vector $(\bar{\boldsymbol{\xi}}_+, \bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{p}}_-)$, where $\bar{\boldsymbol{\xi}}$ is the steady state probability vector of $\bar{T}$. Therefore, if we take the integral of both sides of (1.38), we find the two expressions

$$\bar{\boldsymbol{\xi}}_+ = \bar{\boldsymbol{p}}_-\bar{T}_{-+}(-K)^{-1}C_+^{-1}$$

and

$$\bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{p}}_- = \bar{\boldsymbol{p}}_-\bar{T}_{-+}(-K)^{-1}\Psi|C_-|^{-1}.$$

Substituting the first one into the second, we obtain

$$\bar{\boldsymbol{p}}_- = \bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{\xi}}_+ C_+\Psi|C_-|^{-1}. \tag{1.39}$$

Thus,

$$\begin{aligned}
\bar{\boldsymbol{p}}_-\bar{U} &= (\bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{\xi}}_+C_+\Psi|C_-|^{-1})(\bar{T}_{--} + \bar{T}_{-+}\Psi) \\
&= \bar{\boldsymbol{\xi}}_-\bar{T}_{--} + \bar{\boldsymbol{\xi}}_-\bar{T}_{-+}\Psi \\
&\quad - \bar{\boldsymbol{\xi}}_+C_+\Psi|C_-|^{-1}\bar{T}_{--} - \bar{\boldsymbol{\xi}}_+C_+\Psi|C_-|^{-1}\bar{T}_{-+}\Psi.
\end{aligned}$$

Since $\bar{T} = |C_\bullet|T$ where T is the generator of the fluid queue with net input rates equal to $+1$ or -1, one finds

$$\begin{aligned}\bar{\boldsymbol{p}}_-\bar{U} &= \bar{\boldsymbol{\xi}}_-\bar{T}_{--} + \bar{\boldsymbol{\xi}}_-\bar{T}_{-+}\Psi \\ &\quad - \bar{\boldsymbol{\xi}}_+C_+\Psi T_{--} - \bar{\boldsymbol{\xi}}_+C_+\Psi T_{-+}\Psi.\end{aligned}$$

From the Riccati equation (1.16), we have $\Psi T_{--}+\Psi T_{-+}\Psi = -T_{+-}-T_{++}\Psi$, hence

$$\begin{aligned}\bar{\boldsymbol{p}}_-\bar{U} &= \bar{\boldsymbol{\xi}}_-\bar{T}_{--} + \bar{\boldsymbol{\xi}}_-\bar{T}_{-+}\Psi \\ &\quad + \bar{\boldsymbol{\xi}}_+C_+T_{+-} + \bar{\boldsymbol{\xi}}_+C_+T_{++}\Psi,\end{aligned}$$

which is indeed equal to zero since $C_+T_{+-} = \bar{T}_{+-}$, $C_+T_{++} = \bar{T}_{++}$ and $\bar{\boldsymbol{\xi}}\bar{T} = \mathbf{0}$. The normalizing equation for $\bar{\boldsymbol{p}}_-$ is obtained by imposing that the total probability mass should be equal to one. □

1.9 Performance Measures

We now turn to the determination of some performance measures for the marginal distribution of the fluid level of the process $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$ with arbitrary net input rates. These performance measures may also be obtained using the phase-type representation of the marginal distribution of the fluid level, which will be derived later.

We start by computing the stationary distribution function

$$\tilde{F}(x) = \lim_{t\to\infty} \mathrm{P}[\tilde{X}(t) \le x].$$

We assume throughout that the stability condition is satisfied, that is, we assume that $\mu < 0$, so that the matrix K of the fluid queue with net input rates equal to $+1$ and -1 is nonsingular. As before, the vector $\boldsymbol{p}_-$ and the matrices T and Ψ are, respectively, the steady state probability mass vector, the phase transition generator, and the matrix of first return probabilities to the initial level, for the fluid queue with rates equal to $+1$ and -1.

Proposition 1.9.1 *If $\mu < 0$, the stationary distribution function of the buffer content of the process $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$ is given by*

$$\tilde{F}(x) = \tilde{m}_0 + \gamma\boldsymbol{p}_-T_{-+}(-K)^{-1}(I - e^{Kx})(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-),$$

where $\tilde{m}_0$ is given by (1.32), γ by (1.28) and $\boldsymbol{w}_+$, $\boldsymbol{w}_-$ by (1.30, 1.31).

Proof The stationary distribution $\tilde{F}(x)$ is given by

$$\tilde{F}(x) = \tilde{m}_0 + \int_0^x \mu(u)du.$$

Using (1.29), we obtain that

$$\tilde{F}(x) = \tilde{m}_0 + \gamma \boldsymbol{p}_- T_{-+} \int_0^x e^{Ku}\, du(\boldsymbol{w}_+ + \Psi \boldsymbol{w}_-).$$

By (1.21), $\int_0^x e^{Ku}\, du = -K^{-1}(I - e^{Kx})$ and the result follows. □

The stationary mean and second moment of $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$ are given next.

Proposition 1.9.2 *If $\mu < 0$, the mean M and second moment V of the buffer content of the process $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$ in equilibrium are given by*

$$M = \gamma \boldsymbol{p}_- T_{-+}(-K)^{-2}(\boldsymbol{w}_+ + \Psi \boldsymbol{w}_-) \tag{1.40}$$

and

$$V = 2\gamma \boldsymbol{p}_- T_{-+}(-K)^{-3}(\boldsymbol{w}_+ + \Psi \boldsymbol{w}_-), \tag{1.41}$$

where γ is given by (1.28) and $\boldsymbol{w}_+$, $\boldsymbol{w}_-$ by (1.30, 1.31).

Proof The expected value of the buffer content of $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$ in equilibrium is

$$\begin{aligned} M &= \int_0^\infty x\mu(x)dx \\ &= \gamma \boldsymbol{p}_- T_{-+} \int_0^\infty x e^{Kx} dx(\boldsymbol{w}_+ + \Psi \boldsymbol{w}_-), \end{aligned}$$

using the fact that $\mu(x)$ is given by (1.29). By following the same argument as in Corollary 1.5.4, one shows that the indefinite integral $\int e^{Kx}dx$ is equal to $K^{-1}e^{Kx} + M$ where M is an arbitrary matrix. It is then easy to establish (1.40) through integration by parts.

The proof of (1.41) goes along the same lines, and is therefore omitted. □

We now determine the Laplace-Stieltjes transform of the buffer content in equilibrium. We denote by $\mathcal{R}(s)$ the real part of a complex number s.

Theorem 1.9.3 *If $\mu < 0$, the Laplace-Stieltjes transform $\phi(s)$ of the buffer content of the process $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$ in equilibrium is given by*

$$\phi(s) = \tilde{m}_0 + \gamma \boldsymbol{p}_- T_{-+} A(s)(\boldsymbol{w}_+ + \Psi \boldsymbol{w}_-) \qquad (1.42)$$

for $\mathcal{R}(s) > 0$, where

$$A(s) = -(K - sI)^{-1}, \qquad (1.43)$$

and where $\tilde{m}_0$ is given by (1.32), γ by (1.28) and $\boldsymbol{w}_+$, $\boldsymbol{w}_-$ by (1.30, 1.31).

Proof The Laplace-Stieltjes transform of $\tilde{P}$ is given by

$$\phi(s) = \tilde{m}_0 + \int_0^\infty e^{-sx} \mu(x) dx$$

for $\mathcal{R}(s) > 0$, where $\mu(x)$ is given by (1.29). This is clearly seen to be equivalent to (1.42) where

$$A(s) = \int_0^\infty e^{(K-sI)x} dx.$$

We have seen in Theorem 1.5.3 that all the eigenvalues of K have a strictly negative real part, leading to the conclusion that the eigenvalues of $K - sI$ also have a strictly negative real part since $\mathcal{R}(s) > 0$. Therefore, $K - sI$ is nonsingular and (1.43) is established. □

1.10 A Closely Related Expression

We present in this section the article by Ramaswami [39], which was the starting point of our work. The main difference between this section and the preceding ones lies in the approach used. Ramaswami introduces the arguments based on Markov-renewal theory, but does not use to the fullest the probabilistic interpretations of the matrices K and Ψ.

It is assumed in [39] that the net input rates r_i of fluid into the buffer can take any real value, except zero. We use the same notations as before and consider an infinite buffer fluid queue $\{(\bar{X}(t), \bar{\varphi}(t)) : t \in \mathbb{R}^+\}$ with phase transition generator $\bar{T}$. The set of phases $\mathcal{S}$ is decomposed into $\mathcal{S}_+ \cup \mathcal{S}_-$. We denote again by $\bar{\boldsymbol{\xi}}$ the steady state probability row vector associated with the generator $\bar{T}$.

The next theorem gives the stationary distribution of the buffer content of such a fluid queue. We shall see later that it is equivalent to Theorem 1.8.1.

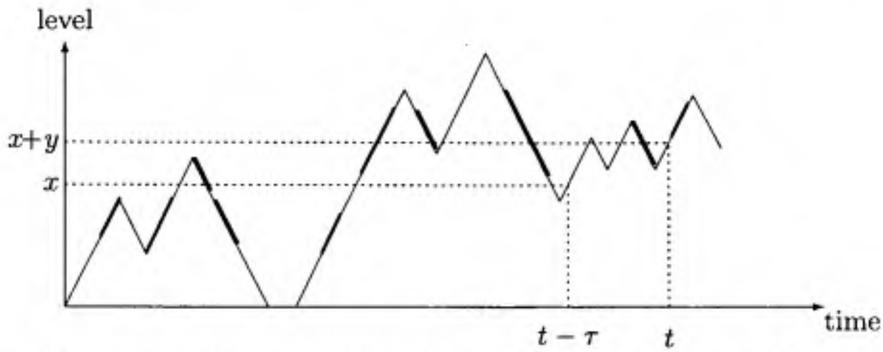


Figure 1.9: The last epoch at which the process crosses the level x before t.

Theorem 1.10.1 *The stationary distribution of the fluid queue has a mass at the level zero and a continuous density for strictly positive values.*

i. There exist a matrix Z of order s_+ and a matrix Υ with dimensions $s_+ \times s_-$ such that the stationary density vector is given by

$$\bar{\boldsymbol{\pi}}_\bullet(x) = -\bar{\boldsymbol{\xi}}_+ Z e^{Zx}[I,\ \Upsilon], \qquad \text{for } x > 0. \tag{1.44}$$

ii. Denote by $\bar{\boldsymbol{p}} = (\bar{p}_i : i \in \mathcal{S})$ the stationary probability mass vector of the empty buffer. It is given by

$$\bar{\boldsymbol{p}} = (\mathbf{0}, \bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{\xi}}_+\Upsilon). \tag{1.45}$$

Proof To prove the first part of the theorem, assume that $\bar{X}(0) = 0$. For $x, y > 0$ and $j \in \mathcal{S}$, we have that $(\bar{X}(t), \bar{\varphi}(t)) = (x + y, j)$ if and only if there exist some time $\tau < t$ and some phase $i \in \mathcal{S}$ such that:

- at time $t - \tau$ the fluid queue is in state (x, i), and
- in the time interval $(t - \tau, t)$, it continuously remains above the level x.

This holds due to the skip-free upward property of the fluid level process and we illustrate it on Figure 1.9.

Denoting by $\bar{f}_j(x;t)$ the density of state (x, j) at time t, we may write that

$$\bar{f}_j(x + y; t) = \int_0^t \sum_{i \in \mathcal{S}} \bar{f}_i(x; t - \tau)[\phi(x, x + y; \tau)]_{ij}\, d\tau \tag{1.46}$$

where $[\phi(x, x+y; \tau)]_{ij}$ is the probability density, given that the process starts in (x, i), of being at $(x+y, j)$ at time τ and remaining above level x in the interval $(0, \tau)$. Since the density $\bar{f}_i(x; t)$ is continuous and goes to zero as t goes to ∞, it is uniformly bounded and thus we can take the limit as $t \to \infty$ in both sides of (1.46) to obtain

$$\bar{\pi}_j(x+y) = \sum_{i \in \mathcal{S}} \bar{\pi}_i(x)[\Phi(x, x+y)]_{ij} \tag{1.47}$$

where $\bar{\pi}_j(x) = \lim_{t\to\infty} \bar{f}_j(x;t)$ and $[\Phi(x, x+y)]_{ij} = \int_0^\infty [\phi(x, x+y; \tau)]_{ij} d\tau$ is the average number of visits to the state $(x+y, j)$ before returning to level x given that the process starts in state (x, i). Equation (1.47) may also be written in matrix notation as

$$\bar{\boldsymbol{\pi}}(x+y) = \bar{\boldsymbol{\pi}}(x)\Phi(x, x+y).$$

The spatial homogeneity of the system implies that

$$\Phi(x, x+y) = \Phi(0, y).$$

By the skip-free upward property again, we know that for $x, y > 0$, the taboo process avoiding level 0 cannot reach level $x + y$ before reaching level x. Hence, we may write

$$\phi(0, x+y; t) = \int_0^t \phi(0, x; t-u)\phi(0, y; u)\, du$$

by conditioning on the last epoch of visit to level x, and this directly yields

$$\Phi(0, x+y) = \Phi(0, x)\Phi(0, y), \qquad x, y > 0. \tag{1.48}$$

The matrix $\Phi(0, x)$ can be written as

$$\Phi(0, x) = \begin{bmatrix} \Phi_{++}(0, x) & \Phi_{+-}(0, x) \\ 0 & 0 \end{bmatrix}, \qquad x > 0. \tag{1.49}$$

The last s_- rows of $\Phi(0, x)$ are equal to zero because starting from level zero and from a phase in $\mathcal{S}_-$, the process stays in level zero, violating the taboo at once.

Putting (1.48) and (1.49) together, we obtain

$$\Phi_{++}(0, x+y) = \Phi_{++}(0, x)\Phi_{++}(0, y)$$

and

$$\Phi_{+-}(0, x+y) = \Phi_{++}(0, x)\Phi_{+-}(0, y),$$

leading to

$$\Phi_{++}(0,x) = e^{Zx} \tag{1.50}$$

since $\Phi_{++}(0,0) = I$, and

$$\Phi_{+-}(0,x) = e^{Zx}\Upsilon \tag{1.51}$$

where $\Upsilon = \lim_{x\to 0^+} \Phi_{+-}(0,x)$. Therefore

$$\bar{\boldsymbol{\pi}}(x) = \boldsymbol{\alpha}[e^{Zx}, e^{Zx}\Upsilon] \tag{1.52}$$

where $\boldsymbol{\alpha}$ is a row vector of order s_+.

To determine $\boldsymbol{\alpha}$, one can integrate the first s_+ components of (1.52) from 0 to ∞; this should be equal to $\bar{\boldsymbol{\xi}}_+$ since the steady state probability of $(0,i)$ is equal to zero for $i \in \mathcal{S}_+$. Thus,

$$\int_0^\infty \boldsymbol{\alpha} e^{Zx} dx = \boldsymbol{\alpha}(-Z)^{-1} = \bar{\boldsymbol{\xi}}_+$$

which gives $\boldsymbol{\alpha} = \bar{\boldsymbol{\xi}}_+ Z$ and completes the proof of the first statement.

To determine $\bar{\boldsymbol{p}}_-$, we integrate the last s_- components of (1.44) and obtain that

$$\begin{aligned}\int_0^\infty (-\bar{\boldsymbol{\xi}}_+ Z)e^{Zx}\Upsilon dx &= -\bar{\boldsymbol{\xi}}_+ Z(-Z)^{-1}\Upsilon \\ &= \bar{\boldsymbol{\xi}}_+\Upsilon.\end{aligned}$$

The steady state probabilities $\bar{\boldsymbol{\xi}}_-$ for the last s_- phases are thus equal to $\bar{\boldsymbol{\xi}}_+\Upsilon + \bar{\boldsymbol{p}}_-$, and we finally obtain

$$\bar{\boldsymbol{p}}_- = \bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{\xi}}_+\Upsilon. \tag{1.53}$$

□

Remark 1.10.2 To obtain the matrix-exponential form of the stationary buffer content, the assumption of finiteness of $\mathcal{S}$ is in fact not necessary; $\mathcal{S}$ can be infinite as long as $\mathcal{S}_+$ is of finite size. Indeed, suppose that $|\mathcal{S}| = \infty$ but $|\mathcal{S}_+| < \infty$. Recall Equation (1.46) and observe that $[\phi(x, x+y;\tau)]_{ij} = 0$ if $i \in \mathcal{S}_-$. Thus, Equation (1.46) becomes

$$\bar{f}_j(x+y;t) = \int_0^t \sum_{i\in\mathcal{S}_+} \bar{f}_i(x;t-\tau)[\phi(x,x+y;\tau)]_{ij}\, d\tau$$

and we are in the same situation as when $\mathcal{S}$ is finite. We can therefore use the same arguments to derive the matrix-exponential form of the stationary density of the system.

Denote by C the diagonal matrix $\mathrm{diag}(r_i : i \in \mathcal{S}_\bullet)$. It will be partitioned in the following way:

$$C = \begin{bmatrix} C_+ & 0 \\ 0 & C_- \end{bmatrix}. \tag{1.54}$$

The following result is a reformulation of Lemma 1.5.1 in the case where the net input rates take arbitrary values except zero. Its proof is therefore omitted.

Lemma 1.10.3 *For $x > 0$, we have*

$$\frac{\partial}{\partial t}\phi_{++}(0,x;t) = \phi_{++}(0,x;t)\bar{T}_{++} + \phi_{+-}(0,x;t)\bar{T}_{-+} - \frac{\partial}{\partial x}\phi_{++}(0,x;t)C_+, \tag{1.55}$$

$$\frac{\partial}{\partial t}\phi_{+-}(0,x;t) = \phi_{++}(0,x;t)\bar{T}_{+-} + \phi_{+-}(0,x;t)\bar{T}_{--} + \frac{\partial}{\partial x}\phi_{+-}(0,x;t)|C_-|. \tag{1.56}$$

□

The next theorem gives expressions for the matrices Z and Υ, similar to (1.19) and (1.17).

Theorem 1.10.4 *The matrices Z and Υ are given by*

$$Z = (\bar{T}_{++} + \Upsilon\bar{T}_{-+})C_+^{-1} \tag{1.57}$$

and

$$\Upsilon = \int_0^\infty e^{Zy}\bar{T}_{+-}|C_-|^{-1}e^{\bar{T}_{--}|C_-|^{-1}y}dy. \tag{1.58}$$

Proof The proof of (1.57) is similar to the proof of (1.19), but we integrate (1.55) instead of (1.18).

To obtain (1.58), we take the integral of both sides of (1.56) for t going from 0 to $+\infty$, and this yields

$$\Phi_{++}(0,x)\bar{T}_{+-} + \Phi_{+-}(0,x)\bar{T}_{--} + \frac{\partial}{\partial x}\Phi_{+-}(0,x)|C_-| = 0.$$

Using (1.50) and (1.51), the equation above reduces to

$$e^{Zx}\bar{T}_{+-} + e^{Zx}\Upsilon\bar{T}_{--} + \frac{\partial}{\partial x}e^{Zx}\Upsilon|C_-| = 0$$

which is equivalent to

$$\bar{T}_{+-} + \Upsilon\bar{T}_{--} + Z\Upsilon|C_-| = 0. \tag{1.59}$$

Now, one can prove that the matrix Υ given by (1.58) is actually a solution of (1.59). Indeed, multiply both sides of (1.58) on the left by Z, and perform integration by parts; this leads to

$$\begin{aligned} Z\Upsilon &= \int_0^\infty Ze^{Zy}\bar{T}_{+-}|C_-|^{-1}e^{\bar{T}_{--}|C_-|^{-1}y}dy \\ &= \left[e^{Zy}\bar{T}_{+-}|C_-|^{-1}e^{\bar{T}_{--}|C_-|^{-1}y}\right]_0^\infty \\ &\quad -\int_0^\infty e^{Zy}\bar{T}_{+-}|C_-|^{-1}e^{\bar{T}_{--}|C_-|^{-1}y}dy\bar{T}_{--}|C_-|^{-1}. \end{aligned} \tag{1.60}$$

The only minor difficulty is to compute the limit as y goes to ∞ of the expression in brackets. In fact, we have that $|C_-|^{-1}\bar{T}_{--}$ is the infinitesimal generator of a transient Markov process, since $(|C_-|^{-1}\bar{T}_{--})\mathbf{1} \leq \mathbf{0}$ and $(|C_-|^{-1}\bar{T}_{--})^{-1}$ exists. Thus,

$$\lim_{y\to\infty} |C_-|^{-1}e^{\bar{T}_{--}|C_-|^{-1}y} = \lim_{y\to\infty} e^{|C_-|^{-1}\bar{T}_{--}y}|C_-|^{-1} = 0.$$

On the other hand, $\lim_{y\to\infty} e^{Zy} = 0$, and therefore the first term on the right-hand side of (1.60) reduces to $-\bar{T}_{+-}|C_-|^{-1}$. Thus,

$$Z\Upsilon = -\bar{T}_{+-}|C_-|^{-1} - \Upsilon\bar{T}_{--}|C_-|^{-1}$$

which is indeed equivalent to (1.59). □

As before, we can write a Riccati equation for the matrix Υ:

$$\bar{T}_{+-}|C_-|^{-1} + \Upsilon\bar{T}_{--}|C_-|^{-1} + \bar{T}_{++}C_+^{-1}\Upsilon + \Upsilon\bar{T}_{-+}C_+^{-1}\Upsilon = 0. \tag{1.61}$$

To see this, it suffices to substitute (1.57) in (1.59).

Remark 1.10.5 We now show how the results in this section relate to Theorem 1.8.1.

Let T be the infinitesimal transition generator of the fluid queue with net input rates equal to $+1$ and -1, and let K and Ψ be the matrices satisfying (1.19) and (1.16), respectively.

We first show that

$$\Upsilon = C_+\Psi|C_-|^{-1} \tag{1.62}$$

and

$$Z = C_+ K C_+^{-1} \tag{1.63}$$

where C is defined in (1.54).

Since $T = |C|^{-1}\bar{T}$, we have by (1.16) that

$$C_+^{-1}\bar{T}_{+-} + \Psi|C_-|^{-1}\bar{T}_{--} + C_+^{-1}\bar{T}_{++}\Psi + \Psi|C_-|^{-1}\bar{T}_{-+}\Psi = 0.$$

Multiplying on the left by C_+ and on the right by $|C_-|^{-1}$, this reduces to

$$\begin{aligned}\bar{T}_{+-}|C_-|^{-1} + C_+\Psi|C_-|^{-1}\bar{T}_{--}|C_-|^{-1}& \\ + \bar{T}_{++}\Psi|C_-|^{-1} + C_+\Psi|C_-|^{-1}\bar{T}_{-+}\Psi|C_-|^{-1} &= 0,\end{aligned}$$

which, using (1.61), gives (1.62).

To prove (1.63), replace Υ by its expression (1.62) in (1.57) to obtain

$$Z = \bar{T}_{--}C_+^{-1} + C_+\Psi|C_-|^{-1}\bar{T}_{-+}C_+^{-1}.$$

Multiplying on the left by C_+^{-1} and on the right by C_+, this yields

$$C_+^{-1}ZC_+ = C_+^{-1}\bar{T}_{--} + \Psi|C_-|^{-1}\bar{T}_{-+}$$

which immediately gives (1.63) by (1.19) and using the relation between T and $\bar{T}$.

Second, note that to verify that (1.45) is in agreement with (1.34), it suffices to consider (1.39) and to see that it is equivalent to (1.45), with Υ given by (1.62).

Finally, we show that equation (1.44) is the same as (1.33). By (1.45), we have that

$$\begin{aligned}\bar{\boldsymbol{p}}_-\bar{T}_{-+} &= (\bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{\xi}}_+\Upsilon)\bar{T}_{-+} \\ &= \bar{\boldsymbol{\xi}}_-\bar{T}_{-+} - \bar{\boldsymbol{\xi}}_+ZC_+ + \bar{\boldsymbol{\xi}}_+\bar{T}_{++}\end{aligned}$$

using (1.57). Since $\bar{\boldsymbol{\xi}}$ is the asymptotic distribution of $\bar{T}$, one finds that

$$\bar{\boldsymbol{p}}_-\bar{T}_{-+} = -\bar{\boldsymbol{\xi}}_+ZC_+.$$

Replacing this in the right-hand side of (1.33) and using (1.62) and (1.63), we obtain

$$\begin{aligned}\bar{\boldsymbol{p}}_-\bar{T}_{-+}e^{Kx}[C_+^{-1},\ \Psi|C_-|^{-1}]& \\ &= -\bar{\boldsymbol{\xi}}_+ZC_+e^{C_+^{-1}ZC_+x}[C_+^{-1},\ C_+^{-1}\Upsilon|C_-||C_-|^{-1}] \\ &= -\bar{\boldsymbol{\xi}}_+Ze^{Zx}[I,\ \Upsilon]\end{aligned}$$

which is indeed the right-hand side of (1.44).

We will need later the following expression

$$\boldsymbol{p}_- = \boldsymbol{\xi}_- - \boldsymbol{\xi}_+ \Psi \tag{1.64}$$

for the steady state probability vector of level zero of a standard fluid queue with net input rates equal to $+1$ and -1 only. To see that this holds, recall that $\boldsymbol{p} = \bar{\alpha}^{-1}\bar{\boldsymbol{p}}|C_\bullet|$ by (1.27), with $\bar{\alpha} = (\boldsymbol{\xi}|C_\bullet|^{-1}\mathbf{1})^{-1}$. Thus,

$$\boldsymbol{p}_- = (\boldsymbol{\xi}|C_\bullet|^{-1}\mathbf{1})(\bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{\xi}}_+\Upsilon)|C_-|$$

using (1.53) and, since $\bar{\boldsymbol{\xi}} = (\boldsymbol{\xi}|C_\bullet|^{-1}\mathbf{1})^{-1}\boldsymbol{\xi}|C_\bullet|^{-1}$ and $\Upsilon = C_+\Psi|C_-|^{-1}$, we find that

$$\begin{aligned}
\boldsymbol{p}_- &= (\boldsymbol{\xi}_-|C_-|^{-1} - \boldsymbol{\xi}_+C_+^{-1}C_+\Psi|C_-|^{-1})|C_-| \\
&= \boldsymbol{\xi}_- - \boldsymbol{\xi}_+\Psi.
\end{aligned}$$

1.11 Phase-type Representation

To obtain a phase-type representation for the stationary distribution of the fluid queue, Ramaswami introduces the *dual fluid queue*, which is defined in the following manner.

Denote by Δ the diagonal matrix $\text{diag}(\bar{\xi}_i : i \in \mathcal{S})$ and by M' the transpose of some matrix M. Define the generator $\check{T} = \Delta^{-1}\bar{T}'\Delta$, and consider a fluid queue with phase transition generator $\check{T}$ and net input rates $-r_i < 0$ for $i \in \mathcal{S}_+$ and $-r_i > 0$ for $i \in \mathcal{S}_-$. This queue is in fact the time reversed stationary version of the fluid process with generator $\bar{T}$; it is still a two-dimensional Markov process, which we denote by $\{(\check{X}(t), \check{\varphi}(t)) : t \in \mathbb{R}^+\}$.

To analyze the dual fluid queue, we adopt a purely algebraic approach.

For $x > 0$, define the matrix $\check{\Phi}(x,0) = \Delta^{-1}(\Phi(0,x))'\Delta$, where $\Phi(0,x)$ is defined in Section 1.10. As usual, we write it as

$$\check{\Phi}(x,0) = \begin{bmatrix} \check{\Phi}_{++}(x,0) & 0 \\ \check{\Phi}_{-+}(x,0) & 0 \end{bmatrix}.$$

The presence of zeros in the last s_- columns of $\check{\Phi}(x,0)$ is justified by the presence of zeros in the last s_- rows of $\Phi(0,x)$, see (1.49). Define also $\Delta_+ = \text{diag}(\bar{\xi}_i : i \in \mathcal{S}_+)$ and $\Delta_- = \text{diag}(\bar{\xi}_i : i \in \mathcal{S}_-)$. The proof of the following result is straightforward.

Theorem 1.11.1 *We have*

$$\check{\Phi}_{++}(x,0) = e^{\check{Z}x}, \tag{1.65}$$

where $\check{Z} = \Delta_+^{-1} Z' \Delta_+$, *and*

$$\check{\Phi}_{-+}(x,0) = \check{\Upsilon} e^{\check{Z}x}, \tag{1.66}$$

where $\check{\Upsilon} = \Delta_-^{-1} \Upsilon' \Delta_+$. *The matrices* $\check{Z}$ *and* $\check{\Upsilon}$ *satisfy the following equations:*

$$\check{Z} = C_+^{-1}(\check{T}_{++} + \check{T}_{+-}\check{\Upsilon}) \tag{1.67}$$

and

$$\check{\Upsilon} = \int_0^\infty e^{|C_-|^{-1}\check{T}_{--}y} |C_-|^{-1} \check{T}_{-+} e^{\check{Z}y} dy. \tag{1.68}$$

Proof Recall that $\Phi_{++}(x,0) = e^{Zx}$. Transpose both sides of this equation and multiply on the left by Δ_+^{-1} and on the right by Δ_+:

$$\begin{aligned} \Delta_+^{-1}(\Phi_{++}(0,x))'\Delta_+ &= \Delta_+^{-1} e^{Z'x} \Delta_+ \\ &= e^{\Delta_+^{-1} Z' \Delta_+ x} \\ &= e^{\check{Z}x}, \end{aligned}$$

which proves (1.65). Similarly, transpose both sides of $\Phi_{+-}(0,x) = e^{Zx}\Upsilon$, multiply on the left by Δ_-^{-1} and on the right by Δ_+ to obtain (1.66).

We next consider (1.57). Transposing both sides of that equation yields

$$Z' = C_+^{-1}\bar{T}'_{++} + C_+^{-1}\bar{T}'_{-+}\Upsilon'.$$

Multiplying again on the left by Δ_+^{-1} and on the right by Δ_+, we get

$$\Delta_+^{-1} Z' \Delta_+ = \Delta_+^{-1} C_+^{-1} \bar{T}'_{++} \Delta_+ + \Delta_+^{-1} C_+^{-1} \bar{T}'_{-+} \Upsilon' \Delta_+$$

which reduces to (1.67) as C_+ and Δ_+ commute since they are diagonal matrices.

Perform the same kind of algebraic manipulations on (1.58) to obtain (1.68). □

Assume that $\check{X}(0) = x$ with $x > 0$ and denote by θ the first epoch when the dual fluid level becomes zero. Thus, $\theta = \inf\{t \geq 0 : \check{X}(t) = 0\}$. Define the matrices $\check{G}_{++}(x,0)$ and $\check{G}_{-+}(x,0)$, with dimensions $s_+ \times s_+$ and $s_- \times s_+$ respectively, by

$$[\check{G}_{++}(x,0)]_{ij} = \mathrm{P}[\theta < \infty \text{ and } \check{\varphi}(\theta) = j \in \mathcal{S}_+ \mid \check{X}(0) = x, \check{\varphi}(0) = i \in \mathcal{S}_+]$$

and

$$[\breve{G}_{-+}(x,0)]_{ij} = \mathrm{P}[\theta < \infty \text{ and } \breve{\varphi}(\theta) = j \in \mathcal{S}_+ \mid \breve{X}(0) = x, \breve{\varphi}(0) = i \in \mathcal{S}_-].$$

The following theorem gives the relationship between the matrices $\breve{G}_{++}(x,0)$ and $\breve{G}_{-+}(x,0)$ on the one hand, and $\check{\Phi}_{++}(x,0)$ and $\check{\Phi}_{-+}(x,0)$ on the other; this leads to a probabilistic interpretation of the latter.

By a change of time scale, we define $\breve{S} = |C|^{-1}\breve{T}$, and observe that $\breve{S}$ is the generator of a Markov process since it has negative diagonal entries, nonnegative off-diagonal entries and its row sums are equal to zero. We consider the fluid queue with phase transition generator $\breve{S}$ and net input rates equal to $+1$ and -1.

Theorem 1.11.2 *The matrices $\breve{G}_{++}(x,0)$ and $\breve{G}_{-+}(x,0)$ are respectively the minimal nonnegative solutions of the equations*

$$\begin{aligned}\breve{G}_{++}(x,0) &= e^{\breve{S}_{++}x} \\ &\quad + \int_0^x e^{\breve{S}_{++}(x-z)}\breve{S}_{+-}\int_0^\infty e^{\breve{S}_{--}y}\breve{S}_{-+}\breve{G}_{++}(z+y,0)dydz\end{aligned} \tag{1.69}$$

and

$$\breve{G}_{-+}(x,0) = \int_0^\infty e^{\breve{S}_{--}z}\breve{S}_{-+}\breve{G}_{++}(x+z,0)dz. \tag{1.70}$$

Moreover, $\breve{G}_{++}(x,0) = \check{\Phi}_{++}(x,0)$ and $\breve{G}_{-+}(x,0) = \check{\Phi}_{-+}(x,0)$.

Proof Due to the skip-free downward property of the fluid level process, $\breve{G}_{++}(x,0)$ has a matrix-exponential form.

In order to obtain (1.69), we condition on the fluid level z at the epoch of first increase and on the quantity y by which the fluid level increases before the queue begins to empty out again. The first term in (1.69) corresponds to the case where the x initial units of fluid are drained out of the buffer without an increase of the content at any time; in the second term, we first have that $x - z$ units of fluid are drained out of the buffer, then the dual modulating Markov process enters in a state of $\mathcal{S}_-$ and y units of fluid arrive into the buffer; the phase changes again to a state in $\mathcal{S}_+$ and we finally have to take into account the probability that the fluid level empties out, starting from level $z + y$, and this probability is given by $\breve{G}_{++}(z+y,0)$.

Multiply both sides of (1.69) by $e^{-\breve{S}_{++}x}$ on the left and take the derivative with respect to x to get

$$\frac{d}{dx}\breve{G}_{++}(x,0) = \breve{S}_{++}\breve{G}_{++}(x,0) + \breve{S}_{+-}\int_0^\infty e^{\breve{S}_{--}z}\breve{S}_{-+}\breve{G}_{++}(x+z,0)dz.$$

This equation admits the solution $\check{G}_{++}(x,0) = e^{\check{Z}x}$ with $\check{Z}$ given by (1.67). Indeed, replacing $\check{G}_{++}(x,0)$ by $e^{\check{Z}x}$ and letting x go to zero in the above equation, we obtain that

$$\lim_{x\to 0}\frac{d}{dx}e^{\check{Z}x} = \check{S}_{++} + \check{S}_{+-}\int_0^\infty e^{\check{S}_{--}z}\check{S}_{-+}e^{\check{Z}z}dz.$$

Since the left-hand side is equal to $\check{Z}$, we find (1.67) and thus we conclude that $\check{G}_{++}(x,0) = \check{\Phi}_{++}(x,0)$.

To obtain (1.70), we condition on the fluid level z that has been added until the first epoch when the dual phase process enters $\mathcal{S}_+$. Next, we take the derivative with respect to x:

$$\begin{aligned}\frac{d}{dx}\check{G}_{-+}(x,0) &= \int_0^\infty e^{\check{S}_{--}z}\check{S}_{-+}\frac{\partial}{\partial x}\check{G}_{++}(x+z,0)dz \\ &= \int_0^\infty e^{\check{S}_{--}z}\check{S}_{-+}e^{\check{Z}z}e^{\check{Z}x}\check{Z}dz \\ &= \check{\Upsilon}e^{\check{Z}x}\check{Z}\end{aligned}$$

and thus we find that $\check{G}_{-+}(x,0) = \check{\Upsilon}e^{\check{Z}x} = \check{\Phi}_{-+}(x,0)$. □

The next result gives a phase-type characterization for the stationary buffer content of the original fluid queue; it allows one to use the machinery available for these distributions and to perform numerical computations with great accuracy.

Theorem 1.11.3 *The stationary distribution of the fluid level of the queue $\{(\bar{X}(t),\bar{\varphi}(t))\}$ is phase-type with representation $(\check{\boldsymbol{\alpha}},\check{Z})$ where $\check{\boldsymbol{\alpha}} = \boldsymbol{\xi}_+ + \boldsymbol{\xi}_-\check{\Upsilon}$.*

Proof First, observe that the matrix $\check{Z}$ in (1.67) is a defective generator. Indeed, it has nonnegative off-diagonal elements and

$$\begin{aligned}\check{Z}\mathbf{1} &= C_+^{-1}\{\check{T}_{++} + \check{T}_{+-}\Delta_-^{-1}\Upsilon'\Delta_+\}\mathbf{1} \\ &= C_+^{-1}\Delta_+^{-1}(\bar{T}'_{++} + \bar{T}'_{-+}\Upsilon')\Delta_+\mathbf{1} \\ &= C_+^{-1}\Delta_+^{-1}[\bar{\boldsymbol{\xi}}_+(\bar{T}_{++} + \Upsilon\bar{T}_{-+})]' \\ &= C_+^{-1}\Delta_+^{-1}[\bar{\boldsymbol{\xi}}_+\bar{T}_{++} + (\bar{\boldsymbol{\xi}}_- - \bar{\boldsymbol{p}})\bar{T}_{-+}]'\end{aligned}$$

by (1.53); since $\bar{\boldsymbol{\xi}}\bar{T} = \mathbf{0}$, we find that

$$\begin{aligned}\check{Z}\mathbf{1} &= -C_+^{-1}\Delta_+^{-1}\bar{T}'_{-+}\bar{\boldsymbol{p}}' \\ &\le \mathbf{0}.\end{aligned}$$

By adding the components of $\bar{\boldsymbol{\pi}}_{\bullet}(x)$ given by (1.44), we obtain the stationary density for the fluid level:

$$\bar{f}(x) = -\bar{\boldsymbol{\xi}}_{+} Z e^{Zx}[\mathbf{1} + \Upsilon \mathbf{1}].$$

Take transposes on both sides of this equation to get

$$\begin{aligned} \bar{f}(x) &= -[\mathbf{1}' + \mathbf{1}'\Upsilon']\Delta_{+}\Delta_{+}^{-1}e^{Z'x}\Delta_{+}\Delta_{+}^{-1}Z'\Delta_{+}\Delta_{+}^{-1}\bar{\boldsymbol{\xi}}_{+}' \\ &= -[\mathbf{1}' + \mathbf{1}'\Upsilon']\Delta_{+}e^{\check{Z}x}\check{Z}\mathbf{1} \\ &= -\check{\boldsymbol{\alpha}}e^{\check{Z}x}\check{Z}\mathbf{1}, \end{aligned}$$

where $\check{\boldsymbol{\alpha}} = [\mathbf{1}' + \mathbf{1}'\Upsilon']\Delta_{+}$. Since $\check{\Upsilon} = \Delta_{-}^{-1}\Upsilon'\Delta_{+}$, we have $\Delta_{-}\check{\Upsilon} = \Upsilon'\Delta_{+}$, so that $\check{\boldsymbol{\alpha}} = \bar{\boldsymbol{\xi}}_{+} + \bar{\boldsymbol{\xi}}_{-}\check{\Upsilon}$, which concludes the proof. □

As was shown in [16], the phase-type representation derived by Ramaswami in [39] and presented above easily extends to the fluid queue with net input rates that can take any real value, including zero.

