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**Nonequilibrium Statistical
Thermodynamics at the Nanoscale**

David Andrieux

Thèse présentée en vue de l'obtention
du grade de Docteur en Sciences
Réalisée sous la direction de

Pierre Gaspard

*Center for Nonlinear Phenomena and Complex Systems
Faculté des Sciences*

Avril 2008

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Abstract

Motivated by the recent advances in nanosciences, we investigate the statistical and thermodynamical properties of mesoscopic systems. In particular, we focus on the recently derived results collectively known as fluctuation theorems. These relations give us predictions on the behavior of various dynamical quantities in far-from-equilibrium situations, while explicitly taking the fluctuations of the time evolution into account.

The first part of our study deals with the relationships between fluctuations and response theory. We start by deriving a functional relation for time-driven quantum systems which generalizes the Jarzynski equality and whose expansion in the driving force can be used to recover results from linear response theory. We then pursue these considerations by looking at nonequilibrium steady states. This is done in the framework of stochastic processes, where we derive a fluctuation relation for the nonequilibrium currents crossing the system. This relation explicitly links the fluctuations to the thermodynamic driving forces, which allows us to work out its consequences at the level of nonlinear response theory. In this way we obtain new relationships between fluctuations and response coefficients in the nonlinear regime.

In the second part of this work, we develop a recent connection relating irreversibility to dynamical randomness. We present experimental evidence showing that entropy production is expressed as the difference between two quantities measuring the symmetry breaking under time reversal at the level of the temporal disorder. We then proceed to study the consequences of this relation in the context of information processing's energetics. In particular, we present a generalization of Landauer's principle for the erasure of information. Likewise, we relate dissipation to information generation by biological entities.

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Introduction

1.1 Motivations

Statistical physics is nowadays challenged by the recent advances in nanosciences. Feynman was actually the first to reveal the potentialities of these scales in his seminal lecture “*There’s plenty of room at the bottom*” [73], given in 1959 at Caltech. Herein, he imagined the possibility of writing the whole 24 volumes of the Encyclopedia Britannica on the head of a pin, the manipulation of single atoms, the miniaturization of computers, the sequencing of human DNA thanks to more powerful microscopes to come, etc. Finally, he issued two seemingly “impossible” challenges, each one backed by a \$1000 prize: the first was to build a working electric motor no larger than a 1/64th-inch cube, and the second was to write a page of a book on an area 1/25,000 times smaller “in such manner that it can be read by an electron microscope.”¹

About fifty years later, microscopy techniques such as Scanning Tunneling Microscopy (STM), Atomic Force Microscopy (AFM), and Transmission Electron Microscopy (TEM) allow us to see and manipulate single atoms [64]. An assembly of 17 molecules has been realized which can process instruction and control several attached molecular machines in parallel [13]. The sequencing of the genome was completed in 2003 and we now live in what has been called the “post-genomic era” in which information about genes expressed in our tissues can be used to diagnose and treat diseases. Eventually, artificial electric motors of the size of about 300 nm with an axis made of a multiwalled carbon nanotube can now be manufactured [71].

Tremendous advances were achieved in other directions as well. We here list a few examples. Electronic transport can now be envisaged in samples where phase coherence is preserved along the structure [72]. At the opposite, conduction in quantum dots and tunnel junctions is governed by the

¹ 11 months later, engineer William McLellan had constructed a 250-microgram 2000-rpm motor out of 13 separate parts that would fit inside a 0.4-mm cube, and collected his reward against the first challenge. The second prize was won in 1985 by Tom Newman, who used e-beam lithography to write the first page of Dickens’ *A Tale of Two Cities* legibly in a 6.25-micron square.

Coulomb interaction and single-electron effects [72]. Also, carbon nanotubes or conducting polymers offer new conduction possibilities [185]. Thanks to the development of fluorescence and force spectroscopy, we can now perform single molecules experiments. For instance, RNA segments are mechanically stretched and their elastic response recorded in real time thanks to AFM and optical tweezers [146]. These techniques are crucial for processes that cannot be fully understood on the bulk level, such as the movement of myosin on actin filaments in muscle tissue or the details of individual local environments in solids. Still at the single molecule level, rotary nanomotors have been constructed by isolating the F_1 -ATPase protein complex from mitochondria and gluing an actin filament or a gold bead to the axis of the motor [173, 232].² In the time domain, femtochemistry allows us to resolve, and ultimately control, chemical reaction pathways at time scales of 10^{-15} s [91]. Also, oscillating reactions involving as few as a hundred molecules are observed on nanometric tips of field electron (FEM) and field ion (FIM) microscopes [224].