To see this, consider again the general fluid queue $\{(\tilde{X}(t), \tilde{\varphi}(t)) : t \in \mathbb{R}^{+}\}$ introduced in Section 1.7, with phase transition generator $\tilde{T}$. The stationary marginal density of its fluid level is given by (1.29), which we recall here:

$$\mu(x) = \gamma \boldsymbol{p}_{-} T_{-+} e^{Kx}(\boldsymbol{w}_{+} + \Psi \boldsymbol{w}_{-}),$$

with γ given by (1.28) and $\boldsymbol{w}_{+}$, $\boldsymbol{w}_{-}$ given by (1.30, 1.31). By (1.64), we know that $\boldsymbol{p}_{-} = \boldsymbol{\xi}_{-} - \boldsymbol{\xi}_{+}\Psi$, thus

$$\begin{aligned} \boldsymbol{p}_{-}T_{-+} &= \boldsymbol{\xi}_{-}T_{-+} - \boldsymbol{\xi}_{+}\Psi T_{-+} \\ &= \boldsymbol{\xi}_{-}T_{-+} - \boldsymbol{\xi}_{+}K + \boldsymbol{\xi}_{+}T_{++} \end{aligned}$$

using the fact that $K = T_{++} + \Psi T_{-+}$. Since $\boldsymbol{\xi}T = \mathbf{0}$, we obtain that

$$\boldsymbol{p}_{-}T_{-+} = -\boldsymbol{\xi}_{+}K, \tag{1.71}$$

and we can rewrite the stationary density of the level of the fluid queue $\{(X(t), \varphi(t))\}$ as

$$\mu(x) = -\boldsymbol{\xi}_{+}Ke^{Kx}(\boldsymbol{w}_{+} + \Psi \boldsymbol{w}_{-}). \tag{1.72}$$

Theorem 1.11.4 *The stationary marginal distribution of the level of the general fluid queue $\{(\tilde{X}(t), \tilde{\varphi}(t))\}$ is phase-type with representation $(\boldsymbol{\beta}, B)$ of order s_{+}, with*

$$\boldsymbol{\beta} = \gamma\,\{\Delta_{+}[\boldsymbol{w}_{+},\ \Psi \boldsymbol{w}_{-}]\}'$$

and

$$B = \Delta_+^{-1} K' \Delta_+,$$

where $\Delta_+ = \mathrm{diag}(\boldsymbol{\xi}_+)$, *and where* γ *is given by (1.28) and* $\boldsymbol{w}_+$, $\boldsymbol{w}_-$ *by (1.30, 1.31).*

Proof We first establish that the matrix B is a defective generator. Indeed, it has nonnegative off-diagonal entries and

$$\begin{aligned} B\mathbf{1} &= \Delta_+^{-1}(T_{++} + \Psi T_{-+})\Delta_+\mathbf{1} \\ &= \Delta_+^{-1}[\boldsymbol{\xi}_+(T_{++} + \Psi T_{-+})]' \\ &= \Delta_+^{-1}[\boldsymbol{\xi}_+ T_{++} + (\boldsymbol{\xi}_- - \boldsymbol{p}_-)T_{-+}]' \end{aligned}$$

by (1.64); since $\boldsymbol{\xi} T = \mathbf{0}$, we obtain

$$B\mathbf{1} = -\Delta_+^{-1} T'_{-+} \boldsymbol{p}'_- \le \mathbf{0}.$$

By transposing both sides of (1.72), we obtain

$$\begin{aligned} \mu(x) &= -\gamma[\boldsymbol{w}_+,\ \Psi\boldsymbol{w}_-]' e^{K'x} K' \boldsymbol{\xi}'_+ \\ &= -\gamma[\boldsymbol{w}_+,\ \Psi\boldsymbol{w}_-]' \Delta_+ \Delta_+^{-1} e^{K'x} \Delta_+ \Delta_+^{-1} K' \Delta_+ \Delta_+^{-1} \boldsymbol{\xi}'_+ \\ &= -\gamma\{\Delta_+[\boldsymbol{w}_+,\ \Psi\boldsymbol{w}_-]\}' e^{Bx} B\mathbf{1}, \end{aligned}$$

which is the announced result. □

1.12 Wiener-Hopf Factorization

Rogers [42] obtains the equation (1.12) through the Wiener-Hopf factorization of finite Markov chains. We summarize his results in this section without giving the proofs, and show how they relate to ours.

Rogers assumes almost throughout his analyzis that the net input rates of fluid into the buffer are equal to $+1$ and -1. We denote by T the infinitesimal generator of the phase process, and by V the square matrix of order $s_+ + s_-$ such that

$$V = \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix},$$

where I_+ and I_- denote identity matrices of orders s_+ and s_-, respectively.

Definition 1.12.1 *A Wiener-Hopf factorization of $V^{-1}T$ is a quadruple $(Z_{-+}, Q_{++}; Z_{+-}, Q_{--})$ such that*

$$V^{-1}T \begin{bmatrix} I_+ & Z_{+-} \\ Z_{-+} & I_- \end{bmatrix} = \begin{bmatrix} I_+ & Z_{+-} \\ Z_{-+} & I_- \end{bmatrix} \begin{bmatrix} Q_{++} & 0 \\ 0 & -Q_{--} \end{bmatrix}, \qquad (1.73)$$

where Z_{-+} and Z_{+-} are matrices of dimensions $s_- \times s_+$ and $s_+ \times s_-$, respectively, and Q_{++}, Q_{--} are square matrices of orders s_+ and s_-, respectively, with nonnegative off-diagonal elements and non positive row sums.

In order to give an expression for the stationary distribution of the fluid queue, Rogers uses two additional Markov processes. Consider the additive functional Y defined in (1.13), which may be seen as describing the evolution of a fluid queue with a bottomless buffer. Let τ_t^+ and τ_t^- denote the following time-changes

$$\tau_t^+ = \inf\{u : Y(u) > t\} \quad \text{and} \quad \tau_t^- = \inf\{u : Y(u) < -t\},$$

and define the time-changed processes

$$D^+(t) = \varphi(\tau_t^+) \quad \text{and} \quad D^-(t) = \varphi(\tau_t^-).$$

Note that $\{D^-(t)\}$ coincides with the process of downward records defined in Section 1.4. The process $\{D^+(t)\}$ can also be interpreted in a similar manner, by considering the level reversed version of $\{D^-(t)\}$; we call it the process of *upward* records.

Theorem 1.12.2

i. The quadruple $(\hat{\Psi}, \hat{U}; \Psi, U)$ is always a Wiener-Hopf factorization of $V^{-1}T$, where

(a) the matrices U and $\hat{U}$ are the infinitesimal transition generators of the Markov processes $\{D^-(t)\}$ and $\{D^+(t)\}$, respectively;

(b) the matrices Ψ and $\hat{\Psi}$ are defined by

$$\Psi = P[\tau_0^- < \infty, \varphi(\tau_0^-) = j | \varphi(0) = i], \quad \text{for } i \in \mathcal{S}_+,\ j \in \mathcal{S}_-,$$

and

$$\hat{\Psi} = P[\tau_0^+ < \infty, \varphi(\tau_0^+) = j | \varphi(0) = i], \quad \text{for } i \in \mathcal{S}_-,\ j \in \mathcal{S}_+.$$

ii. If the phase process is transient, then the Wiener-Hopf factorization is unique.

□

The reason for the choice of the notations U and Ψ in Theorem 1.12.2 above comes from the fact that these matrices have the same probabilistic interpretation as the matrices U and Ψ defined in Sections 1.4 and 1.3. Moreover, using (1.73), Theorem 1.12.2 leads to the four equations

$$\begin{aligned} T_{--} + T_{-+}\Psi &= U, & (1.74)\\ T_{+-} + T_{++}\Psi + \Psi T_{--} + \Psi T_{-+}\Psi &= 0, & (1.75)\\ T_{++} + T_{+-}\hat{\Psi} &= \hat{U}, & (1.76)\\ T_{-+} + T_{--}\hat{\Psi} + \hat{\Psi}T_{++} + \hat{\Psi}T_{+-}\hat{\Psi} &= 0. & (1.77)\end{aligned}$$

Equations (1.74, 1.75) are the same as (1.14, 1.16). While we obtain these expressions through the probabilistic interpretation of the quantities involved, Rogers obtains them through a purely algebraic approach. Similar comments apply to equations (1.76, 1.77), as we shall see later.

The next result gives the characterization of all possible solutions of (1.76, 1.77) in the case where the phase process is not transient.

Theorem 1.12.3 *Suppose that the phase process is recurrent and that*

$$V^{-1}T\begin{bmatrix} I_+ \\ Z_{-+}\end{bmatrix} = \begin{bmatrix} I_+ \\ Z_{-+}\end{bmatrix} Q_{++}$$

for some matrix Z_{-+} of dimension $s_- \times s_+$, and some square matrix Q_{++} of order s_+, with nonnegative off-diagonal elements and non positive row sums.

i. If Q_{++} is transient, then $Q_{++} = \hat{U}$ and $Z_{-+} = \hat{\Psi}$.

ii. If Q_{++} is recurrent and $\hat{U}$ is recurrent, then $Q_{++} = \hat{U}$ and $Z_{-+} = \hat{\Psi}$.

iii. If Q_{++} is recurrent and $\hat{U}$ is transient, then $Q_{++} = \hat{U} - (\hat{U}\mathbf{1})\boldsymbol{u}$, where $\boldsymbol{u}$ is the left eigenvector of $\hat{U}$ whose eigenvalue has largest real part, and $\boldsymbol{u}$ is normalized by the condition $\boldsymbol{u}\mathbf{1} = 1$.

□

Consider now the dual fluid queue $\{(\check{X}(t), \check{\varphi}(t)) : t \in \mathbb{R}^+\}$, already defined in Section 1.11 except that now the net input rates r_i are equal to $+1$ for i in $\mathcal{S}_-$ and to -1 for i in $\mathcal{S}_+$. Its phase process has generator $\check{T} = \Delta^{-1}T'\Delta$, where $\Delta = \mathrm{diag}(\boldsymbol{\xi})$. Define the matrices $\check{U} = \Delta_-^{-1}U'\Delta_-$ and $\check{\Psi} = \Delta_-^{-1}\Psi'\Delta_+$; they have the same interpretation for the dual process as U and Ψ for the original fluid queue. It is a matter of simple manipulations to verify that

$$\check{U} = \check{T}_{++} + \check{T}_{+-}\check{\Psi}$$

and

$$\check{T}_{-+} + \check{\Psi}\check{T}_{++} + \check{T}_{--}\check{\Psi} + \check{\Psi}\check{T}_{+-}\check{\Psi} = 0.$$

Rogers obtains the following expression for the stationary distribution of the fluid queue.

Theorem 1.12.4 *For $x \geq 0$ and j in $\mathcal{S}$,*

$$\lim_{t\to\infty} P[X(t) > x, \varphi(t) = j] = \begin{cases} \xi_j(e^{\check{U}x}\mathbf{1})_j, & j \in \mathcal{S}_+, \\ \xi_j(\check{\Psi}e^{\check{U}x}\mathbf{1})_j, & j \in \mathcal{S}_-. \end{cases}$$

□

This gives the following expression for the jth component of the stationary density vector $\boldsymbol{\pi}(x)$:

$$\pi_j(x) = \begin{cases} \xi_j(\check{U}e^{\check{U}x}\mathbf{1})_j, & j \in \mathcal{S}_+, \\ \xi_j(\check{\Psi}\check{U}e^{\check{U}x}\mathbf{1})_j, & j \in \mathcal{S}_-. \end{cases}$$

Let us show that this is in fact equivalent to (1.12). We only consider the case where j is in $\mathcal{S}_+$ because the computations are similar in the other case.

First, we show that $\check{U} = \Delta_+^{-1}K'\Delta_+$, where K is given by (1.19). Indeed,

$$\begin{aligned} \check{U} &= \check{T}_{++} + \check{T}_{+-}\check{\Psi} \\ &= \Delta_+^{-1}T'_{++}\Delta_+ + \Delta_+^{-1}T'_{-+}\Delta_-\Delta_-^{-1}\Psi'\Delta_+ \end{aligned}$$

by definition of $\check{T}$ and $\check{\Psi}$, and therefore

$$\check{U} = \Delta_+^{-1}(T'_{++} + \Psi T_{-+})'\Delta_+ = \Delta_+^{-1}K'\Delta_+$$

by (1.19). Thus, for $j \in \mathcal{S}_+$, we have

$$\begin{aligned} \pi_j(x) &= -\xi_j(\check{U}e^{\check{U}x}\mathbf{1})_j \\ &= -\xi_j(\Delta_+^{-1}K'\Delta_+\Delta_+^{-1}e^{K'x}\Delta_+\mathbf{1})_j. \end{aligned}$$

Transposing both sides of this equation, we obtain

$$\pi_j(x) = -\xi_j(\boldsymbol{\xi}_+ K e^{Kx} \Delta_+^{-1})_j$$

using the facts that $\Delta_+ \mathbf{1} = \boldsymbol{\xi}'_+$ and that K and e^{Kx} commute. Since Δ_+^{-1} is a diagonal matrix, we can write that

$$\begin{aligned} \pi_j(x) &= -\xi_j(\boldsymbol{\xi}_+ K e^{Kx})_j (\Delta_+^{-1})_{jj} \\ &= -(\boldsymbol{\xi}_+ K e^{Kx})_j \end{aligned}$$

because $\xi_j = (\Delta_+)_{jj}$. Using (1.71), we find that

$$\pi_j(x) = (\boldsymbol{p}_- T_{-+} e^{Kx})_j,$$

which is indeed the jth component of (1.12), for j in $\mathcal{S}_+$.

2

Algorithms

We recall in the first section the definition of a QBD process and the matrix-geometric form of its stationary distribution, as well as some key quantities for the analysis of QBDs. We point out the similarities between the stationary distributions of fluid queues on the one hand, and of QBD processes on the other hand.

In Section 2.2, we present the very efficient algorithmic procedure introduced by Ramaswami [39], which is based on QBD processes and which allows to compute the solution of the Riccati equation (1.16), leading to the complete stationary distribution of fluid queues with infinite buffers. We give the probabilistic interpretation of Ramaswami's algorithm, and we give in Section 2.3 a generalization of this algorithm which is interesting from a theoretical point of view, but which does not really improve the efficiency of the original procedure, as we see in Section 2.4 through a numerical example and in Section 2.5 through a brief convergence analysis.

The results exposed in Section 2.2 were presented in da Silva Soares and Latouche [16].

We conclude this chapter by presenting some other algorithms for solving the Riccati equation (1.16) and their probabilistic interpretation on the fluid flow setting; this material comes from Bean et al. [10].

2.1 Discrete-Time Homogeneous QBDs

A discrete-time QBD process is a Markov chain $\{(L_t, J_t) : t \in \mathbb{N}\}$ on the two-dimensional state space $\{(n, j) : n \in \mathbb{N}, 1 \leq j \leq m\}$, where m may be either finite or infinite. The first component L_t is called the *level*, and the second one J_t is called the *phase*. We denote by $\ell(n)$ the subset of states in level n, that is, $\ell(n) = \{(n, j) : 1 \leq j \leq m\}$. For $n \in \mathbb{N}$ and for $1 \leq j, j' \leq m$, the only transitions allowed are:

- from state (n, j) to state (n, j');
- from state (n, j) to state $(n+1, j')$;
- from state (n, j) to state $(n-1, j')$, provided that $n \geq 1$.

The process is said to be homogeneous because we assume that the transition probabilities are independent of the level except for the transitions starting from level zero. The transition matrix thus has the following block tridiagonal form:

$$P = \begin{bmatrix} B & A_0 & 0 & 0 & \dots \\ A_2 & A_1 & A_0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & \dots \\ 0 & 0 & A_2 & A_1 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.1}$$

where B, A_0, A_1 and A_2 are $m \times m$ matrices.

Assume that the process is aperiodic and positive recurrent. We denote by $\boldsymbol{\pi}$ its stationary probability vector; it is the unique solution of the system $\boldsymbol{\pi} P = \boldsymbol{\pi}$, $\boldsymbol{\pi}\mathbf{1} = 1$. We partition the vector $\boldsymbol{\pi}$ by levels as $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_n, \dots)$ where each subvector $\boldsymbol{\pi}_i$, $i \geq 0$, has m components. The following result is known as the matrix-geometric property of the equilibrium vector $\boldsymbol{\pi}$. We refer the reader to [29, Chapter 6] for the proofs of the results presented in this section.

Theorem 2.1.1 *If the QBD is positive recurrent, then*

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R^n \qquad \text{for } n \geq 0,$$

where R is the matrix such that, for any $n \geq 0$ and $1 \leq i, j \leq m$, R_{ij} is the expected number of visits to $(n+1, j)$, starting from (n, i), before a return to $\ell(0) \cup \ldots \cup \ell(n)$.

□

Remark 2.1.2 For $1 \leq i, j \leq m$, the nth power R^n is such that its (i, j)th entry gives the expected number of visits to (n, j), starting from $(0, i)$, before a return to level zero. Thus, the stationary probability vector $\boldsymbol{\pi}_n$ of level n is expressed in terms of the stationary probability vector $\boldsymbol{\pi}_0$ of level zero and of the matrix R^n recording the expected number of visits to level n, starting from level zero, under the taboo of this initial level.

Recall the form (1.12) of the stationary density vector $\boldsymbol{\pi}(x)$ of a fluid queue. It is also expressed in terms of the steady state probability vector of level zero and of a matrix power which records the expected number of visits to level x, starting from level zero, before returning to the initial level. The similarity between the stationary distributions of the two processes is striking.

Together with the matrix R, there are two other matrices, denoted by U and G, which are closely related to the dynamics of the QBD process, the matrix G being of particular interest for us, as we shall see later.

Assume that the QBD process starts from level one. Define τ as the first epoch of visit to the level zero and θ as the first epoch of return to the level one; thus

$$\tau = \inf\{t \geq 0 : L_t = 0\}$$

and

$$\theta = \inf\{t \geq 1 : L_t = 1\}.$$

The matrix U records the probability that, starting from level 1, the process returns to level 1 before visiting level 0:

$$U_{ij} = \mathrm{P}[\theta < \tau, (L_\theta, J_\theta) = (1, j) | (L_0, J_0) = (1, i)].$$

The matrix G records the probability that, starting from level 1, the process visits level 0 in a finite time:

$$G_{ij} = \mathrm{P}[\tau < \infty, (L_\tau, J_\tau) = (0, j) | (L_0, J_0) = (1, i)].$$

The matrices R, U and G are related to each other and, once one knows one of the three, one can determine the other two. This is expressed in the next theorem.

Theorem 2.1.3 *If any one of the matrices U, G, or R is known, then we may determine the other two by applying one of the following equations:*

$$\begin{aligned} R &= A_0(I-U)^{-1}, \\ G &= (I-U)^{-1}A_2, \\ U &= A_1 + A_0 G, \\ U &= A_1 + RA_2. \end{aligned}$$

□

The following result is a direct consequence.

Theorem 2.1.4 *The matrices U, G and R, respectively, satisfy the following equations:*

$$\begin{aligned} U &= A_1 + A_0(I-U)^{-1}A_2, \\ G &= A_2 + A_1 G + A_0 G^2, \\ R &= A_0 + RA_1 + R^2 A_2. \end{aligned} \tag{2.2}$$

□

There exist simple and very efficient computational algorithms for the determination of G, and thus also for the determination of U and R; it is the case, for instance, of the Logarithmic-Reduction algorithm of Latouche and Ramaswami [29, Section 8.4]. This algorithm is iterative, easy to implement, numerically stable and it converges quadratically fast (Bini, Latouche and Meini [11], Guo [25], Meini [34, 35]).

In order to have the complete stationary distribution of the QBD, one needs to determine $\boldsymbol{\pi}_0$; this is the object of the next theorem.

Theorem 2.1.5 *The stationary distribution $\boldsymbol{\pi}_0$ of the boundary level is the unique solution of the system*

$$\begin{aligned} \boldsymbol{\pi}_0(B + A_0 G) &= \boldsymbol{\pi}_0 \\ \boldsymbol{\pi}_0(I-R)^{-1}\mathbf{1} &= 1. \end{aligned}$$

□

Remark 2.1.6 We observe again a close similarity between fluid queues and QBDs. The vector $\boldsymbol{\pi}_0$ is proportional to the steady state probability vector of the process with transition matrix $B + A_0G$, which is in fact the censored process on the states of level zero. This is also the case for a fluid queue, for which we saw in Theorem 1.6.1 that the steady state probability mass vector $\boldsymbol{p}_-$ of level zero is proportional to the stationary vector of the restricted process to the states of level zero, with generator $U = T_{--} + T_{-+}\Psi$.

Theorem 2.1.1 is conditioned on the fact that the QBD process is positive recurrent. We give next two necessary and sufficient conditions for the positive recurrence of the QBD, in the case where the number m of phases is finite. We denote by $\boldsymbol{\alpha}$ the row vector which is the unique solution of the system $\boldsymbol{\alpha}A = \boldsymbol{\alpha}$, $\boldsymbol{\alpha}\mathbf{1} = 1$, with $A = A_0 + A_1 + A_2$. The notation $\mathrm{sp}(M)$ stands for the spectral radius of some matrix M.

Theorem 2.1.7 *For a QBD with transition matrix (2.1), we have the following characterizations.*

i. If m is finite, then the process is positive recurrent if and only if $\mathrm{sp}(R) < 1$.

ii. If m is finite, if the QBD is irreducible, and if the stochastic matrix A is irreducible, then the process is positive recurrent if and only if $\mu = \boldsymbol{\alpha}A_0\mathbf{1} - \boldsymbol{\alpha}A_2\mathbf{1} < 0$. The process is null recurrent if $\mu = 0$, and it is transient if $\mu > 0$.

□

2.2 Uniformization and Interpretation

We shall now return to the fluid setting and describe the computational procedure proposed in [39] for the determination of the matrix Ψ of first passage probabilities to the initial level, which, as we have seen, is the key quantity for obtaining the stationary distribution of any given fluid queue. We also expose the probabilistic interpretation of this algorithm, which we presented in [16].

We consider a fluid queue, with phase transition generator T and with net input rates equal to $+1$ and -1. This is not at all restrictive, since we have already seen how to obtain the stationary distribution of a completely general fluid queue once we have the stationary distribution of the simpler one.

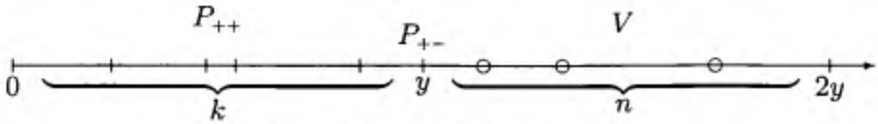


Figure 2.1: Uniformization.

The starting point is the uniformization of the equation relating Ψ and the generator $U = T_{--} + T_{-+}\Psi$ of the process of downward records:

$$\Psi = \int_0^\infty e^{T_{++}y} T_{+-} e^{Uy} dy. \tag{2.3}$$

First, we transform the phase process by uniformization and we define $P = I + 1/\mu T$, where $\mu \geq \max_{i \in \mathcal{S}} |T_{ii}|$; we decompose P in a manner conformant to the partition of T. With these, we have that

$$e^{T_{++}y} = \sum_{k \geq 0} e^{-\mu y} \frac{(\mu y)^k}{k!} P_{++}^k.$$

Since $U \geq T_{--}$, we may use the same parameter μ to discretize the process of downward records and write that

$$e^{Uy} = \sum_{n \geq 0} e^{-\mu y} \frac{(\mu y)^n}{n!} V^n$$

where

$$\begin{aligned} V &= I + \frac{1}{\mu} U \\ &= P_{--} + P_{-+}\Psi, \end{aligned} \tag{2.4}$$

using (1.14). We write that V is the transition matrix of the *discretized* process of downward records.

Using this uniformization, (2.3) becomes

$$\Psi = \int_0^\infty \sum_{k \geq 0} e^{-\mu y} \frac{(\mu y)^k}{k!} \mu \sum_{n \geq 0} e^{-\mu y} \frac{(\mu y)^n}{n!} P_{++}^k P_{+-} V^n dy \tag{2.5}$$

and the right-hand side is a discretized version of the fluid/phase process, which we interpret as follows. One considers the epochs of a Poisson process with rate μ, and a phase process which starts in $\mathcal{S}_+$. Equation (2.5) states that Ψ is equal to the probability matrix of the following event (see Figure 2.1 for an illustration): there exist y, k and n such that

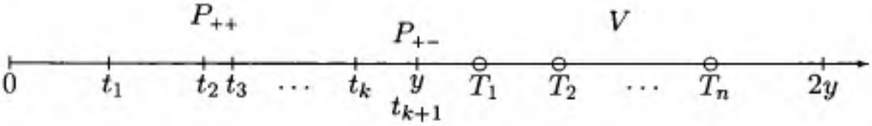


Figure 2.2: The Poisson epochs $\{t_1, t_2, \ldots\}$ before the first passage to $\mathcal{S}_-$, and the Poisson epochs $\{T_1, T_2, \ldots\}$ afterwards.

- a Poisson epoch occurs at time y, k epochs occur in $(0, y)$ and n epochs occur in $(y, 2y)$;
- the epoch at time y is the first at which the phase enters $\mathcal{S}_-$;
- at each epoch in $(0, y)$, a transition occurs from $\mathcal{S}_+$ to $\mathcal{S}_+$ with probabilities given by transition matrix P_{++};
- at each epoch in $(y, 2y)$, a transition occurs from $\mathcal{S}_-$ to $\mathcal{S}_-$ with probabilities given by transition matrix V.

Next, one writes (2.5) as

$$\Psi = \sum_{k,n \geq 0} \gamma_{kn} P_{++}^k P_{+-} V^n, \tag{2.6}$$

where

$$\gamma_{kn} = \int_0^\infty e^{-2\mu y} \frac{(\mu y)^{k+n}}{k!n!} \mu dy = \frac{\mu^{k+n+1}}{k!n!} \int_0^\infty e^{-2\mu y} y^{k+n} dy.$$

Perform $k + n + 1$ integrations by parts to obtain

$$\gamma_{kn} = \frac{(k+n)!}{k!n!} \mu^{k+n+1} \left(\frac{1}{2\mu}\right)^{k+n+1} = \binom{k+n}{n} \left(\frac{1}{2}\right)^{k+n+1},$$

which is the probability of n failures before the k + 1st success in a Bernoulli sequence with probability $1/2$ of success.

This, in turn, may be interpreted as follows. Denote by $\{t_1, t_2, \ldots\}$ the Poisson epochs before the first passage to $\mathcal{S}_-$ and by $\{T_1, T_2, \ldots\}$ the Poisson epochs afterwards; this is depicted in Figure 2.2. Since they occupy the non overlapping intervals $(0, y)$ and $(y, 2y)$, they are independent. Therefore, one may replace the Poisson process over two disjoint intervals by two independent processes over the same interval

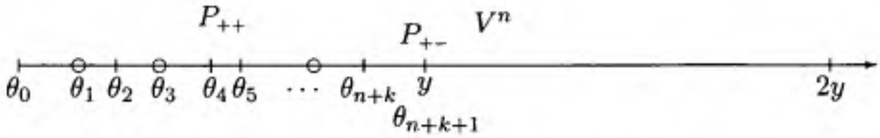


Figure 2.3: The Poisson process $\mathcal{P}_t \cup \mathcal{P}_T = \{\theta_i : i \geq 0\}$.

and consider $\mathcal{P}_t = \{t_i : i \geq 0\}$ and $\mathcal{P}_T = \{T_i : i \geq 0\}$, both with intensity μ and with $t_0 = T_0 = 0$.

The superposition $\mathcal{P}_t \cup \mathcal{P}_T = \{\theta_i : i \geq 0\}$ forms a Poisson process with intensity 2μ which is illustrated in Figure 2.3; each epoch θ_i belongs to $\mathcal{P}_t$ or to $\mathcal{P}_T$ with probability $1/2$, independently of the others.

In (2.6), we count the number n of epochs of $\mathcal{P}_T$ which occur before the epoch t_{k+1} which marks the first passage to $\mathcal{S}_-$.

The second transformation consists in completely disconnecting the discretized process from any reference to the fluid buffer. Here, we write (2.6) as

$$\Psi = \sum_{n \geq 0} \{ \sum_{k \geq 0} \gamma_{kn} P_{++}^k \} P_{+-} V^n$$

and the nth term in the right-hand side is interpreted as follows. We consider a Bernoulli process with probability $1/2$ of success, we start with a phase in $\mathcal{S}_+$ and a counter D initialized to 1, and we perform the operations described below:

- in the case of a failure, we increase the counter D by 1 and do not change the phase;
- in the case of a success, either we make a transition to $\mathcal{S}_+$ with probabilities given by the elements of the matrix P_{++} and we keep D constant,
- or we make a transition to $\mathcal{S}_-$ with probabilities given by the elements of P_{+-} and we decrement D by 1;
- once the phase has moved to $\mathcal{S}_-$, we stop the Bernoulli process, we systematically apply the transition matrix V and we decrement D by 1 at each step until it becomes zero.

The counter and the phase now evolve like in a discrete time QBD process with transition matrices

$$A_0 = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{1}{2}P_{++} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & \frac{1}{2}P_{+-} \\ 0 & V \end{bmatrix}.$$

Here, Ψ is a matrix of first passage probabilities to lower levels; specifically, Ψ_{ij} is the conditional probability of eventually reaching $(0, j)$, before any other state in $(0, \mathcal{S})$, given that the process starts from $(1, i)$ at time 0, with i in $\mathcal{S}_+$ and j in $\mathcal{S}_-$.

Of course, we do not know V, so that we need to pursue the matter a little further. In view of the interpretation we have given to Ψ, (2.4) tells us that there are two ways to reduce D by one, starting from $\mathcal{S}_-$:

- either one does it directly, using the transition probabilities given by the elements of P_{--},
- or a transition is made to $\mathcal{S}_+$, with probabilities given by the elements of P_{-+}, in which case one must recursively apply the same procedure in order to eventually reduce D by one, with probability matrix Ψ.

Thus, we finally interpret Ψ as the matrix of first passage probabilities from $(1, \mathcal{S}_+)$ to $(0, \mathcal{S}_-)$ for the QBD with transition matrices

$$A_0 = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{1}{2}P_{++} & 0 \\ P_{-+} & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & \frac{1}{2}P_{+-} \\ 0 & P_{--} \end{bmatrix} \tag{2.7}$$

and we have that

$$G = \begin{bmatrix} 0 & \Psi \\ 0 & V \end{bmatrix}, \tag{2.8}$$

where G is the matrix of first passage probabilities to lower levels, solution of (2.2), for the QBD process defined by (2.7). As already stated in Section 2.1, one can compute G using the Logarithmic-Reduction algorithm, and thus obtain the matrix Ψ through this very efficient procedure.

Note that the Riccati equation (1.16) for Ψ,

$$T_{++}\Psi + \Psi T_{--} + T_{+-} + \Psi T_{-+}\Psi = 0,$$

together with (1.14), is equivalent to the statement that G is a solution of (2.2); indeed, if we replace G by its expression (2.8) and use A_0, A_1 and A_2 given by (2.7), we obtain the equations

$$\begin{aligned} \Psi &= \frac{1}{2}P_{+-} + \frac{1}{2}P_{++}\Psi + \frac{1}{2}\Psi V \\ V &= P_{--} + P_{-+}\Psi \end{aligned}$$

which are equivalent to (1.16), using $P = I + 1/\mu T$.

2.3 Uniformization Using Different Parameters

The computational procedure described in Section 2.2 brings up a natural and interesting question: is it really necessary to perform the uniformizations of T and U using the same parameter? The answer is negative and we show that the preceding development holds true if ones uses different parameters for the two uniformisations.

We choose μ_1 such that $\mu_1 \geq \max_{i\in\mathcal{S}_+} |T_{ii}|$ to uniformize the phase process and define $P = I + 1/\mu_1 T$. Thus,

$$e^{T_{++}y} = \sum_{k\geq 0} e^{-\mu_1 y}\frac{(\mu_1 y)^k}{k!}P_{++}^k.$$

Next, we discretize the process of downward records with a parameter μ_2 such that $\mu_2 \geq \max_{i\in\mathcal{S}_-} |U_{ii}|$ and write

$$e^{Uy} = \sum_{n\geq 0} e^{-\mu_2 y}\frac{(\mu_2 y)^n}{n!}V^n$$

where $V = I + 1/\mu_2 U$. Therefore, defining $W = I + 1/\mu_2 T$, we have

$$V = W_{--} + W_{-+}\Psi. \tag{2.9}$$

Equation (1.15) becomes

$$\Psi = \int_0^\infty \sum_{k\geq 0} e^{-\mu_1 y}\frac{(\mu_1 y)^k}{k!}\mu_1 \sum_{n\geq 0} e^{-\mu_2 y}\frac{(\mu_2 y)^n}{n!}P_{++}^k P_{+-}V^n dy. \tag{2.10}$$

A probabilistic interpretation of this equation goes along the same lines as the interpretation given in Section 2.2. We consider two Poisson processes $\{N_1(t) : t \in \mathbb{R}^+\}$ and $\{N_2(t) : t \in \mathbb{R}^+\}$ with rates μ_1 and μ_2, respectively, and a phase process which starts in $\mathcal{S}_+$. Equation (2.10) states that Ψ is the matrix recording the probability of the following event: there exist y, k and n such that

- an epoch of $\{N_1(t)\}$ occurs at time y, k epochs of $\{N_1(t)\}$ occur in $(0, y)$ and n epochs of $\{N_2(t)\}$ occur in $(y, 2y)$;
- the epoch at time y is the first at which the phase enters $\mathcal{S}_-$;
- at each epoch of N_1 in $(0, y)$, a transition occurs from $\mathcal{S}_+$ to $\mathcal{S}_+$ with probabilities given by the entries of the transition matrix P_{++};

- at each epoch of N_2 in $(y, 2y)$, a transition occurs from $\mathcal{S}_-$ to $\mathcal{S}_-$ with transition matrix V.

One may write (2.10) as

$$\Psi = \sum_{k,n \geq 0} \gamma_{kn} P_{++}^k P_{+-} V^n, \tag{2.11}$$

where

$$\begin{aligned}\gamma_{kn} &= \int_0^\infty e^{-\mu_1 y} \frac{(\mu_1 y)^k}{k!} \mu_1 e^{-\mu_2 y} \frac{(\mu_2 y)^n}{n!} dy \\ &= \frac{(k+n)!}{k!n!} \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)^{k+1} \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^n,\end{aligned}$$

and is the probability of n failures before the $k+1$st success in a Bernoulli sequence with probability $\mu_1/(\mu_1 + \mu_2)$ of success.

Once again, we denote by $\{t_1, t_2, \ldots\}$ the Poisson epochs before the first passage to $\mathcal{S}_-$ and by $\{T_1, T_2, \ldots\}$ the Poisson epochs afterwards. By the same arguments as before, we may replace these Poisson epochs by two independent processes over the same interval and consider $\mathcal{P}_t = \{t_i : i \geq 0\}$ with intensity μ_1 and $\mathcal{P}_T = \{T_i : i \geq 0\}$ with intensity μ_2, and with $t_0 = T_0 = 0$.

The process $\mathcal{P}_t \cup \mathcal{P}_T = \{\theta_i : i \geq 0\}$ is Poisson with intensity $\mu_1 + \mu_2$ and each epoch θ_i belongs to $\mathcal{P}_t$ with probability $\mu_1/(\mu_1 + \mu_2)$ or to $\mathcal{P}_T$ with probability $\mu_2/(\mu_1 + \mu_2)$, independently of the others.

In (2.11), we count the number n of epochs of $\mathcal{P}_T$ which occur before the epoch t_{k+1} which marks the first passage to $\mathcal{S}_-$.

Just as we did previously, we disconnect the discretized process from the fluid queue and write (2.11) as

$$\Psi = \sum_{n \geq 0} \{\sum_{k \geq 0} \gamma_{kn} P_{++}^k\} P_{+-} V^n. \tag{2.12}$$

We consider a Bernoulli process with probability $\mu_1/(\mu_1 + \mu_2)$ of success and a counter D initialized to 1. By performing the same operations as in Section 2.2, we can interpret the nth term in the right-hand side of (2.12) exactly as before.

Since $V = W_{--} + W_{-+}\Psi$, and since the probabilities of success and failure of this Bernoulli process are $\mu_1/(\mu_1 + \mu_2)$ and $\mu_2/(\mu_1 + \mu_2)$, respectively, we find that Ψ is the matrix of first passage probabilities from

$(1, \mathcal{S}_+)$ to $(0, \mathcal{S}_-)$ for the QBD process with transition matrices:

$$A_0 = \begin{bmatrix} \frac{\mu_2}{\mu_1+\mu_2} I & 0 \\ 0 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} \frac{\mu_1}{\mu_1+\mu_2} P_{++} & 0 \\ W_{-+} & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & \frac{\mu_1}{\mu_1+\mu_2} P_{+-} \\ 0 & W_{--} \end{bmatrix} \tag{2.13}$$

and we have that

$$G = \begin{bmatrix} 0 & \Psi \\ 0 & V \end{bmatrix}, \tag{2.14}$$

where G is the matrix of first passage probabilities to lower levels for the QBD process defined by (2.13).

Here, (2.2) turns into the pair of equations

$$0 = T_{+-} + T_{++}\Psi + \Psi T_{--} + \Psi T_{-+}\Psi$$

and

$$V = I + \frac{1}{\mu_2}(T_{--} + T_{-+}\Psi).$$

The first equation is the same as (1.16) and it is therefore perfectly clear that the determination of the matrix Ψ is independent of the parameters μ_1 and μ_2, hence on the way the uniformization is carried out. On the other hand, one can observe that the matrix V only depends on the parameter μ_2 and not on μ_1; thus, μ_1 can take any real value provided that it satisfies $\mu_1 \geq \max_{i \in \mathcal{S}_+} |T_{ii}|$, leaving V unchanged.

Another observation is that the matrix V converges to the identity matrix I as μ_2 goes to infinity. In Section 2.5 we will show how the parameter μ_2 determines the rate of convergence of the Logarithmic-Reduction algorithm.

2.4 Numerical Illustration

We consider a random environment which cycles through three periods: one during which the fluid builds up at the constant rate c, followed by one where the fluid level remains constant and finally the third period during which the fluid decreases at the constant rate 0.5. After the third period, the cycle repeats. The first period lasts 1 unit of time, on average, and the second and third periods last 2 units of time each, on average. The traffic intensity ρ is the ratio of the amount of fluid going in the buffer to the amount going out. The process is positive recurrent if $\rho < 1$; this is equivalent to the stability condition $\boldsymbol{\xi r} < 0$, which reduces here to $c < 1$.

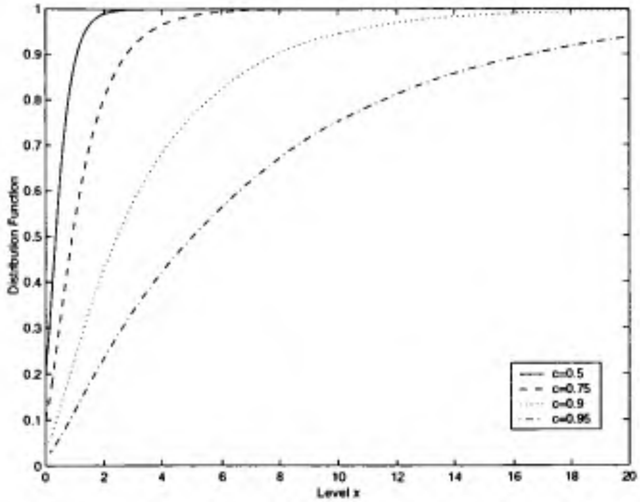


Figure 2.4: Distribution function of the stationary buffer content for a fluid queue driven by a Markov process with parameters $s_+ = 2$ and $s_0 = s_- = 4$. The traffic intensity c varies from 0.5 to 0.95.

The generator T has the following structure

$$T = \begin{bmatrix} -\lambda_1 & \lambda_1 & & & \\ & -\lambda_2 & \lambda_2 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{s-1} & \lambda_{s-1} \\ \lambda_s & & & & -\lambda_s \end{bmatrix}$$

where $s = s_0 + s_+ + s_-$, and the λ_i's and r_i's are defined as follows:

$$\begin{array}{lll} \lambda_i = s_+, & r_i = c, & \text{for } 1 \le i \le s_+, \\ \lambda_i = s_0/2, & r_i = 0, & \text{for } s_+ + 1 \le i \le s_+ + s_0, \\ \lambda_i = s_-/2, & r_i = -0.5, & \text{for } s_+ + s_0 + 1 \le i \le s_+ + s_0 + s_-. \end{array}$$

The system is thus fully parameterized by s_0, s_+, s_- and c.