The typical sizes of these systems range from the micrometer down to the nanometer. Such systems are thus of size intermediate between the microscopic world where the atomic features are apparent and the macroscopic world containing a number of particles of the order of the Avogadro number. Many intriguing questions arise in this context.

Indeed, nanosystems are large enough for the onset of statistical behaviors such as friction, diffusion, viscosity, and sustained nonequilibrium motions to appear. Many studies show that statistical behavior starts to emerge already in relatively small systems containing dozen, hundred, or thousand particles depending on the property of concern. For instance, the transport coefficients can be calculated in hard-ball systems with a few dozen particles [221] while dynamical friction between double-walled nanotubes [200] or Rayleigh-Bénard convection rolls [153] can be simulated by molecular dynamics in systems of several thousand particles. Nanosystems are thus of prime importance to gain profound insights into the transition from reversible equations of motion to statistical and irreversible behaviors.

Another fascinating result from the nanosciences is that self-organization already manifests itself at the scale of a few nanometers and this under both equilibrium or nonequilibrium conditions. The Glansdorff-Prigogine dissipative structures [100] showed that self-organization is possible in macroscopic systems under nonequilibrium constraints, whereas recent discoveries show that self-assembly and self-organization of matter already start at the nanoscale. The formation of micelles of about 10 nm in liquid mixtures or of nanoclusters of similar sizes at solid-gas or solid-liquid interfaces is an example of equilibrium self-organization at the nanoscale. On the other hand, the biological world functions under nonequilibrium conditions. This results from the need to possess a metabolism maintaining the autonomy of the organism with respect to the environment. Accordingly, biology presents a large variety of self-organization: biomolecules, proteins, RNA, DNA self-assemble into supramolecular structures

² In this thesis, multiple references are organized in chronological order.

such as ribosomes, nanomotors, membranes, virus, organelles, bacteria, etc. This shows that structuration and complexification of matter already starts at the nanoscale.

Finally, observation reveals that molecular fluctuations play a crucial role in these systems. This entails that we cannot engineer nanodevices based on macroscopic designs: they might not work at all due to the relatively large thermal fluctuations present at these scales. Accordingly, nanosystems must take advantage of the molecular fluctuations in order to present coherent space-time behaviors. This is accomplished thanks to the nonequilibrium conditions which break time-reversal symmetry and allow for the emergence of a dynamical order. The fact is that, despite these fluctuations, the efficiency and robustness of these nanosystems turn out to be quite impressive. For instance, conversion of chemical energy into mechanical work by ATP consumption has been reported to be close to 100% efficiency in certain nanomotors [126]. In this context, some general mechanisms to obtain directed motion from thermal fluctuations have now been studied in detail [186, 198] but a general understanding of transport or energy transduction (chemical reaction cycles, error rates, speed of operation, thermodynamic efficiencies, etc) in nanodevices is still lacking.

In summary, nanosystems are of technological importance and the developments of statistical tools able to cope with the equilibrium and nonequilibrium properties of these intermediate structures is of practical and fundamental interest. It is in this recent context that this thesis is written and we summarize below some relevant works in this direction.

1.2 Past and recent developments

Since Newton, the motion has been conceived as regular and reversible. The description of the celestial motion of planets around the sun by Laplace along with the observations of Kepler were a brilliant verification of the mechanical doctrine of these years. Still, in the early nineteen century, the French mathematician Fourier, motivated by the problem of heat conduction, came up with an equation which presented a significant feature: unlike Newton's laws of motion, Fourier's equation is irreversible. In 1824 Sadi Carnot published his work *Reflections on the Motive Power of Fire* [36], a discourse on heat, power, and engine efficiency. Herein, Carnot introduced us to the first modern day definition of "work": weight lifted through a height. He concluded that a difference of temperature between two bodies could be used to perform work. In 1850 the mathematical physicist Rudolf Clausius understood that heat and work are actually two different forms of the same quantity: energy. For a closed system, the transformation of energy into heat and mechanical work is an expression of the *first law of thermodynamics*. However, badly designed engines loose the possibility to perform work, pointing out the existence of dissipation and irreversible operations. Clausius introduced the quantity dQ/T as a measure of the work lost during transfer of heat from a hot to a cold body. Thanks to this new quantity, which he called the entropy, the *second law of thermodynamics* can be easily stated: the entropy of an isolated system

always tends to increase. Building on these foundations, this new discipline, named thermodynamics, brought these engine concepts into the thoroughfare of almost every modern-day branch of science.

At the same time, the idea of indivisible particles naturally led to the introduction of statistics and fluctuations. In the study of gases, Maxwell introduced the idea of random motion for the molecules. He suggested that, instead of tending to equalize the velocities of all molecules, the successive collisions would produce a statistical distribution of velocities. At thermal equilibrium, he derived his famous distribution function for the velocities of the gas particles. In 1872 Boltzmann established its famous equation for the evolution of the single particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$, which describes the probability density to find a number of molecules f with a velocity $\mathbf{v}$ at position $\mathbf{r}$ and time t . Its time evolution is governed by a binary collision term, which is a good approximation for dilute gases. In this approximation and in order to close its equation, Boltzmann made the fundamental assumption that the velocities of colliding particles are uncorrelated. This is the so-called *Stosszahlansatz*, which became later known as the assumption of *molecular chaos*. He also noted that this collision term is responsible for the irreversibility: he derived his famous *H-theorem*, which shows that the quantity $H = - \int f \ln f \, d\mathbf{r} d\mathbf{v}$ always decreases in time until a Maxwell distribution is reached. The Maxwell distribution plays a special role as it is the more likely distribution to be found at equilibrium because it corresponds to the largest number of microstates. With this understanding, Boltzmann concluded that the approach to equilibrium corresponds to a transition from less probable microstates to more probable microstates. These insights led him to propose a novel expression for the entropy of a macrostate: $S = k_B \ln \Omega$, where Ω is the number of possible microstates corresponding to the macroscopic state of the system. Subsequently, Gibbs expanded these considerations by introducing the concept of statistical ensembles. Since Boltzmann, a systematic expansion based on the reduced n -particle distribution function has been developed in the form of the BBGKY hierarchy. For instance, kinetic equations were derived for plasmas by Landau, Vlasov, and more recently by Balescu and Lenard who included the Debye screening of Coulomb interaction [12]. However, each such kinetic equation involves a stochastic assumption which is introduced by some truncation of the evolution equation in the low-density or weak-coupling limit. Furthermore, the derivation of kinetic equations and transport coefficients in the high-density limit leads to new difficulties in the form of diverging series expansions [54]. This will be one of the motivations in introducing the hypothesis of microscopic chaos and the study of chaotic systems, as discussed below.

In kinetic theories, statistics enter as a description of the probable states but do not have an incidence on the dynamics. In contrast, Robert Brown observed in 1827 the very irregular motion of colloidal particles in a microscope [32]. He first observed minute particles within vacuoles in the pollen grains executing

a continuous jittery motion. He then observed the same motion in particles of dust, enabling him to rule out the hypothesis that the motion was due to pollen being alive. In the early twentieth century, Einstein attributed these fluctuations to the collisions of the Brownian particle with the discrete atoms of the fluid [65]. He expressed the diffusion coefficient of the Brownian particle in terms of the mean-square of its displacement:

$$D = \lim_{t \rightarrow \infty} \frac{\langle [x(t) - x(0)]^2 \rangle}{2t}. \quad (1.1)$$

He also noted that the same random forces which cause the erratic motion of a particle in Brownian motion would also cause drag if the particle were pulled through the fluid, obtaining in this way an early form of the fluctuation-dissipation theorem.³ For electrical systems, Schottky [192] observed that the thermionic current in a vacuum tube presents rapid and irregular changes in magnitude, due to the random emission from the cathode. Johnson [123, 124] demonstrated experimentally that an electrical resistor automatically generated fluctuations of electrical voltage and concluded that the thermal agitation of the electric charges in the conductor is the cause of this phenomenon. Nyquist's theoretical work [174] related these equilibrium voltage fluctuations to the sample's resistance, in complete accordance with Johnson's experiments. In 1931 Onsager exploited stochastic processes to discuss irreversible thermodynamics in the linear regime and derive his reciprocity relations [175]. The link between fluctuation and dissipation was further developed by Callen and Welton [34], who derived a general form of the fluctuation-dissipation theorem. These different works contributed to the establishment of the linear-response theory further developed in the fifties by Green [105], Kubo [131], and Mori [159], who related the transport coefficients to the integral of time auto-correlation functions. For instance, the diffusion coefficient reads

$$D = \frac{1}{2} \int_{-\infty}^{+\infty} \langle v(t)v(0) \rangle dt. \quad (1.2)$$

In parallel, Onsager and Machlup proposed a functional formula for the probability density of a stochastic process [176]. Their expression extends Boltzmann's fundamental relationship between the entropy and the probability to the time domain. In summary, the main message of these works would be that equilibrium notions can be used to probe the nonequilibrium behavior of a system.