We use the results of Section 1.9 to compute several performance measures. We show on Figure 2.4 the steady state distribution function for the fluid queue with $s_+ = 2$ and $s_0 = s_- = 4$ in four different cases: $c = 0.5$, $c = 0.75$, $c = 0.9$ and $c = 0.95$. We observe three effects resulting from increasing the rate c: the probability mass moves to the right and is spread over a larger interval, both resulting from the fact that the fluid reaches higher values at the end of the first period, and,

c	0.50	0.75	0.90	0.95
$\tilde{m}_0$	0.2000	0.1000	0.0400	0.0200
M	0.4395	1.2133	3.4768	7.2312
V	0.4036	2.9162	23.8791	103.7814

Table 2.1: Stationary probability mass $\tilde{m}_0$ of level zero and stationary mean M and second moment V of the buffer content for a fluid queue driven by a Markov process with parameters $s_+ = 2$ and $s_0 = s_- = 4$. The traffic intensity c varies from 0.5 to 0.95.

case	a	b	c	d
M	8.6400	4.5042	3.4768	2.4680
V	155.5200	40.6570	23.8791	11.3009

Table 2.2: Stationary mean M and second moment V of the buffer content for fluid queues driven by Markov processes with increasingly regular cycles. The traffic intensity c is equal to 0.9.

furthermore, the probability of an empty buffer decreases, because of shorter intervals at the end of each cycle where the fluid has returned to zero.

The stationary probability mass $\tilde{m}_0$ of level zero, as well as the first two moments M and V in steady state are given in Table 2.1.

For the examples in Figure 2.5, we fix $c = 0.9$ and analyze four fluid queues for which we vary the number of phases:

case	a	b	c	d
s_+	1	2	2	4
s_0	1	2	4	4
s_-	1	2	4	4

By increasing the s_i's while keeping the average lengths of the three periods constant, we make the system more regular (the probability distribution of the intervals of time spent in $\mathcal{S}_+$, $\mathcal{S}_-$ and $\mathcal{S}_0$ is more concentrated around the mean). We observe that the effect is to make the fluid density more concentrated around its mean as well. The first two moments are reported in Table 2.2. One may also notice that the mean decreases when the parameters s_i increase. In fact, our numerical in-

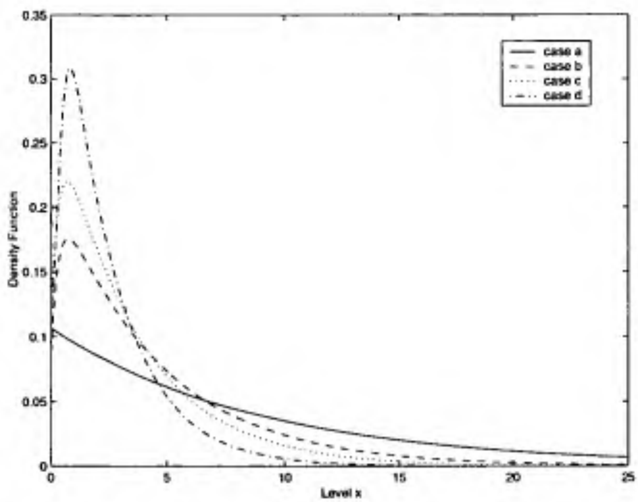


Figure 2.5: Density function of the stationary buffer content for fluid queues driven by Markov processes with increasingly regular cycles. The traffic intensity c is equal to 0.9.

vestigations lead to the observation that the more we increase the s_i's, the more the mean gets close to the value 0.614 that we would find in a purely deterministic system.

2.5 Convergence Analysis

We now analyse the speed of convergence of the Logarithmic-Reduction algorithm as a function of the uniformization parameters. As already stated in Section 2.3, μ_1 does not influence the computation of the matrix G, thus we only need to take into account the value of μ_2.

We have observed on a few examples that the number of iterations increases with μ_2. Consider case c of the preceding example. If $\mu_1 = \mu_2 = 4$, the number of iterations of the algorithm is equal to 10. Fix μ_1 and define v as the minimum possible value for μ_2, which is 4 in this example. The number of iterations of the algorithm versus the values of μ_2 are given below:

μ_2	v	$2v$	$5v$	$10v$	$100v$	$1000v$
Iterations	10	11	12	13	13	20

This is a typical example. Actually, it seems that this is not only true for this particular example, but that it is a general property holding in all cases. Unfortunately, we have not proved it, but we have an argument to support our claim that the number of iterations increases with μ_2.

As discussed in [29, Section 8.1], the number of iterations needed to compute the matrix G of first passage probabilities to lower levels of a QBD process is closely related to the maximum level reached by the QBD when it starts from level 1 and before it goes down to level 0. In the case of the Logarithmic-Reduction algorithm, the number of iterations is equal to K if the maximum level visited is the level 2^K with significantly high probability. Thus, K is an increasing function of the maximum level reached by the process.

Let (L_t, J_t) denote the level and the phase at time t for the QBD with transition matrices A_0, A_1 and A_2 given by (2.13). Recall that, for $t \in \mathbb{N}$, we have $L_t \in \mathbb{N}$ and $J_t \in \mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$. Define $\gamma(n)$, $n \geq 0$, as the first passage time to level n and

$$N = \max\{L_t : 0 \leq t \leq \gamma(0)\}$$

as the maximum level reached by the QBD process before the first visit to level zero. Also, define the matrix $G(n)$ by

$$\begin{aligned}(G(n)\mathbf{1})_i &= \mathrm{P}[\gamma(n) > \gamma(0) \mid L_0 = 1,\ J_0 = i] \\ &= \mathrm{P}[N < n \mid L_0 = 1,\ J_0 = i],\end{aligned}$$

for all $i \in \mathcal{S}$. One may find in Latouche and Ramaswami [29, Section 8.1] that the simplest iterative algorithm stops when

$$\max_{i \in \mathcal{S}} (\mathbf{1} - G(\kappa)\mathbf{1})_i \leq \varepsilon < \max_{i \in \mathcal{S}} (\mathbf{1} - G(\kappa - 1)\mathbf{1})_i.$$

The number of iterations K of the Logarithmic-Reduction algorithm is $\log_2 \kappa$, and κ is determined by the two following conditions:

$$\forall i \in \mathcal{S},\ (G(\kappa)\mathbf{1})_i \geq 1 - \varepsilon$$

and

$$\exists i \in \mathcal{S} \text{ such that } (G(\kappa - 1)\mathbf{1})_i < 1 - \varepsilon.$$

These are equivalent to

$$\forall i \in \mathcal{S},\ \mathrm{P}[N < \kappa \mid J_0 = i] \geq 1 - \varepsilon$$

and

$$\exists i \in \mathcal{S} \text{ such that } \mathrm{P}[N < \kappa - 1 \mid J_0 = i] < 1 - \varepsilon.$$

For $i \in \mathcal{S}$, define B_i through the inequalities

$$\mathrm{P}[N < B_i - 1 \mid J_0 = i] < 1 - \varepsilon \leq \mathrm{P}[N < B_i \mid J_0 = i].$$

The number of iterations κ is then determined by $\max_{i \in \mathcal{S}} B_i$.

If we could show that $\mathrm{P}[N \leq x \mid J_0 = i]$, which is a function of i, x and μ_2, is a decreasing function of μ_2 for any given x and i, then the B_i's would increase with μ_2 and so would κ. We now present our arguments supporting the claim that the number of iterations increases with μ_2.

First, we show that $\max_{i \in \mathcal{S}} B_i$ is reached for $i \in \mathcal{S}_+$, by proving that

$$\exists i \in \mathcal{S}_+ \text{ such that } \mathrm{P}[N \leq x \mid J_0 = i] \leq \mathrm{P}[N \leq x \mid J_0 = j], \ \forall j \in \mathcal{S}_-. \tag{2.15}$$

To prove (2.15), take j in $\mathcal{S}_-$ and write

$$\mathrm{P}[N \leq x \mid J_0 = j] = ((A_2)_{--}\mathbf{1})_j + \sum_{k \in \mathcal{S}_+} ((A_1)_{-+})_{jk} \, \mathrm{P}[N \leq x \mid J_0 = k].$$

This holds because, starting from a phase in $\mathcal{S}_-$, the process either stays in a phase of $\mathcal{S}_-$ and therefore makes a transition to level zero with the transition probabilities in A_2, or the phase changes to $\mathcal{S}_+$ and then we have to take into account the probability that the maximum level reached is below level x starting from a phase in $\mathcal{S}_+$. Now, suppose that there exists some j in $\mathcal{S}_-$ such that $\mathrm{P}[N \leq x \mid J_0 = i] > \mathrm{P}[N \leq x \mid J_0 = j]$ for all i in $\mathcal{S}_+$. Then, for this value of j and for all i in $\mathcal{S}_+$,

$$\begin{aligned}
\mathrm{P}[N \leq x \mid J_0 = i] &> ((A_2)_{--}\mathbf{1})_j + \sum_{k \in \mathcal{S}_+} ((A_1)_{-+})_{jk} \, \mathrm{P}[N \leq x \mid J_0 = k] \\
&= (\mathbf{1} + \frac{1}{\mu_2} T_{--}\mathbf{1})_j + \sum_{k \in \mathcal{S}_+} (\frac{1}{\mu_2} T_{-+})_{jk} \mathrm{P}[N \leq x \mid J_0 = k],
\end{aligned}$$

by definition of A_1 and A_2. Now, choose i_* in $\mathcal{S}_+$ such that it achieves the minimum of the quantity $\mathrm{P}[N \leq x \mid J_0 = k]$ among all k in $\mathcal{S}_+$. Thus,

$$\mathrm{P}[N \leq x \mid J_0 = i] > (\mathbf{1} + \frac{1}{\mu_2} T_{--}\mathbf{1})_j + (\frac{1}{\mu_2} T_{-+}\mathbf{1})_j \mathrm{P}[N \leq x \mid J_0 = i_*]$$

for all i in $\mathcal{S}_+$. In particular, the inequality holds for i_*, and it follows that

$$\mathrm{P}[N \leq x \mid J_0 = i_*] > (\mathbf{1} + \frac{1}{\mu_2} T_{--}\mathbf{1})_j + (\frac{1}{\mu_2} T_{-+}\mathbf{1})_j \mathrm{P}[N \leq x \mid J_0 = i_*].$$

Using the fact that $T_{-+}\mathbf{1} = -T_{--}\mathbf{1}$, we find that

$$(\mathbf{1} + \frac{1}{\mu_2} T_{--}\mathbf{1})_j \mathrm{P}[N \leq x \mid J_0 = i_*] > (\mathbf{1} + \frac{1}{\mu_2} T_{--}\mathbf{1})_j$$

leading to

$$\mathrm{P}[N \leq x \mid J_0 = i_*] > 1$$

which contradicts the fact that the left-hand side is a probability. The statement (2.15) is thus proved.

Now, let us concentrate on the case where J_0 is in $\mathcal{S}_+$. Recall the form of the QBD transition matrices A_0, A_1 and A_2 defined by (2.13). Starting from a phase in $\mathcal{S}_+$, the process can make transitions

- to upper levels, with probabilities given by the elements of the matrix $\mu_2/(\mu_1 + \mu_2)I$,
- to the same level, with probabilities given by the elements of the matrix $\mu_1/(\mu_1 + \mu_2)P_{++}$,
- or to lower levels, with probabilities given by the elements of the matrix $\mu_1/(\mu_1 + \mu_2)P_{+-}$.

The transition probabilities to upper levels increase with μ_2, while the others decrease. One thus expects that the process will reach higher and higher levels if the value of μ_2 increases, and it is reasonable to believe that the number of iterations should increase with μ_2.

2.6 Other Algorithms

We briefly describe in this section some other algorithms available to numerically solve the Riccati equation (1.16) for the matrix Ψ of first passage probabilities to the initial level in a fluid queue; we also describe their physical interpretations. The material presented in this section can be found in Bean et al. [10]. In that paper, the authors compare several algorithms, including the one based on the Logarithmic-Reduction algorithm described in Section 2.2. They conclude that, in principle, Newton's method described in Section 2.6.2 below is the most reliable, but its implementation is more difficult. The performance of the algorithms depend on the physical properties of the processes considered, and the authors give some recommendations concerning which method is best in which circumstances.

2.6.1 First-Exit Algorithm

First, note that the Riccati equation (1.16) may be written as

$$T_{++}\Psi + \Psi(T_{--} + T_{-+}\Psi) = -T_{+-},$$

which gives the following functional iteration: starting with $\acute{\Psi}_0 = 0$, the equation

$$T_{++}\acute{\Psi}_{n+1} + \acute{\Psi}_{n+1}(T_{--} + T_{-+}\acute{\Psi}_n) = -T_{+-}$$

allows to compute the successive values of $\acute{\Psi}_n$, for $n \geq 1$. In [10], the authors show that this iteration is equivalent to

$$\acute{\Psi}_{n+1} = \int_0^\infty e^{T_{++}y} T_{+-} e^{(T_{--}+T_{-+}\acute{\Psi}_n)y} dy.$$

The matrix $\acute{\Psi}_n$ records the first passage probabilities from level zero back to level zero under a restricted set of sample paths. Denote by $\acute{\Omega}_n$ the set of sample paths which contribute to $\acute{\Psi}_n$. The matrix $\acute{\Psi}_1$ gives the probability that the process returns to its initial level in finite time, and that there is exactly one transition from $\mathcal{S}_+$ to $\mathcal{S}_-$. For i in $\mathcal{S}_+$ and j in $\mathcal{S}_-$, the (i,j)th entry of $\acute{\Psi}_{n+1}$ has the following interpretation: starting from level zero in phase i at time 0,

- an upward process with generator T_{++} takes place, until the fluid reaches some level y;
- next, a transition from $\mathcal{S}_+$ to $\mathcal{S}_-$ occurs;
- finally, a downward process with generator $T_{--} + T_{-+}\acute{\Psi}_n$ brings the fluid down to level zero in some phase j in $\mathcal{S}_-$. This process can include a transition from $\mathcal{S}_-$ to $\mathcal{S}_+$, say at some level x, but the sample path between this point and the subsequent return to level x must be in $\acute{\Omega}_n$.

In [10], the authors introduce another algorithm, the *Last-Entrance Algorithm*, which is very similar to this one; the physical interpretation of its nth iteration is obtained via time reversal of the interpretation of the nth iteration of the First-Exit algorithm.

2.6.2 Newton's Method

Finding the solution of the Riccati equation (1.16) is equivalent to solving the equation $F(X) = 0$, where $F(X) = T_{++}X + XT_{--} + XT_{-+}X + T_{+-}$.

Newton's method can be used to approximate the solution of $F(X) = 0$, using the iteration

$$X_{k+1} = X_k - [F'(X_k)]^{-1} F(X_k).$$

Guo shows in [24] that this is equivalent to the iteration

$$(T_{++} + \Psi_n T_{-+})\Psi_{n+1} + \Psi_{n+1}(T_{--} + T_{-+}\Psi_n) = -T_{+-} + \Psi_n T_{-+} \Psi_n,$$

and it is shown in [10] that this is equivalent to

$$\Psi_{n+1} = \int_0^\infty e^{(T_{++}+\Psi_n T_{-+})y} (T_{+-} - \Psi_n T_{-+} \Psi_n) e^{(T_{--}+T_{-+}\Psi_n)y} dy.$$

As for the First-Exit algorithm, we have that starting from $\Psi_0 = 0$, the first iteration Ψ_1 gives the probability that the process returns to its initial level in finite time, and that there is exactly one transition from $\mathcal{S}_+$ to $\mathcal{S}_-$. For i in $\mathcal{S}_+$ and j in $\mathcal{S}_-$, the probabilistic interpretation of the (i,j)th entry of Ψ_{n+1} is the following: $(\Psi_{n+1})_{ij}$ is the probability mass of all *distinct* sample paths contributing to Ψ_{ij} in which, starting from level zero in phase i,

- first, an upward process with generator $T_{++} + \Psi_n T_{-+}$ takes place, and the fluid moves to some level y;
- then, a phase transition from $\mathcal{S}_+$ to $\mathcal{S}_-$ occurs;
- finally, a downward process with generator $T_{--} + T_{-+}\Psi_n$ brings the level down to zero, in some phase j in $\mathcal{S}_-$.

3

Fluid Queues with Finite Buffers

Some domains of application of fluid models need buffers of finite capacity; it is the case, for instance, of manufacturing systems. To analyze finite buffer fluid queues, we adopt again a matrix-analytic approach, combined with Markov-renewal type arguments. Our motivation comes from the observation that the stationary distribution of a fluid queue with an infinite buffer is similar to that of a QBD process with infinitely many levels, as we have seen in the previous chapters. It is known that the stationary distribution of a finite QBD may be expressed as a linear combination of two matrix-geometric vectors; the question is whether we can derive an expression for the equilibrium distribution of a fluid queue with a finite buffer as a linear combination of two matrix-exponential terms. The answer turns out to be positive, as we show in this chapter.

We start by defining a QBD process with a finite number of levels, and we give the expression of its stationary distribution. Its basic elements are the steady state probability vectors of the boundary levels, and two matrices recording taboo expected number of visits.

After these preliminaries, we define in Section 3.2 a finite buffer fluid queue and, using the same kind of renewal arguments as in Chapter 1, we derive an expression for its stationary distribution, which is expressed in terms of the steady state probability mass vectors of the boundary levels of the buffer, determined in Section 3.5, and of two exponentials of matrices giving expected number of visits to some levels, under taboo of the boundary levels, determined in Section 3.4. We first assume that the net input rates of fluid into the buffer are equal to $+1$ and -1 only and show in Section 3.6 how to return to the general setting.

The results exposed in Sections 3.2 to 3.6 were presented in da Silva Soares and Latouche [18].

We conclude this chapter by determining some performance measures in Section 3.7 and by providing a numerical illustration in Section 3.8.

3.1 Finite QBDs

A finite QBD process is similar to an infinite one, except that its state space is restricted to the set $\{(n,j) : 0 \leq n \leq M, 1 \leq j \leq m\}$, where n takes integer values and M is finite. For $0 \leq n \leq M$ and for $1 \leq j, j' \leq m$, the only transitions allowed are:

- from state (n,j) to state (n,j');
- from state (n,j) to state $(n+1,j')$, provided that $n \leq M-1$;
- from state (n,j) to state $(n-1,j')$, provided that $n \geq 1$.

We assume that the transition probabilities do not depend on the level, except near the boundary levels 0 and M, so that the process is said to be homogeneous. In the discrete-time case, its transition matrix has the following block tridiagonal form:

$$P = \begin{bmatrix} B_0 & A_0 & 0 & \cdots & 0 & 0 & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 & 0 & 0 \\ 0 & A_2 & A_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 & A_0 & 0 \\ 0 & 0 & 0 & \cdots & A_2 & A_1 & A_0 \\ 0 & 0 & 0 & \cdots & 0 & A_2 & B_M \end{bmatrix} \tag{3.1}$$

where B_0, B_M, A_0, A_1 and A_2 are $m \times m$ matrices.

We denote by $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_M)$ the stationary probability vector of the QBD; to give its expression, we need to introduce another QBD, called the *level-reversed* process. Like the QBD defined in Section 2.1, the level-reversed process has infinitely many levels, and its transition matrix is given by

$$\hat{P} = \begin{bmatrix} B & A_2 & 0 & 0 & \dots \\ A_0 & A_1 & A_2 & 0 & \dots \\ 0 & A_0 & A_1 & A_2 & \dots \\ 0 & 0 & A_0 & A_1 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

so that the events which lead to an increase of level for the original process with transition matrix (2.1) lead here to a decrease and vice versa. We define $\hat{R}$, $\hat{U}$ and $\hat{G}$ as the matrices having the same interpretation for the level-reversed process as R, U and G for the original QBD. Thus $\hat{R}$ and $\hat{G}$ satisfy the following equations

$$\hat{R} = A_2 + \hat{R}A_1 + \hat{R}^2 A_0$$

and

$$\hat{G} = A_0 + A_1\hat{G} + A_2\hat{G}^2,$$

respectively, and $\hat{U}$ is defined as $\hat{U} = A_1 + \hat{R}A_0 = A_1 + A_2\hat{G}$.

Recall the definition of the vector $\boldsymbol{\alpha}$: $\boldsymbol{\alpha}A = \boldsymbol{\alpha}$, $\boldsymbol{\alpha}\mathbf{1} = 1$, with $A = A_0 + A_1 + A_2$.

Theorem 3.1.1 *If $\boldsymbol{\alpha}A_0\mathbf{1} \neq \boldsymbol{\alpha}A_2\mathbf{1}$, then the stationary distribution of the QBD with transition matrix (3.1) is given by*

$$\boldsymbol{\pi}_i = \boldsymbol{x}_0 R^i + \boldsymbol{x}_M \hat{R}^{M-i}, \quad 0 \leq i \leq M, \tag{3.2}$$

where $(\boldsymbol{x}_0, \boldsymbol{x}_M)$ is the solution of the system

$$(\boldsymbol{x}_0, \boldsymbol{x}_M) \begin{bmatrix} B_0 + RA_2 - I & R^M(B_M - I) + R^{M-1}A_0 \\ \hat{R}^M(B_0 - I) + \hat{R}^{M-1}A_2 & B_M + \hat{R}A_0 - I \end{bmatrix} = \mathbf{0}$$

and

$$\boldsymbol{x}_0 \sum_{0 \leq i \leq M} R^i \mathbf{1} + \boldsymbol{x}_M \sum_{0 \leq i \leq M} \hat{R}^i \mathbf{1} = 1.$$

□

We refer the reader to Hajek [26] or Latouche and Ramaswami [29] for a detailed proof of this result.

By (3.2), we may write that

$$\boldsymbol{\pi}_0 = \boldsymbol{x}_0 + \boldsymbol{x}_M \hat{R}^M \quad \text{and} \quad \boldsymbol{\pi}_M = \boldsymbol{x}_0 R^M + \boldsymbol{x}_M.$$

Under the assumption that $\boldsymbol{\alpha} A_0 \mathbf{1} \neq \boldsymbol{\alpha} A_2 \mathbf{1}$, the matrix

$$\begin{bmatrix} I & R^M \\ \hat{R}^M & I \end{bmatrix}$$

is nonsingular, and we thus have the following expression relating the vectors $(\boldsymbol{x}_0, \boldsymbol{x}_M)$ and $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_M)$:

$$(\boldsymbol{x}_0, \boldsymbol{x}_M) = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_M) \begin{bmatrix} I & R^M \\ \hat{R}^M & I \end{bmatrix}^{-1}. \tag{3.3}$$

With (3.2) and (3.3), we see that the stationary distribution of a finite QBD is a linear combination of two matrix-geometric vectors; it is expressed via the stationary probability vectors $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_M$ of the boundary levels, and via the powers of the matrices R and $\hat{R}$ recording expected number of visits to some level above zero, starting from level zero, before returning to the initial level, for the original QBD and the level-reversed one, respectively.

3.2 Finite Buffer Fluid Queues: Background

We consider a fluid queue with a finite buffer, of which the maximum capacity is b. We denote by $\{(X^{(b)}(t), \varphi(t)) : t \in \mathbb{R}^+\}$ the resulting Markov process; its state space is $[0, b] \times \mathcal{S}$, where $\mathcal{S}$ is a finite set. The evolution of the level $X^{(b)}(t)$ is controlled by the phase $\varphi(t)$ in the following way: during the intervals of time when $\varphi(t)$ is constant and equal to some i in $\mathcal{S}$, we have

$$\frac{dX^{(b)}(t)}{dt} = \begin{cases} r_i, & \text{if } 0 < X^{(b)}(t) < b, \\ \max(0, r_i), & \text{if } X^{(b)}(t) = 0, \\ \min(0, r_i), & \text{if } X^{(b)}(t) = b. \end{cases}$$

The evolution of the level of a finite fluid queue is therefore similar to the evolution of the fluid queues we have studied so far, except that when the level reaches b, if the phase at that time belongs to $\mathcal{S}_+$, then the level remains equal to b. Once more, it is very useful to assume that the net input rates r_i are all equal to $+1$ or -1; this assumption is without loss of generality, as we show in Section 3.6. Figure 3.1 depicts one possible trajectory for the evolution of the fluid queue with finite capacity; this figure is to be interpreted like the figures in Chapter 1.

We use the same notations as before; we denote by T the infinitesimal transition generator of the phase process $\{\varphi(t) : t \in \mathbb{R}^+\}$ and by $\boldsymbol{\xi}$ the

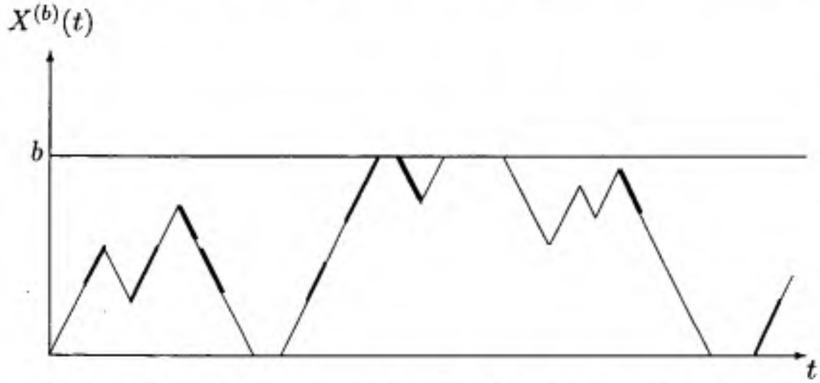


Figure 3.1: Possible evolution of the buffer content for a finite fluid queue with net input rates equal to +1 or −1.

corresponding steady state probability vector. Since the buffer is of finite capacity, we have that the fluid queue is positive recurrent for any value of the drift $\mu = \boldsymbol{\xi}_+\mathbf{1} - \boldsymbol{\xi}_-\mathbf{1}$.

For $j \in \mathcal{S}$ and for $0 \leq x \leq b$, we define the joint distribution of the level and the phase at time t by

$$F_j^{(b)}(x;t) = \mathrm{P}[X^{(b)}(t) \leq x, \varphi(t) = j],$$

and its density by

$$f_j^{(b)}(x;t) = \frac{\partial}{\partial x} F_j^{(b)}(x;t)$$

for $0 < x < b$, with

$$f_j^{(b)}(0;t) = \lim_{x \to 0^+} f_j^{(b)}(x;t) \text{ and } f_j^{(b)}(b;t) = \lim_{x \to b^-} f_j^{(b)}(x;t)$$

being defined by continuity. We are mainly interested in the state of the system when it reaches equilibrium, and we denote by

$$\boldsymbol{\pi}^{(b)}(x) = (\pi_j^{(b)}(x) : j \in \mathcal{S})$$

the stationary density vector of the level of the fluid buffer, with

$$\pi_j^{(b)}(x) = \lim_{t \to \infty} f_j^{(b)}(x;t).$$

The stationary probability mass vector of the empty buffer is denoted by $\boldsymbol{p}^{(0)} = \lim_{t\to\infty} \mathrm{P}[X^{(b)}(t) = 0]$. As in the infinite buffer case, the fluid queue cannot remain at level zero with a phase in $\mathcal{S}_+$, and thus we have $\boldsymbol{p}^{(0)} = (\mathbf{0}, \boldsymbol{p}_-^{(0)})$. Similarly, $\boldsymbol{p}^{(b)} = \lim_{t\to\infty} \mathrm{P}[X^{(b)}(t) = b]$ is the probability mass vector of level b in the stationary regime, and $\boldsymbol{p}^{(b)} = (\boldsymbol{p}_+^{(b)}, \mathbf{0})$ since it is not possible to keep a full buffer with a phase in $\mathcal{S}_-$.

3.3 Stationary Density

We derive in this section an expression for the stationary density vector $\boldsymbol{\pi}^{(b)}(x)$ of the buffer content of a finite capacity fluid queue. As announced previously, it is expressed in terms of the steady state probability mass vectors of the boundary levels of the buffer and of matrices recording certain expected number of visits. We thus need to introduce the following quantities: for $i, j \in \mathcal{S}$, $0 \le x \le b$ and $0 < y < b$, $N_{ij}^{(b)}(x, y)$ is the expected number of crossings of level y in phase j, starting from (x, i), before the first visit either to level zero or to level b. We group separately these quantities into the matrices $N_{+}^{(b)}(x, y)$, for $i \in \mathcal{S}_+$, and $N_{-}^{(b)}(x, y)$, for $i \in \mathcal{S}_-$. Note that $N_{+}^{(b)}(x, y)$ and $N_{-}^{(b)}(x, y)$ have dimensions $s_+ \times s$ and $s_- \times s$, respectively. We observe that $N_{+}^{(b)}(b, y) = 0$ because if the process starts from (b, i) at time 0, with $i \in \mathcal{S}_+$, it remains at level b, violating the taboo; similarly, $N_{-}^{(b)}(0, y) = 0$.

Theorem 3.3.1 *For $0 < x < b$, the stationary density vector $\boldsymbol{\pi}^{(b)}(x)$ of the buffer content of the finite fluid queue $\{(X^{(b)}(t), \varphi(t))\}$ is given by*

$$\boldsymbol{\pi}^{(b)}(x) = (\boldsymbol{p}_{+}^{(b)}, \boldsymbol{p}_{-}^{(0)}) \begin{bmatrix} 0 & T_{+-} \\ T_{-+} & 0 \end{bmatrix} \begin{bmatrix} N_{+}^{(b)}(0, x) \\ N_{-}^{(b)}(b, x) \end{bmatrix}. \tag{3.4}$$

Proof We use the same argument as in Theorem 1.3.1, with the difference that we must now take two boundary levels into consideration (zero and b) instead of one.

Assume without loss of generality that $X^{(b)}(0) = 0$. The state at time t is (x, k) with $0 < x < b$ and $k \in \mathcal{S}$ in one of the following cases:

- either the fluid queue is in state $(0, i)$ with $i \in \mathcal{S}_-$ at some time $t - \tau < t$, when there is a phase transition from $i \in \mathcal{S}_-$ to $j \in \mathcal{S}_+$, and during the interval $(t - \tau, t)$ the process goes from state $(0, j)$ to state (x, k), avoiding both levels zero and b (see Figure 3.2),
- or the fluid queue is in state (b, i) with $i \in \mathcal{S}_+$ at some time $t - \tau < t$, when there is a phase transition from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$, and in the interval $(t - \tau, t)$ the process goes from state (b, j) to state (x, k), avoiding both levels zero and b (see Figure 3.3),
- or in the interval $[0, t)$, the fluid queue continuously remains between levels zero and b, which may only occur if $\varphi(0)$ is in $\mathcal{S}_+$ (see Figure 3.4).

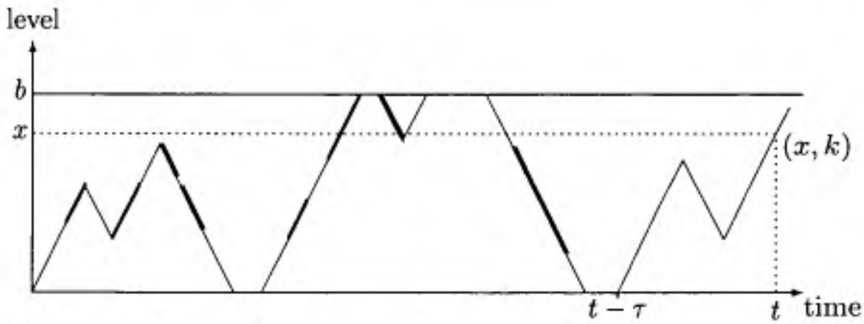


Figure 3.2: The last visit to the boundary levels occurs at level zero.

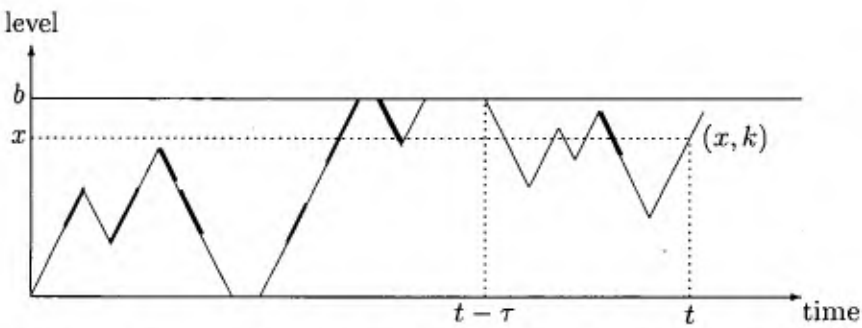


Figure 3.3: The last visit to the boundary levels occurs at level b.

Thus, the density function at time t is such that

$$
\begin{aligned}
f_k^{(b)}(x;t) &= \int_0^t \sum_{\substack{i\in\mathcal{S}_-\\ j\in\mathcal{S}_+}} F_i^{(b)}(0;t-\tau)(T_{-+})_{ij}\gamma_{jk}^{(b)}(0,x;\tau)d\tau \\
&\quad + \int_0^t \sum_{\substack{i\in\mathcal{S}_+\\ j\in\mathcal{S}_-}} G_i^{(b)}(b;t-\tau)(T_{+-})_{ij}\gamma_{jk}^{(b)}(b,x;\tau)d\tau \\
&\quad + \sum_{i\in\mathcal{S}_+} \mathrm{P}[\varphi(0)=i]\gamma_{ik}^{(b)}(0,x;t)
\end{aligned}
$$

where, for $i \in \mathcal{S}_+$, $G_i^{(b)}(b;t) = \mathrm{P}[X^{(b)}(t) = b, \varphi^{(b)}(t) = i]$, and $\gamma_{jk}^{(b)}(y,x;\tau)$ is the taboo conditional density of (x,k) at time τ, avoiding both levels zero and b in $(0,\tau)$, given that the initial state is (y,j).

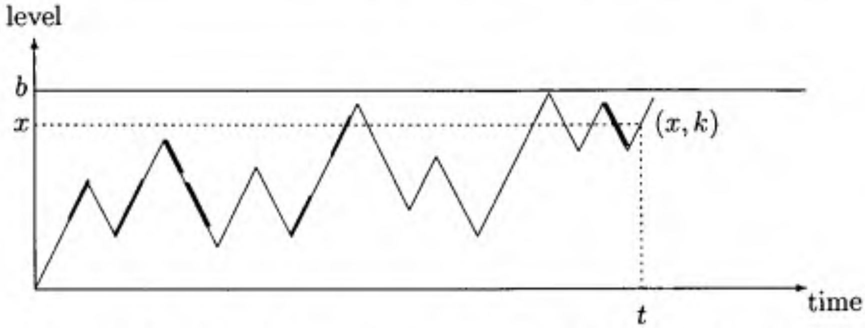


Figure 3.4: The fluid queue remains strictly between levels zero and b in the interval $[0, t)$.

Since we are interested in the system in equilibrium, we take the limit as $t \to \infty$ in the above expression, use the same argument as in the proof of Theorem 1.3.1, and eventually establish that the stationary density of (x, k) is

$$\pi_k^{(b)}(x) = \sum_{\substack{i \in \mathcal{S}_- \\ j \in \mathcal{S}_+}} p_i^{(0)}(T_{-+})_{ij}(N_+^{(b)}(0,x))_{jk} + \sum_{\substack{i \in \mathcal{S}_+ \\ j \in \mathcal{S}_-}} p_i^{(b)}(T_{+-})_{ij}(N_-^{(b)}(b,x))_{jk}$$

where $N_{ij}^{(b)}(y,x) = \int_0^\infty \gamma_{ij}^{(b)}(y,x;\tau)d\tau$ is the expected number of crossings of level x in phase j, avoiding both levels zero and b, given that the initial state is (y, i). Writing this equation in matrix form yields (3.4). $\square$

If we rewrite (3.4) as

$$\boldsymbol{\pi}^{(b)}(x) = \boldsymbol{p}_-^{(0)} T_{-+} N_+^{(b)}(0,x) + \boldsymbol{p}_+^{(b)} T_{+-} N_-^{(b)}(b,x),$$

we note the similarity between the first term of this expression and the stationary density (1.7) of the infinite buffer fluid queue. Our approach thus leads to a unified characterization of the equilibrium distribution of both finite and infinite capacity fluid queues.

3.4 Expected Number of Crossings

In order to derive an expression for the matrices $N_+^{(b)}(0,x)$ and $N_-^{(b)}(b,x)$ of taboo expected number of crossings, we need to introduce another fluid queue, called the *level-reversed* fluid queue.

Consider the standard fluid queue $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$ with an infinite capacity buffer, defined in Section 1.3. Its phase process has transition generator T and its net flow rates are $r_i = +1$ for i in $\mathcal{S}_+$ and $r_i = -1$ for i in $\mathcal{S}_-$. The level-reversed fluid queue, which we denote by $\{(\hat{X}(t), \varphi(t)) : t \in \mathbb{R}^+\}$, has the same phase transition generator T, but its net flow rates are $\hat{r}_i = -r_i$, for all i in $\mathcal{S}$. Thus, $\hat{X}(t)$ increases during the intervals of time where $\varphi(t)$ is in $\mathcal{S}_-$, and decreases when $\varphi(t)$ is in $\mathcal{S}_+$.

For the fluid queue $\{(\hat{X}(t), \varphi(t))\}$, we define the matrices $\hat{\Psi}$, $\hat{K}$ and $\hat{U}$ with the same interpretation for the level-reversed process as Ψ, K and U for the original fluid queue; we obviously have that

$$\hat{K} = T_{--} + \hat{\Psi} T_{+-}, \tag{3.5}$$
$$\hat{U} = T_{++} + T_{+-}\hat{\Psi} \tag{3.6}$$

and $\hat{\Psi}$ is the solution of the following Riccati equation:

$$T_{-+} + \hat{\Psi} T_{++} + T_{--}\hat{\Psi} + \hat{\Psi} T_{+-} \Psi = 0. \tag{3.7}$$

Note that $\hat{U}$ and $\hat{\Psi}$ were already defined in Theorem 1.12.2. With these, the matrices $N_+^{(b)}(0, x)$ and $N_-^{(b)}(b, x)$ can be computed using the next two lemmas.

Lemma 3.4.1 *For $0 < x < b$, the matrices $N_+^{(b)}(0, x)$ and $N_-^{(b)}(b, x)$ satisfy the following system of equations:*

$$\begin{bmatrix} I & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & I \end{bmatrix} \begin{bmatrix} N_+^{(b)}(0,x) \\ N_-^{(b)}(b,x) \end{bmatrix} = \begin{bmatrix} e^{Kx} & 0 \\ 0 & e^{\hat{K}(b-x)} \end{bmatrix} \begin{bmatrix} I & \Psi \\ \hat{\Psi} & I \end{bmatrix}. \tag{3.8}$$

Proof We only focus on the first equation

$$N_+^{(b)}(0, x) + e^{Kb}\Psi N_-^{(b)}(b, x) = e^{Kx}[I,\ \Psi] \tag{3.9}$$

since the second one can be derived in a similar manner by considering the level-reversed process. In order to simplify our equations, we shall use the notations

$$\mathrm{P}_{(x,i)}[\cdot] = \mathrm{P}[\cdot | X(0) = x, \varphi(0) = i]$$

and

$$\mathrm{E}_{(x,i)}[\cdot] = \mathrm{E}[\cdot | X(0) = x, \varphi(0) = i].$$

Consider the standard *infinite* buffer fluid queue $\{(X(t), \varphi(t))\}$, and assume that it starts from a state in $(0, \mathcal{S}_+)$. Let x and j be arbitrary but fixed, with $0 < x < b$ and $j \in \mathcal{S}$, and let Z_j be the number of visits to (x, j) in the interval $(0, \tau)$, where $\tau = \inf\{t > 0 : X(t) = 0\}$ is the epoch of first passage to level zero after time 0. For i in $\mathcal{S}_+$ and j in $\mathcal{S}$, $\mathrm{E}_{(0,i)}[Z_j]$ is the expected number of visits to the state (x, j), starting from $(0, i)$, before the first return to the level zero, and

$$\mathrm{E}_{(0,i)}[Z_j] = (e^{Kx}[I, \ \Psi])_{ij}$$

by Theorem 1.3.2. This is on the right hand-side of (3.9).

One may organize the visits to $(x, \mathcal{S})$ into several groups: the visits which occur before the first passage to $(b, \mathcal{S}_+)$, those which occur after the first but before the second passage through $(b, \mathcal{S}_+)$, the visits which occur after the second but before the third passages, and so on. Formally, we define $0 < \theta_1 < \theta_2 < \ldots$ as the successive epochs of visit to a state in $(b, \mathcal{S}_+)$: starting with $\theta_0 = 0$,

$$\theta_i = \inf\{t > \theta_{i-1} : X(t) = b, \varphi(t) \in \mathcal{S}_+\} \quad \text{for } i \geq 1.$$

Let $Y_j^{(n)}$ be the number of visits to (x, j) in the interval $(\theta_n, \min(\tau, \theta_{n+1}))$, with $Y_j^{(n)} = 0$ if $\theta_n > \tau$.

We may write that

$$Z_j = \sum_{n \geq 0} I\{\theta_n < \tau\} Y_j^{(n)} = Y_j^{(0)} + \sum_{n \geq 1} I\{\theta_n < \tau\} Y_j^{(n)}$$

since $\theta_0 = 0 < \tau$. This equation may also be written as

$$Z_j = Y_j^{(0)} + \sum_{n \geq 1} \sum_{s \in \mathcal{S}_+} I\{\theta_n < \tau, \varphi(\theta_n) = s\} Y_j^{(n)}$$

and therefore, for any i in $\mathcal{S}_+$,

$$\begin{aligned}\mathrm{E}_{(0,i)}[Z_j] = {} & \mathrm{E}_{(0,i)}[Y_j^{(0)}] \\ & + \sum_{n \geq 1} \sum_{s \in \mathcal{S}_+} \mathrm{P}_{(0,i)}[\theta_n < \tau, \varphi(\theta_n) = s] \mathrm{E}_{(0,i)}[Y_j^{(n)} | \theta_n < \tau, \varphi(\theta_n) = s].\end{aligned}$$

By the strong Markov property, we have that

$$\mathrm{E}_{(0,i)}[Y_j^{(n)} | \theta_n < \tau, \varphi(\theta_n) = s] = \mathrm{E}_{(b,s)}[W_j],$$

where W_j is the number of visits to (x, j) before $\min(\tau, \theta_1)$, and we obtain

$$\begin{aligned}
\mathrm{E}_{(0,i)}[Z_j] &= \mathrm{E}_{(0,i)}[W_j] + \sum_{n\geq 1}\sum_{s\in\mathcal{S}_+} \mathrm{P}_{(0,i)}[\theta_n < \tau, \varphi(\theta_n) = s]\mathrm{E}_{(b,s)}[W_j] \\
&= \mathrm{E}_{(0,i)}[W_j] + \sum_{s\in\mathcal{S}_+} \mathrm{E}_{(0,i)}[\sum_{n\geq 1} I\{\theta_n < \tau, \varphi(\theta_n) = s\}]\mathrm{E}_{(b,s)}[W_j] \\
&= \mathrm{E}_{(0,i)}[W_j] + \sum_{s\in\mathcal{S}_+} \mathrm{E}_{(0,i)}[V_s]\mathrm{E}_{(b,s)}[W_j] \qquad (3.10)
\end{aligned}$$

where V_s is the number of visits to (b, s) in the interval $(0, \tau)$, that is, before the first return to level zero.

The matrix Ψ gives the conditional distribution of the first state visited in $(b, \mathcal{S}_-)$, given that the process starts in a state of $(b, \mathcal{S}_+)$. Therefore, $\mathrm{E}_{(b,s)}[W_j] = (\Psi N_-^{(b)}(b, x))_{sj}$: indeed, starting from a state in $(b, \mathcal{S}_+)$, the queue first needs to return to level b with a phase in $\mathcal{S}_-$ before we start counting visits to level x.

Since $\mathrm{E}_{(0,i)}[W_j] = N_{ij}^{(b)}(0, x)$ by definition and $\mathrm{E}_{(0,i)}[V_s] = (e^{Kb})_{is}$ by Theorem 1.3.2, we may rewrite (3.10) as

$$(e^{Kx}[I,\ \Psi])_{ij} = N_{ij}^{(b)}(0, x) + \sum_{s\in\mathcal{S}_+} (e^{Kb})_{is}(\Psi N_-^{(b)}(b, x))_{sj},$$

which, in matrix notations, is (3.9). □

In the sequel, we shall need various first passage probabilities between the levels zero and b:

- $(\Lambda_{++}^{(b)})_{ij}$ is the probability, starting from $(0, i)$ with i in $\mathcal{S}_+$, of reaching level b in phase j in $\mathcal{S}_+$, before returning to level zero,
- and $(\Psi_{+-}^{(b)})_{ik}$, with k in $\mathcal{S}_-$, is the probability of returning to level zero in phase k, without reaching level b.
- Similarly, $(\hat{\Lambda}_{--}^{(b)})_{ik}$, with i and k in $\mathcal{S}_-$, is the probability, starting from (b, i), of reaching down to $(0, k)$ without returning to level b
- and $(\hat{\Psi}_{-+}^{(b)})_{ij}$, with j in $\mathcal{S}_+$, is the probability of returning to (b, j) before reaching down to level zero.

The following lemma is purely technical and will be useful in the sequel. Namely, it shows that the coefficient matrix of the system (3.8) is nonsingular if the drift μ of the fluid queue is different from zero, leading in that case to a complete characterization of the expected number of crossings $N_+^{(b)}(0,x)$ and $N_-^{(b)}(b,x)$.

Lemma 3.4.2 *If $\mu \neq 0$, then the following properties hold.*

i. The series $\Sigma = \sum_{k\geq 0}(e^{Ub}\hat{\Psi}e^{\hat{U}b}\Psi)^k$ converges.

ii. The matrix

$$M = \begin{bmatrix} I & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & I \end{bmatrix} \tag{3.11}$$

is nonsingular.

iii. The matrix

$$M' = \begin{bmatrix} I & \Psi e^{Ub} \\ \hat{\Psi} e^{\hat{U}b} & I \end{bmatrix}$$

is nonsingular.

Proof We only prove the three assertions of the lemma under the assumption that $\mu < 0$, the arguments being easily adapted to the reversed inequality.

Consider a fluid model for which the level is allowed to range from $-\infty$ to $+\infty$. Take i in $\mathcal{S}_-$ and j in $\mathcal{S}_+$. In view of Remark 1.4.2, we have that $(e^{Ub}\hat{\Psi})_{ij}$ is the probability that, starting from (b,i), the process eventually reaches down to level zero, spends some time in the negative levels and then returns to level zero in $(0,j)$.

Similarly, for j in $\mathcal{S}_+$ and k in $\mathcal{S}_-$, $(e^{\hat{U}b}\Psi)_{jk}$ is the probability that, starting from $(0,j)$, the process eventually moves above level b and then returns to (b,k).

Thus, $e^{Ub}\hat{\Psi}e^{\hat{U}b}\Psi$ records the probabilities of a first return to $(b,\mathcal{S}_-)$, starting from $(b,\mathcal{S}_-)$, *after* a passage through level zero, and the series Σ gives the expected number of such returns. If $\mu < 0$, meaning that the drift goes to the negative levels, then Σ is finite because the doubly infinite process is transient and the *total* number of visits to any state is finite. The first statement is thus established.

To prove the second statement, first note that

$$\begin{bmatrix} I & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & I \end{bmatrix}^{-1} = \sum_{k\geq 0}(-1)^k \begin{bmatrix} 0 & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & 0 \end{bmatrix}^k$$

if the series converges, which occurs if and only if the series

$$S = \sum_{k\geq 0} (e^{Kb}\Psi e^{\hat{K}b}\hat{\Psi})^k \tag{3.12}$$

converges.

The matrix $e^{Kb}\Psi$ records the expected number of visits to the states in $(b, \mathcal{S}_-)$, before the first return to level zero, starting from a state in $(0, \mathcal{S}_+)$. It may be expressed as follows in terms of first passage probabilities:

$$e^{Kb}\Psi = \Lambda^{(b)}_{++}\Psi[I + \hat{\Psi}^{(b)}_{-+}\Psi + (\hat{\Psi}^{(b)}_{-+}\Psi)^2 + \ldots]. \tag{3.13}$$

To justify this, we observe that the first factor gives the probability that, starting from $(0, \mathcal{S}_+)$, the process does reach $(b, \mathcal{S}_+)$ before returning to level zero. The second factor gives the probability, thereafter, of returning in finite time to $(b, \mathcal{S}_-)$. Similarly, $\hat{\Psi}^{(b)}_{-+}\Psi$ gives the probability, starting from $(b, \mathcal{S}_-)$, of returning to $(b, \mathcal{S}_-)$, after passing through $(b, \mathcal{S}_+)$ but before visiting level zero; the expected number of such returns is $I + \hat{\Psi}^{(b)}_{-+}\Psi + (\hat{\Psi}^{(b)}_{-+}\Psi)^2 + \ldots$, which justifies the third factor in (3.13).

Since $e^{Kb}\Psi$ is finite, the series in (3.13) converges and we have $e^{Kb}\Psi = \Lambda^{(b)}_{++}\Psi(I - \hat{\Psi}^{(b)}_{-+}\Psi)^{-1}$. Similarly, $e^{\hat{K}b}\hat{\Psi} = \hat{\Lambda}^{(b)}_{--}\hat{\Psi}(I - \Psi^{(b)}_{+-}\hat{\Psi})^{-1}$, so that

$$\begin{aligned}
(e^{Kb}\Psi e^{\hat{K}b}\hat{\Psi})^k &= [\Lambda^{(b)}_{++}\Psi(I - \hat{\Psi}^{(b)}_{-+}\Psi)^{-1}\hat{\Lambda}^{(b)}_{--}\hat{\Psi}(I - \Psi^{(b)}_{+-}\hat{\Psi})^{-1}]^k \\
&= \Lambda^{(b)}_{++}\Psi(GH)^{k-1}G(I - \Psi^{(b)}_{+-}\hat{\Psi})^{-1}
\end{aligned} \tag{3.14}$$

where $G = (I - \hat{\Psi}^{(b)}_{-+}\Psi)^{-1}\hat{\Lambda}^{(b)}_{--}\hat{\Psi}$ and $H = (I - \Psi^{(b)}_{+-}\hat{\Psi})^{-1}\Lambda^{(b)}_{++}\Psi$. Thus, the series S in (3.12) converges if and only if the series Σ converges, which is the case by the first statement of the lemma if $\mu \neq 0$.

The proof of the third statement follows the same steps as the proof of the second one and is therefore omitted. □

In the case not covered by Lemma 3.4.2, that is, when $\mu = 0$, the matrices e^{Ub} and $e^{\hat{U}b}$ are both stochastic, $\Psi\mathbf{1} = \mathbf{1}$ and $\hat{\Psi}\mathbf{1} = \mathbf{1}$, and one shows that the matrix M defined in (3.11) is singular, so that the system (3.8) does not completely characterize the matrices $N^{(b)}_+(0, x)$ and $N^{(b)}_-(b, x)$.

The claim that M is singular when $\mu = 0$ follows from the observation that zero is an eigenvalue of M:

$$M \begin{bmatrix} \Lambda^{(b)}_{++}\Psi\mathbf{1} \\ -e^{\hat{K}b}\hat{\Psi}\Lambda^{(b)}_{++}\Psi\mathbf{1} \end{bmatrix} = \begin{bmatrix} \Lambda^{(b)}_{++}\Psi\mathbf{1} - e^{Kb}\Psi e^{\hat{K}b}\hat{\Psi}\Lambda^{(b)}_{++}\Psi\mathbf{1} \\ e^{\hat{K}b}\hat{\Psi}\Lambda^{(b)}_{++}\Psi\mathbf{1} - e^{\hat{K}b}\hat{\Psi}\Lambda^{(b)}_{++}\Psi\mathbf{1} \end{bmatrix} = \mathbf{0}.$$

Indeed,

$$\begin{aligned}\Lambda_{++}^{(b)}\Psi\mathbf{1} - e^{Kb}\Psi e^{\hat{K}b}\hat{\Psi}\Lambda_{++}^{(b)}\Psi\mathbf{1} &= \Lambda_{++}^{(b)}\Psi\mathbf{1} - \Lambda_{++}^{(b)}\Psi G(I - \Psi_{+-}^{(b)}\hat{\Psi})^{-1}\Lambda_{++}^{(b)}\Psi\mathbf{1} \\ &= \Lambda_{++}^{(b)}\Psi\mathbf{1} - \Lambda_{++}^{(b)}\Psi G H\mathbf{1}. \qquad (3.15)\end{aligned}$$

The first equality follows from (3.14) and the second one from the definition of H. Since $\mu = 0$, G and H are both stochastic matrices, and (3.15) is equal to $\mathbf{0}$.

We summarize the two lemmas above in one theorem, showing that the density vector is a mixture of two matrix exponential terms. The simple, short proof is omitted.

Theorem 3.4.3 *If $\mu \neq 0$, the stationary density of the finite buffer fluid queue $\{(X^{(b)}(t), \varphi(t))\}$ is given by*

$$(\boldsymbol{\pi}_+^{(b)}(x), \boldsymbol{\pi}_-^{(b)}(x)) = (\boldsymbol{v}_+, \boldsymbol{v}_-)\begin{bmatrix} e^{Kx} & e^{Kx}\Psi \\ e^{\hat{K}(b-x)}\hat{\Psi} & e^{\hat{K}(b-x)} \end{bmatrix}$$

for $0 < x < b$, where

$$(\boldsymbol{v}_+, \boldsymbol{v}_-) = (\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})\begin{bmatrix} 0 & T_{+-} \\ T_{-+} & 0 \end{bmatrix}\begin{bmatrix} I & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & I \end{bmatrix}^{-1}. \qquad (3.16)$$

□

3.5 Boundary Probability Vectors

We now concentrate on determining the steady state probability mass vectors $\boldsymbol{p}_+^{(b)}$ and $\boldsymbol{p}_-^{(0)}$ of the boundary levels. Once we have these vectors, we are able to compute the whole stationary distribution of the buffer content of the finite capacity fluid queue $\{(X^{(b)}(t), \varphi(t))\}$. As in the infinite buffer case, we obtain the stationary probability vectors of the boundary levels by considering the censored process which only sees the sojourn intervals in the boundary states. The main difference is that now we have to consider the boundary states $(0, \mathcal{S}_-)$ and $(b, \mathcal{S}_+)$, instead of $(0, \mathcal{S}_-)$ only.

Theorem 3.5.1 *The vector* $(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})$ *is the unique solution of the system*

$$(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})W = \mathbf{0} \tag{3.17}$$

and

$$\boldsymbol{p}_+^{(b)}\mathbf{1} - \boldsymbol{p}_-^{(0)}\mathbf{1} = \boldsymbol{\xi}_+\mathbf{1} - \boldsymbol{\xi}_-\mathbf{1}, \tag{3.18}$$

where

$$W = \begin{bmatrix} T_{++} + T_{+-}\hat{\Psi}_{-+}^{(b)} & T_{+-}\hat{\Lambda}_{--}^{(b)} \\ T_{-+}\Lambda_{++}^{(b)} & T_{--} + T_{-+}\Psi_{+-}^{(b)} \end{bmatrix}.$$

Proof To determine $\boldsymbol{p}_+^{(b)}$ and $\boldsymbol{p}_-^{(0)}$, we consider the censored process which only sees the sojourn intervals in the boundary states $(0, \mathcal{S}_-)$ and $(b, \mathcal{S}_+)$. Its equilibrium probability vector is proportional to $(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})$, and its infinitesimal generator is easily seen to be given by W, with $\Psi_{+-}^{(b)}$, $\hat{\Psi}_{-+}^{(b)}$, $\Lambda_{++}^{(b)}$ and $\hat{\Lambda}_{--}^{(b)}$ defined in Section 3.4. We have thus proved (3.17).

We need an additional equation in order to properly normalize the stationary distribution. Firstly, we note that

$$\boldsymbol{\xi}_+ = \int_0^b \boldsymbol{\pi}_+^{(b)}(x)dx + \boldsymbol{p}_+^{(b)}, \qquad \text{and} \qquad \boldsymbol{\xi}_- = \boldsymbol{p}_-^{(0)} + \int_0^b \boldsymbol{\pi}_-^{(b)}(x)dx.$$

Secondly, we have

$$\int_0^b \boldsymbol{\pi}_+^{(b)}(x)\mathbf{1}dx = \int_0^b \boldsymbol{\pi}_-^{(b)}(x)\mathbf{1}dx$$

since the fluid queue spends as much time going up in the intermediary levels as going down. A simple subtraction yields (3.18), which completes the proof. □

Remark 3.5.2 Let us denote by $\boldsymbol{\pi}^{(b)}(0)$ the right limit of $\boldsymbol{\pi}^{(b)}(x)$ as x goes to zero. By Theorem 3.3.1, we have

$$(\boldsymbol{\pi}_+^{(b)}(0), \boldsymbol{\pi}_-^{(b)}(0)) = (\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) \begin{bmatrix} 0 & T_{+-} \\ T_{-+} & 0 \end{bmatrix} \begin{bmatrix} N_+^{(b)}(0,0) \\ N_-^{(b)}(b,0) \end{bmatrix},$$

where $N_+^{(b)}(0,0) = \lim_{x\to 0^+} N_+^{(b)}(0,x)$ and $N_-^{(b)}(b,0) = \lim_{x\to 0^+} N_-^{(b)}(b,x)$. It is quite easy to see that $N_+^{(b)}(0,0) = [I,\ \Psi_{+-}^{(b)}]$ and that $N_-^{(b)}(b,0) = [0,\ \hat{\Lambda}_{--}^{(b)}]$. Therefore,

$$(\boldsymbol{\pi}_+^{(b)}(0), \boldsymbol{\pi}_-^{(b)}(0)) = (\boldsymbol{p}_-^{(0)}T_{-+}, \boldsymbol{p}_-^{(0)}T_{-+}\Psi_{+-}^{(b)} + \boldsymbol{p}_+^{(b)}T_{+-}\hat{\Lambda}_{--}^{(b)}).$$

Similarly, with $\boldsymbol{\pi}^{(b)}(b) = \lim_{x\to b^-} \boldsymbol{\pi}(x)$, we find that

$$(\boldsymbol{\pi}_+^{(b)}(b), \boldsymbol{\pi}_-^{(b)}(b)) = (\boldsymbol{p}_-^{(0)}T_{-+}\Lambda_{++}^{(b)} + \boldsymbol{p}_+^{(b)}T_{+-}\hat{\Psi}_{-+}^{(b)}, \boldsymbol{p}_+^{(b)}T_{+-}),$$

using the facts that $N_+^{(b)}(0,b) = \lim_{x\to b^-} N_+^{(b)}(0,x) = [\Lambda_{++}^{(b)},\ 0]$ and $N_-^{(b)}(b,b) = \lim_{x\to b^-} N_-^{(b)}(b,x) = [\hat{\Psi}_{-+}^{(b)},\ I]$. By (3.17), we get

$$(\boldsymbol{\pi}_+^{(b)}(0), \boldsymbol{\pi}_-^{(b)}(0)) = (\boldsymbol{p}_-^{(0)}T_{-+}, -\boldsymbol{p}_-^{(0)}T_{--}) \tag{3.19}$$

and

$$(\boldsymbol{\pi}_+^{(b)}(b), \boldsymbol{\pi}_-^{(b)}(b)) = (-\boldsymbol{p}_+^{(b)}T_{++}, \boldsymbol{p}_+^{(b)}T_{+-}), \tag{3.20}$$

as was already observed in [42].

To see the analogy with the finite QBD case, rewrite the steady state density of the finite fluid buffer as

$$\boldsymbol{\pi}^{(b)}(x) = \boldsymbol{v}_+ e^{Kx}[I, \Psi] + \boldsymbol{v}_- e^{\hat{K}(b-x)}[\hat{\Psi}, I], \tag{3.21}$$

and use (3.19), (3.20) to rewrite (3.16) as

$$(\boldsymbol{v}_+, \boldsymbol{v}_-) = (\boldsymbol{\pi}_+^{(b)}(0), \boldsymbol{\pi}_-^{(b)}(b)) \begin{bmatrix} I & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & I \end{bmatrix}^{-1}. \tag{3.22}$$

The similarity between (3.21, 3.22) on the one hand, and (3.2, 3.3) on the other hand is striking.

In order to solve the system (3.17), we need to know the matrices $\Psi_{+-}^{(b)}$, $\hat{\Psi}_{-+}^{(b)}$, $\Lambda_{++}^{(b)}$ and $\hat{\Lambda}_{--}^{(b)}$. Using probabilistic arguments, we now derive some expressions for these matrices.

Theorem 3.5.3 *The matrices of first passage probabilities $\Psi_{+-}^{(b)}$, $\hat{\Psi}_{-+}^{(b)}$, $\Lambda_{++}^{(b)}$ and $\hat{\Lambda}_{--}^{(b)}$ satisfy the following system:*

$$\begin{bmatrix} \Lambda_{++}^{(b)} & \Psi_{+-}^{(b)} \\ \hat{\Psi}_{-+}^{(b)} & \hat{\Lambda}_{--}^{(b)} \end{bmatrix} \begin{bmatrix} I & \Psi e^{Ub} \\ \hat{\Psi} e^{\hat{U}b} & I \end{bmatrix} = \begin{bmatrix} e^{\hat{U}b} & \Psi \\ \hat{\Psi} & e^{Ub} \end{bmatrix}, \tag{3.23}$$

the coefficient matrix of the system being nonsingular if $\mu \neq 0$.

Proof The matrix Ψ gives the distribution of the phase in $\mathcal{S}_-$, upon return to the level zero, starting from $(0, \mathcal{S}_+)$, for the standard fluid queue. Two cases are possible: either this return occurs before the first passage to level b, or it occurs later. Therefore, we have

$$\Psi = \Psi_{+-}^{(b)} + \Lambda_{++}^{(b)} G_{+-}^{(b)}$$

where $G_{+-}^{(b)}$ records the probability of eventually being in a state in $(0, \mathcal{S}_-)$ given that the initial state is in $(b, \mathcal{S}_+)$. Now, starting from $(b, \mathcal{S}_+)$, the queue must first return to $(b, \mathcal{S}_-)$, with probability given by Ψ, it must then move down to level zero, with probability given by e^{Ub}, as we mentioned in Remark 1.4.2. Altogether, we find that

$$\Psi = \Psi_{+-}^{(b)} + \Lambda_{++}^{(b)} \Psi e^{Ub}. \tag{3.24}$$

Next we consider e^{Ub}. There are two ways of visiting $(0, \mathcal{S}_-)$, starting from $(b, \mathcal{S}_-)$: either the visit occurs before a return to level b, or it occurs later. This decomposition gives us

$$e^{Ub} = \hat{\Lambda}_{--}^{(b)} + \hat{\Psi}_{-+}^{(b)} \Psi e^{Ub}. \tag{3.25}$$

Together, (3.24, 3.25) justify the equations corresponding to the last column in (3.23); the other equations are obtained by applying the same arguments to the level-reversed queue.

By Lemma 3.4.2, the coefficient matrix is nonsingular in case μ is different from zero. □

We note for future reference that (3.24) and (3.25) together lead to the following equation:

$$\Psi = \Psi_{+-}^{(b)} + \Lambda_{++}^{(b)} \Psi (I - \hat{\Psi}_{-+}^{(b)} \Psi)^{-1} \hat{\Lambda}_{--}^{(b)}. \tag{3.26}$$

Some algebraic manipulations of the system in (3.23) lead to explicit expressions for the matrices of first passage probabilities.

Corollary 3.5.4 *If $\mu \neq 0$, then the matrices $\Psi_{+-}^{(b)}$, $\hat{\Psi}_{-+}^{(b)}$, $\Lambda_{++}^{(b)}$ and $\hat{\Lambda}_{--}^{(b)}$ are given by*

$$\begin{aligned}
\Psi_{+-}^{(b)} &= (\Psi - e^{\hat{U}b} \Psi e^{Ub})(I - \hat{\Psi} e^{\hat{U}b} \Psi e^{Ub})^{-1} && (3.27)\\
\hat{\Psi}_{-+}^{(b)} &= (\hat{\Psi} - e^{Ub} \hat{\Psi} e^{\hat{U}b})(I - \Psi e^{Ub} \hat{\Psi} e^{\hat{U}b})^{-1} && (3.28)\\
\Lambda_{++}^{(b)} &= (I - \Psi \hat{\Psi}) e^{\hat{U}b} (I - \Psi e^{Ub} \hat{\Psi} e^{\hat{U}b})^{-1} && (3.29)\\
\hat{\Lambda}_{--}^{(b)} &= (I - \hat{\Psi} \Psi) e^{Ub} (I - \hat{\Psi} e^{\hat{U}b} \Psi e^{Ub})^{-1}. && (3.30)
\end{aligned}$$

Proof We only prove the first equation, the others being obtained in a similar manner. From (3.23) we have

$$\Lambda_{++}^{(b)} + \Psi_{+-}^{(b)} \hat{\Psi} e^{\hat{U}b} = e^{\hat{U}b}.$$

We post-multiply by Ψe^{Ub}, use (3.24), and obtain

$$\Psi_{+-}^{(b)}(I - \hat{\Psi} e^{\hat{U}b} \Psi e^{Ub}) = \Psi - e^{\hat{U}b} \Psi e^{Ub}. \tag{3.31}$$

We need to show that $(I - \hat{\Psi} e^{\hat{U}b} \Psi e^{Ub})$ is nonsingular. From Lemma 3.4.2, we know that the series Σ converges, so that the series

$$\begin{aligned}\sum_{k \geq 0} (\hat{\Psi} e^{\hat{U}b} \Psi e^{Ub})^k &= I + \sum_{k \geq 1} (\hat{\Psi} e^{\hat{U}b} \Psi e^{Ub})^k \\ &= I + \hat{\Psi} e^{\hat{U}b} \Psi \Sigma e^{Ub}\end{aligned}$$

also converges, which implies that $\sum_{k \geq 0} (\hat{\Psi} e^{\hat{U}b} \Psi e^{Ub})^k = (I - \hat{\Psi} e^{\hat{U}b} \Psi e^{Ub})^{-1}$ and we see that (3.27) follows from (3.31). □

The next result is equivalent to Rogers [42, equations (4.4)] but our proof is different. Note that Rogers also needs the assumption $\mu \neq 0$.

Theorem 3.5.5 *If $\mu \neq 0$, then the steady state probability mass vectors $\boldsymbol{p}_-^{(0)}$ and $\boldsymbol{p}_+^{(b)}$ of the boundary levels 0 and b, are given by*

$$\boldsymbol{p}_+^{(b)} = \boldsymbol{\xi}_+ \Lambda_{++}^{(b)} \tag{3.32}$$

and

$$\boldsymbol{p}_-^{(0)} = \boldsymbol{\xi}_- \hat{\Lambda}_{--}^{(b)}. \tag{3.33}$$

Proof We start from (3.17) and write the system as

$$(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) \left\{ \begin{bmatrix} T_{++} & 0 \\ 0 & T_{--} \end{bmatrix} + \begin{bmatrix} 0 & T_{+-} \\ T_{-+} & 0 \end{bmatrix} \begin{bmatrix} \Lambda_{++}^{(b)} & \Psi_{+-}^{(b)} \\ \hat{\Psi}_{-+}^{(b)} & \hat{\Lambda}_{--}^{(b)} \end{bmatrix} \right\} = \mathbf{0}.$$

We post-multiply this equation by the matrix

$$\begin{bmatrix} I & \Psi e^{Ub} \\ \hat{\Psi} e^{\hat{U}b} & I \end{bmatrix}$$

and obtain, by Theorem 3.5.3, the system

$$(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) \left\{ \begin{bmatrix} T_{++} & 0 \\ 0 & T_{--} \end{bmatrix} \begin{bmatrix} I & \Psi e^{Ub} \\ \hat{\Psi} e^{\hat{U}b} & I \end{bmatrix} + \begin{bmatrix} 0 & T_{+-} \\ T_{-+} & 0 \end{bmatrix} \begin{bmatrix} e^{\hat{U}b} & \Psi \\ \hat{\Psi} & e^{Ub} \end{bmatrix} \right\} = \mathbf{0}$$

or

$$(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) \begin{bmatrix} T_{++} + T_{+-}\hat{\Psi} & (T_{+-} + T_{++}\Psi)e^{Ub} \\ (T_{-+} + T_{--}\hat{\Psi})e^{\hat{U}b} & T_{--} + T_{-+}\Psi \end{bmatrix} = \mathbf{0}. \tag{3.34}$$

By (1.14) and (1.16), we have that

$$T_{+-} + T_{++}\Psi = -\Psi U$$

and, for the level-reversed queue,

$$T_{-+} + T_{--}\hat{\Psi} = -\hat{\Psi}\hat{U}. \tag{3.35}$$

Thus, (3.34) becomes

$$\begin{aligned} (\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) & \begin{bmatrix} \hat{U} & -\Psi U e^{Ub} \\ -\hat{\Psi}\hat{U} e^{\hat{U}b} & U \end{bmatrix} \\ &= (\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) \begin{bmatrix} I & -\Psi e^{Ub} \\ -\hat{\Psi} e^{\hat{U}b} & I \end{bmatrix} \begin{bmatrix} \hat{U} & 0 \\ 0 & U \end{bmatrix} = \mathbf{0} \end{aligned}$$

since M and e^M commute for any matrix M. We write separately the two equations:

$$(\boldsymbol{p}_+^{(b)} - \boldsymbol{p}_-^{(0)}\hat{\Psi} e^{\hat{U}b})\hat{U} = \mathbf{0} \tag{3.36}$$

$$(\boldsymbol{p}_-^{(0)} - \boldsymbol{p}_+^{(b)}\Psi e^{Ub})U = \mathbf{0}. \tag{3.37}$$

Assume that $\mu < 0$. The infinite buffer fluid queue is recurrent, and, by Remark 1.4.2, the generator U of the process of downward records is nonsingular. Since the phase process is irreducible, U is also irreducible, and thus has a unique eigenvector for the eigenvalue zero, up to a multiplicative constant. This implies that, by Theorem 1.6.1, $\boldsymbol{p}_-^{(0)} - \boldsymbol{p}_+^{(b)}\Psi e^{Ub} = c\boldsymbol{p}_-$ for some constant c, where $\boldsymbol{p}_-$ is the steady state probability mass vector of level zero in the standard fluid queue. Moreover, by Remark 1.4.2, e^{Ub} is stochastic; on the other hand, $\Psi\mathbf{1} = \mathbf{1}$. Thus,

$$c\boldsymbol{p}_{-}\mathbf{1} = (\boldsymbol{p}_{-}^{(0)} - \boldsymbol{p}_{+}^{(b)}\Psi e^{Ub})\mathbf{1} = \boldsymbol{p}_{-}^{(0)}\mathbf{1} - \boldsymbol{p}_{+}^{(b)}\mathbf{1} = \boldsymbol{\xi}_{-}\mathbf{1} - \boldsymbol{\xi}_{+}\mathbf{1}$$

by (3.18). Since $\boldsymbol{p}_{-} = \boldsymbol{\xi}_{-} - \boldsymbol{\xi}_{+}\Psi$ by (1.64), we conclude that $c = 1$ and that

$$\boldsymbol{p}_{-}^{(0)} - \boldsymbol{p}_{+}^{(b)}\Psi e^{Ub} = \boldsymbol{p}_{-} = \boldsymbol{\xi}_{-} - \boldsymbol{\xi}_{+}\Psi. \tag{3.38}$$

The level-reversed process is transient, so that $\hat{U}$ is nonsingular (see Remark 1.4.2) and, therefore, (3.36) implies that

$$\boldsymbol{p}_{+}^{(b)} = \boldsymbol{p}_{-}^{(0)}\hat{\Psi}e^{\hat{U}b}. \tag{3.39}$$

Post-multiplying both sides of (3.39) by Ψe^{Ub} and replacing the resulting expression in the left-hand side of (3.38), we obtain that

$$\boldsymbol{p}_{-}^{(0)}(I - \hat{\Psi}e^{\hat{U}b}\Psi e^{Ub}) = \boldsymbol{\xi}_{-} - \boldsymbol{\xi}_{+}\Psi,$$

so that

$$\boldsymbol{p}_{-}^{(0)} = (\boldsymbol{\xi}_{-} - \boldsymbol{\xi}_{+}\Psi)(I - \hat{\Psi}e^{\hat{U}b}\Psi e^{Ub})^{-1}, \tag{3.40}$$

since $I - \hat{\Psi}e^{\hat{U}b}\Psi e^{Ub}$ is nonsingular (see the proof of Corollary 3.5.4). Using the fact that $\boldsymbol{p}_{-}U = \mathbf{0}$, we have $\boldsymbol{p}_{-}e^{Ub} = \boldsymbol{p}_{-}$ and the equation above may be written as

$$\boldsymbol{p}_{-}^{(0)} = (\boldsymbol{\xi}_{-} - \boldsymbol{\xi}_{+}\Psi)e^{Ub}(I - \hat{\Psi}e^{\hat{U}b}\Psi e^{Ub})^{-1}. \tag{3.41}$$

Observe that

$$\begin{aligned}
\boldsymbol{\xi}_{+}\hat{U} &= \boldsymbol{\xi}_{+}T_{++} + \boldsymbol{\xi}_{+}T_{+-}\hat{\Psi} && \text{by (3.6)}, \\
&= -\boldsymbol{\xi}_{-}T_{-+} - \boldsymbol{\xi}_{-}T_{--}\hat{\Psi} && \text{since } \boldsymbol{\xi}T = \mathbf{0}, \\
&= \boldsymbol{\xi}_{-}\hat{\Psi}\hat{U}
\end{aligned}$$

by (3.35), so that $(\boldsymbol{\xi}_{+} - \boldsymbol{\xi}_{-}\hat{\Psi})\hat{U} = \mathbf{0}$. The matrix $\hat{U}$ being nonsingular, this proves that $\boldsymbol{\xi}_{+} = \boldsymbol{\xi}_{-}\hat{\Psi}$. Using this relation, (3.41) becomes

$$\boldsymbol{p}_{-}^{(0)} = \boldsymbol{\xi}_{-}(I - \hat{\Psi}\Psi)e^{Ub}(I - \hat{\Psi}e^{\hat{U}b}\Psi e^{Ub})^{-1},$$

which, combined with (3.30), gives (3.33).

We replace $\boldsymbol{p}_-^{(0)}$ in (3.39) by the right-hand side of (3.40) and obtain that

$$\begin{aligned}
\boldsymbol{p}_+^{(b)} &= \boldsymbol{\xi}_-(I-\hat{\Psi}\Psi)(I-\hat{\Psi}e^{\hat{U}b}\Psi e^{Ub})^{-1}\hat{\Psi}e^{\hat{U}b} \\
&= \boldsymbol{\xi}_-(I-\hat{\Psi}\Psi)\hat{\Psi}e^{\hat{U}b}(I-\Psi e^{Ub}\hat{\Psi}e^{\hat{U}b})^{-1} \\
&= \boldsymbol{\xi}_-\hat{\Psi}(I-\Psi\hat{\Psi})e^{\hat{U}b}(I-\Psi e^{Ub}\hat{\Psi}e^{\hat{U}b})^{-1} \\
&= \boldsymbol{\xi}_+(I-\Psi\hat{\Psi})e^{\hat{U}b}(I-\Psi e^{Ub}\hat{\Psi}e^{\hat{U}b})^{-1}
\end{aligned}$$

which, together with (3.29), proves (3.32).

The argument is easily adapted to the case where $\mu > 0$. □

3.6 General Fluid Input Rates

We briefly indicate in this section how to obtain the solution for finite buffer fluid queues with arbitrary net input rates once we have the solution using net input rates equal to $+1$ or -1.

Denote by $\{(\tilde{X}^{(b)}(t), \tilde{\varphi}(t))\}$ a finite buffer fluid queue with arbitrary real-valued net input rates $\tilde{r}_i$ and with phase transition generator $\tilde{T}$.

We follow the same steps as in Section 1.7 and obtain the following expressions for the probability mass vectors $\tilde{\boldsymbol{p}}^{(0)} = (\tilde{\boldsymbol{p}}_0^{(0)}, \mathbf{0}, \tilde{\boldsymbol{p}}_-^{(0)})$ and $\tilde{\boldsymbol{p}}^{(b)} = (\tilde{\boldsymbol{p}}_0^{(b)}, \tilde{\boldsymbol{p}}_+^{(b)}, \mathbf{0})$:

$$\tilde{\boldsymbol{p}}^{(0)} = \gamma \boldsymbol{p}_-^{(0)}(|C_-|^{-1}\tilde{T}_{-0}(-\tilde{T}_{00})^{-1}, \mathbf{0}, |C_-|^{-1})$$

and

$$\tilde{\boldsymbol{p}}^{(b)} = \gamma \boldsymbol{p}_+^{(b)}(C_+^{-1}\tilde{T}_{+0}(-\tilde{T}_{00})^{-1}, C_+^{-1}, \mathbf{0}),$$

where γ is given by (1.28) and the vectors $\boldsymbol{p}_-^{(0)}$ and $\boldsymbol{p}_+^{(b)}$ are a solution of the system (3.17), normalized by (3.18). The stationary density $\tilde{\boldsymbol{\pi}}^{(b)}(x) = (\tilde{\boldsymbol{\pi}}_0^{(b)}(x), \tilde{\boldsymbol{\pi}}_+^{(b)}(x), \tilde{\boldsymbol{\pi}}_-^{(b)}(x))$ is such that

$$\begin{aligned}
\tilde{\boldsymbol{\pi}}_+^{(b)}(x) &= \gamma \boldsymbol{v}_+ e^{Kx} C_+^{-1} + \gamma \boldsymbol{v}_- e^{\hat{K}(b-x)}\hat{\Psi} C_+^{-1}, \\
\tilde{\boldsymbol{\pi}}_-^{(b)}(x) &= \gamma \boldsymbol{v}_+ e^{Kx}\Psi |C_-|^{-1} + \gamma \boldsymbol{v}_- e^{\hat{K}(b-x)}|C_-|^{-1}
\end{aligned}$$

and

$$\tilde{\boldsymbol{\pi}}_0^{(b)}(x) = \tilde{\boldsymbol{\pi}}_+^{(b)}(x)\tilde{T}_{+0}(-\tilde{T}_{00})^{-1} + \tilde{\boldsymbol{\pi}}_-^{(b)}(x)\tilde{T}_{-0}(-\tilde{T}_{00})^{-1},$$

where $(\boldsymbol{v}_+, \boldsymbol{v}_-)$ is given by (3.16). As before, the matrix Ψ gives the first return probabilities to the initial level for an infinite buffer fluid queue with phase transition generator T and corresponding net input rates equal to $+1$ and -1, and solves the Riccati equation (1.16); the matrix K is defined by $K = T_{++} + \Psi T_{-+}$. The matrices $\hat{\Psi}$ and $\hat{K}$ have the same interpretation as Ψ and K but for the level-reversed fluid queue.

3.7 Performance Measures

We are interested in computing some performance measures for the marginal distribution of the fluid level, in a system with a finite buffer and arbitrary net input rates, denoted as before by $\{(\tilde{X}^{(b)}(t), \tilde{\varphi}(t))\}$.

For this purpose, we will need the following expressions: the probability masses $\tilde{m}_0$ at level zero and $\tilde{m}_b$ at level b are

$$\tilde{m}_0 = \tilde{\boldsymbol{p}}^{(0)}\mathbf{1} = \gamma \boldsymbol{p}_-^{(0)}\boldsymbol{w}_- \qquad \text{and} \qquad \tilde{m}_b = \tilde{\boldsymbol{p}}^{(b)}\mathbf{1} = \gamma \boldsymbol{p}_+^{(b)}\boldsymbol{w}_+, \tag{3.42}$$

with

$$\boldsymbol{w}_- = |C_-|^{-1}\{\mathbf{1} + \tilde{T}_{-0}(-\tilde{T}_{00})^{-1}\mathbf{1}\}$$

and

$$\boldsymbol{w}_+ = C_+^{-1}\{\mathbf{1} + \tilde{T}_{+0}(-\tilde{T}_{00})^{-1}\mathbf{1}\},$$

and the density $\mu^{(b)}(x)$ between zero and b is

$$\mu^{(b)}(x) = \tilde{\boldsymbol{\pi}}^{(b)}(x)\mathbf{1} = \gamma\{\boldsymbol{\pi}_+^{(b)}(x)\boldsymbol{w}_+ + \boldsymbol{\pi}_-{}^{(b)}(x)\boldsymbol{w}_-\}, \tag{3.43}$$

where $\boldsymbol{\pi}^{(b)}(x)$, $\boldsymbol{p}_+^{(b)}$ and $\boldsymbol{p}_-^{(0)}$ are determined in Sections 3.3 to 3.5.