Yet, the aforementioned works exclusively deal with systems in the so-called linear regime, i.e. systems close enough to equilibrium for them to respond linearly to the thermodynamic forces. Surprisingly perhaps, the concept of thermodynamic equilibrium itself has gradually evolved over the years. Thermodynamic equilibrium was first defined as a state where no macroscopic changes occur. However, in the context of chemical reactions, R. Wegscheider [227] pointed

³ This method was actually used by Perrin [179] in order to obtain Avogadro's number, giving evidence for the atomic hypothesis.

out that the condition of vanishing rate does not necessarily coincide with the thermodynamic equilibrium condition. Indeed, stationarity does not rule out the possibility of "circular" processes, characterized by the presence of fluxes. This situation, which became known as Wegscheider's paradox, was invoked by Lewis [143] as part of the justification for his general "law of entire equilibrium", which requires that every elementary process shall have a reverse process, and that their rates must balance at equilibrium. Nonetheless, Tolman [214] pointed out that the existence of a balanced reversed process has already been assumed in several works, for instance by Langmuir in 1916 in connection with the problem of evaporation and condensation [136] or by Einstein in deducing Planck's radiation law. These assumptions were subsequently named *microscopic reversibility* principle or *detailed balance* conditions. Detailed balance also appeared in the study of collisions in particle physics, where the symmetries of the S matrix account for the principle of reversibility [43, 229].

However, it was not until the sixties that the violation of detailed balance was quantitatively associated with nonequilibrium properties. Hill [112], in the context of nonequilibrium stochastic models, expressed an irreversible entropy production in terms of fluxes and affinities, in analogy with macroscopic thermodynamics. These fluxes and affinities measure the breaking of detailed balance along the cyclic trajectories of the system [112, 115, 113]. Schnakenberg [190] studied this form of the entropy production from a statistical point of view, starting from the Gibbs entropy. These works extend a result of Kolmogorov [129], who proved that a Markov process is reversible if and only if all cyclic trajectories satisfy the detailed balance conditions. Later, these considerations were further developed by the Qians [121]. In all these analyses, graph theory plays a central role. In 1847 already, Kirchhoff [127] expressed the steady state solution for the currents in an electrical circuit in terms of the maximal trees of the circuit. The analogy between networks and macroscopic thermodynamics was further studied by Oster and coworkers [177, 178].

In parallel, the general research on far-from-equilibrium systems was pioneered by Haken [106, 107] and Prigogine [100, 169]. In the linear regime, equilibrium states are modified by the constraints preventing the system to reach equilibrium but no new structure appears. However, this situation changes drastically in far-from-equilibrium situations, where coherent space-time behaviors can appear. The most remarkable observation is thus that nonequilibrium can be a source of order. Spectacular examples are oscillatory Turing patterns in the Belousov-Zhabotinsky reaction or convection rolls in the Rayleigh-Bénard instability. The term *dissipative structures* has been coined in order to designate these self-organizing processes. The fact is that Boltzmann's principle cannot account for these structures since it would associate a virtually zero probability to the occurrence of a coherent pattern involving a macroscopic number of molecules. The appearance of these new structures is actually accomplished via a succession of bifurcations. The mechanism put forward by the Brussels school is *order through fluctuations*: small fluctuations are amplified up to a macroscopic current by an instability reached after a certain threshold in the parameter space. These macroscopic fluctuations are in turn

stabilized by the exchange of energy and matter with the outside world, which absorbs the internal entropy produced by the system, eventually leading to a coherent space-time behavior. But nonequilibrium situations can also be at the source of other order mechanisms. For instance, long-range correlations appear out of equilibrium, either in the time or spatial domains [166, 128]. Anyhow, this new paradigm resolves the apparent contradiction of the “thermal death” of the universe that could be anticipated on the basis of linear thermodynamics.

Our perspective about irreversibility and the second law of thermodynamics has further evolved under the influence of the new concepts coming from dynamical systems theory [61]. One major challenge is to get rid of stochastic assumptions such as Boltzmann’s Stosszahlansatz in the derivation of kinetic theories. Despite their justification in some scaling limit [204], a more fundamental derivation of transport processes in terms of the intrinsic properties of the underlying dynamics is searched for. A fruitful approach is based on the hypothesis of *microscopic chaos*.