A first performance measure is the probability that the buffer overflows in the stationary regime, it is equal to $\tilde{m}_b$ given by (3.42).

In order to compute other performance measures, like the stationary distribution function or the moments, we will need to express the integrals $\int_0^b e^{Kx}dx$ and $\int_0^b e^{\hat{K}(b-x)}dx$ in a simple manner. This is the reason why we now introduce generalized matrix inverses, because either K or $\hat{K}$ or both are singular as we show in Theorem 3.7.2 below. We use the group inverse, which we now define.

Definition 3.7.1 *The group inverse of a matrix M, when it exists, is the unique matrix $M^{\#}$ such that $MM^{\#}M = M$, $M^{\#}MM^{\#} = M^{\#}$ and $MM^{\#} = M^{\#}M$.*

A useful property, which makes it easy to compute, is that $M^{\#}$, when it exists, is the unique solution of the system

$$\left\{ \begin{array}{rcl} M^{\#}M & = & I - \boldsymbol{v}\boldsymbol{u} \\ M^{\#}\boldsymbol{v} & = & \mathbf{0} \end{array} \right. \tag{3.44}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively denote the left and right eigenvectors of M for the eigenvalue 0, normalized by $\boldsymbol{u}\boldsymbol{v} = 1$, $\boldsymbol{u}\mathbf{1} = 1$. We refer the reader to Campbell and Meyer [12] for details and for more properties about generalized inverses.

Theorem 3.7.2

i. *If $\mu < 0$, the matrix K is nonsingular and the matrix $\hat{K}$ is singular. The group inverse of $\hat{K}$ exists.*

ii. *If $\mu > 0$, the matrix $\hat{K}$ is nonsingular and the matrix K is singular. The group inverse of K exists.*

iii. *If $\mu = 0$, then both K and $\hat{K}$ are singular matrices and their group inverse exist.*

Proof We only prove the first assertion because the other two are obtained in a similar manner. We have already proved in Theorem 1.5.3 that if $\mu < 0$, then K is nonsingular. Nevertheless, we provide here a different proof for this result, based on the relationship between fluid queues and QBD processes.

Consider the QBD process obtained by restricting the infinite buffer fluid queue to those epochs when the level is a multiple of b. Its transition matrices are

$$A_0 = \begin{bmatrix} \Lambda_{++}^{(b)} & 0 \\ 0 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & \Psi_{+-}^{(b)} \\ \hat{\Psi}_{-+}^{(b)} & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\Lambda}_{--}^{(b)} \end{bmatrix}, \tag{3.45}$$

where $\Lambda_{++}^{(b)}$, $\hat{\Lambda}_{--}^{(b)}$, $\Psi_{+-}^{(b)}$ and $\hat{\Psi}_{-+}^{(b)}$ are defined in Section 3.4. The R matrix of this QBD, that is, the matrix which records the expected number of visits to level one, starting from level zero, before the first return to level zero, is given by

$$R = \begin{bmatrix} e^{Kb} & e^{Kb}\Psi \\ 0 & 0 \end{bmatrix}$$

because e^{Kb} is the matrix of expected number of visits to level b, for the infinite buffer fluid queue, starting from level zero, before the first return to level zero (see Theorem 1.3.2). Since $\mu < 0$, the drift is downwards and it follows from Theorem 2.1.7 that the spectral radius of R, denoted by $\mathrm{sp}(R)$, is strictly less than one. Therefore, $\mathrm{sp}(e^{Kb}) < 1$. This implies that all the eigenvalues of K must have a strictly negative real part, leading us to the conclusion that K is nonsingular.

To show that $\hat{K}$ is singular and that its group inverse exists, consider the QBD process obtained from (3.45) by reversing the levels. For this process, the matrix $\hat{R}$ of expected number of visits is given by

$$\hat{R} = \begin{bmatrix} 0 & 0 \\ e^{\hat{K}b}\hat{\Psi} & e^{\hat{K}b} \end{bmatrix}.$$

For the level-reversed process, the drift is upwards, and it follows from [29, Corollary 7.1.2 and Theorem 7.2.2] that $\mathrm{sp}(\hat{R}) = 1$ and that the eigenvalue 1 has multiplicity one. Therefore, $\mathrm{sp}(e^{\hat{K}b}) = 1$, zero is an eigenvalue of $\hat{K}$ and $\hat{K}$ is singular; all the other eigenvalues of $\hat{K}$ have a strictly negative real part. Since the eigenvalue 0 has multiplicity one, the group inverse of $\hat{K}$ exists (see [12, Section 7.2]). □

The following indefinite integrals will be useful in the sequel.

Lemma 3.7.3 *If M is nonsingular, then*

$$\int e^{Mx}dx = M^{-1}e^{Mx} + M_1 \tag{3.46}$$

where M_1 is an arbitrary matrix.

If M is singular and its group inverse exists, then

$$\int e^{Mx}dx = M^{\#}e^{Mx} + x\boldsymbol{v}\boldsymbol{u} + M_2 \tag{3.47}$$

where M_2 is an arbitrary matrix and $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively denote the left and right eigenvectors of M for the eigenvalue 0, normalized by $\boldsymbol{uv} = 1$, $\boldsymbol{u}\mathbf{1} = 1$.

Proof To prove (3.46), we may write that

$$\begin{aligned} \int e^{Mx}dx &= \int \sum_{n\geq 0} \frac{M^n}{n!}x^n dx \\ &= \sum_{n\geq 0} M^n \frac{x^{n+1}}{(n+1)!} + C \end{aligned}$$

for some arbitrary matrix C. If M is invertible, then

$$\begin{aligned}
\int e^{Mx}dx &= \sum_{n\geq 0} M^{-1}\frac{(Mx)^{n+1}}{(n+1)!} + C \\
&= M^{-1}\sum_{n\geq 1}\frac{(Mx)^n}{n!} + C \\
&= M^{-1}(e^{Mx} - I) + C = M^{-1}e^{Mx} + M_1
\end{aligned}$$

where M_1 is an arbitrary matrix.

If M is singular and its group inverses exists, we have by (3.44) that $I = MM^{\#} + \boldsymbol{vu}$ and we may write that

$$\begin{aligned}
\int e^{Mx}dx &= \sum_{n\geq 0}\frac{x^{n+1}}{(n+1)!}M^{n+1}M^{\#} + \sum_{n\geq 0}\frac{x^{n+1}}{(n+1)!}M^n\boldsymbol{vu} + C \\
&= \sum_{n\geq 1}\frac{(xM)^n}{n!}M^{\#} + x\boldsymbol{vu} + C
\end{aligned}$$

since $M\boldsymbol{v} = \mathbf{0}$. This finally reduces to

$$\begin{aligned}
\int e^{Mx}dx &= (e^{Mx} - I)M^{\#} + x\boldsymbol{vu} + C \\
&= M^{\#}e^{Mx} + x\boldsymbol{vu} + M_2,
\end{aligned}$$

where M_2 is an arbitrary matrix, and (3.47) is established. □

We now have all the preliminary results needed to compute the stationary distribution function $\tilde{F}^{(b)}(x) = \lim_{t\to\infty} \mathrm{P}[\tilde{X}^{(b)}(t) \leq x]$. We assume throughout that $\mu < 0$, so that K is nonsingular and $\hat{K}$ is singular; the equations have to be changed in an obvious manner in case $\mu > 0$. The proof is similar to the proof of Lemma 3.7.3 and is omitted.

Proposition 3.7.4 *If $\mu < 0$, the stationary distribution function of the buffer content of the process $\{(\tilde{X}^{(b)}(t), \tilde{\varphi}(t))\}$ is given by*

$$\tilde{F}^{(b)}(x) = \tilde{m}_0 + \gamma\left\{\boldsymbol{v}_+ A(x)(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-) + \boldsymbol{v}_- B(x)(\hat{\Psi}\boldsymbol{w}_+ + \boldsymbol{w}_-)\right\} \tag{3.48}$$

for $0 \leq x < b$, where

$$A(x) = (I - e^{Kx})(-K)^{-1} \tag{3.49}$$

and

$$B(x) = e^{\hat{K}(b-x)}(I - e^{\hat{K}x})(-\hat{K})^{\#} + x\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}, \tag{3.50}$$

and where $\hat{\boldsymbol{u}}$ *and* $\hat{\boldsymbol{v}}$ *respectively denote the left and right eigenvectors of* $\hat{K}$ *for the eigenvalue 0, normalized by* $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}} = 1$, $\hat{\boldsymbol{u}}\mathbf{1} = 1$.

□

The stationary mean and second moment of $\{(\tilde{X}^{(b)}(t), \tilde{\varphi}(t))\}$ are given next.

Proposition 3.7.5 *If* $\mu < 0$, *the mean* M *and second moment* V *of* $\tilde{X}^{(b)}$ *in stationary regime are given by*

$$M = b\tilde{m}_b + \gamma\left\{\boldsymbol{v}_+ C(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-) + \boldsymbol{v}_- D(\hat{\Psi}\boldsymbol{w}_+ + \boldsymbol{w}_-)\right\} \tag{3.51}$$

and

$$V = b^2\tilde{m}_b + \gamma\left\{\boldsymbol{v}_+ E(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-) + \boldsymbol{v}_- F(\hat{\Psi}\boldsymbol{w}_+ + \boldsymbol{w}_-)\right\} \tag{3.52}$$

where

$$\begin{aligned}
C &= (-K)^{-1}[(-K)^{-1}(I - e^{Kb}) - be^{Kb}] && (3.53)\\
D &= (-\hat{K})^{\#}[bI - (I - e^{\hat{K}b})(-\hat{K})^{\#}] + \frac{b^2}{2}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}} && (3.54)\\
E &= 2(-K)^{-1}C - (-K)^{-1}b^2e^{Kb}\\
F &= 2\hat{K}^{\#}D + b^2(-\hat{K})^{\#} + \frac{b^3}{3}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}},
\end{aligned}$$

and where $\hat{\boldsymbol{u}}$ *and* $\hat{\boldsymbol{v}}$ *are defined in Proposition 3.7.4.*

Proof The expected value of the stationary buffer content is

$$M = b\tilde{m}_b + \int_0^b x\mu^{(b)}(x)dx.$$

Since

$$(\boldsymbol{\pi}_+^{(b)}(x), \boldsymbol{\pi}_-^{(b)}(x)) = (\boldsymbol{v}_+, \boldsymbol{v}_-)\begin{bmatrix} e^{Kx} & e^{Kx}\Psi \\ e^{\hat{K}(b-x)}\hat{\Psi} & e^{\hat{K}(b-x)} \end{bmatrix}, \tag{3.55}$$

and by (3.43), we have that M is given by the right-hand side of (3.51), where $C = \int_0^b xe^{Kx}dx$ and $D = \int_0^b xe^{\hat{K}(b-x)}dx$.

Equations (3.53, 3.54) are easily proved by part integration, using Lemma 3.7.3, and keeping in mind that $\hat{K}^{\#}\hat{\boldsymbol{v}} = \mathbf{0}$.

One proves (3.52) in a similar manner. □

We finally turn to the Laplace-Stieltjes transform of the buffer content in equilibrium. We need to define the set $\hat{\sigma}$ of eigenvalues of $\hat{K}$ and the set $\hat{\Lambda} = \{-\lambda : \lambda \in \hat{\sigma}, \lambda \neq 0\}$.

Theorem 3.7.6 *If $\mu < 0$, the Laplace-Stieltjes transform $\phi(s)$ of the buffer content in equilibrium is given by*

$$\begin{aligned}\phi(s) &= \tilde{m}_0 + e^{-sb}\tilde{m}_b \\ &\quad + \gamma\left\{\boldsymbol{v}_+ C(s)(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-) + \boldsymbol{v}_- D(s)(\hat{\Psi}\boldsymbol{w}_+ + \boldsymbol{w}_-)\right\},\end{aligned} \tag{3.56}$$

for $\mathcal{R}(s) > 0$, where

$$C(s) = -(K - sI)^{-1}(I - e^{(K-sI)b}) \tag{3.57}$$

and

$$D(s) = (-\hat{K})^{\#}(e^{-sb}I - e^{\hat{K}b})(I + s\hat{K}^{\#})^{-1} + \frac{1}{s}(I - e^{-sb}I)\hat{\boldsymbol{v}}\hat{\boldsymbol{u}} \tag{3.58}$$

if $s \notin \hat{\Lambda}$, $D(s)$ being defined by continuity on $\hat{\Lambda}$.

Proof The Laplace-Stieltjes transform of the stationary buffer content is given by

$$\phi(s) = \tilde{m}_0 + e^{-sb}\tilde{m}_b + \int_0^b e^{-sx}\mu^{(b)}(x)dx$$

which, by (3.43) and (3.55), is clearly seen to be equivalent to (3.56), where

$$\begin{aligned} C(s) &= \int_0^b e^{(K-sI)x}dx \\ D(s) &= \int_0^b e^{-s(b-x)}e^{\hat{K}x}dx. \end{aligned}$$

We have seen in the proof of Theorem 3.7.2 that all the eigenvalues of K have a strictly negative real part, leading to the conclusion that the eigenvalues of $K - sI$ also have a strictly negative real part since $\mathcal{R}(s) > 0$. Therefore, $K - sI$ is nonsingular and (3.57) is proved.

We integrate by parts the expression for $D(s)$ and, keeping in mind the fact that $\hat{K}$ and $\hat{K}^{\#}$ commute, we find after some simple but tedious manipulations that

$$D(s)(I + s\hat{K}^{\#}) = (-\hat{K})^{\#}(e^{-sb}I - e^{\hat{K}b}) + \frac{1}{s}(I - e^{-sb}I)\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}.$$

If s is not in $\hat{\Lambda}$, then $I + s\hat{K}^{\#}$ is invertible and, using the fact that $\hat{K}$ and $\hat{K}^{\#}$ commute and $\hat{\boldsymbol{u}}(I + s\hat{K}^{\#})^{-1} = \hat{\boldsymbol{u}}$, we finally obtain (3.58). □

3.8 Numerical Illustration

To illustrate the results obtained in Section 3.7, we consider the same example as in Section 2.4, consisting in a random environment cycling through three periods, except that we now impose that the buffer has a finite capacity b. We only consider the case where $s_+ = 2$ and $s_0 = s_- = 4$, and we first fix the traffic intensity c to be equal to 0.9. In Figures 3.5

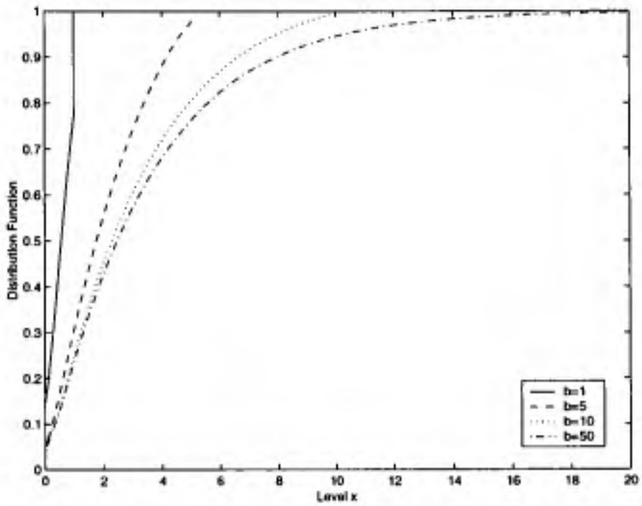


Figure 3.5: Distribution function of the stationary buffer content for finite fluid queues with capacities $b = 1, 5, 10, 50$. The value of c is 0.9.

b	1	5	10	50
$\tilde{m}_0$	0.1448	0.0516	0.0422	0.0400
$\tilde{m}_b$	0.2283	0.0259	0.0049	3.7904e-08
M	0.5446	1.9931	2.9217	3.4768
V	0.4342	6.0089	14.4139	23.8779

Table 3.1: Steady state probability masses of levels zero and b, mean M and second moment V of the reservoir content, for finite fluid queues with capacities $b = 1, 5, 10, 50$. The value of c is 0.9.

and 3.6 we illustrate the stationary distribution function $\tilde{F}^{(b)}(x)$ and the stationary density function $\mu^{(b)}(x)$, respectively, for four different values of the buffer capacity: $b = 1$, $b = 5$, $b = 10$ and $b = 50$. Note that, for the sake of clarity, we have scaled the horizontal axis of Figure 3.6

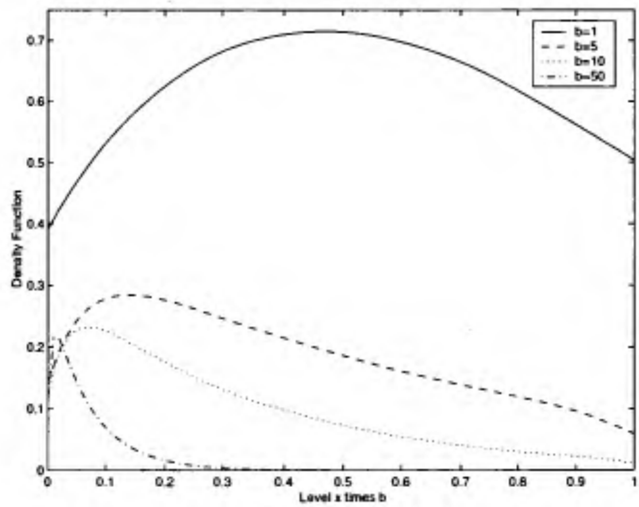


Figure 3.6: Density function of the stationary buffer content for finite fluid queues with capacities $b = 1, 5, 10, 50$, plotted in units of the buffer size. The value of c is 0.9.

by the factor b. The steady state probability masses of the boundary levels and the first two moments are given in Table 3.1. Comparing these results with those presented in Table 2.1 and corresponding to the case $c = 0.90$, we observe, as expected, that we obtain the same values when $b = 50$ as in the infinite buffer case. Also, we observe that the probability masses of levels zero and b decrease when we increase the size of the buffer. In Figures 3.7 and 3.8 we illustrate the stationary

b	1	5	10	50
$\tilde{m}_0$	0.1362	0.0312	0.0168	0.0054
$\tilde{m}_b$	0.2587	0.0544	0.0256	0.0028
M	0.5704	2.5474	4.8828	19.6518
V	0.4642	8.7380	32.4000	577.7344

Table 3.2: Values of $\tilde{m}_0$, $\tilde{m}_b$, M and V for finite fluid queues with capacities $b = 1, 5, 10, 50$. The value of c is 0.99.

distribution and density functions for the same values of b in the case where $c = 0.99$, and the steady state probability masses of the boundary levels, as well as the first two moments, are reported in Table 3.2. We see that one of the effects of increasing the traffic intensity c is that the fluid reaches higher levels, and the steady state probability mass of the full buffer when $b = 50$ is greater than in the case where $c = 0.9$. Also, with $c = 0.99$, the stationary density is nearly uniform over the interval $(0, b)$. This is due to the fact that, if b were infinite, the queue would be close to be transient; with b finite, the stationary distribution tends to spread evenly over the whole state space. Note that the horizontal axis of Figure 3.8 is also scaled by the factor b.

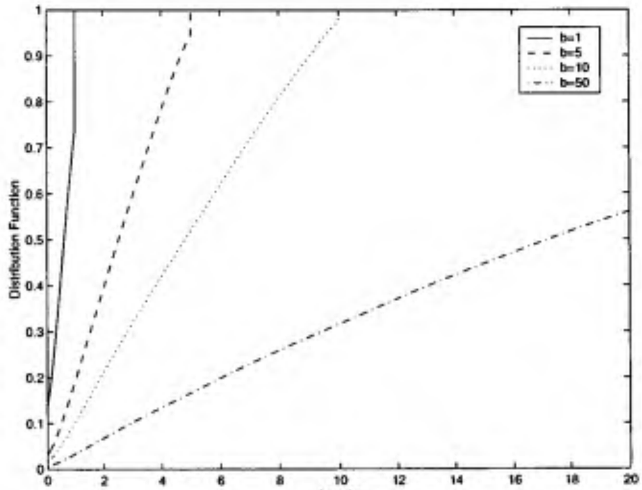


Figure 3.7: Distribution function for fluid queues with capacities $b = 1, 5, 10, 50$. The value of c is 0.99.

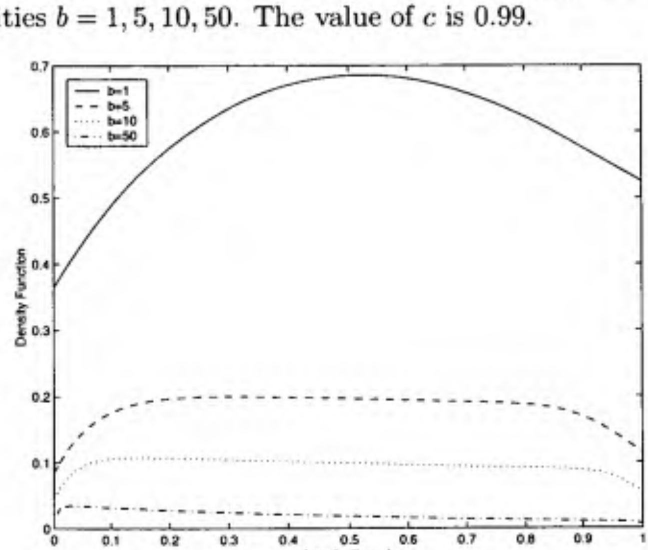


Figure 3.8: Density function for fluid queues with capacities $b = 1, 5, 10, 50$, plotted in units of the buffer size. The value of c is 0.99.

4

Feedback Fluid Queues

We focus in this chapter on the analysis of different classes of fluid queues, in which the behaviour of the phase process may change according to the value of the level of the buffer.

Our interest for such processes first came from a fluid model of an Internet protocol (see van Foreest et al. [52], [53]), that we explain in Section 4.1. We do not concentrate on this practical model of which the description serves only as a motivation for feedback fluid queues and, after this first preliminary section, we analyze the processes from a general mathematical point of view.

We first consider in Section 4.2 an infinite buffer fluid queue, in which we change the behaviour of the phase process at level zero, and determine its stationary density vector. Our approach developed in Chapter 1 shows here its great tractability since we can determine the stationary distribution of the modified fluid queue from that of the original, practically at no cost.

We then treat in Section 4.3 the finite buffer case, by changing the behaviour of the phase process at the two boundary levels. Again, we assume in Sections 4.2 and 4.3 that the net input rates of fluid into the buffer are equal to $+1$ and -1 only, and show in Section 4.4 how to determine the stationary distribution of the general model once one has the stationary distribution of the simplified process. We present in Section 4.5 a numerical illustration of the results obtained in this context. The results for both the infinite and the finite buffer cases were presented in da Silva Soares and Latouche [17].

Finally, the last two sections are devoted to the analysis of fluid queues in which the phase process changes each time the buffer level reaches certain thresholds, the change being carried on in two different manners. The models considered become more and more complex, and we treat them successively from the simplest to the more complex, in order to allow the reader to detect the ingredients of the solution. Note that similar processes were studied in Elwalid and Mitra [21]; there, the authors model a system with loss priorities using a finite buffer fluid queue with a finite number of thresholds, such that the net input rates change their values from one region of the buffer to another. Transient results for similar models as those considered in Section 4.7 have been obtained by Chen, Hong and Trivedi in [13].

Observe that the fluid queues constructed in this chapter are not of the canonical type, since the marginal phase processes are not continuous time Markov chains.

4.1 Fluid Model of TCP

Our motivation for studying feedback fluid queues is the analysis of an Internet congestion control protocol, namely the *Transport Control Protocol* (TCP), which is based on feedback. We briefly describe in this section the main characteristics of TCP Reno, which is one of the most popular implementations in the Internet today. Our description is based on van Foreest et al. [52]: the authors construct a fluid model for the mathematical analysis of TCP, however, their approach is based on spectral methods, leading to the necessity of numerically computing an eigenvector, with eigenvalue zero, of an ill-conditioned matrix. When buffers are large, or when there is a large number of states, the numerical evaluation becomes unstable. On the contrary, as for the fluid models in the first three chapters of this work, our approach leads to a stable and very efficient computational procedure.

The Internet transfers data packets from sources to destinations over a network of links and routers with buffers. The main role of TCP is to adapt the sending rate of a source to the capacity of the network. The flow control of TCP is based on a window mechanism, which consists in limiting the number of packets sent by the source and not acknowledged yet by the receiver. The control algorithm roughly works as follows.

The sender maintains a state variable, the *congestion window*, which bounds the number of packets between the source and the destination. The congestion window is initialized to one, and a packet is sent. When the destination receives it correctly, it sends an acknowledgement (ack). When the sender receives the ack, it increments the congestion window from one to two, and two packets can be sent. When each of these two packets are acknowledged, the congestion window is increased to four. Thus, the window size is doubled every round-trip time (RTT), that is, every time interval between the sending of a packet and the reception of the corresponding ack. This first phase of the algorithm is called *slow start*.

At some point in time, buffers along the network path start to fill and may overflow, resulting in the loss of packets. The sender detects this loss by means of either *timeouts* or *duplicate acks*, and it has to slow down its sending rate. If the congestion is indicated by a timeout, the sender sets the window size to one, and performs the slow start phase until half of the congestion window previous to the loss is reached. After this, the system enters in the *congestion avoidance* phase, during which the sender may only increase the window at a linear rate. The congestion avoidance phase will also be entered after reception of three duplicate acks; this is known as *fast recovery*. The sender reduces the window size by half, instead of setting it to one as in the slow start phase.

The fluid model proposed in [52] to analyze a single TCP source is the following. The state of the source that transmits fluid into a buffer is described by a Markov process $\{\varphi(t) : t \in \mathbb{R}^+\}$ on the state space $\mathcal{S} = \{1, 2, \ldots, N\}$. The buffer content at time t is denoted by $X(t)$ and it takes its values in the bounded interval $[0, b]$. When $\varphi(t) = i$, the source sends fluid into the buffer at the rate ir; the buffer empties out at the rate l. Therefore, the net input rate vector is $(r-l, 2r-l, \ldots, Nr-l)$. To ensure that the buffer is not continuously overloaded, we assume that $r < l$; on the other hand, in order to ensure that congestion occasionally occurs, we assume that $l < Nr$.

If $X(t) < b$, that is, if the buffer is not full, it sends positive signals to the source. When the source receives such a signal, $\varphi(t)$ increases by one, provided that it is strictly smaller than N, leading to an increase of the sending rate. When $X(t) = b$, the buffer sends negative signals notifying that fluid is lost. As a response, when $\varphi(t) > 1$, it becomes $\lfloor \varphi(t)/2 \rfloor$ to indicate that the source decreases its rate by half. The time intervals between two consecutive positive and negative signals are assumed to be independent and identically distributed exponential random variables, with parameters λ and μ, respectively.

The source process $\{\varphi(t)\}$ has two generators. When $X(t) < b$, the source transition generator, denoted by T, implements linear increase; it is given by

$$T = \begin{bmatrix} -\lambda & \lambda & & & & \\ & -\lambda & \lambda & & & \\ & & \ddots & \ddots & & \\ & & & -\lambda & \lambda & \\ 0 & & \dots & & 0 & 0 \end{bmatrix}.$$

When $X(t) = b$, the source makes multiplicative decrements according to a generator $T^{(b)}$, which is such that

- $T^{(b)}_{ii} = -\mu$ for $1 < i \leq N$,
- $T^{(b)}_{ij} = \mu$ for $1 < i \leq N$ and $j = \lfloor i/2 \rfloor$,

all the other entries being equal to zero. As an example, for a source with $N = 5$, we have that

$$T^{(b)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu & -\mu & 0 & 0 & 0 \\ \mu & 0 & -\mu & 0 & 0 \\ 0 & \mu & 0 & -\mu & 0 \\ 0 & \mu & 0 & 0 & -\mu \end{bmatrix}.$$

This model can be justified in the TCP context in the following way. The parameter r corresponds to the increase of the congestion window during the congestion avoidance phase. The source state N may then be interpreted in two ways: either it may correspond to the physical capacity Nr of the access link, or it is the maximum window size which determines the peak rate of the source. The random time between two consecutive positive or negative signals models the sum of the transmission, propagation and queueing delays of the packets at other routers in the network, and randomness of the operating systems at the sender and receiver. Thus, $1/\lambda$ is the average round-trip time. In [52], the authors consider two types of negative feedback: either $\mu = \lambda$ or $1/\mu = 1/\lambda + b/l$. In the first case, it is assumed that the RTT is not affected by buffer overflow, while in the second, the negative rate is an explicit function of the buffer size, and is chosen to be the largest possible RTT.

4.2 Infinite Buffer and Feedback Control

Let $\{(X_f(t), \varphi_f(t)) : t \in \mathbb{R}^+\}$ be an infinite buffer fluid queue with feedback; we suppose that the generator of the phase process $\{\varphi_f(t)\}$ is

T when $X_f(t) > 0$, but that it is $T^{(0)}$ when the buffer is empty, with

$$T^{(0)} = [T^{(0)}_{-+},\ T^{(0)}_{--}].$$

Observe that we do not need to define $T^{(0)}_{ij}$ for i in $\mathcal{S}_+$ since $X_f(t)$ cannot remain equal to zero when the input rate r_i is strictly positive. Apart from the fact that the phase process has two generators, the characteristics of the fluid queue are the same as in Section 1.3.

We assume again without loss of generality that the net input rates are all equal to $+1$ or -1, and we show in Section 4.4 how to return to the general setting. The mean drift of fluid into the buffer $\mu = \boldsymbol{\xi}_+\mathbf{1} - \boldsymbol{\xi}_-\mathbf{1}$ is assumed to be negative, so that the stationary density $\boldsymbol{\pi}(x)$ of the buffer content exists. We denote by $\boldsymbol{p} = (\mathbf{0}, \boldsymbol{p}_-)$ the steady state probability mass vector of the empty buffer.

Theorem 4.2.1 *If $\mu < 0$, then the stationary density of the buffer content of the process $\{(X_f(t), \varphi_f(t))\}$ is given by*

$$\boldsymbol{\pi}(x) = \boldsymbol{p}_- T^{(0)}_{-+} N(x) \tag{4.1}$$

for $x > 0$, where $(N(x))_{ij}$ is the expected number of visits to (x, j), under taboo of level zero, starting from $(0, i)$, for all j and for i in $\mathcal{S}_+$. This may also be written as

$$\boldsymbol{\pi}(x) = \boldsymbol{p}_- T^{(0)}_{-+} e^{Kx}[I,\ \Psi] \tag{4.2}$$

where Ψ and K are given by (1.16, 1.19), respectively. The vector $\mathbf{p}_-$ is the unique solution of the system

$$\begin{aligned} \boldsymbol{p}_- U^{(0)} &= \mathbf{0} \\ \boldsymbol{p}_-(\mathbf{1} - 2T^{(0)}_{-+}K^{-1}\mathbf{1}) &= 1 \end{aligned} \tag{4.3}$$

where

$$U^{(0)} = T^{(0)}_{--} + T^{(0)}_{-+}\Psi$$

is the infinitesimal generator of the censored Markov process on the states $(0, \mathcal{S}_-)$.

Proof To prove (4.1), we repeat the same sequence of arguments as in Theorem 1.3.1, keeping in mind the fact that the transition rates at level zero are not the same as elsewhere. The matrix $N(x)$ only deals with the system behaviour when the level is strictly above zero; therefore, (4.2) immediately follows by Theorem 1.3.2. To obtain (4.3), we mimic the proof of Theorem 1.6.1, again using the fact that the transition generator at level zero is $T^{(0)}$ instead of T. □

We would like to point out the fact that the stationary distribution of the feedback fluid queue is again completely determined once we have the matrix Ψ of first passage probabilities, and this matrix does not depend on $T^{(0)}$. Therefore, we may change the behaviour at level zero of an infinite buffer fluid queue and obtain the new stationary distribution at no cost.

4.3 Finite Buffer and Feedback Control

We now consider a feedback fluid queue $\{(X_f^{(b)}(t), \varphi_f(t)) : t \in \mathbb{R}^+\}$ with a buffer of finite capacity b. It differs from the process defined in Section 3.2 by the fact that the rates of the phase transitions change each time the buffer is either empty or full.

- When the buffer is empty, that is, when $X_f^{(b)}(t) = 0$ and $\varphi_f(t)$ is in $\mathcal{S}_-$, the phase transition generator is $T^{(0)} = [T_{-+}^{(0)},\ T_{--}^{(0)}]$;
- when it is full, that is, when $X_f^{(b)}(t) = b$ and $\varphi_f(t)$ is in $\mathcal{S}_+$, the generator is $T^{(b)} = [T_{++}^{(b)},\ T_{+-}^{(b)}]$.

Note that we need not define $T_{ij}^{(0)}$ for i in $\mathcal{S}_+$ since it is not possible that the buffer remains empty when the input rate r_i is strictly positive. Similarly, it is not necessary to define $T_{ij}^{(b)}$ for i in $\mathcal{S}_-$ because $X_f(t)$ cannot remain equal to b when the input rate r_i is strictly negative.

We denote by $\boldsymbol{p}^{(0)}$ and $\boldsymbol{p}^{(b)}$ the stationary probability vectors of levels zero and b for this process. As before, $\boldsymbol{p}^{(0)} = (\mathbf{0}, \boldsymbol{p}_-^{(0)})$ and $\boldsymbol{p}^{(b)} = (\boldsymbol{p}_+^{(b)}, \mathbf{0})$. The stationary density vector of the buffer content is denoted by $\boldsymbol{\pi}^{(b)}(x) = (\pi_i^{(b)}(x) : i \in \mathcal{S})$; since the buffer is of finite capacity, $\boldsymbol{\pi}^{(b)}(x)$ exists for any value of the drift μ.

Theorem 4.3.1 *For $0 < x < b$, the stationary density vector of the finite buffer fluid queue with feedback is given by*

$$\boldsymbol{\pi}^{(b)}(x) = (\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) \begin{bmatrix} 0 & T_{+-}^{(b)} \\ T_{-+}^{(0)} & 0 \end{bmatrix} \begin{bmatrix} N_+^{(b)}(0,x) \\ N_-^{(b)}(b,x) \end{bmatrix}. \tag{4.4}$$

If $\mu \neq 0$, this may also be written as

$$\boldsymbol{\pi}^{(b)}(x) = (\boldsymbol{v}_+, \boldsymbol{v}_-) \begin{bmatrix} e^{Kx} & e^{Kx}\Psi \\ e^{\hat{K}(b-x)}\hat{\Psi} & e^{\hat{K}(b-x)} \end{bmatrix} \tag{4.5}$$

where

$$(\boldsymbol{v}_+, \boldsymbol{v}_-) = (\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)}) \begin{bmatrix} 0 & T_{+-}^{(b)} \\ T_{-+}^{(0)} & 0 \end{bmatrix} \begin{bmatrix} I & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & I \end{bmatrix}^{-1}, \tag{4.6}$$

the matrices K, Ψ are given by (1.19, 1.16), and $\hat{K}$, $\hat{\Psi}$ are given by (3.5, 3.7).

Proof To prove (4.4), we follow the steps in Theorem 3.3.1. The matrices $N_+^{(b)}(0,x)$ and $N_-^{(b)}(b,x)$ only depend on the system behaviour when the buffer content is strictly between levels zero and b; therefore, they are the same as for the process $\{(X^{(b)}(t), \varphi(t))\}$ defined in Section 3.2, and we apply Lemmas 3.4.1 and 3.4.2 to conclude the proof. □

The following theorem gives a procedure to determine the steady state probability vectors $(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})$ of the states $(b, \mathcal{S}_+)$ and $(0, \mathcal{S}_-)$ when the drift μ is strictly negative. The result has to be adapted in a obvious manner in case μ is strictly positive.

Theorem 4.3.2 *The vector $(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})$ is equal to $c(\boldsymbol{x}_+, \boldsymbol{x}_-)$, where c is a normalizing constant and $(\boldsymbol{x}_+, \boldsymbol{x}_-)$ is the unique solution of the system*

$$\begin{cases} (\boldsymbol{x}_+, \boldsymbol{x}_-)W &= \mathbf{0} \\ \boldsymbol{x}_+\mathbf{1} + \boldsymbol{x}_-\mathbf{1} &= 1 \end{cases} \tag{4.7}$$

with

$$W = \begin{bmatrix} T_{++}^{(b)} + T_{+-}^{(b)}\hat{\Psi}_{-+}^{(b)} & T_{+-}^{(b)}\hat{\Lambda}_{--}^{(b)} \\ T_{-+}^{(0)}\Lambda_{++}^{(b)} & T_{--}^{(0)} + T_{-+}^{(0)}\Psi_{+-}^{(b)} \end{bmatrix},$$

where $\Psi_{+-}^{(b)}$, $\hat{\Psi}_{-+}^{(b)}$, $\Lambda_{++}^{(b)}$ and $\hat{\Lambda}_{--}^{(b)}$ are defined in Section 3.4.

If $\mu < 0$, then the normalizing factor c is given by

$$c = \{\boldsymbol{x}_+\mathbf{1} + \boldsymbol{x}_-\mathbf{1} + \boldsymbol{z}_+A(b)\mathbf{1} + \boldsymbol{z}_-B(b)\mathbf{1}\}^{-1}$$

where $A(\cdot)$, $B(\cdot)$ are given by (3.49, 3.50), and where

$$(\boldsymbol{z}_+, \boldsymbol{z}_-) = (\boldsymbol{x}_+, \boldsymbol{x}_-) \begin{bmatrix} 0 & T_{+-}^{(b)} \\ T_{-+}^{(0)} & 0 \end{bmatrix} \begin{bmatrix} I & e^{Kb}\Psi \\ e^{\hat{K}b}\hat{\Psi} & I \end{bmatrix}^{-1}. \tag{4.8}$$

Proof We mimic the proof of Theorem 3.5.1. The vector $(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})$ is proportional to the steady state probability vector of the restricted process obtained by observing the feedback fluid queue $\{(X_f^{(b)}(t), \varphi_f(t))\}$ only at those intervals of time where it is in $(0, \mathcal{S}_-)$ or in $(b, \mathcal{S}_+)$. It is easy to see, using the definition of the matrices $\Psi_{+-}^{(b)}$, $\hat{\Psi}_{-+}^{(b)}$, $\Lambda_{++}^{(b)}$ and $\hat{\Lambda}_{--}^{(b)}$, that the generator of the censored process is W.

The normalizing factor c is given by

$$c = \{\boldsymbol{x}_+\mathbf{1} + \boldsymbol{x}_-\mathbf{1} + \int_0^b \boldsymbol{y}(x)\mathbf{1}dx\}^{-1}$$

where, using (4.5, 4.6),

$$\boldsymbol{y}(x) = (\boldsymbol{z}_+, \boldsymbol{z}_-)\begin{bmatrix} e^{Kx} & e^{Kx}\Psi \\ e^{\hat{K}(b-x)}\hat{\Psi} & e^{\hat{K}(b-x)} \end{bmatrix}, \tag{4.9}$$

with $(\boldsymbol{z}_+, \boldsymbol{z}_-)$ defined in (4.8). Thus,

$$c = \{\boldsymbol{x}_+\mathbf{1} + \boldsymbol{x}_-\mathbf{1} + \boldsymbol{z}_+\int_0^b e^{Kx}dx[I,\ \Psi]\mathbf{1} + \boldsymbol{z}_-\int_0^b e^{\hat{K}(b-x)}dx[\hat{\Psi},\ I]\mathbf{1}\}^{-1}.$$

By Theorem 3.7.2, if $\mu < 0$ we have that K is nonsingular and $\hat{K}$ is singular. Using Lemma 3.7.3, we obtain the announced result. □

We point out the fact that the simple, direct expressions for the vectors $(\boldsymbol{p}_+^{(b)}, \boldsymbol{p}_-^{(0)})$ obtained in Theorem 3.5.5 are no longer available in this context.