The study of chaotic systems in the eighties and early nineties has introduced new types of relationships within nonequilibrium physics. Following the work of Shannon [202] in information theory, it became clear that dynamical systems could be as random as a coin-tossing process. The Kolmogorov-Sinai entropy per unit time h_{KS} characterizes the temporal disorder by analogy to the entropy per unit volume which characterizes the spatial disorder. This notion opened the way to the large deviation formalism of dynamical chaos introduced by Ruelle and Bowen [187], also called *thermodynamic formalism*. This formalism is based on the formal analogy between time randomness of chaotic trajectories and space randomness of a system configuration in equilibrium statistical mechanics. Temporal randomness originates from a dynamical instability inducing an exponential separation of two nearby trajectories. The Lyapunov exponents λ_i characterize this sensitivity to initial conditions while the Kolmogorov-Sinai entropy per unit time measures the degree of dynamical randomness developed by the trajectories of the system during their time evolution. In this way, we can understand that dynamical instability is linked to the KS entropy via Pesin’s theorem

$$h_{KS} = \sum_{\lambda_i > 0} \lambda_i. \quad (1.3)$$

This sensitivity to initial conditions provides a natural justification for the introduction of statistical ensembles.

In this regard, chaotic systems may present strong mixing properties, in which case the phase-space density will evolve towards an equilibrium distribution. This relaxation process is characterized by the eigenvalues or decay rates of the time-evolution operator which are known under the name of Pollicott-Ruelle resonances. On the other hand, transport processes such as diffusion can be conceived in terms of chaotic scattering, which leads to the so-called escape-rate formulas [95] relating the diffusion coefficient to the Lyapounov exponents and the KS entropy:

$$D \left(\frac{\pi}{L} \right)^2 \simeq \gamma = \left(\sum_{\lambda_i > 0} \lambda_i - h_{KS} \right)_L, \quad (1.4)$$

where L is the size of the system and γ is the leading Pollicott-Ruelle resonance. We remark that this formula extends Pesin's theorem to open systems. The generalization of this escape-rate formula to other transport processes can be found in Refs. [55, 93]. Along these lines, the hydrodynamic modes of diffusion as well as the nonequilibrium steady states have been calculated directly from the Liouvillian dynamics in phase space. This approach establishes a direct connection between the phase-space dynamics and hydrodynamical laws [82]. In particular, an *ab initio* derivation of the transport properties and the entropy production of nonequilibrium thermodynamics can be achieved from the underlying Hamiltonian dynamics [83, 98]. The positivity of the entropy production is here associated with the fractality of the phase-space density due to the nonequilibrium boundary conditions. These new relationships link the large deviations or large fluctuations that the dynamical properties of a system may undergo during the time evolution to nonequilibrium statistical properties.

Large deviations are at the basis of important advances in nonequilibrium statistical mechanics during the last fifteen years [53]. In 1993 Evans, Cohen, and Morriss [69] found in numerical simulations a symmetry property of the invariant measure in nonequilibrium steady states. The first mathematical formulation of this symmetry, the *fluctuation theorem*, was obtained in 1995 by Gallavotti and Cohen under strong chaoticity assumptions [79]. They proposed the *chaotic hypothesis* according to which, for the purpose of studying macroscopic properties, the time evolution of a many-body system in a stationary state can be regarded as a transitive Anosov system, i.e., a fully chaotic system. These authors suggest that this hypothesis constitutes the natural extension of the ergodic hypothesis in equilibrium statistical physics to out-of-equilibrium systems. In this context, the fluctuation theorem appears as a test of the chaotic hypothesis.

The fluctuation theorem can be roughly summarized as follows. The key feature is to consider a trajectory-dependent quantity S_t that can be interpreted as the entropy irreversibly produced during the random evolution of the system. This interpretation is supported by the fact that the average value $\langle S_t \rangle / t$ can be identified with the thermodynamic entropy production. Now, if we denote by $P(S_t/t = \zeta)$ the probability density that the quantity S_t/t takes a given value ζ , the fluctuation theorem reads

$$P(S_t/t = \zeta) \sim e^{t\zeta} P(S_t/t = -\zeta) \quad (1.5)$$

at long times. For the nonequilibrium stationary states considered by Gallavotti and Cohen, S_t is expressed as the contraction rate in phase space integrated along the trajectory but, depending on the situation, it is better called the dissipated work [133] or the entropy current [149]. This relation has the remarkable feature to be valid arbitrarily far from equilibrium. Moreover, it expresses a symmetry property of the large deviations since the fluctuating quantity ζ can

significantly deviate from the statistical average. Finally, Eq. (1.5) shows that it is exponentially more likely to observe a positive dissipation rather than a negative one, providing a more fundamental justification of the second law of thermodynamics.