4.4 General Fluid Input Rates

The expressions relating the stationary distribution of the feedback fluid queue with a finite buffer and with net input rates equal to $+1$ or -1 to that of the same process with arbitrary net input rates are obviously very similar to those exposed in Section 3.6 for the finite buffer fluid queue without feedback. It is straightforward to obtain that the probability mass vectors $\tilde{\boldsymbol{p}}^{(0)} = (\tilde{\boldsymbol{p}}_0^{(0)}, \mathbf{0}, \tilde{\boldsymbol{p}}_-^{(0)})$ and $\tilde{\boldsymbol{p}}^{(b)} = (\tilde{\boldsymbol{p}}_0^{(b)}, \tilde{\boldsymbol{p}}_+^{(b)}, \mathbf{0})$ and the density $\tilde{\boldsymbol{\pi}}^{(b)}(x) = (\tilde{\boldsymbol{\pi}}_0^{(b)}(x), \tilde{\boldsymbol{\pi}}_+^{(b)}(x), \tilde{\boldsymbol{\pi}}_-^{(b)}(x))$ in the stationary regime are given by:

$$\tilde{\boldsymbol{p}}^{(0)} = \gamma\boldsymbol{x}_-(|C_-|^{-1}\tilde{T}_{-0}^{(0)}(-\tilde{T}_{00}^{(0)})^{-1},\ \mathbf{0},\ |C_-|^{-1}),$$
$$\tilde{\boldsymbol{p}}^{(b)} = \gamma\boldsymbol{x}_+(C_+^{-1}\tilde{T}_{+0}^{(b)}(-\tilde{T}_{00}^{(b)})^{-1},\ C_+^{-1},\ \mathbf{0}),$$

$$\tilde{\boldsymbol{\pi}}_+^{(b)}(x) = \gamma \boldsymbol{y}_+(x) C_+^{-1}, \quad \tilde{\boldsymbol{\pi}}_-^{(b)}(x) = \gamma \boldsymbol{y}_-(x) |C_-|^{-1}$$

and

$$\tilde{\boldsymbol{\pi}}_0^{(b)}(x) = \tilde{\boldsymbol{\pi}}_+^{(b)}(x)\tilde{T}_{+0}(-\tilde{T}_{00})^{-1} + \tilde{\boldsymbol{\pi}}_-^{(b)}(x)\tilde{T}_{-0}(-\tilde{T}_{00})^{-1},$$

where $\boldsymbol{x}_+$, $\boldsymbol{x}_-$ solve the system (4.7) and where $\boldsymbol{y}(x)$ is given by (4.9).

The normalizing constant γ is given by

$$\gamma = \{\boldsymbol{x}_+ \boldsymbol{w}_+^{(b)} + \boldsymbol{x}_- \boldsymbol{w}_-^{(0)} + \int_0^b (\boldsymbol{y}_+(x)\boldsymbol{w}_+ + \boldsymbol{y}_-(x)\boldsymbol{w}_-)dx\}^{-1} \tag{4.10}$$

where

$$\begin{aligned}
\boldsymbol{w}_+^{(b)} &= C_+^{-1}\{\mathbf{1} + \tilde{T}_{+0}^{(b)}(-\tilde{T}_{00}^{(b)})^{-1}\mathbf{1}\} \\
\boldsymbol{w}_-^{(0)} &= |C_-|^{-1}\{\mathbf{1} + \tilde{T}_{-0}^{(0)}(-\tilde{T}_{00}^{(0)})^{-1}\mathbf{1}\} \\
\boldsymbol{w}_+ &= C_+^{-1}\{\mathbf{1} + \tilde{T}_{+0}(-\tilde{T}_{00})^{-1}\mathbf{1}\} \qquad (4.11)\\
\boldsymbol{w}_- &= |C_-|^{-1}\{\mathbf{1} + \tilde{T}_{-0}(-\tilde{T}_{00})^{-1}\mathbf{1}\}. \qquad (4.12)
\end{aligned}$$

Note the similarity to c in Theorem 4.3.2. To integrate $\boldsymbol{y}(x)$, one uses again Theorem 3.7.2 and Lemma 3.7.3.

The following expressions are useful for computational purposes: the probability masses $\tilde{m}_0$ at level zero and $\tilde{m}_b$ at level b are

$$\tilde{m}_0 = \tilde{\boldsymbol{p}}^{(0)}\mathbf{1} = \gamma \boldsymbol{x}_- \boldsymbol{w}_-^{(0)} \tag{4.13}$$

and

$$\tilde{m}_b = \tilde{\boldsymbol{p}}^{(b)}\mathbf{1} = \gamma \boldsymbol{x}_+ \boldsymbol{w}_+^{(b)}, \tag{4.14}$$

and the density $\mu^{(b)}(x)$ for $0 < x < b$ is

$$\mu^{(b)}(x) = \tilde{\boldsymbol{\pi}}^{(b)}(x)\mathbf{1} = \gamma\{\boldsymbol{y}_+(x)\boldsymbol{w}_+ + \boldsymbol{y}_-(x)\boldsymbol{w}_-\} \tag{4.15}$$

where $\boldsymbol{y}(x)$ is given by (4.9).

In the infinite buffer case, the density is

$$\mu(x) = \alpha\{\boldsymbol{\pi}_+(x)\boldsymbol{w}_+ + \boldsymbol{\pi}_-(x)\boldsymbol{w}_-\}$$

and the probability mass at level zero is

$$\tilde{m}_0 = \alpha \boldsymbol{p}_- \boldsymbol{w}_-^{(0)},$$

where $\boldsymbol{\pi}(x)$ and $\boldsymbol{p}_-$ are given in Theorem 4.2.1. The normalizing factor α is given by

$$\alpha = \left(\boldsymbol{p}_- \boldsymbol{w}_-^{(0)} + \int_0^\infty (\boldsymbol{\pi}_+(x)\boldsymbol{w}_+ + \boldsymbol{\pi}_-(x)\boldsymbol{w}_-)dx\right)^{-1},$$

with $\boldsymbol{w}_+$ and $\boldsymbol{w}_-$ being defined in (4.11, 4.12).

4.5 Numerical Illustration

We illustrate the results obtained in the last section for a feedback fluid queue with a finite buffer of capacity b. We compute the stationary probability masses $\tilde{m}_0$ at level zero and $\tilde{m}_b$ at level b given by (4.13, 4.14), and the stationary density function given by (4.15). We adapt the performance measures obtained in Section 3.7 and illustrate the stationary distribution function, as well as the mean and second moment in the stationary regime.

The system that we choose to illustrate has a stationary mean drift μ strictly positive. In this case, we obtain that the stationary distribution function is given by

$$\tilde{F}^{(b)}(x) = \tilde{m}_0 + \gamma\left\{\boldsymbol{z}_+\mathcal{A}(x)(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-) + \boldsymbol{z}_-\mathcal{B}(x)(\hat{\Psi}\boldsymbol{w}_+ + \boldsymbol{w}_-)\right\}$$

for $0 \leq x < b$, where

$$\mathcal{A}(x) = e^{K(b-x)}(I - e^{Kx})(-K)^{\#} + x\boldsymbol{v}\boldsymbol{u}$$

and

$$\mathcal{B}(x) = (I - e^{\hat{K}x})(-\hat{K})^{-1},$$

and where $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively denote the left and right eigenvectors of K for the eigenvalue 0, normalized by $\boldsymbol{u}\boldsymbol{v} = 1$, $\boldsymbol{u}\boldsymbol{1} = 1$. The normalizing constant γ is given by (4.10), $\tilde{m}_0$ is given by (4.13), $(\boldsymbol{z}_+, \boldsymbol{z}_-)$ are given by (4.8) and $\boldsymbol{w}_+$, $\boldsymbol{w}_-$ are given by (4.11, 4.12).

By the same arguments as in Proposition 3.7.5, we find that the stationary mean M and second moment V are given by

$$M = b\tilde{m}_b + \gamma\left\{\boldsymbol{z}_+\mathcal{C}(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-) + \boldsymbol{z}_-\mathcal{D}(\hat{\Psi}\boldsymbol{w}_+ + \boldsymbol{w}_-)\right\}$$

and

$$V = b^2\tilde{m}_b + \gamma\left\{\boldsymbol{z}_+\mathcal{E}(\boldsymbol{w}_+ + \Psi\boldsymbol{w}_-) + \boldsymbol{z}_-\mathcal{F}(\hat{\Psi}\boldsymbol{w}_+ + \boldsymbol{w}_-)\right\}$$

where

$$\begin{aligned}
\mathcal{C} &= (-K)^{\#}[bI - (I - e^{Kb})(-K)^{\#}] + \frac{b^2}{2}\boldsymbol{v}\boldsymbol{u} \\
\mathcal{D} &= (-\hat{K})^{-1}[(-\hat{K})^{-1}(I - e^{\hat{K}b}) - be^{\hat{K}b}] \\
\mathcal{E} &= 2K^{\#}\mathcal{C} + b^2(-K)^{\#} + \frac{b^3}{3}\boldsymbol{v}\boldsymbol{u} \\
\mathcal{F} &= 2(-\hat{K})^{-1}\mathcal{D} - (-\hat{K})^{-1}b^2e^{\hat{K}b},
\end{aligned}$$

and where $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined above.

Consider a water reservoir of capacity b. There are N processes which consume water from the reservoir. Each process is either idle or it taps water at a constant rate c; it stays in the idle state for an exponentially distributed amount of time, with parameter β, then it enters in the active state and taps a quantity of water which is exponentially distributed, with parameter α/c. Under normal circumstances, therefore, a process leaves its active state at the constant rate α.

When the level of water is strictly between zero and b, the reservoir is normally filled at a constant rate R.

When the reservoir is full, it overflows and the excess water is wasted; there is a trigger which reacts after a random interval of time, and which reduces the rate at which the reservoir is filled, from R to $R/2$. If the reservoir remains full, a second trigger reacts, the input rate is reduced to zero and the reservoir stops being filled. The filling rate returns to R when another trigger reacts to the fact that the reservoir is not full anymore. The reaction time of each trigger is exponential with parameter γ.

For the reservoir to be empty, it is necessary that the number i of active processes should be such that $i > R/c$. At such a time, we assume that the incoming water is equally shared among the active processes. The consequence is that each process needs more time to accumulate the amount of water that it requests and the rate of transition to the idle state is $(\alpha R)/(ic)$ for each active process. When a probe detects this situation, the input rate is increased and becomes aR, where $a > 1$. At a later time, the level becomes positive again, and the filling rate returns to R when a trigger reacts to the fact that the reservoir has started filling.

We illustrate four different cases:

1. There is no feedback effect, the matrices $T^{(0)}$ and $T^{(b)}$ are equal to T.

2. Each process reduces its tapping rate at level zero, and there is no probe to detect that the reservoir is empty or full. In this case, $T^{(0)}$ is different but T and $T^{(b)}$ are still equal.

3. This is the same as Case 2, but now there are probes to detect that the reservoir is full. In this case, T, $T^{(0)}$ and $T^{(b)}$ are different.

4. This is the same as Case 3, with an additional probe to detect that the reservoir is empty.

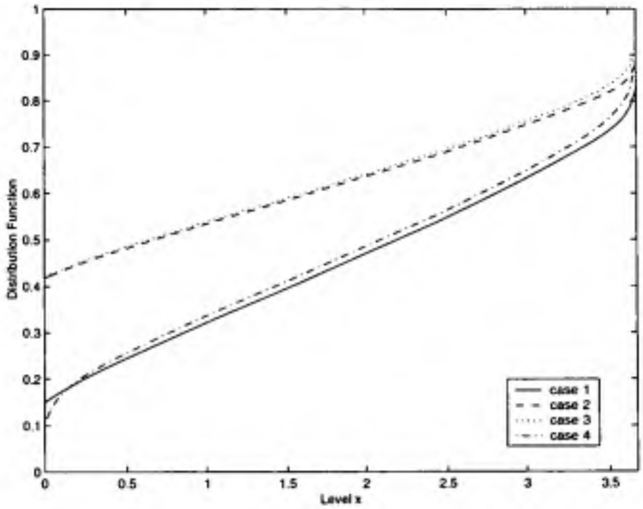


Figure 4.1: Stationary distribution function

The parameters chosen are the following: $\alpha = 1$ and $c = 1$, so that the unit of volume is fixed to be the expected quantity of water which is taken by a process and the unit of time is the expected duration of the active state, under normal conditions; the number of processes is $N = 20$; we set $R = 1.01N\beta c/(\alpha + \beta)$ so that the normal filling rate of the reservoir is slightly above what is required by the processes, on the average, and the drift μ is positive; finally, $\beta = 0.1$, $\gamma = 10$, $a = 2$ and $b = 2R$.

The probability masses $\tilde{m}_0$ and $\tilde{m}_b$ of levels zero and b, and the first two moments are given in Table 4.1 below, and the distribution function of the content of the reservoir in stationary regime is given in Figure 4.1.

The difference between the first two cases is due to the fact that, at

Case	1	2	3	4
$\tilde{m}_0$	0.1500	0.4171	0.4202	0.1040
$\tilde{m}_b$	0.1764	0.1210	0.0784	0.1126
M	1.7574	1.2053	1.0607	1.7460
V	5.1293	3.5177	2.9635	5.0432

Table 4.1: Steady State Probability Masses of Levels 0 and b, Mean M and Second Moment V of the Reservoir Content

level zero, each process receives water at a lower rate and this creates a

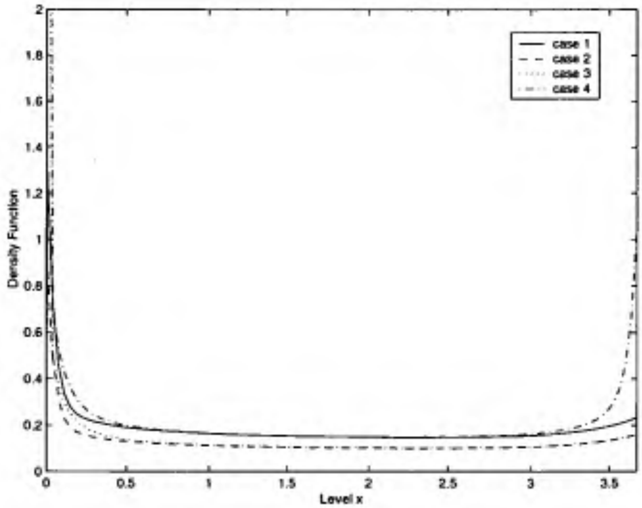


Figure 4.2: Stationary density function

positive feedback loop: the processes remain active longer, this in turns give more opportunities for idle processes to become active, which further increases the time spent by each process in its active state, etc.

In Case 4, the probe at level zero is fast ($\gamma = 10$) and the input rate to the reservoir is quickly doubled, so that the system spends little time at level zero. By increasing a, we may force $\tilde{m}_0$ to be even smaller.

The comparison of Cases 2 and 3 shows, as expected, that the probability of overflowing decreases when there are probes to detect that the reservoir is full. It appears that the idea of reducing the inflow by one half before cutting it altogether might not be very efficient.

The density function of the reservoir content in stationary regime is given in Figure 4.2. One clearly sees that the density is nearly uniform over most of the interval $(0, b)$.

4.6 Fluid Queues with Thresholds

We consider now a more complex fluid model $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$ in which the behaviour of the background phase process changes when the content of the buffer reaches certain thresholds; more specifically, it is the transition generator of the phase process that changes when the level of the buffer crosses the thresholds. Assume that the buffer is of infinite capacity and, for $0 \leq i \leq n$, let c_i denote the values of the thresholds,

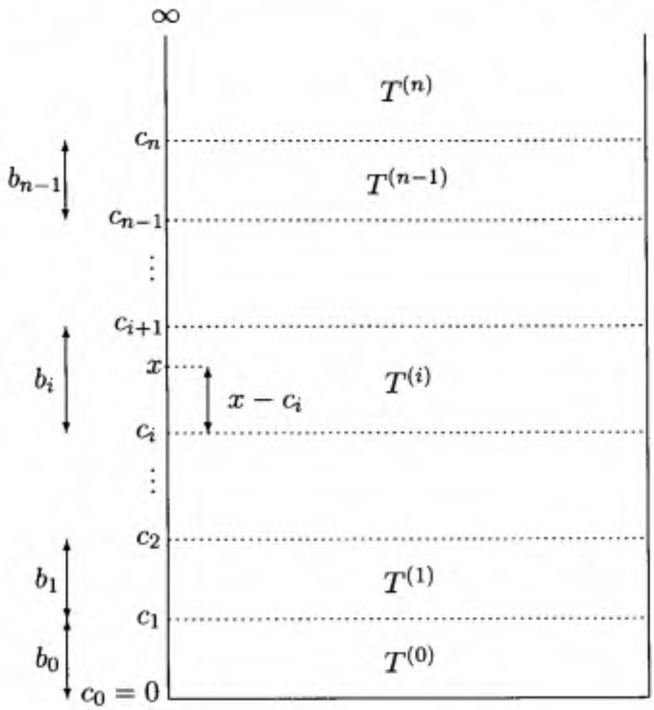


Figure 4.3: The fluid buffer with thresholds.

with

$$0 = c_0 < c_1 < c_2 < \ldots < c_n < \infty.$$

The transition generators of the phase process $\{\varphi(t)\}$ are denoted by

- $T^{(i)}$ when $c_i \le X(t) < c_{i+1}$ for $0 \le i \le n-1$,
- and $T^{(n)}$ when $X(t) \ge c_n$,

where $X(t)$ is the buffer content at time t (see Figure 4.3). We define b_i as the difference $c_{i+1} - c_i$, for $0 \le i \le n-1$.

In order to analyze such a system, we need to consider several other fluid processes and to introduce new notations. In order to simplify the presentation, we assume throughout that the net input rates of all the fluid queues considered are equal to $+1$ or -1.

For $0 \le i \le n$, we denote by $\{(X_i(t), \varphi_i(t)) : t \in \mathbb{R}^+\}$ the standard fluid queue with phase transition generator $T^{(i)}$. The matrix $\Psi^{(i)}$ gives the first return probabilities to the initial level; by the results of Section 1.4, $\Psi^{(i)}$ is the solution of the Riccati equation

$$\Psi^{(i)} T^{(i)}_{-+} \Psi^{(i)} + T^{(i)}_{++} \Psi^{(i)} + T^{(i)}_{--} \Psi^{(i)} + T^{(i)}_{+-} = 0,$$

and we have presented in Chapter 2 several procedures to solve it numerically. The matrix $N^{(i)}(x)$ recording the expected number of visits to level $x > 0$, starting from level zero, before returning to the initial level, for the fluid queue $\{(X_i(t), \varphi_i(t))\}$ is equal to

$$N^{(i)}(x) = e^{K^{(i)}x}[I,\ \Psi^{(i)}], \tag{4.16}$$

by Theorem 1.3.2, where $K^{(i)} = T^{(i)}_{++} + \Psi^{(i)}T^{(i)}_{-+}$. The generator of the process of downward records of this fluid queue is $U^{(i)} = T^{(i)}_{--} + T^{(i)}_{-+}\Psi^{(i)}$. We denote by $\hat{\Psi}^{(i)}$, $\hat{K}^{(i)}$ and $\hat{U}^{(i)}$ the matrices having the same interpretations as $\Psi^{(i)}$, $K^{(i)}$ and $U^{(i)}$ but for the level-reversed version of this process.

For $0 \le i \le n-1$, let $\{(X^{(b_i)}(t), \varphi_i(t)) : t \in \mathbb{R}^+\}$ be the fluid queue with a finite buffer of capacity $b_i = c_{i+1} - c_i$ and with phase transition generator $T^{(i)}$, independently of the buffer content. We respectively denote by $N^{(i)}_+(0, x)$ and $N^{(i)}_-(b_i, x)$ the matrices recording the expected number of visits to level x in between zero and b_i, starting from a state in $(0, \mathcal{S}_+)$ or $(b_i, \mathcal{S}_-)$, under the taboo of both levels zero and b_i. By Lemmas 3.4.1 and 3.4.2, we have that

$$\begin{bmatrix} N^{(i)}_+(0,x) \\ N^{(i)}_-(b_i,x) \end{bmatrix} =$$
$$\begin{bmatrix} I & e^{K^{(i)}b_i}\Psi^{(i)} \\ e^{\hat{K}^{(i)}b_i}\hat{\Psi}^{(i)} & I \end{bmatrix}^{-1} \begin{bmatrix} e^{K^{(i)}x} & e^{K^{(i)}x}\Psi^{(i)} \\ e^{\hat{K}^{(i)}(b_i-x)}\hat{\Psi}^{(i)} & e^{\hat{K}^{(i)}(b_i-x)} \end{bmatrix}, \tag{4.17}$$

under the condition that the drift of the ith fluid queue is different from zero. We will also need the matrices of first passage probabilities $\Psi^{(b_i)}_{+-}$, $\hat{\Psi}^{(b_i)}_{-+}$, $\Lambda^{(b_i)}_{++}$ and $\hat{\Lambda}^{(b_i)}_{--}$ which may be determined using (3.27 – 3.30) by respectively replacing Ψ, $\hat{\Psi}$, U and $\hat{U}$ by $\Psi^{(i)}$, $\hat{\Psi}^{(i)}$, $U^{(i)}$ and $\hat{U}^{(i)}$.

The buffer content of the fluid queue with thresholds has a mass at level $c_0 = 0$ and a continuous density for strictly positive values of the buffer content. The next theorem gives the expression for this stationary density; it is expressed in terms of matrices recording expected numbers of visits and of the stationary density vectors evaluated at the threshold levels. These will be determined afterwards.

Theorem 4.6.1 *The stationary density of the buffer content of the fluid queue $\{(X(t), \varphi(t))\}$ with thresholds $0 = c_0 < c_1 < c_2 < \ldots < c_n < \infty$ is given by*

$$\boldsymbol{\pi}(x) = \boldsymbol{\pi}_+(c_n)N^{(n)}(x - c_n), \tag{4.18}$$

for $x > c_n$, *and*

$$\boldsymbol{\pi}(x) = \boldsymbol{\pi}_+(c_i)N_+^{(i)}(0, x - c_i) + \boldsymbol{\pi}_-(c_{i+1})N_-^{(i)}(b_i, x - c_i), \tag{4.19}$$

for $c_i < x < c_{i+1}$ *and* $0 \le i \le n-1$.

Proof Assume throughout the proof and without loss of generality that $X(0) = 0$.

Take $x > c_n$ and j in $\mathcal{S}$. By decomposition based on the last crossing of level c_n from below, we have that $X(t) \in (x, x+h)$ and $\varphi(t) = j$ if and only if there exist some time $\tau < t$ and some phase i in $\mathcal{S}_+$ such that the process crosses level c_n at time $t - \tau$ with $\varphi(t-\tau) = i$, and it continuously remains above level c_n in the interval $(t - \tau, t)$. Thus,

$$f_j(x;t)h = \int_0^t \sum_{i \in \mathcal{S}_+} f_i(c_n; t-\tau) d\tau [\phi^{(n)}(x;\tau)]_{ij} h + o(h)$$

where $[\phi^{(n)}(x;\tau)]_{ij}h$ is the conditional probability, given that the process starts in (c_n, i), that it remains above level c_n in the interval $(0, \tau)$ and that, at time τ, the level is in $(x, x+h)$ and the phase is j. Dividing both sides of the above expression by h, taking the limits as h goes to zero and as t goes to infinity, and using the same kind of arguments as in Theorem 1.3.1, we find

$$\pi_j(x) = \sum_{i \in \mathcal{S}_+} \pi_i(c_n) \int_0^\infty [\phi^{(n)}(x;\tau)]_{ij} d\tau$$

where the integral is the expected number of crossings of level x in phase j, starting from state (c_n, i), before the first return to level c_n. This number of crossings has the same distribution as the number of crossings of level $x - c_n$, starting from level zero, under taboo of level zero, for the fluid queue $\{(X_n(t), \varphi_n(t))\}$. Therefore, its expected value is $N^{(n)}(x - c_n)$ given by (4.16); thus, (4.18) is proved.

Now, take $c_i < x < c_{i+1}$ with $0 \le i \le n-1$, and j in $\mathcal{S}$. We have that $X(t) \in (x, x+h)$ and $\varphi(t) = j$ in one of two cases:

- either there exist some time $\tau < t$ and some k in $\mathcal{S}_+$ such that the process crosses level c_i from below at time $t - \tau$, with $\varphi(t-\tau) = k$,
- or there exist some time $\tau < t$ and some k in $\mathcal{S}_-$ such that the process crosses level c_{i+1} from above at time $t - \tau$, with $\varphi(t-\tau) = k$,

and in the interval $(t - \tau, t)$ it continuously remains between levels c_i and c_{i+1} without crossing either. Following the same steps as in the first part of the proof, we eventually obtain

$$\pi_j(x) = \sum_{k \in \mathcal{S}_+} \pi_k(c_i)[\Phi^{(i)}(c_i, x)]_{kj} + \sum_{k \in \mathcal{S}_-} \pi_k(c_{i+1})[\Phi^{(i)}(c_{i+1}, x)]_{kj}$$

where $[\Phi^{(i)}(c_i, x)]_{kj}$ denotes the expected number of visits to state (x, j), starting from state (c_i, k), remaining strictly between levels c_i and c_{i+1}. This number of visits has the same distribution as the number of visits to $(x - c_i, j)$, starting from $(0, k)$, under taboo of levels zero and b_i, in the fluid queue $\{(X^{(b_i)}(t), \varphi_i(t))\}$ with finite capacity b_i. Therefore, $\Phi^{(i)}(c_i, x) = N_+^{(i)}(0, x - c_i)$ and, similarly, $\Phi^{(i)}(c_{i+1}, x) = N_-^{(i)}(b_i, x - c_i)$. Expressions for these matrices are given in (4.17), and the proof is complete. □

In the sequel, we will need the following matrices of first return probabilities. For $0 \le i \le n$, we denote by $\Pi_{+-}^{(i)}$ the matrix giving the first return probability to level c_i for the fluid queue $\{(X(t), \varphi(t))\}$. More precisely, denoting by θ_i the first return time to level c_i,

$$\theta_i = \inf\{t > 0 : X(t) = c_i, X(t-) \neq c_i\},$$

which is finite almost surely in the present context, we have that, for j in $\mathcal{S}_+$ and k in $\mathcal{S}_-$,

$$\Pi_{jk}^{(i)} = \mathrm{P}[\theta_i < \infty \text{ and } (X(\theta_i), \varphi(\theta_i)) = (c_i, k) | (X(0), \varphi(0)) = (c_i, j)].$$

The following lemma states how to determine the matrices $\Pi_{+-}^{(i)}$. Note the similarity between equation (4.20) below and equation (3.26); we also point out that the two equations are obtained by a similar sequence of arguments.

Lemma 4.6.2 *For $0 \le i \le n-1$, the matrices $\Pi_{+-}^{(i)}$ are given by*

$$\Pi_{+-}^{(i)} = \Psi_{+-}^{(b_i)} + \Lambda_{++}^{(b_i)} \Pi_{+-}^{(i+1)} (I - \hat{\Psi}_{-+}^{(b_i)} \Pi_{+-}^{(i+1)})^{-1} \hat{\Lambda}_{--}^{(b_i)} \qquad (4.20)$$

and

$$\Pi_{+-}^{(n)} = \Psi^{(n)}.$$

Proof The fact that $\Pi^{(n)}_{+-} = \Psi^{(n)}$ is obvious.

To determine $\Pi^{(i)}_{+-}$, for $0 \le i \le n-1$, we consider two cases: either the return to level c_i occurs before the first passage to level c_{i+1}, or it occurs later. Therefore, we have that

$$\Pi^{(i)}_{+-} = \Psi^{(b_i)}_{+-} + \Lambda^{(b_i)}_{++} G^{(i+1)}_{+-}$$

where $G^{(i+1)}_{+-}$ records the probability of eventually being in a state in $(c_i, \mathcal{S}_-)$ given that the initial state is in $(c_{i+1}, \mathcal{S}_+)$. Now, starting from $(c_{i+1}, \mathcal{S}_+)$, the queue must first return to $(c_{i+1}, \mathcal{S}_-)$, with probability given by $\Pi^{(i+1)}_{+-}$, and it must then move down to level c_i. Thus,

$$\Pi^{(i)}_{+-} = \Psi^{(b_i)}_{+-} + \Lambda^{(b_i)}_{++} \Pi^{(i+1)}_{+-} G^{(i+1)}_{--} \tag{4.21}$$

with $G^{(i+1)}_{--}$ giving the probability of eventually being in $(c_i, \mathcal{S}_-)$, starting from $(c_{i+1}, \mathcal{S}_-)$. There are two ways of visiting $(c_i, \mathcal{S}_-)$, starting from $(c_{i+1}, \mathcal{S}_-)$: either the visit occurs before a return to level c_{i+1}, or it occurs after. This decomposition leads to

$$G^{(i+1)}_{--} = \hat{\Lambda}^{(b_i)}_{--} + \hat{\Psi}^{(b_i)}_{-+} \Pi^{(i+1)}_{+-} G^{(i+1)}_{--},$$

which is equivalent to

$$G^{(i+1)}_{--} = (I - \hat{\Psi}^{(b_i)}_{-+} \Pi^{(i+1)}_{+-})^{-1} \hat{\Lambda}^{(b_i)}_{--} \tag{4.22}$$

since $\hat{\Psi}^{(b_i)}_{-+} \Pi^{(i+1)}_{+-}$ is a sub-stochastic matrix. Equations (4.21) and (4.22) together give the announced result. □

To determine the stationary density of the buffer content, it remains for us to determine the vectors $(\boldsymbol{\pi}_+(c_i), \boldsymbol{\pi}_-(c_i))$ for $0 \le i \le n$. These vectors are obtained by solving the system of equations given in the next theorem.

Theorem 4.6.3 *The vectors $(\boldsymbol{\pi}_+(c_i), \boldsymbol{\pi}_-(c_i))$ for $0 \le i \le n$ are the solution of the system*

$$\boldsymbol{\pi}_-(c_n) = \boldsymbol{\pi}_+(c_n) \Psi^{(n)} \tag{4.23}$$

$$\boldsymbol{\pi}_+(c_i) = \boldsymbol{\pi}_+(c_{i-1}) \Lambda^{(b_{i-1})}_{++} + \boldsymbol{\pi}_-(c_i) \hat{\Psi}^{(b_{i-1})}_{-+}, \quad 1 \le i \le n \tag{4.24}$$

$$\boldsymbol{\pi}_-(c_i) = \boldsymbol{\pi}_+(c_i) \Psi^{(b_i)}_{+-} + \boldsymbol{\pi}_-(c_{i+1}) \hat{\Lambda}^{(b_i)}_{--}, \quad 1 \le i \le n-1 \tag{4.25}$$

and

$$\pi_+(0) = p_-^{(0)} T_{-+}^{(0)}, \tag{4.26}$$

where $p_-^{(0)}$ *is the steady state probability mass vector of the states in* $(0, \mathcal{S}_-)$. *It is equal to* $c x_-$, *where* x_- *is the unique solution of the system*

$$\begin{aligned} x_-(T_{--}^{(0)} + T_{-+}^{(0)} \Pi_{+-}^{(0)}) &= \mathbf{0} \\ x_- \mathbf{1} &= 1. \end{aligned}$$

The matrix $T_{--}^{(0)} + T_{-+}^{(0)} \Pi_{+-}^{(0)}$ *is the generator of the censored process to the states in* $(0, \mathcal{S}_-)$. *The normalizing factor* c *is given by*

$$c = \{x_- \mathbf{1} + \int_0^\infty y(x) \mathbf{1} dx\}^{-1}$$

where $y(x)$ *is obtained using Theorem 4.6.1 and replacing* $p_-^{(0)}$ *by* x_- *in (4.26).*

Proof The proof is again based on level crossing arguments. To prove (4.23) we condition on the last crossing of level c_n; we have that $X(t) \in (c_n, c_n + h)$ and $\varphi(t) = j \in \mathcal{S}_-$ if and only if there exist a time $\tau < t$ and $k \in \mathcal{S}_+$ such that the process crosses level c_n from below at time $t - \tau$, with $\varphi(t - \tau) = k$, and continuously remains above level c_n in the interval $(t - \tau, t)$. Thus,

$$f_j(c_n; t) h = \int_0^t \sum_{k \in \mathcal{S}_+} f_k(c_n; t - \tau) h \Theta_{kj}^{(n)}(d\tau) + o(h)$$

where

$$\Theta_{kj}^{(n)}(\tau) = \mathrm{P}[\theta_n \le \tau, \varphi(\theta_n) = j | X(0) = c_n, \varphi(0) = k] \tag{4.27}$$

with θ_n being the first return time to level c_n. Dividing by h and taking the limits as h goes to zero and as t goes to infinity, we eventually find that

$$\pi_j(c_n) = \sum_{k \in \mathcal{S}_+} \pi_k(c_n) \int_0^\infty \Theta_{kj}^{(n)}(d\tau)$$

where the integral gives the first return probability to level c_n and is therefore equal to $\Psi^{(n)}$.

Next, take $j \in \mathcal{S}_+$. We have that $X(t) \in (c_i, c_i + h)$ and $\varphi(t) = j$ if and only if

- either there exist some time $\tau < t$ and k in $\mathcal{S}_+$ such that the process crosses level c_{i-1} from below at time $t-\tau$, with $\varphi(t-\tau) = k$,
- or there exist some time $\tau < t$ and k in $\mathcal{S}_-$ such that the process crosses level c_i from above at time $t-\tau$, with $\varphi(t-\tau) = k$,

and in the interval $(t-\tau, t)$ it continuously remains between levels c_{i-1} and c_i without crossing either. Thus,

$$\begin{aligned} f_j(c_i;t)h &= \int_0^t \sum_{k\in\mathcal{S}_+} f_k(c_{i-1};t-\tau)h\Gamma_{kj}^{(i)}(d\tau) \\ &\quad + \int_0^t \sum_{k\in\mathcal{S}_-} f_k(c_i;t-\tau)h\Theta_{kj}^{(i)}(d\tau) + o(h) \end{aligned}$$

where

$$\Gamma_{kj}^{(i)}(\tau) = \mathrm{P}[\theta_i \le \tau, \varphi(\theta_i) = j | X(0) = c_{i-1}, \varphi(0) = k]$$

with θ_i being the first passage time to level c_i, and $\Theta_{kj}^{(i)}(\tau)$ is defined like in (4.27). By the same sequence of steps as in the first part of the proof, we obtain

$$\begin{aligned} \pi_j(c_i) &= \sum_{k\in\mathcal{S}_+} \pi_k(c_{i-1}) \int_0^\infty \Gamma_{kj}^{(i)}(d\tau) + \sum_{k\in\mathcal{S}_-} \pi_k(c_i) \int_0^\infty \Theta_{kj}^{(i)}(d\tau) \\ &= \sum_{k\in\mathcal{S}_+} \pi_k(c_{i-1})\Lambda_{kj}^{(b_i-1)} + \sum_{k\in\mathcal{S}_-} \pi_k(c_i)\hat{\Psi}_{kj}^{(b_i-1)} \end{aligned}$$

since the integrals $\int_0^\infty \Gamma^{(i)}(d\tau)$ and $\int_0^\infty \Theta^{(i)}(d\tau)$ give respectively the first passage probability to level c_i, starting from level c_{i-1}, without returning to this initial level, and the first return probability to level c_i, starting from level c_i, under taboo of level c_{i-1}.

Equation (4.25) is obtained along the same lines and we omit the proof. To prove (4.26), we write

$$f_j(0;t)h = \sum_{k\in\mathcal{S}_-} F_k(0;t)T_{kj}^{(0)}h + o(h)$$

which is easily seen to give the announced result.

The remainder of the proof follows from the same kind of arguments which lead to the determination of the steady state probability mass vector in Theorem 4.2.1, with a slight and obvious modification concerning the generator $U^{(0)}$.

In order to show that the solution of the system (4.23–4.25) is unique, we show that its coefficient matrix Ω is sub-stochastic. For $1 \le i \le n$ and $1 \le j \le n$, denote by $\Omega((c_i, \mathcal{S}_*); (c_j, \mathcal{S}'_*))$ the sub-matrix of Ω containing the entries $\Omega_{(c_i,k),(c_j,l)}$ for k in $\mathcal{S}_*$ and l in $\mathcal{S}'_*$, where $\mathcal{S}_*$ and $\mathcal{S}'_*$ are either $\mathcal{S}_+$ or $\mathcal{S}_-$.

For $1 \le i \le n-1$, we have that

$$\Omega((c_i, \mathcal{S}_+); (c_i, \mathcal{S}_-)) = \Psi^{(b_i)}_{+-} \quad \text{and} \quad \Omega((c_i, \mathcal{S}_+); (c_{i+1}, \mathcal{S}_+)) = \Lambda^{(b_i)}_{++},$$

all the other entries $\Omega((c_i, \mathcal{S}_+); y)$ are equal to zero, for $y \neq (c_i, \mathcal{S}_-)$ and $(c_{i+1}, \mathcal{S}_+)$.

Similarly, for $1 \le i \le n-1$,

$$\Omega((c_{i+1}, \mathcal{S}_-); (c_i, \mathcal{S}_-)) = \hat{\Lambda}^{(b_i)}_{--} \quad \text{and} \quad \Omega((c_{i+1}, \mathcal{S}_-); (c_{i+1}, \mathcal{S}_+)) = \hat{\Psi}^{(b_i)}_{-+},$$

all the other entries $\Omega((c_{i+1}, \mathcal{S}_-); y)$ are equal to zero, for $y \neq (c_i, \mathcal{S}_-)$ and $(c_{i+1}, \mathcal{S}_+)$.

For the states in $(c_n, \mathcal{S}_+)$, we have that

$$\Omega((c_n, \mathcal{S}_+); (c_n, \mathcal{S}_-)) = \Psi^{(n)},$$

the entries $\Omega((c_n, \mathcal{S}_+); y)$ being equal to zero for $y \neq (c_n, \mathcal{S}_-)$.

Finally, for the states in $(c_1, \mathcal{S}_-)$, we find

$$\Omega((c_1, \mathcal{S}_-); (c_1, \mathcal{S}_+)) = \hat{\Psi}^{(b_0)}_{-+}$$

and all the other entries $\Omega((c_1, \mathcal{S}_+); y) = 0$ for $y \neq (c_1, \mathcal{S}_+)$.

As an example, if $n = 3$, the matrix Ω has the following structure

$$\begin{array}{c} \\ (c_1,\mathcal{S}_-) \\ (c_1,\mathcal{S}_+) \\ (c_2,\mathcal{S}_-) \\ (c_2,\mathcal{S}_+) \\ (c_3,\mathcal{S}_-) \\ (c_3,\mathcal{S}_+) \end{array}
\begin{array}{c}
\begin{array}{cccccc} (c_1,\mathcal{S}_-) & (c_1,\mathcal{S}_+) & (c_2,\mathcal{S}_-) & (c_2,\mathcal{S}_+) & (c_3,\mathcal{S}_-) & (c_3,\mathcal{S}_+) \end{array} \\
\begin{bmatrix}
0 & \hat{\Psi}^{(b_0)}_{-+} & 0 & 0 & 0 & 0 \\
\Psi^{(b_1)}_{+-} & 0 & 0 & \Lambda^{(b_1)}_{++} & 0 & 0 \\
\hat{\Lambda}^{(b_1)}_{--} & 0 & 0 & \hat{\Psi}^{(b_1)}_{-+} & 0 & 0 \\
0 & 0 & \Psi^{(b_2)}_{+-} & 0 & 0 & \Lambda^{(b_2)}_{++} \\
0 & 0 & \hat{\Lambda}^{(b_2)}_{--} & 0 & 0 & \hat{\Psi}^{(b_2)}_{-+} \\
0 & 0 & 0 & 0 & \Psi^{(3)} & 0
\end{bmatrix}
\end{array}.$$

Post-multiplying the matrix Ω by a column vector $\mathbf{1}$, we obtain that

$$\Omega((c_i, \mathcal{S}_+); \cdot)\mathbf{1} = \Psi^{(b_i)}_{+-}\mathbf{1} + \Lambda^{(b_i)}_{++}\mathbf{1} = \mathbf{1},$$

and

$$\Omega((c_{i+1}, \mathcal{S}_-); \cdot)\mathbf{1} = \hat{\Psi}^{(b_i)}_{-+}\mathbf{1} + \hat{\Lambda}^{(b_i)}_{--}\mathbf{1} = \mathbf{1},$$

for $1 \leq i \leq n-1$. For the states in $(c_n, \mathcal{S}_+)$, we have that

$$\Omega((c_n, \mathcal{S}_+); \cdot)\mathbf{1} = \Psi^{(n)}\mathbf{1} = \mathbf{1}.$$

Lastly, for the states in $(c_1, \mathcal{S}_-)$, we find

$$\Omega((c_1, \mathcal{S}_-); \cdot)\mathbf{1} = \hat{\Psi}^{(b_0)}_{-+}\mathbf{1} < \mathbf{1}.$$

The matrix Ω is thus sub-stochastic and is the transition matrix of a defective Markov chain, since the fluid queue is irreducible; therefore, there is a path to level zero from any state in the levels c_1 to c_n, and we conclude that $I - \Omega$ is non singular. The proof is therefore completed. □

4.7 Sticky and Repulsive Thresholds

We make the model of Section 4.6 somewhat more complex by considering a fluid queue in which the behaviour of the phase process changes in two ways when the buffer content reaches certain thresholds: first, there are different phase transition generators when the level is in between different thresholds, as in Section 4.6; second, the rate associated to any given phase may change, so that it is possible that it switches from making the level increase *below* the threshold to making the level increase *above* the threshold, and vice-versa. Again, in order to simplify the presentation, we assume that the net input rates are equal to $+1$ and -1.