In 1998 Kurchan showed that the fluctuation theorem holds for a class of diffusion processes in the form of a symmetry property of the evolution operator [133]. Lebowitz and Spohn extended Kurchan's result to general Markov processes [139] while Maes thought of the fluctuation theorem as a property of space-time Gibbs measure [148, 149]. The entropy production at the level of stochastic trajectories was discussed by Seifert [195, 189]. Transient versions of the fluctuation theorem were also derived [70, 194]. Gallavotti [78] and Evans [11] proposed a local version of the fluctuation theorem. These relations have been observed in different experimental setups such as turbulent flows [41], electric circuits [81] or driven Brownian particles [226, 35], among others [212]. A general feature of these works is that the dissipation can be identified with the odd part under time-reversal of the action functional. The study of the even part was carried out by Maes and van Wieren [150]. A more detailed account of the many related works can be found in the recent reviews by Harris et al [109] or by Sevick et al [201].

Another family of recently derived relationships are the *nonequilibrium work relations*. In this setup, the system is controlled thanks to an external parameter λ . The system starts with a parameter value $\lambda_0 = \lambda_A$ at time $t = 0$. The system is then driven in a time-dependent way by varying the parameter λ in time following an arbitrary schedule that ends at the value $\lambda_t = \lambda_B$ after a time t . For systems starting at equilibrium, Jarzynski [118, 119] derived the following equality:

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}. \quad (1.6)$$

Here, W is the work done during the process, which is a random variable depending on the initial condition of the trajectory. The average is taken over a canonical ensemble of initial conditions at inverse temperature $\beta = 1/k_B T$. $\Delta F = F_B - F_A$ is the free energy difference between a fictive system at equilibrium with the parameter $\lambda = \lambda_B$ and the initial system at equilibrium with the parameter $\lambda = \lambda_A$. This result is independent of both the size of the system and the schedule for perturbing it.

Many works appeared in its wake, although we note that Bochkov and Kuzovlev derived similar relations in the eighties [25, 26]. Crooks showed that Jarzynski's equality (1.6) can be recovered from a more detailed relation relating the probability of the work done during the forward protocol to its opposite value done while performing the time-reversed protocol [50]. In the same setup, Kawai and coworkers expressed the mean value of the dissipated work as the logarithm of the ratio between two phase-space probability densities associated with the forward and backward processes, respectively [125]. The transition between nonequilibrium states was studied by Hatano and Sasa [110]. Quantum versions were considered by Kurchan [134], Monnai [156], and Hänggi and coworkers [208, 207], among others [160, 120].

These relations have several important consequences. For instance, equation (1.6) relates the work W performed on a system during a nonequilibrium process to the change in the value of a thermodynamic state function ΔF . Nonequilibrium properties can thus be used to extract equilibrium informations as well as potentials of mean force [117]. A remarkable result is that Jarzynski's equality (1.6) implies Clausius inequality

$$\langle W \rangle \geq \Delta F, \quad (1.7)$$

interpreted statistically. This relation results from Jensen's inequality applied to $\exp x$.⁴ We can also use (1.6) to derive a more stringent inequality. For a given schedule λ_t , let $P(W < X_n)$ denote the probability of observing a work value no greater than $X_n \equiv \Delta F - nk_B T$, where n is an arbitrary positive constant. We can think of this as the probability of observing an apparent violation of the second law, where the magnitude of the violation is at least n units of $k_B T$. Jarzynski's equality, combined with the general bound (A.2) presented in Appendix A, leads us to the following inequality:

$$P(W < X_n) \leq \exp(-n). \quad (1.8)$$

It reveals that the likelihood of an apparent violation of the second law diminishes at least exponentially with the degree of violation: no matter how we drive the system, it is extremely unlikely that we will observe a violation whose magnitude is, e.g., at least $15 k_B T$ since $P \leq \exp(-15) \approx 3.1 \cdot 10^{-7}$. In a nanoengineering perspective, this would correspond to setting bounds on the probability for a device to operate "in reverse".

Nonequilibrium work relations were observed experimentally in different setups. The Jarzynski equality was verified by Ciliberto and coworkers for a torsion pendulum [56] and for an inertial oscillator [57], while integral fluctuation relations were observed by Schuler et al [193]. Jarzynski and Crooks relations were verified in single molecule experiments for the folding and unfolding of RNA segments [145, 47]. In Ref. [215], Hatano-Sasa's relation was verified as well.

All these relationships share the common mathematical structure that they give an irreversible property as the difference between two decay rates of mesoscopic or microscopic properties [87, 88]. In this regard, they belong to the same family of large-deviation properties as the escape-rate and chaos-transport formulas [95, 55, 92]. Here, the dissipation appears as a fluctuating quantity that can be positive or negative. It is only by averaging over all trajectories that we recover a positive dissipation in accord with the second law of thermodynamics.

The statistical nature of the second law was already envisaged by Maxwell. In his 1871 work [154], he discussed the problem of irreversibility in relation with molecular fluctuations. He conceived his famous "demon" challenging the second law of thermodynamics:

⁴ Jensen's inequality states that, for any convex function ϕ , we have $\langle \phi(x) \rangle \geq \phi(\langle x \rangle)$.

“... if we conceive of a being whose faculties are so sharpened that he can follow every molecule in its course, such a being, whose attributes are as essentially finite as our own, would be able to do what is impossible to us. For we have seen that molecules in a vessel full of air at uniform temperature are moving with velocities by no means uniform, though the mean velocity of any great number of them, arbitrarily selected, is almost exactly uniform. Now let us suppose that such a vessel is divided into two portions, A and B, by a division in which there is a small hole, and that a being, who can see the individual molecules, opens and closes this hole, so as to allow only the swifter molecules to pass from A to B, and only the slower molecules to pass from B to A. He will thus, without expenditure of work, raise the temperature of B and lower that of A, in contradiction to the second law of thermodynamics.”

The demon, thanks to its ability to manipulate single molecules, can create temperature gradients from an initial isothermal condition.⁵ In 1929 Szilard [206] argued on a mechanical analogue of the demon that any gained work is compensated by an equivalent loss that he attributed to the measurement process. Brillouin [31] came to a similar conclusion, and argued further that the dissipation must increase with the reliability of the measurement apparatus. In 1961 Landauer correctly identified logical irreversibility as the source of the dissipation [135]. Logical irreversibility is a feature of many-to-one maps, i.e., non-invertible operations. For instance, during the erasure of random bits, two states 0 and 1 are mapped indistinctively to the unique state 0. The entropy change $\Delta S = -\ln 2$ during this operation must be compensated by an entropy increase $\geq \ln 2$ in the environment, leading to a minimal dissipation of $k_B T \ln 2$ for an environment at temperature T .⁶ von Neumann [223] had also envisaged a minimal dissipation of $k_B T \ln 2$ per “elementary act of information” but failed to associate it to information erasure. Subsequently, Bennett [17] showed that measurement could be performed at zero cost and attributed the energy cost to the erasure of information, in concordance with Landauer. Maxwell’s demon can now be exorcised. Indeed, the demon must collect and store information about the molecules. If the demon has a finite memory capacity, he cannot continue to cool the gas indefinitely; eventually, information must be erased.

Along these lines, Bennett found in 1973 that any computation can be performed using only reversible steps, and so in principle requires no dissipation and no power expenditure [15]. Interestingly, the simplest model of reversible computation can be accomplished using a billiard model [17]. However, the slightest perturbation will prevent the functioning of the computing device due to dynamical instabilities or chaos. Another family of models tries to exploit

⁵ The demon reappeared under different forms over the years. For example, the rectification of thermal fluctuations by a diode became known as Brillouin’s paradox [30], and is closely related to the difficulty of consistently taking into account internal noise in nonlinear systems [217]. More recently, it was discussed in the context of Brownian motors and the ratchet effect [186, 216, 199].

⁶ Circa the year 2000, computers dissipated roughly $500 k_B T \ln 2$ per elementary logical operation.

the presence of thermal noise in the form of Brownian computers [17]. The computational pathway is here traveled as a random walk with nearly equal forward and backward transitions. A small driving force is sufficient to put the system on the right computational track, hence the dissipation can be made arbitrary small if a correspondingly small computing speed is tolerated. However, in the presence of intrinsic hardware errors, a near zero dissipation can not be achieved without incurring excessive errors. Error correction is another example of logically irreversible computation. We can here understand the constructive role of dissipation, which is necessary in order to prevent and correct errors in presence of noise.

1.3 Outline

The purpose of the present work is to study the thermodynamics of mesoscopic systems. At this scale, molecular fluctuations play an important role and nonequilibrium conditions are the norm. Previous arguments suggest that relationships between fluctuations, dissipation, and organization exist and are fundamental for nonequilibrium statistical mechanics and for the development of nanosciences. Precise relationships can be established at different levels.

First, the nonequilibrium fluctuation relations can be conceived as extensions of the fluctuation-dissipation theorem to the nonequilibrium realm. In particular, such relations lead to new symmetry relations for the nonlinear response coefficients and the nonequilibrium fluctuations [5].