Consider a fluid queue $\{(X(t), \varphi(t))\}$ with a buffer of infinite capacity, and assume again that there are n thresholds such that $0 = c_0 < c_1 < \ldots < c_n < \infty$. As in Section 4.6, the phase transition generators of the process $\{\varphi(t)\}$ are denoted by $T^{(i)}$ when $c_i \leq X(t) < c_{i+1}$, for $0 \leq i \leq n-1$, and by $T^{(n)}$ when $X(t) > c_n$. The state space of the phase process is denoted by $\mathcal{S}$.

When $X(t) = 0$, $\mathcal{S}$ is decomposed into two disjoint subsets: $\mathcal{S}^{(0)}_u$ which contains the phases that make the level increase and $\mathcal{S}^{(0)}_s$ which contains the phases that make the level decrease, and thus force the level to remain equal to zero.

For $0 < X(t) < c_1$, $\mathcal{S}$ is decomposed into $\mathcal{S}^{(0)}_+ \cup \mathcal{S}^{(0)}_-$, where $\mathcal{S}^{(0)}_+$ and $\mathcal{S}^{(0)}_-$ contain the phases corresponding to positive and negative fluid net input rates, respectively.

At the threshold c_1, the fluid net input rates corresponding to some of the phases in $\mathcal{S}_+^{(0)}$ may become negative, and the rates corresponding to some of the phases in $\mathcal{S}_-^{(0)}$ may become positive. Thus, at the threshold c_1, $\mathcal{S}$ is potentially decomposed into four disjoint subsets of phases:

- $\mathcal{S}_u^{(1)}$ which contains the phases such that the corresponding net input rates are equal to $+1$ both below and above the threshold c_1; the fluid queue is pushed upward through the threshold c_1;
- $\mathcal{S}_d^{(1)}$ which contains the phases such that the corresponding net input rates are equal to -1 both below and above c_1; the fluid queue is pushed downward through the threshold c_1;
- $\mathcal{S}_r^{(1)}$ which contains the phases such that the corresponding net input rates are equal to -1 below c_1 and equal to $+1$ above c_1; these phases are called repulsive at the threshold c_1 since they cannot be reached either from above or from below;
- $\mathcal{S}_s^{(1)}$ which contains the phases such that the corresponding net input rates are equal to $+1$ below c_1 and equal to -1 above c_1; the fluid queue remains stuck at the threshold c_1 until there is a change of phase to either of the sets $\mathcal{S}_u^{(1)}$, $\mathcal{S}_d^{(1)}$ or $\mathcal{S}_r^{(1)}$.

Some of these sets may be empty. In between the thresholds c_1 and c_2, the state space $\mathcal{S}$ is decomposed as usual into the two disjoint subsets $\mathcal{S}_+^{(1)}$ and $\mathcal{S}_-^{(1)}$. We repeat this construction at each threshold, and thus we have the following decompositions of $\mathcal{S}$ in terms of the buffer level:

- $\mathcal{S} = \mathcal{S}_u^{(0)} \cup \mathcal{S}_s^{(0)}$ at level zero;
- $\mathcal{S} = \mathcal{S}_u^{(i)} \cup \mathcal{S}_d^{(i)} \cup \mathcal{S}_s^{(i)} \cup \mathcal{S}_r^{(i)}$ at each threshold c_i, for $1 \leq i \leq n$;
- $\mathcal{S} = \mathcal{S}_+^{(i)} \cup \mathcal{S}_-^{(i)}$ for $c_i < X(t) < c_{i+1}$, for $0 \leq i \leq n-1$;
- $\mathcal{S} = \mathcal{S}_+^{(n)} \cup \mathcal{S}_-^{(n)}$ above level c_n.

This is illustrated in Figure 4.4 in the case where $n = 2$. The symbols $\circ, \bullet$, $\diamond$ and $\blacklozenge$ respectively indicate phases belonging to $\mathcal{S}_u^{(i)}$, $\mathcal{S}_s^{(i)}$, $\mathcal{S}_r^{(i)}$ and $\mathcal{S}_d^{(i)}$, for some i.

If the process reaches the state (c_i, j) with j in $\mathcal{S}_s^{(i)}$ from above or from below, it stays there as long as the phase remains in $\mathcal{S}_s^{(i)}$. The net input rates and the generators are defined by continuity from below,

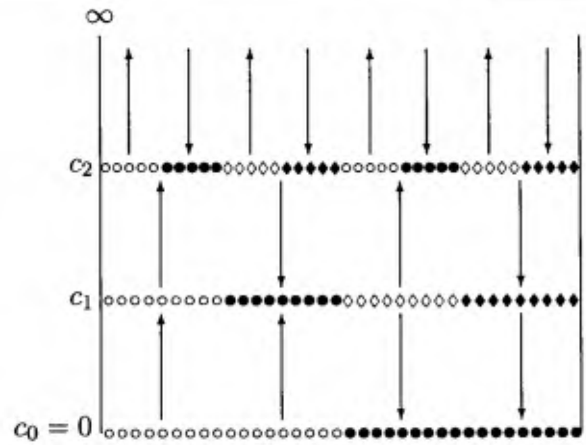


Figure 4.4: The fluid buffer with sticky and repulsive thresholds, for $n = 2$.

thus, for example, if there is a phase transition from some j in $\mathcal{S}_s^{(i)}$ to some j' in $\mathcal{S}_r^{(i)}$, then the fluid level increases.

The stationary distribution of this system has probability masses associated to the states $(c_i, \mathcal{S}_s^{(i)})$, for $0 \le i \le n$, and a continuous density for the other states. We denote by $\boldsymbol{p}_s^{(i)}$ the steady state probability mass vectors of the states $(c_i, \mathcal{S}_s^{(i)})$, for $0 \le i \le n$, and by $\boldsymbol{\pi}(x)$ the stationary density vector of the queue. This vector is decomposed into $\boldsymbol{\pi}(x) = (\boldsymbol{\pi}_+(x), \boldsymbol{\pi}_-(x))$ for values of x different from c_i, $1 \le i \le n$. At the thresholds, $\boldsymbol{\pi}(c_i) = (\boldsymbol{\pi}_+(c_i), \boldsymbol{\pi}_-(c_i))$ for $1 \le i \le n$, denoting by $\mathcal{S}_+^{(i)}$ the subset $\mathcal{S}_u^{(i)} \cup \mathcal{S}_r^{(i)}$ and by $\mathcal{S}_-^{(i)}$ the subset $\mathcal{S}_d^{(i)} \cup \mathcal{S}_r^{(i)}$.

We define again $\{(X_i(t), \varphi_i(t)) : t \in \mathbb{R}^+\}$, for $0 \le i \le n$, as the standard fluid queue with phase transition generator $T^{(i)}$, and, for $0 \le i \le n-1$, $\{(X^{(b_i)}(t), \varphi_i(t)) : t \in \mathbb{R}^+\}$ as the finite buffer fluid queue with capacity $b_i = c_{i+1} - c_i$. We use similar notations as in Section 4.6, replacing the indices $_+$ and $_-$ by the indices u, d, s and r, where applicable.

We define

$$\boldsymbol{\pi}(y+0) = \lim_{\substack{x \to y \\ x > y}} \boldsymbol{\pi}(x)$$

and, similarly,

$$\boldsymbol{\pi}(y-0) = \lim_{\substack{x \to y \\ x < y}} \boldsymbol{\pi}(x).$$

At the thresholds c_i with $1 \le i \le n$, we may have that $\boldsymbol{\pi}(c_i+0) \ne \boldsymbol{\pi}(c_i-0)$, due to the influence of the states in $\mathcal{S}_s^{(i)}$. We give in the next theorem the density vector of the fluid queue in equilibrium, expressed in terms of $\boldsymbol{\pi}_u(c_i+0)$ and $\boldsymbol{\pi}_d(c_i-0)$, for $1 \le i \le n$.

Theorem 4.7.1 *The stationary density vector of the fluid queue with sticky and repulsive thresholds is given by*

$$\boldsymbol{\pi}(x) = \boldsymbol{\pi}_u(c_n+0)N_u^{(n)}(x-c_n) + \boldsymbol{p}_s^{(n)}T_{sr}^{(n)}N_r^{(n)}(x-c_n), \tag{4.28}$$

for $x > c_n$,

$$\begin{aligned}\boldsymbol{\pi}(x) = {} & \boldsymbol{\pi}_u(c_i+0)N_u^{(i)}(0,x-c_i) + \boldsymbol{p}_s^{(i)}T_{sr}^{(i)}N_r^{(i)}(0,x-c_i) \\ & + \boldsymbol{\pi}_d(c_{i+1}-0)N_d^{(i)}(b_i,x-c_i),\end{aligned} \tag{4.29}$$

for $c_i < x < c_{i+1}$ *and* $1 \le i \le n-1$, *and*

$$\boldsymbol{\pi}(x) = \boldsymbol{p}_s^{(0)}T_{su}^{(0)}N_u^{(0)}(0,x) + \boldsymbol{\pi}_d(c_1-0)N_d^{(0)}(b_0,x),$$

for $0 < x < c_1$.

Proof The arguments are similar to those of the proof of Theorem 4.6.1, but we now have to take into consideration the fact that the stationary density vector may no longer be continuous and that there are states at the threshold levels, corresponding to the phases in $\mathcal{S}_s^{(i)}$, where the process may remain for a certain amount of time.

To determine the stationary density vector $\boldsymbol{\pi}(x)$ for $x > c_n$, we condition on the last visit to level c_n. This visit might have happened in two different ways: either the process starts in level c_n with a phase of $\mathcal{S}_u^{(n)}$, which eventually leads to the first term in (4.28), with left-most factor $\boldsymbol{\pi}_u(c_n+0)$; or the process is in a sticky state of $(c_n, \mathcal{S}_s^{(n)})$, which eventually leads to the second term in (4.28), with left-most factor $\boldsymbol{p}_s^{(n)}$, the probability mass vector of these states.

For $c_i < x < c_{i+1}$ with $1 \le i \le n-1$, we condition on the last visit to levels c_i and c_{i+1}. We have three cases: either the process starts in level c_i with a phase of $\mathcal{S}_u^{(i)}$, or it is glued in a state of $(c_i, \mathcal{S}_s^{(i)})$, or it starts in level c_{i+1} with a phase of $\mathcal{S}_d^{(i+1)}$.

Finally, for $0 < x < c_1$, we condition on the last visit to levels zero and c_1. Either the process is at level zero with a phase of $\mathcal{S}_s^{(0)}$, or it starts in level c_1 with a phase of $\mathcal{S}_d^{(1)}$. □

The expressions for the vectors $\boldsymbol{\pi}_u(c_i+0)$ and $\boldsymbol{\pi}_d(c_i-0)$, for $1 \le i \le n$, are given next.

Theorem 4.7.2 *The vectors $\boldsymbol{\pi}_u(c_i+0)$ and $\boldsymbol{\pi}_d(c_i-0)$, for $1 \le i \le n$, are the solution of the system*

$$\begin{aligned}
\boldsymbol{\pi}_d(c_n-0) &= \boldsymbol{p}_s^{(n)}T_{sd}^{(n)} + \boldsymbol{p}_s^{(n)}T_{sr}^{(n)}\Psi_{rd}^{(n)} + \boldsymbol{\pi}_u(c_n+0)\Psi_{ud}^{(n)} \\
\boldsymbol{\pi}_u(c_i+0) &= \boldsymbol{p}_s^{(i)}T_{su}^{(i)} + \boldsymbol{p}_s^{(i-1)}T_{sr}^{(i-1)}\Lambda_{ru}^{(b_{i-1})} \qquad (4.30)\\
&\quad + \boldsymbol{\pi}_u(c_{i-1}+0)\Lambda_{uu}^{(b_{i-1})} + \boldsymbol{\pi}_d(c_i-0)\hat{\Psi}_{du}^{(b_{i-1})}, \ \textit{for } 2 \le i \le n \\
\boldsymbol{\pi}_d(c_i-0) &= \boldsymbol{p}_s^{(i)}T_{sd}^{(i)} + \boldsymbol{p}_s^{(i)}T_{sr}^{(i)}\Psi_{rd}^{(b_i)} \\
&\quad + \boldsymbol{\pi}_u(c_i+0)\Psi_{ud}^{(b_i)} + \boldsymbol{\pi}_d(c_{i+1}-0)\hat{\Lambda}_{dd}^{(b_i)}, \ \textit{for } 1 \le i \le n-1
\end{aligned}$$

and

$$\boldsymbol{\pi}_u(c_1+0) = \boldsymbol{p}_s^{(1)}T_{su}^{(1)} + \boldsymbol{p}_s^{(0)}T_{su}^{(0)}\Lambda_{uu}^{(b_0)} + \boldsymbol{\pi}_d(c_1-0)\hat{\Psi}_{du}^{(b_0)}.$$

The expressions for $\boldsymbol{p}_s^{(i)}$, $0 \le i \le n$, will be given later.

Proof We only derive (4.30) because the other equations are obtained using similar arguments.

First note that

$$\boldsymbol{\pi}_u(c_i+0) = \boldsymbol{p}_s^{(i)}T_{su}^{(i)} + \boldsymbol{\pi}_u(c_i-0). \qquad (4.31)$$

This is obtained by observing that, in order to be in c_i+0 with an increasing phase, either the process was in a state of $(c_i, \mathcal{S}_s^{(i)})$ and then there was a phase transition from $\mathcal{S}_s^{(i)}$ to $\mathcal{S}_u^{(i)}$, or the process crossed level c_i coming from below, which gives rise to the term $\boldsymbol{\pi}_u(c_i-0)$.

To determine $\boldsymbol{\pi}_u(c_i-0)$, we take the limit as x goes to c_i, $x < c_i$, in (4.29). We find

$$\begin{aligned}
\boldsymbol{\pi}_u(c_i-0) &= \boldsymbol{\pi}_u(c_{i-1}+0)N_{uu}^{(i-1)}(0,b_{i-1}) + \boldsymbol{p}_s^{(i-1)}T_{sr}^{(i-1)}N_{ru}^{(i-1)}(0,b_{i-1}) \\
&\quad +\boldsymbol{\pi}_d(c_i-0)N_{du}^{(i-1)}(b_{i-1},b_{i-1}) \qquad (4.32)\\
&= \boldsymbol{\pi}_u(c_{i-1}+0)\Lambda_{uu}^{(b_{i-1})} + \boldsymbol{p}_s^{(i-1)}T_{sr}^{(i-1)}\Lambda_{ru}^{(b_{i-1})} \\
&\quad +\boldsymbol{\pi}_d(c_i-0)\hat{\Psi}_{du}^{(b_{i-1})}
\end{aligned}$$

by Remark 3.5.2. The equations (4.31, 4.32) together lead to the announced result. □

To completely characterize the stationary distribution of the system, it remains for us to give expressions for the vectors $\boldsymbol{p}_s^{(i)}$, for $0 \le i \le n$;

these are obtained through the construction of the censored Markov chain which only sees the transitions among the states $(c_i, \mathcal{S}_s^{(i)})$, $0 \le i \le n$. We now proceed with such a construction.

First, consider the jump Markov chain on the states (c_i, j), for $0 \le i \le n$, where j belongs to $\mathcal{S}_u^{(0)} \cup \mathcal{S}_s^{(0)}$ if $i = 0$, and to $\mathcal{S}_u^{(i)} \cup \mathcal{S}_s^{(i)} \cup \mathcal{S}_r^{(i)} \cup \mathcal{S}_d^{(i)}$ if $1 \le i \le n$. Let Φ denote its transition matrix and, for $0 \le i \le n$, define $P_{ss}^{(i)} = I + \Delta_i^{-1} T_{ss}^{(i)}$, $P_{su}^{(i)} = \Delta_i^{-1} T_{su}^{(i)}$, $P_{sr}^{(i)} = \Delta_i^{-1} T_{sr}^{(i)}$ and $P_{sd}^{(i)} = \Delta_i^{-1} T_{sd}^{(i)}$, with Δ_i being the diagonal matrix $\mathrm{diag}(-T_{ss}^{(i)})$. For $i = 0$, we obviously only need to define $P_{ss}^{(0)}$ and $P_{su}^{(0)}$. We recall that when the process starts from a state in $(c_i, \mathcal{S}_r^{(i)})$, $1 \le i \le n$, it is repelled upwards, which justifies the entries corresponding to these states in the matrix Φ. Using the same notations as in the preceding section for the blocks of the matrix Φ, we have the following non-null blocks.

For $1 \le i \le n-1$,

$$\Phi((c_i, \mathcal{S}_u^{(i)}); (c_i, \mathcal{S}_s^{(i)})) = \Psi_{us}^{(b_i)}, \ \Phi((c_i, \mathcal{S}_u^{(i)}); (c_i, \mathcal{S}_d^{(i)})) = \Psi_{ud}^{(b_i)},$$

$$\Phi((c_i, \mathcal{S}_u^{(i)}); (c_{i+1}, \mathcal{S}_u^{(i+1)})) = \Lambda_{uu}^{(b_i)}, \ \Phi((c_i, \mathcal{S}_u^{(i)}); (c_{i+1}, \mathcal{S}_s^{(i+1)})) = \Lambda_{us}^{(b_i)},$$

$$\Phi((c_i, \mathcal{S}_s^{(i)}); (c_i, \mathcal{S}_*^{(i)})) = P_{s*}^{(i)}$$

where the subscript $*$ is either u, s, r or d,

$$\Phi((c_i, \mathcal{S}_r^{(i)}); (c_i, \mathcal{S}_s^{(i)})) = \Psi_{rs}^{(b_i)}, \ \Phi((c_i, \mathcal{S}_r^{(i)}); (c_i, \mathcal{S}_d^{(i)})) = \Psi_{rd}^{(b_i)},$$

$$\Phi((c_i, \mathcal{S}_r^{(i)}); (c_{i+1}, \mathcal{S}_u^{(i+1)})) = \Lambda_{ru}^{(b_i)}, \ \Phi((c_i, \mathcal{S}_r^{(i)}); (c_{i+1}, \mathcal{S}_s^{(i+1)})) = \Lambda_{rs}^{(b_i)},$$

$$\Phi((c_i, \mathcal{S}_d^{(i)}); (c_i, \mathcal{S}_u^{(i)})) = \hat{\Psi}_{du}^{(b_{i-1})}, \ \Phi((c_i, \mathcal{S}_d^{(i)}); (c_i, \mathcal{S}_s^{(i)})) = \Psi_{ds}^{(b_{i-1})},$$

$$\Phi((c_i, \mathcal{S}_d^{(i)}); (c_{i-1}, \mathcal{S}_s^{(i-1)})) = \hat{\Lambda}_{ds}^{(b_{i-1})}$$

and

$$\Phi((c_i, \mathcal{S}_d^{(i)}); (c_{i-1}, \mathcal{S}_d^{(i-1)})) = \hat{\Lambda}_{dd}^{(b_{i-1})},$$

provided that $i > 1$ in the last case.

For $i = 0$, we have that

$$\Phi((c_0, \mathcal{S}_u^{(0)}); (c_0, \mathcal{S}_s^{(0)})) = \Psi_{us}^{(b_0)},$$

$$\Phi((c_0, \mathcal{S}_u^{(0)}); (c_1, \mathcal{S}_u^{(1)})) = \Lambda_{uu}^{(b_0)}, \ \Phi((c_0, \mathcal{S}_u^{(0)}); (c_1, \mathcal{S}_s^{(1)})) = \Lambda_{us}^{(b_0)},$$

$$\Phi((c_0, \mathcal{S}_s^{(0)}); (c_0, \mathcal{S}_u^{(0)})) = P_{su}^{(0)}, \ \Phi((c_0, \mathcal{S}_s^{(0)}); (c_0, \mathcal{S}_s^{(0)})) = P_{ss}^{(0)}.$$

Finally, for $i = n$, we find that

$$\Phi((c_n, \mathcal{S}_u^{(n)}); (c_n, \mathcal{S}_s^{(n)})) = \Psi_{us}^{(n)}, \ \Phi((c_n, \mathcal{S}_u^{(n)}); (c_n, \mathcal{S}_d^{(n)})) = \Psi_{ud}^{(n)},$$

$$\Phi((c_n, \mathcal{S}_s^{(n)}); (c_n, \mathcal{S}_*^{(n)})) = P_{s*}^{(n)},$$

where the subscript $*$ is either u, s, r or d,

$$\Phi((c_n, \mathcal{S}_r^{(n)}); (c_n, \mathcal{S}_s^{(n)})) = \Psi_{rs}^{(n)}, \ \Phi((c_n, \mathcal{S}_r^{(n)}); (c_n, \mathcal{S}_d^{(n)})) = \Psi_{rd}^{(n)},$$

$$\Phi((c_n, \mathcal{S}_d^{(n)}); (c_n, \mathcal{S}_u^{(n)})) = \hat{\Psi}_{du}^{(b_{n-1})}, \ \Phi((c_n, \mathcal{S}_d^{(n)}); (c_n, \mathcal{S}_s^{(n)})) = \hat{\Psi}_{ds}^{(b_{n-1})},$$

$$\Phi((c_n, \mathcal{S}_d^{(n)}); (c_{n-1}, \mathcal{S}_s^{(n-1)})) = \hat{\Lambda}_{ds}^{(b_{n-1})}$$

and

$$\Phi((c_n, \mathcal{S}_d^{(n)}); (c_{n-1}, \mathcal{S}_d^{(n-1)})) = \hat{\Lambda}_{dd}^{(b_{n-1})}.$$

As an example, if $n = 2$, we have that

$$\Phi = \left[\begin{array}{cc|cccc|cccc}
0 & \Psi_{us}^{(b_0)} & \Lambda_{uu}^{(b_0)} & \Lambda_{us}^{(b_0)} & 0 & 0 & 0 & 0 & 0 & 0 \\
P_{su}^{(0)} & P_{ss}^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \Psi_{us}^{(b_1)} & 0 & \Psi_{ud}^{(b_1)} & \Lambda_{uu}^{(b_1)} & \Lambda_{us}^{(b_1)} & 0 & 0 \\
0 & 0 & P_{su}^{(1)} & P_{ss}^{(1)} & P_{sr}^{(1)} & P_{sd}^{(1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Psi_{rs}^{(b_1)} & 0 & \Psi_{rd}^{(b_1)} & \Lambda_{ru}^{(b_1)} & \Lambda_{rs}^{(b_1)} & 0 & 0 \\
0 & \hat{\Lambda}_{ds}^{(b_0)} & \hat{\Psi}_{du}^{(b_0)} & \hat{\Psi}_{ds}^{(b_0)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{us}^{(2)} & 0 & \Psi_{ud}^{(2)} \\
0 & 0 & 0 & 0 & 0 & 0 & P_{su}^{(2)} & P_{ss}^{(2)} & P_{sr}^{(2)} & P_{sd}^{(2)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Psi_{rs}^{(2)} & 0 & \Psi_{rd}^{(2)} \\
0 & 0 & 0 & \hat{\Lambda}_{ds}^{(b_1)} & 0 & \hat{\Lambda}_{dd}^{(b_1)} & \hat{\Psi}_{du}^{(b_1)} & \hat{\Psi}_{ds}^{(b_1)} & 0 & 0
\end{array}\right].$$

Next, we censor out the states in $E_n = \mathcal{S}_r^{(n)} \cup \mathcal{S}_u^{(n)} \cup \mathcal{S}_d^{(n)}$ and obtain an embedded Markov chain with transition matrix

$$\Omega^{(n-1)} = \Omega_{E_n^C}^{(n)} + \Omega_{E_n^C E_n}^{(n)} (I - \Omega_{E_n}^{(n)})^{-1} \Omega_{E_n E_n^C}^{(n)}$$

where $\Omega^{(n)} = \Phi$ and where E_n^C is the complementary of E_n and contains all the states at the thresholds c_0 to c_{n-1} plus the sticky states at level c_n.

We then censor out the states in $E_{n-1} = \mathcal{S}_r^{(n-1)} \cup \mathcal{S}_u^{(n-1)} \cup \mathcal{S}_d^{(n-1)}$ and obtain another embedded Markov chain with transition matrix

$$\Omega^{(n-2)} = \Omega_{E_{n-1}^C}^{(n-1)} + \Omega_{E_{n-1}^C E_{n-1}}^{(n-1)} (I - \Omega_{E_{n-1}}^{(n-1)})^{-1} \Omega_{E_{n-1} E_{n-1}^C}^{(n-1)}.$$

Now, E_{n-1}^C contains all the states at the thresholds c_0 to c_{n-2} plus the sticky states at levels c_{n-1} and c_n.

We repeat this construction until we are left with a Markov chain on the states $\mathcal{S}_s^{(0)} \cup \mathcal{S}_s^{(1)} \cup \ldots \cup \mathcal{S}_s^{(n)}$ only. Its transition matrix is denoted by $\Omega^{(0)}$ and is easily obtained by repeating the steps above.

Finally, in order to obtain the steady state probability mass vector

$$\boldsymbol{m} = (\boldsymbol{p}_s^{(0)}, \boldsymbol{p}_s^{(1)}, \ldots, \boldsymbol{p}_s^{(n)})$$

of the fluid queue, we need first to compute the invariant vector

$$\boldsymbol{x} = (\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)})$$

of the discrete time Markov chain with transition matrix $\Omega^{(0)}$; this vector is such that

$$\boldsymbol{x}\Omega^{(0)} = \boldsymbol{x}.$$

The fluid queue being time continuous, we have the following result, which gives the relationship between $\boldsymbol{x}$ and $\boldsymbol{m}$.

Theorem 4.7.3 *The probability mass vector $\boldsymbol{m}$ of the fluid queue with sticky and repulsive thresholds is equal to $c\boldsymbol{x}\Delta^{-1}$, where c is a normalizing factor and Δ is the block diagonal matrix*

$$\Delta = \begin{bmatrix} -T_{ss}^{(0)} & 0 & \ldots & 0 \\ 0 & -T_{ss}^{(1)} & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & -T_{ss}^{(n)} \end{bmatrix}.$$

The factor c is obtained via normalization: $\boldsymbol{m}\mathbf{1} + \int_0^\infty \boldsymbol{\pi}(x)\mathbf{1}\, dx = 1$.

□

5

Level-Phase Independence

We examine in this chapter whether it is possible to design a fluid queue such that its two components, the level and the phase, are independent under the stationary distribution. The answer turns out to be positive, for any given fluid queue, provided that one changes in a very specific manner its transition rules at the boundary level.

Our interest in this question arises from a similar result for QBD processes: Latouche and Taylor [31] show that it is always possible to define the boundary transition probabilities of a QBD process in such a way that the level and the phase are independent under the stationary distribution.

We show in Section 5.2 that the level and the phase of a standard fluid queue are asymptotically independent. In order to obtain the exact independence in Section 5.4, we need to eliminate the probability mass of level zero, as we explain in Section 5.3. Finally, we construct in Section 5.5 a fluid process that has the desired level-phase independence property.

The results of this chapter may also be found in da Silva Soares and Latouche [15].

5.1 Introduction

Consider a feedback fluid queue $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$ with an infinite capacity buffer. When the phase process $\{\varphi(t)\}$ is in state i, the net rate of input into the fluid buffer is given by r_i, which can take any real value except zero. At the end of Section 5.5, we shall formulate the main result of this chapter in the general case where the net input rates are allowed to take the value zero. Let C denote the diagonal matrix $\mathrm{diag}(\boldsymbol{r})$, with $r_i \neq 0$ for all i. We assume that the infinitesimal transition generator of the phase process is Q when $X(t) > 0$, and that it is $Q^{(0)}$ when $X(t) = 0$ and $\varphi(t)$ is in $\mathcal{S}_-$. Recall that the stationary density vector $\boldsymbol{\pi}(x)$ of the buffer content of this fluid queue satisfies the set of differential equations

$$-r_j \frac{d}{dx}\pi_j(x) + \sum_{i\in\mathcal{S}} \pi_i(x) Q_{ij} = 0; \tag{5.1}$$

it exists if and only if the drift $\mu = \boldsymbol{\xi}_+ \boldsymbol{r}_+ + \boldsymbol{\xi}_- \boldsymbol{r}_-$ of the fluid queue is negative. This condition is assumed throughout the chapter.

As we have seen in Sections 4.2 and 4.4, $\boldsymbol{\pi}(x)$ is given by

$$\boldsymbol{\pi}(x) = \boldsymbol{p}_- Q^{(0)}_{-+} e^{Kx} [C_+^{-1},\ \Psi |C_-|^{-1}], \tag{5.2}$$

for $x > 0$, where Ψ solves the Riccati equation

$$\Psi T_{-+} \Psi + T_{++}\Psi + \Psi T_{--} + T_{+-} = 0 \tag{5.3}$$

and gives the first return probabilities to the initial level, and where

$$K = T_{++} + \Psi T_{-+}, \tag{5.4}$$

with $T = C^{-1}Q$ being the phase transition generator of the restricted process with net input rates equal to $+1$ and -1. The steady state probability mass vector $\boldsymbol{p}_-$ of level zero is the unique solution of the system

$$\begin{aligned} \boldsymbol{p}_-(Q^{(0)}_{--} + Q^{(0)}_{-+}\Psi) &= \mathbf{0} \\ \boldsymbol{p}_-(\mathbf{1} - Q^{(0)}_{-+} K^{-1} [C_+^{-1},\ \Psi |C_-|^{-1}]\mathbf{1}) &= 1, \end{aligned} \tag{5.5}$$

where $Q^{(0)}_{--} + Q^{(0)}_{-+}\Psi$ is the infinitesimal generator of the censored Markov process obtained by observing $\{(X(t), \varphi(t))\}$ only when it is in $(0, \mathcal{S}_-)$.

We choose in this chapter to directly analyze a fluid queue with arbitrary non-null net input rates because general net input rates do not complicate matters much in this context. On the one hand, there are matrices that have a probabilistic interpretation, like K and Ψ, for example, and play a major role; on the other hand, there are matrices that are only present for normalization purposes, like C_+ and $|C_-|$, and are not much intrusive.

5.2 Asymptotic Independence

To prove the main result of this section, we first need a preliminary lemma.

Lemma 5.2.1 *There exists an eigenvalue ζ of K, which is real, strictly negative and which has geometric and algebraic multiplicities equal to one. It is maximal in the sense that every other eigenvalue of K has a real part strictly less than ζ.*

Furthermore, there exist real, strictly positive, left and right eigenvectors of K for the eigenvalue ζ, which we denote by $\boldsymbol{w}$ and $\boldsymbol{z}$ respectively, and which are normalized by $\boldsymbol{w}\mathbf{1} = \boldsymbol{w}\boldsymbol{z} = 1$.

Proof Consider the QBD process defined in the proof of Theorem 3.7.2 and obtained by restricting the infinite buffer fluid queue to those epochs when the level is a multiple of some quantity b. We have seen there that the R matrix of this QBD is

$$R = \begin{bmatrix} e^{Kb} & e^{Kb}\Psi \\ 0 & 0 \end{bmatrix} \tag{5.6}$$

and that, since $\mu < 0$, R has a maximal eigenvalue which is strictly less than one. Also, since the phase process of the fluid queue is irreducible, the matrix $A_0 + A_1 + A_2$, with A_0, A_1 and A_2 given by (3.45), is also irreducible, and the maximal eigenvalue of R has geometric and algebraic multiplicities equal to one (see Neuts [37] and Latouche and Taylor [31]). Therefore, by (5.6), the matrix e^{Kb} also has a maximal eigenvalue which is real, strictly less than one and which has geometric and algebraic multiplicities equal to one.

The Perron-Frobenius theory ensures the existence of the vectors $\boldsymbol{w}$ and $\boldsymbol{z}$ with the stated property; this completes the proof of the lemma. □

We immediately conclude that we may write

$$e^{Kx} = e^{\zeta x}\boldsymbol{z}\boldsymbol{w} + o(e^{\zeta x})$$

asymptotically as x goes to infinity (see, for example, Seneta [44, Theorem 2.7]). Replacing this expression in (5.2), the next theorem readily follows.

Theorem 5.2.2 *For the fluid queue* $\{(X(t), \varphi(t))\}$, *we have*

$$\boldsymbol{\pi}(x) = (\boldsymbol{p}_- Q^{(0)}_{-+}\boldsymbol{z})e^{\zeta x}\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}] + o(e^{\zeta x})$$

asymptotically as x goes to infinity, where ζ is the eigenvalue of K defined in Lemma 5.2.1, and $\boldsymbol{w}$, $\boldsymbol{z}$ are the corresponding left and right eigenvectors.

□

This means that the level is asymptotically independent of the phase as x goes to infinity: the conditional density vector $(\boldsymbol{\pi}(x)\mathbf{1})^{-1}\boldsymbol{\pi}(x)$ of the phase, given the level x, is asymptotically equal to

$$\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}]/(\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}]\mathbf{1}),$$

independently of x, as x becomes large.

5.3 Censoring Out Level Zero

Note that we cannot expect to have independence between the level and the phase under the stationary distribution if there is some steady state probability mass at level zero; indeed, when the level is zero, the phase cannot be in $\mathcal{S}_+$, it must be in $\mathcal{S}_-$.

Thus, we need to eliminate that mass and, in order to better understand what happens in such circumstances, we begin by censor out the intervals of time spent at level zero. We consider a standard fluid queue with an infinite buffer and we define the sequences of epochs $\{a_n\}$ at which the buffer becomes empty and $\{d_n\}$ at which it starts filling up:

$$a_0 = \inf\{t \geq 0 : X(t) = 0\},$$
$$d_n = \inf\{t > a_n : X(t) > 0\},$$

and

$$a_{n+1} = \inf\{t > d_n : X(t) = 0\},$$

for $n \geq 0$. We define the matrix Π_{-+} which records the conditional distribution of $\varphi(d_n)$, given $\varphi(a_n)$. Thus, for i in $\mathcal{S}_-$ and j in $\mathcal{S}_+$, we have that

$$\begin{aligned}[]
[\Pi_{-+}]_{ij} &= \mathrm{P}[\varphi(d_n) = j | \varphi(a_n) = i] \\
&= [(-Q^{(0)}_{--})^{-1}Q^{(0)}_{-+}]_{ij}. \qquad (5.7)
\end{aligned}$$

Next, we observe that the process $\{Y_n = \varphi(d_n) : n \geq 0\}$ is a discrete-time homogeneous Markov chain on the states of $\mathcal{S}_+$. The (i,j)th entry of its transition matrix P is given by

$$\begin{aligned} P_{ij} &= \mathrm{P}[Y_{n+1} = j | Y_n = i] \\ &= \sum_{k \in \mathcal{S}_-} \mathrm{P}[\varphi(a_{n+1}) = k | Y_n = i]\, \mathrm{P}[Y_{n+1} = j | \varphi(a_{n+1}) = k, Y_n = i] \\ &= \sum_{k \in \mathcal{S}_-} \Psi_{ik} [(-Q^{(0)}_{--})^{-1} Q^{(0)}_{-+}]_{kj} = (\Psi \Pi_{-+})_{ij} \end{aligned}$$

where Ψ is given by (5.3). The matrix P might not be irreducible but, owing to the assumption that the fluid queue is irreducible, it has a unique irreducible class, so that it has a unique stationary probability vector. Here, however, we only need an invariant measure, which we denote by $\boldsymbol{\alpha}$; it is the solution of the system $\boldsymbol{\alpha} P = \boldsymbol{\alpha}$ and is defined up to a multiplicative constant.

We can easily show that $\boldsymbol{p}_- Q^{(0)}_{-+}$, the left-most factor in the right-hand side of (5.2) is proportional to $\boldsymbol{\alpha}$, as follows. By (5.5), it follows that $\boldsymbol{p}_- Q^{(0)}_{-+} \Psi \Pi_{-+} = -\boldsymbol{p}_- Q^{(0)}_{--} \Pi_{-+}$ and, using (5.7), we immediately obtain

$$\boldsymbol{p}_- Q^{(0)}_{-+} \Psi \Pi_{-+} = \boldsymbol{p}_- Q^{(0)}_{-+}.$$

This is an important observation for what follows because it shows that the vector $\boldsymbol{p}_- Q^{(0)}_{-+}$ depends only on the matrix Π_{-+} and not on the specific behaviour of the phases at level zero; we show in Section 5.5 that we may choose that matrix (or, equivalently, the boundary behaviour) in a way which leads to the independence between the level and the phase of the fluid queue.

To censor out the intervals of time spent at level zero is equivalent to instantaneously switching to a phase in $\mathcal{S}_+$ as soon as the fluid queue reaches level zero, with probabilities given by the transition matrix Π_{-+}, making the level increase immediately.

5.4 Exact Independence

We denote by $\{(X^*(t), \varphi^*(t)) : t \in \mathbb{R}^+\}$ a fluid queue such that, when the process reaches level zero, with a phase in $\mathcal{S}_-$, the phase immediately switches to $\mathcal{S}_+$, with probabilities given by some matrix Υ_{-+}. It is a process without probability mass at level zero. Note that the constructed fluid queue is not of the canonical type in that the marginal phase process is not a continuous time Markov chain; also, for the special case where $\Upsilon_{-+} = \Pi_{-+}$, $\{(X^*(t), \varphi^*(t))\}$ is equivalent to the conditional fluid process, given that the level is strictly positive. We give the stationary distribution of $\{(X^*(t), \varphi^*(t))\}$ in the following lemma.

Lemma 5.4.1 *Denote by $\{(X^*(t), \varphi^*(t)) : t \in \mathbb{R}^+\}$ the fluid process with rate vector $\boldsymbol{r}$ and infinitesimal generator Q, which is instantaneously restarted at level zero with the matrix Υ_{-+}.*

Its stationary distribution has no mass at zero and its stationary density is given by

$$\boldsymbol{\pi}(x) = \boldsymbol{\alpha} e^{Kx}[C_+^{-1},\ \Psi|C_-|^{-1}],$$

for $x \geq 0$, with

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}\Psi\Upsilon_{-+}$$

normalized by $\boldsymbol{\alpha}(-K)^{-1}[C_+^{-1},\ \Psi|C_-|^{-1}]\mathbf{1} = 1$.

□

We now give a necessary and sufficient condition for the level-phase independence of the fluid queue without probability mass at level zero.

Theorem 5.4.2 *For the fluid queue $\{(X^*(t), \varphi^*(t))\}$, the level and the phase are independent under the stationary distribution and*

$$\boldsymbol{\pi}(x) = \zeta e^{\zeta x}\boldsymbol{l},$$

for some row vector $\boldsymbol{l}$, if and only if $\boldsymbol{\alpha} = c\boldsymbol{w}$, where c is a constant, $\boldsymbol{\alpha}$ is a stationary measure of the matrix $\Psi\Upsilon_{-+}$ and $\boldsymbol{w}$ is the left eigenvector of K corresponding to its maximal eigenvalue ζ. The constant c is equal to $-\zeta/(\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}]\mathbf{1})$.

Proof Recall that the process $\{(X^*(t), \varphi^*(t))\}$ has no probability mass at level zero, thus $\boldsymbol{p}_- = \mathbf{0}$.

It is easy to verify that, if $\boldsymbol{\alpha}$ is proportional to $\boldsymbol{w}$, then

$$\begin{aligned} \boldsymbol{\pi}(x) &= c\boldsymbol{w}e^{Kx}[C_+^{-1},\ \Psi|C_-|^{-1}] \\ &= ce^{\zeta x}\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}] \end{aligned}$$

for some constant c, which shows that the level is independent of the phase. To compute the factor c, we integrate $\boldsymbol{\pi}(x)\mathbf{1}$ with respect to x over $(0, +\infty)$; this integral must be equal to one, and hence

$$c\int_0^\infty e^{\zeta x}dx\ \boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}]\mathbf{1} = 1.$$

The integral converges because $\zeta < 0$ (see Lemma 5.2.1) and thus we obtain $c = -\zeta/(\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}]\mathbf{1})$ and we have proved the sufficiency part of the claim.

Consider now the measure $\boldsymbol{\tau}(x) = ce^{\zeta x}\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}]$, for which the level is independent of the phase. We show that it is an invariant measure for the fluid queue, that is, it satisfies the set of differential equations (5.1).

Replacing $\boldsymbol{\pi}(x)$ by $\boldsymbol{\tau}(x)$ in the left-hand side of (5.1) yields the expression

$$-r_j\zeta e^{\zeta x}(\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}])_j + \sum_{i\in\mathcal{S}} e^{\zeta x}(\boldsymbol{w}[C_+^{-1},\ \Psi|C_-|^{-1}])_i Q_{ij},$$

which is equal to zero if and only if

$$-[\zeta\boldsymbol{w}C_+^{-1},\ \zeta\boldsymbol{w}\Psi|C_-|^{-1}]\mathrm{diag}(\boldsymbol{r}) + [\boldsymbol{w}C_+^{-1},\ \boldsymbol{w}\Psi|C_-|^{-1}]Q = \mathbf{0}.$$

This is equivalent to

$$[-\zeta\boldsymbol{w},\ \zeta\boldsymbol{w}\Psi] + [\boldsymbol{w}C_+^{-1},\ \boldsymbol{w}\Psi|C_-|^{-1}]Q = \mathbf{0}$$

since $C = \mathrm{diag}(\boldsymbol{r})$. Using the fact that $T = C^{-1}Q$, we see that this holds if and only if the two vectors

$$-\zeta\boldsymbol{w} + \boldsymbol{w}T_{++} + \boldsymbol{w}\Psi T_{-+}$$

and

$$\zeta\boldsymbol{w}\Psi + \boldsymbol{w}T_{+-} + \boldsymbol{w}\Psi T_{--}$$

are equal to zero. The first one is equal to zero since $(\zeta, \boldsymbol{w})$ is an (eigenvalue, eigenvector) pair of K by assumption. The second vector may be written as $\boldsymbol{w}(T_{+-} + \Psi T_{--} + K\Psi)$, which is equal to zero by (5.3, 5.4). The proof is therefore complete. □

If one should take $\Upsilon_{-+} = \mathbf{1}\boldsymbol{w}$, then clearly $\boldsymbol{\alpha}$ would be equal to $\boldsymbol{w}$ and Theorem 5.4.2 would apply. In that case, successive excursions above level zero would be independent since the phase upon restart is selected anew according to the distribution $\boldsymbol{w}$.

In the next section, we give another, less elementary example, where the phases before and after the restart are related in a way which is, to the extent possible, determined by the dynamics of the original fluid queue.

5.5 Construction

We now construct a fluid queue with no probability mass associated to the boundary level, and with a behaviour at level zero which makes the level independent of the phase. This construction is based on the one in Latouche and Taylor [31], and we start by recalling their construction, which serves as our inspiration.

Consider a homogeneous QBD process, with transition matrix

$$\begin{bmatrix} B & A_0 & 0 & 0 & \dots \\ A_2 & A_1 & A_0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & \dots \\ 0 & 0 & A_2 & A_1 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Let R be the matrix recording the expected number of visits to level 1 before a return to level zero, given that the process starts from level zero, already defined in Section 2.1. Assumption 1 in [31] requires that the transition matrix

$$\begin{bmatrix} \ddots & \ddots & & & & \\ \dots & A_1 & A_0 & 0 & 0 & \\ \dots & A_2 & A_1 & A_0 & 0 & \\ \dots & 0 & A_2 & A_1 & A_0 & \\ \dots & 0 & 0 & A_2 & A_1 & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5.8}$$

of the doubly infinite process on the set $\{(n,j) : n \in \mathbb{Z}, 1 \leq j \leq m\}$ is irreducible. This assumption suffices to ensure the existence of a maximal eigenvalue η for the matrix R, such that $0 < \eta < 1$ and with algebraic and geometric multiplicities equal to one. We call η the *Perron-Frobenius* eigenvalue of R, and we denote by $\boldsymbol{u}$ the unique normalized left eigenvector of R, corresponding to η.

It is shown in Ramaswami and Taylor [40] that, for a QBD process, the level is exactly independent of the phase if and only if

$$\boldsymbol{\pi}_0 = (1-\eta)\boldsymbol{u},$$

where $\boldsymbol{\pi}_0$ is the steady state probability row vector of level zero. This is equivalent to our Theorem 5.4.2.

We have recalled in Theorem 2.1.5, that $\boldsymbol{\pi}_0$ is a solution of the system $\boldsymbol{\pi}_0(B + A_0G) = \boldsymbol{\pi}_0$ where G is the minimal nonnegative solution of the matrix equation

$$G = A_2 + A_1G + A_0G^2.$$

Thus, we have the exact level-phase independence if and only if

$$\boldsymbol{u}(B + A_0G) = \boldsymbol{u}.$$

This will be used in the proof of the next theorem.

In [31], the QBD process for which the level is independent of the phase is obtained by choosing the transition matrix B within the states of level zero in the following way:

$$B = A_1 + A_2[H + (I - H)\mathbf{1}\boldsymbol{h}],$$

where H is the minimal nonnegative solution of

$$H = A_0 + A_1H + A_2H^2 \tag{5.9}$$

and $\boldsymbol{h}$ is the normalized Perron-Frobenius eigenvector of H, with the same maximal eigenvalue η as R. The matrix H has the following interpretation: assume that the level of the QBD is allowed to take negative as well as positive values and that its transition matrix is given by (5.8); the matrix H records the first passage probabilities from level -1 to level zero.

Now, take the QBD process associated in Chapter 2 to our fluid queue. Its transition matrices are

$$A_0 = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} \frac{1}{2}P_{++} & 0 \\ P_{-+} & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & \frac{1}{2}P_{+-} \\ 0 & P_{--} \end{bmatrix}, \tag{5.10}$$

where $P = I + 1/\mu T$ for some $\mu \geq \max_{i \in \mathcal{S}} |T_{ii}|$. We know from Section 2.2 that the matrix G of this process is

$$G = \begin{bmatrix} 0 & \Psi \\ 0 & V \end{bmatrix}, \tag{5.11}$$

where Ψ is defined by (5.3), and $V = P_{--} + P_{-+}\Psi$. The matrix H of this QBD is given in the following lemma.

Lemma 5.5.1 *For the QBD process with transition matrices (5.10), we have that*

$$H = \begin{bmatrix} W & 0 \\ \Gamma & 0 \end{bmatrix} \tag{5.12}$$

where Γ *is the minimal nonnegative solution of*

$$T_{-+} + T_{--}\Gamma + \Gamma T_{++} + \Gamma T_{+-}\Gamma = 0 \tag{5.13}$$

and

$$W = 1/2\,[I - 1/2\,P_{++} - 1/2\,P_{+-}\Gamma]^{-1}. \tag{5.14}$$

The Perron-Frobenius eigenvector $\boldsymbol{h}$ *of* H *is given by* $\boldsymbol{h} = (\boldsymbol{h}_+, \mathbf{0})$, *where* $\boldsymbol{h}_+$ *is the Perron-Frobenius eigenvector of* W *for the eigenvalue* η, *that is,* $\boldsymbol{h}_+ W = \eta \boldsymbol{h}_+$.

Proof In view of the value of A_0, it is obvious that the columns of H with indices in $\mathcal{S}_-$ must be equal to zero. This proves that H has the structure (5.12).

If we replace H by its expression (5.12) in (5.9), we find after some simple manipulations that

$$\begin{aligned} [I - 1/2\,P_{++} - 1/2\,P_{+-}\Gamma]W &= 1/2\,I && (5.15)\\ \Gamma &= P_{-+}W + P_{--}\Gamma W. && (5.16) \end{aligned}$$

It now suffices to verify that Γ and W given by (5.13, 5.14) are indeed a solution of (5.15, 5.16). We omit the details and only recall that $P_{++} + P_{+-}\Gamma$ is a sub-stochastic matrix, so that its spectral radius is at most 1, which implies that $I - 1/2\,P_{++} - 1/2\,P_{+-}\Gamma$ is nonsingular.

The structure of $\boldsymbol{h}$ is an immediate consequence of the structure of H, and simple computations lead to $\boldsymbol{h}_+ W = \eta \boldsymbol{h}_+$. □

We can now state the main result of this chapter.

Theorem 5.5.2 *Let* Ψ *and* Γ *be the minimal nonnegative solutions of the equations (5.3) and (5.13), respectively. Define the matrix*

$$\Gamma^* = \Gamma + (\mathbf{1} - \Gamma\mathbf{1})\boldsymbol{h}_+ \tag{5.17}$$

where $\boldsymbol{h}_+$ *is such that* $\boldsymbol{h}_+ W = \eta \boldsymbol{h}_+$ *and* $\boldsymbol{h}_+\mathbf{1} = 1$.

Consider the fluid queue $\{(X^*(t), \varphi^*(t))\}$ *with phase transition generator* Q, *which is instantaneously restarted at level zero with the matrix* $\Upsilon_{-+} = \Gamma^*$. *The level and the phase of the process* $\{(X^*(t), \varphi^*(t))\}$ *are independent under the stationary distribution.*

Proof By Lemma 5.4.1 and Theorem 5.4.2, we need to show that the eigenvector $\boldsymbol{\alpha}$ of $\Psi\Gamma^*$ is also an eigenvector of K. This we do by computing the Perron-Frobenius eigenvector $\boldsymbol{u} = (\boldsymbol{u}_+, \boldsymbol{u}_-)$ of the matrix R

for the QBD with transition matrices (5.10), and by proving that $\boldsymbol{u}_+$ is an eigenvector of both K and $\Psi\Gamma^*$.

By Theorem 2.1.3, the matrix R is given by $R = A_0(I - A_1 - A_0G)^{-1}$. Using (5.10, 5.11), we obtain that

$$I - A_1 - A_0G = \begin{bmatrix} I - \frac{1}{2}P_{++} & -\frac{1}{2}\Psi \\ -P_{-+} & I \end{bmatrix}.$$

To compute its inverse, we use the fact that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & D^{-1} \end{bmatrix} + \begin{bmatrix} I \\ -D^{-1}C \end{bmatrix} \Delta_D^{-1} \begin{bmatrix} I & -BD^{-1} \end{bmatrix},$$

where $\Delta_D = A - BD^{-1}C$ is called the Schur complement of D (see [27, Section A.1]) and we obtain, after simple computations, that

$$(I - A_1 - A_0G)^{-1} = \begin{bmatrix} (I - \frac{1}{2}(P_{++} + \Psi P_{-+}))^{-1} & \frac{1}{2}(I - \frac{1}{2}(P_{++} + \Psi P_{-+}))^{-1}\Psi \\ P_{-+}(I - \frac{1}{2}(P_{++} + \Psi P_{-+}))^{-1} & I + \frac{1}{2}P_{-+}(I - \frac{1}{2}(P_{++} + \Psi P_{-+}))^{-1}\Psi \end{bmatrix}.$$

Pre-multiplying by A_0, we finally get

$$R = \begin{bmatrix} (I - \frac{1}{\mu}K)^{-1} & \frac{1}{2}(I - \frac{1}{\mu}K)^{-1}\Psi \\ 0 & 0 \end{bmatrix}, \tag{5.18}$$

using the fact that

$$P_{++} + \Psi P_{-+} = I + 1/\mu K \tag{5.19}$$

by (5.4) and since $P = I + 1/\mu T$.

The equation $\boldsymbol{u}R = \eta\boldsymbol{u}$, with $\boldsymbol{u} = (\boldsymbol{u}_+, \boldsymbol{u}_-)$, is decomposed into two equations: $\boldsymbol{u}_+ R_{++} = \eta\boldsymbol{u}_+$, where

$$R_{++} = (I - 1/\mu K)^{-1}, \tag{5.20}$$

which shows that $\boldsymbol{u}_+$ is also a left eigenvector of K, and

$$\frac{1}{2}\boldsymbol{u}_+\Psi = \boldsymbol{u}_- \tag{5.21}$$

which we will use later.

Take the QBD process with transition matrices (5.10) and choose B such that

$$\begin{aligned} B &= A_1 + A_2[H + (I - H)\mathbf{1}\boldsymbol{h}] \\ &= \begin{bmatrix} \frac{1}{2}P_{++} & 0 \\ P_{-+} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}P_{+-} \\ 0 & P_{--} \end{bmatrix} \begin{bmatrix} W + (\mathbf{1} - W\mathbf{1})\boldsymbol{h}_+ & 0 \\ \Gamma + (\mathbf{1} - \Gamma\mathbf{1})\boldsymbol{h}_+ & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(P_{++} + P_{+-}\Gamma^*) & 0 \\ P_{-+} + P_{--}\Gamma^* & 0 \end{bmatrix}. \end{aligned}$$

This QBD process has the level-phase independence property and, therefore, we know that $\boldsymbol{u}$ satisfies $\boldsymbol{u}(B + A_0G) = \boldsymbol{u}$, with

$$B + A_0G = \begin{bmatrix} \frac{1}{2}(P_{++} + P_{+-}\Gamma^*) & \frac{1}{2}\Psi \\ P_{-+} + P_{--}\Gamma^* & 0 \end{bmatrix}$$

in the present context. We thus obtain a system of two equations, one of which is

$$\frac{1}{2}\boldsymbol{u}_+(P_{++} + P_{+-}\Gamma^*) + \boldsymbol{u}_-(P_{-+} + P_{--}\Gamma^*) = \boldsymbol{u}_+ \tag{5.22}$$

and the other is (5.21) which we had before. After replacing $\boldsymbol{u}_-$ in (5.22) by the left-hand side of (5.21) and rearranging some terms, one gets

$$\boldsymbol{u}_+[P_{++} + \Psi P_{-+} + (P_{+-} + \Psi P_{--})\Gamma^*] = 2\boldsymbol{u}_+.$$

This is equivalent to

$$\frac{1}{\eta}\boldsymbol{u}_+\left[R_{++}(P_{++} + \Psi P_{-+}) + R_{++}(P_{+-} + \Psi P_{--})\Gamma^*\right] = 2\boldsymbol{u}_+, \tag{5.23}$$

using the fact that $\boldsymbol{u}_+ = 1/\eta\boldsymbol{u}_+R_{++}$. From (5.19, 5.20), we obtain

$$\begin{aligned} R_{++}(P_{++} + \Psi P_{-+}) &= (I - \frac{1}{\mu}K)^{-1} + (I - \frac{1}{\mu}K)^{-1}\frac{1}{\mu}K \\ &= 2(I - \frac{1}{\mu}K)^{-1} - I \\ &= 2R_{++} - I. \end{aligned} \tag{5.24}$$

Furthermore, $RA_2 = A_0G$ by Theorem 2.1.3, which gives us the equation

$$R_{++}(P_{+-} + \Psi P_{--}) = \Psi. \tag{5.25}$$

Combining (5.23, 5.24, 5.25), we find that

$$\frac{1}{\eta}\boldsymbol{u}_+(2R_{++} - I + \Psi\Gamma^*) = 2\boldsymbol{u}_+,$$

which leads to

$$\boldsymbol{u}_+\Psi\Gamma^* = \boldsymbol{u}_+. \tag{5.26}$$

This completes the proof. □

In order to take care of the general case, we must allow in addition to $\mathcal{S}_+$ and $\mathcal{S}_-$ a subset $\mathcal{S}_0$ of phases such that the fluid remains constant: $r_i = 0$ for i in $\mathcal{S}_0$. The infinitesimal generator for the phase process away from the boundary now becomes

$$Q = \begin{bmatrix} Q_{++} & Q_{+-} & Q_{+0} \\ Q_{-+} & Q_{--} & Q_{-0} \\ Q_{0+} & Q_{0-} & Q_{00} \end{bmatrix}. \tag{5.27}$$

Except for minor changes, the previous results are preserved and we state the following general version of our main theorem.

Theorem 5.5.3 *Consider the fluid queue with phase transition generator (5.27) and net input rates $r_i > 0$ for i in $\mathcal{S}_+$, $r_i < 0$ for i in $\mathcal{S}_-$, and $r_i = 0$ for i in $\mathcal{S}_0$. Define the matrix T on $\mathcal{S}_+ \cup \mathcal{S}_-$ as*

$$T = \begin{bmatrix} C_+^{-1} & 0 \\ 0 & |C_-|^{-1} \end{bmatrix} \begin{bmatrix} Q_{++} + Q_{+0}(-Q_{00})^{-1}Q_{0+} & Q_{+-} + Q_{+0}(-Q_{00})^{-1}Q_{0-} \\ Q_{-+} + Q_{-0}(-Q_{00})^{-1}Q_{0+} & Q_{--} + Q_{-0}(-Q_{00})^{-1}Q_{0-} \end{bmatrix},$$

where $C = \mathrm{diag}(r_i : i \in \mathcal{S}_+ \cup \mathcal{S}_-)$. Define the matrices K, Ψ, Γ, W and Γ^* respectively by (5.4, 5.3, 5.13, 5.14) and (5.17).

If, whenever the process reaches level zero, the phase immediately switches to $\mathcal{S}_+$ with probabilities given by Γ^*, then the level and the phase are independent under the stationary distribution and its stationary density is given by

$$\begin{aligned} &\left[\boldsymbol{\pi}_+(x), \quad \boldsymbol{\pi}_-(x), \quad \boldsymbol{\pi}_0(x)\right] \\ &\quad = \quad e^{\zeta x}\boldsymbol{\alpha}\left[C_+^{-1}, \quad \Psi|C_-|^{-1}, \quad [C_+^{-1}Q_{+0} + \Psi|C_-|^{-1}Q_{-0}](-Q_{00})^{-1}\right], \end{aligned}$$

where ζ is the eigenvalue of maximal real part of K and $\boldsymbol{\alpha}$ is the stationary probability vector of the matrix $\Psi\Gamma^*$, normalized by

$$-\frac{1}{\zeta}\boldsymbol{\alpha}\{C_+^{-1}\mathbf{1} + \Psi|C_-|^{-1}\mathbf{1} + [C_+^{-1}Q_{+0} + \Psi|C_-|^{-1}Q_{-0}](-Q_{00})^{-1}\mathbf{1}\} = 1.$$

Proof The proof is quite simple. By censoring out the phases in $\mathcal{S}_0$, we find that

$$\boldsymbol{\pi}_0(x) = [\boldsymbol{\pi}_+(x)Q_{+0} + \boldsymbol{\pi}_-(x)Q_{-0}](-Q_{00})^{-1}$$

so that, if the vectors $\boldsymbol{\pi}_+(x)$ and $\boldsymbol{\pi}_-(x)$ are factored into a scalar function of x multiplied by a constant vector, so is $\boldsymbol{\pi}_0(x)$. Then, we only need to replace, in the definition of T, the matrix Q by the transition matrix of the censored process. □

Appendix A

Markov Processes and PH Distributions

We recall here the definitions of both discrete time and continuous time Markov processes, as well as some related properties which are often used throughout the text. We also define PH random variables. The interested reader may find material on this topics in Latouche and Ramaswami [29], Norris [38] or Resnick [41], among others.

A.1 Markov Processes

Consider a discrete time stochastic process $\{X_n : n \in \mathbb{N}\}$; it is a family of random variables, indexed by $\mathbb{N}$ and such that $X_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. The set $\mathcal{S}$ is called the state space of the process and is assumed to be denumerable.

Definition A.1.1 *The stochastic process $\{X_n : n \in \mathbb{N}\}$ is a Markov chain if and only if it satisfies the Markov property*

$$P[X_{n+1} = j | X_0, X_1, \ldots, X_n] = P[X_{n+1} = j | X_n]$$

for all $n \in \mathbb{N}$ and $j \in \mathcal{S}$.

The Markov chain is homogeneous if

$$P[X_{n+1} = j | X_n = i] = P[X_1 = j | X_0 = i]$$

for all $n \in \mathbb{N}$ and $i, j \in \mathcal{S}$.

In the continuous time case, we consider a collection of random variables $\{X(t) : t \in \mathbb{R}^+\}$ where for all $t \in \mathbb{R}^+$, $X(t) \in \mathcal{S}$. The state space $\mathcal{S}$ is assumed to be denumerable.

Definition A.1.2 *The stochastic process $\{X(t) : t \in \mathbb{R}^+\}$ is a Markov process if and only if it satisfies the Markov property*

$$P[X(t+s) = j|X(u), 0 \leq u \leq t] = P[X(t+s) = j|X(t)]$$

for all $s, t \in \mathbb{R}^+$ and $j \in \mathcal{S}$. It is homogeneous if, in addition,

$$P[X(t+s) = j|X(s) = i] = P[X(t) = j|X(0) = i]$$

for all $s, t \in \mathbb{R}^+$ and $i, j \in \mathcal{S}$.

Throughout this work, we essentially deal with continuous time Markov processes, and they are all homogeneous. This is the reason why we only consider this case from now on.

Let $\{X(t) : t \in \mathbb{R}^+\}$ be a homogeneous Markov process. The vector $\boldsymbol{\alpha} = (\alpha_i : i \in \mathcal{S})$ such that $\alpha_i = \mathrm{P}[X(0) = i]$ for i in $\mathcal{S}$ is called the *initial probability vector* of the process.

For i, j in $\mathcal{S}$ and $t \in \mathbb{R}^+$, the transition functions

$$P_{ij}(t) = \mathrm{P}[X(t) = j|X(0) = i]$$

are the solution of the *forward Kolmogorov* equations:

$$\frac{dP(t)}{dt} = P(t)Q,$$

with $P(0) = I$, where the matrix $P(t)$ contains the elements $P_{ij}(t)$ for all $i, j \in \mathcal{S}$ and where the coefficient matrix Q is called the *infinitesimal transition generator* of the process. The interpretation of the entries of the matrix Q is the following.

- For $i \neq j$, Q_{ij} is the instantaneous transition rate from state i to state j. In other words,

  $$Q_{ij}h = \mathrm{P}[X(t+h) = j|X(t) = i] + o(h),$$

  where $o(h)$ has the usual meaning $\lim_{h\to 0} o(h)/h = 0$, thus $Q_{ij}h$ is the probability that the process leaves state i before time $t+h$, starting from state i at time t, and enters j. The entries Q_{ij} are nonnegative and are strictly positive if it is possible to move from i to j in one step.

- The process stays is state i during an interval of time which is exponentially distributed with parameter $q_i = -Q_{ii}$. If $q_i = 0$, then the process stays forever in state i once it has reached this state; i is then called an *absorbing* state. On the diagonal, $Q_{ii} = -\sum_{j \in \mathcal{S}, j \neq i} Q_{ij}$.

Denote by $N_{t,t+h}$ the number of state changes of the Markov process $\{X(t)\}$ in the interval $[t, t+h]$; one has the following characterization:

$$\begin{aligned} \mathrm{P}[N_{t,t+h} = 0 | X(t) = i] &= 1 - q_i h + o(h), \\ \mathrm{P}[N_{t,t+h} = 1 | X(t) = i] &= q_i h + o(h) \end{aligned}$$

and

$$\mathrm{P}[N_{t,t+h} \geq 2 | X(t) = i] = o(h),$$

for some small h and for any i in $\mathcal{S}$. These are easily proved using the fact that the sojourn time in any state i is exponentially distributed with parameter q_i.

We associate an *oriented transition graph* to a Markov process in the following way. The nodes of the graph represent the states of the Markov process, and there is a directed arc from state i to state j, denoted by (i, j), if and only if $Q_{ij} > 0$. A *path* from i to j is a finite sequence of directed arcs $(i, i_1), (i_1, i_2), \ldots, (i_n, j)$.

Example A.1.3 *Birth-and-Death process*
We give here an example of a Markov process on the state space $\mathcal{S} = \mathbb{N}$. Consider a population such that, when there are i individuals present in the population:

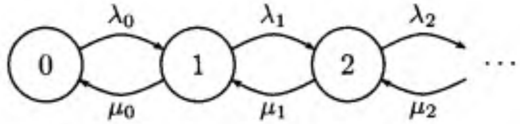
- a new individual is born after a random interval of time exponentially distributed with parameter λ_i,

- one individual will die after an interval of time which is random and exponentially distributed with parameter μ_i; we set $\mu_0 = 0$.

The infinitesimal transition generator Q of such a Markov process is

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \ldots & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \ldots & \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \ldots & \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

One may associate with this process the following oriented transition graph:



We say that state l is *accessible* from state k if there is a path that leads from k to l in the transition graph.

Definition A.1.4 *The Markov process $\{X(t) : t \in \mathbb{R}^+\}$ on the state space $\mathcal{S}$ is irreducible if, for all i, j in $\mathcal{S}$, i is accessible from j and j is accessible from i.*

For j in $\mathcal{S}$, we denote by T_j the first passage time to state j, that is,

$$T_j = \inf\{t > 0 : X(t) = j, X(t-) \neq j\}$$

where $X(t-) = \lim_{u \to t,\ u<t} X(u)$. The first passage probability from state i to state j is

$$F_{ij} = \mathrm{P}[T_j < \infty | X(0) = i].$$

Definition A.1.5 *A state j is positive recurrent if and only if $F_{jj} = 1$ and $E[T_j|X(0) = j] < \infty$.*

Corollary A.1.6 *If the Markov process $\{X(t) : t \in \mathbb{R}^+\}$ is irreducible and if one of its states is positive recurrent, then all the states are positive recurrent and $F_{ij} = 1$ for all i, j in $\mathcal{S}$. In that case, the process itself is said to be positive recurrent.*

□

Remark A.1.7 A positive recurrent Markov process is sometimes also called *ergodic*.

The following theorem gives a necessary and sufficient condition for the process to be positive recurrent.

Theorem A.1.8 *Assume that the Markov process $\{X(t) : t \in \mathbb{R}^+\}$ is irreducible and non exploding, and denote by Q its transition generator. It is positive recurrent if and only if there exists a probability row vector $\boldsymbol{\pi}$ such that $\pi_i > 0$ for all i in $\mathcal{S}$, and such that $\boldsymbol{\pi}$ solves the system*

$$\begin{cases} \boldsymbol{\pi} Q &= \mathbf{0} \\ \boldsymbol{\pi} \mathbf{1} &= 1 \end{cases}.$$

This vector $\boldsymbol{\pi}$ is unique and is such that

$$\pi_j = \lim_{t\to\infty} P[X(t) = j | X(0) = i],$$

for all i and j in $\mathcal{S}$.

□

The vector $\boldsymbol{\pi}$ is called the *stationary* probability vector of the process. Other terms used are *invariant, steady state, equilibrium* or *asymptotic.*

In case the process is not positive recurrent, we have that

$$\lim_{t\to\infty} \mathrm{P}[X(t) = j | X(0) = i] = 0$$

for all $i, j \in \mathcal{S}$.

A.2 Poisson Processes and the M/M/1 Queue

Consider a continuous-time stochastic process $\{N(t) : t \in \mathbb{R}^+\}$ with state space $\mathcal{S} = \mathbb{N}$. One definition of a *Poisson process $\{N(t)\}$ with parameter* λ is that it is a Markov process such that

- $N(0) = 0$,
- the only transitions allowed are from some state i to $i + 1$, with $i \in \mathbb{N}$,
- the sojourn times in each state are exponentially distributed with parameter λ, independently of the state.

The generator of $\{N(t)\}$ is thus given by

$$\begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \\ & & & \end{bmatrix}.$$

An important feature of the Poisson process is that the number of events of the process in the interval $(0, t)$ is Poisson distributed. More specifically, for the process $\{N(t)\}$, we have that

$$\mathrm{P}[N(t) = k] = e^{-\lambda t}\frac{(\lambda t)^k}{k!},$$

for $k \geq 0$, thus the number of events in $(0, t)$ is Poisson distributed with parameter λt.

The Poisson process is widely used to model customer arrivals to a queueing system. The simplest one is the so-called M/M/1 queue which can be described as follows. Customers arrive in the system according to a Poisson process with rate λ. There is one server who serves at rate μ, that is, service times are independent and identically distributed exponential random variables with parameter μ, independent of the arrival process. The generator of the M/M/1 queue is

$$\begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \mu & -(\lambda+\mu) & \lambda & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}.$$

Note that it is a special case of the Birth-and-Death process defined in Example A.1.3.

A.3 Censored Markov Processes

Consider now two proper subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$, that is, $\mathcal{S} = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. We partition the generator Q accordingly as

$$Q = \begin{bmatrix} Q_{\mathcal{AA}} & Q_{\mathcal{AB}} \\ Q_{\mathcal{BA}} & Q_{\mathcal{BB}} \end{bmatrix}.$$

Thus, $Q_{\mathcal{AA}}$ contains the components Q_{ij} such that $i, j \in \mathcal{A}$, $Q_{\mathcal{AB}}$ contains the components Q_{ij} such that $i \in \mathcal{A}$ and $j \in \mathcal{B}$, and so forth. We also partition the stationary probability vector $\boldsymbol{\pi}$ according to the decomposition of $\mathcal{S}$, and write that $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\mathcal{A}}, \boldsymbol{\pi}_{\mathcal{B}})$ where $\boldsymbol{\pi}_{\mathcal{A}} = (\pi_i : i \in \mathcal{A})$ and $\boldsymbol{\pi}_{\mathcal{B}} = (\pi_i : i \in \mathcal{B})$.

Theorem A.3.1 *Assume that the Markov process $\{X(t) : t \in \mathbb{R}^+\}$ is irreducible and let $\boldsymbol{\pi}$ be its stationary probability vector. Let $\mathcal{A}$ and $\mathcal{B}$ be* *two proper sets of $\mathcal{S}$. One has*

$$\boldsymbol{\pi}_{\mathcal{B}} = \boldsymbol{\pi}_{\mathcal{A}} Q_{\mathcal{AB}} N_{\mathcal{B}}$$

where $N_{\mathcal{B}}$ records the expected sojourn time in the states of $\mathcal{B}$ given an initial state in $\mathcal{B}$, before the first visit to $\mathcal{A}$.

□

Definition A.3.2 *The censored process restricted to the set $\mathcal{A}$ is obtained by removing from the original Markov process all the intervals of time during which it is in $\mathcal{B}$; it is denoted by $\{X^{\mathcal{A}}(t) : t \in \mathbb{R}^+\}$.*

The following theorem gives the characterization of the censored process restricted to the set $\mathcal{A}$.

Theorem A.3.3 *Let $\{X(t) : t \in \mathbb{R}^+\}$ be an irreducible and positive recurrent Markov process on the state space $\mathcal{S}$, with generator Q. Let $\mathcal{A}$ and $\mathcal{B}$ be two proper subsets of $\mathcal{S}$.*

The restricted process $\{X^{\mathcal{A}}(t) : t \in \mathbb{R}^+\}$ is an irreducible and positive recurrent Markov process on the states of $\mathcal{A}$. Its generator is given by

$$Q^* = Q_{\mathcal{AA}} + Q_{\mathcal{AB}}(-Q_{\mathcal{BB}})^{-1} Q_{\mathcal{BA}}.$$

Its stationary probability vector is proportional to $\boldsymbol{\pi}_{\mathcal{A}}$, thus

$$\boldsymbol{\pi}_{\mathcal{A}} Q^* = \mathbf{0}.$$

□

A.4 Phase-Type Distributions

Consider a Markov process on the state space $\mathcal{S} = \{0, 1, \ldots, n\}$ with initial probability vector $(\tau_0, \boldsymbol{\tau})$ and infinitesimal generator

$$Q = \begin{bmatrix} 0 & \mathbf{0} \\ \boldsymbol{t} & T \end{bmatrix}$$

where $\boldsymbol{\tau}$ is $1 \times n$, T is $n \times n$ and $\boldsymbol{t}$ is $n \times 1$. The state 0 is absorbant, while all the other states are transient and lead to 0.

Definition A.4.1 *The distribution of the time X until absorption into the absorbing state 0 is called a phase-type distribution with representation $(\boldsymbol{\tau}, T)$. We say that X is PH$(\boldsymbol{\tau}, T)$.*

The following theorem gives the distribution and density functions of a PH random variable, as well as its moments.

Theorem A.4.2 *Assume that X is PH$(\boldsymbol{\tau}, T)$. Its distribution function is given by*

$$F(x) = 1 - \boldsymbol{\tau} e^{Tx}\mathbf{1}, \quad \text{for } x \geq 0,$$

and its density function is given by

$$f(x) = \boldsymbol{\tau} e^{Tx}\mathbf{t}, \quad \text{for } x > 0.$$

Its moments are given by

$$E[X^k] = k!\boldsymbol{\tau}(-T^{-1})^k\mathbf{1}, \quad \text{for } k \geq 1.$$

□

Remark A.4.3 The matrix exponential is defined as $e^M = \sum_{n\geq 0} \frac{M^n}{n!}$.

Conclusion and Perspectives

It is now apparent that the renewal approach to fluid queues, based on the analogy with QBD processes, allows for the unified and straightforward analysis of models which exhibit a variety of features.

Very interesting questions arise from the results presented in this work. The first one is to study feedback fluid queues with thresholds, as those presented in Sections 4.6 and 4.7, but allowing the net input rates to take any real value. The development is quite straightforward regarding the model of Section 4.6, but somewhat more involved for the model of Section 4.7.

In Chapter 5, we constructed a fluid queue with stationary independence between the level and the phase; we obtained this at the cost of removing the probability mass at level zero. One can construct other fluid queues which may exhibit the same independence property, for example by adding some probability mass on the states $(0, \mathcal{S}_+)$. This construction requires a major modification of fluid queues.

We have exploited in [6] and [48] the relationship between risk processes and fluid queues, and obtained efficient algorithms for computing the probability of ruin under different scenarios. The method developed there also applies to more general jump processes: we have shown in Dzial et al. [19] how to associate a fluid queue with a jump process and derive some interesting results about the latter, at least in cases where transitions are level-independent. It would be interesting to investigate more complex situations.

A line of enquiring which has recently opened and which we have not mentioned earlier is the so-called *transient analysis* of fluid queues, whereby one analyses quantities such as first passage time distributions.

Finally, even the analogy to QBD processes is far from being thoroughly exploited; one might for instance analyse fluid queues with a positive drift and determine if there exists an invariant measure for the differential equations, following a similar development for QBD processes.

Notations

$X(\cdot)$: level of a fluid queue

$X^{(b)}(\cdot)$: level of a fluid queue with finite capacity b

$\{\varphi(t) : t \in \mathbb{R}^+\}$: Markovian phase process

T : infinitesimal transition generator of $\{\varphi(t)\}$

$\boldsymbol{\xi}$: steady state probability vector corresponding to the generator T

$\mathcal{S}$: state space of $\{\varphi(t)\}$

r_i: net input rate of fluid when the phase is i

$\boldsymbol{r} = (r_i : i \in \mathcal{S})$

$\mu = \boldsymbol{\xi}\boldsymbol{r}$: mean stationary drift

$\mathcal{S}_0 = \{i \in \mathcal{S} : r_i = 0\}$

$\mathcal{S}_+ = \{i \in \mathcal{S} : r_i > 0\}$

$\mathcal{S}_- = \{i \in \mathcal{S} : r_i < 0\}$

$\mathcal{S}_\bullet = \mathcal{S}_+ \cup \mathcal{S}_-$

s_0, s_+, s_- : cardinalities of $\mathcal{S}_0$, $\mathcal{S}_+$ and $\mathcal{S}_-$, respectively

$F_i(x;t)$: joint distribution function of state (x, i) at time t

$f_i(x;t)$: joint density function of state (x, i) at time t

$\pi_i(x) = \lim_{t\to\infty} f_i(x;t)$: stationary density of state (x, i)

$\boldsymbol{\pi}(x) = (\pi_i(x) : i \in \mathcal{S})$

$\mu(x) = \boldsymbol{\pi}(x)\mathbf{1}$: stationary density of level x

$\boldsymbol{p}$, $\boldsymbol{p}^{(0)}$: steady state probability mass vector of the empty buffer

$\boldsymbol{p}^{(b)}$: steady state probability mass vector of the full buffer

e^{Kx} : expected number of crossings of $(x, \mathcal{S}_+)$, starting from $(0, \mathcal{S}_+)$, before returning to level zero

Ψ : matrix of first return probabilities to the initial level

$\Psi^{(b)}_{+-}$: first return probabilities to the initial level, before reaching level b

$\Lambda^{(b)}_{++}$: first passage probabilities to level b, starting from level zero, before returning to the initial level

$N^{(b)}_{+}(0, x)$: expected number of visits to level x, starting from $(0, \mathcal{S}_+)$, before visiting levels 0 and b

$N^{(b)}_{-}(b, x)$: expected number of visits to level x, starting from $(b, \mathcal{S}_-)$, before visiting levels 0 and b

$\{D(t)\}$: Markov process of downward records

U : infinitesimal transition generator of $\{D(t)\}$

$\hat{K}$, $\hat{\Psi}$, $\hat{U}$: equivalent to K, Ψ, U for the level-reversed fluid queue

$\hat{\Psi}^{(b)}_{-+}$: equivalent to $\Psi^{(b)}_{+-}$ for the level-reversed fluid queue

$\hat{\Lambda}^{(b)}_{--}$: equivalent to $\Lambda^{(b)}_{++}$ for the level-reversed fluid queue

A_0, A_1, A_2 : transition matrices of a QBD process

G : first passage probabilities to lower levels for a QBD process

R : expected number of visits to level one, starting from level zero, before returning to level zero, for a QBD process

$\hat{G}$, $\hat{R}$: equivalent to G, R for the level-reversed QBD

$\mathrm{sp}(M)$: spectral radius of a matrix M

$M^{\#}$: group inverse of a matrix M

$\mathcal{R}(s)$: real part of a complex number s

$\mathbf{0}$: vector of zeros

$\mathbf{1}$: vector of ones

I : identity matrix

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FLUID QUEUES
BUILDING UPON THE ANALOGY WITH QBD PROCESSES

ANA DA SILVA SOARES

Markov modulated fluid queues are two-dimensional Markov processes, of which the first component, the *level*, represents the content of a reservoir and takes real values, and the second component, the *phase*, is the state of some Markov process evolving in the background. The level varies linearly with time, at a rate which depends on the phase.

In this thesis, we apply Markov renewal techniques to fluid queues and we explore the analogy with Quasi Birth-and-Death (QBD) processes. Our approach leads to a unified and straightforward analysis of all the fluid models considered, and is always combined with a very efficient computational procedure.

We first consider a fluid queue with an infinite capacity buffer, and we determine its stationary distribution; we observe that it is very similar to that of a QBD process. We further exploit this similarity by determining the stationary distribution of a finite capacity fluid queue following the same kind of arguments.

We then consider more complex models, of either finite or infinite capacities, in which the behaviour of the phase process may change whenever the buffer is empty or full, or when it reaches certain thresholds. We show that the techniques developed for the simpler models can be extended quite naturally in this context.

Finally, we study the necessary and sufficient conditions that lead to the stationary independence between the level and the phase of an infinite capacity fluid queue. These results are based on similar developments for QBD processes.