Second, the entropy production turns out to be directly related to the time-reversal symmetry breaking on a quantity that measures the randomness of the temporal evolution [86]. Thereby, out of equilibrium, the trajectories of a system are more ordered in time than at equilibrium. These results have been observed experimentally in two fluctuating mesoscopic systems [9, 10]. This changes our perspective on the second law since it now appears to play a constructive role in the evolution of the system. In particular, these concepts link in a fundamental way the fluctuations and the dissipation to the physics of information.

The thesis is organized as follows. Chapter 2 is concerned with the quantum aspects of the nonequilibrium work relations. We study the time-reversal symmetry for a quantum system driven by a time-dependent perturbation. We obtain a generalization of the quantum Jarzynski relation which can be used to recover the linear response theory. In this way, we can already understand how nonequilibrium relations reveal the interplay between fluctuations and dissipation in out-of-equilibrium systems and generalize the near-equilibrium results. The study of nonequilibrium steady states is carried out in Chapter 3. Such steady states are obtained via a coupling of the system with reservoirs of energy and particles. We thus introduce the stochastic description, as well as its formulation in terms of cycles that allows us to link the mesoscopic description to the macroscopic one. In this framework, we demonstrate a fluctuation relation for the macroscopic currents crossing the system in a nonequilibrium stationary

state. This relation is then applied to several systems of interests such as chemical reaction networks, electronic transport in tunnel junctions, and the recently observed rotary molecular motors. In Chapter 4 we develop the consequences of this new symmetry at the level of the nonlinear response theory. In particular, we show that Onsager's symmetry along with the Green-Kubo formula are readily obtained at the level of linear response. The higher-order relations relate the response coefficients to the nonequilibrium behavior of the fluctuations, leading to new symmetries generalizing Onsager's relations. Chapter 5 addresses the problem of the thermodynamic entropy production in nonequilibrium stationary states. We introduce the concepts of forward and time-reversed entropies per unit time, which characterize the dynamical randomness of the time evolution respectively in the forward and backward direction of time. The entropy production is equal to the difference between these two quantities, showing that irreversibility occurs as a spontaneous time-symmetry breaking under nonequilibrium conditions. We present experimental evidence for a trapped Brownian particle and a current-driven RC circuit showing that the entropy production is related to the time-asymmetry in the temporal disorder. This work is the result of a collaboration with Prof. Ciliberto and his group at the Ecole Normale Supérieure de Lyon. Chapter 6 explores the perspectives generated by this new understanding of the second law of thermodynamics. We consider Landauer's principle from the viewpoint of dynamical randomness, which allows us to generalize it to the case where the information to be erased is arbitrarily correlated. Furthermore, we study the entropy production in a model of copolymerization with an underlying template, as it is the case in DNA replication or transcription. As a matter of fact, the entropy production is closely related to the information about the template acquired by the copolymer. Conclusions and perspectives are drawn in Chapter 7.

Quantum Work Relations

Nonequilibrium work relations have recently attracted much interest. They provide relations for the work dissipated in time-dependent driven systems, independently of the form of the driving. They are therefore of theoretical interest but they also provide new tools to study nanosystems. In the nanoscopic world, the extension of these classical relations to quantum systems is of particular importance and different approaches have been proposed.

A first scheme was introduced by Kurchan [134]. In this framework, a measurement of the system state is performed at the initial time. In the sequel, the system is perturbed by a time-dependent Hamiltonian before performing another measurement at the final time. The random work performed on the system is associated with the energy difference between the final and initial eigenstates. This setup leads to the quantum extension of Jarzynski's equality and Crooks fluctuation theorem [210, 156, 208, 207]. Another possibility is to introduce a quantum work operator which measures the energy difference [157], in which cases quantum corrections to the fluctuation theorem must be taken into account. On the other hand, quantum fluctuation theorems have been obtained in suitable limits where the dynamics admits a Markovian description, allowing in particular the applications to nonequilibrium steady states [160, 52, 68, 108, 67]. Yet, the connection between the quantum work relations and response theory is still an open question even in the linear regime.

The purpose of the present chapter is to derive a new type of work relations which involves a functional of an arbitrary observable. This generating functional can be related to another functional but averaged over the time-reversed process. This new work relation turns out to be of great generality since we can recover known results such as Jarzynski equality as special cases. Furthermore, this universal work relation allows us to formulate the response theory, to derive the quantum linear response functions, the quantum Green-Kubo relations [105, 131], as well as the Casimir-Onsager reciprocity relations [175, 37] in the regime close to the thermodynamic equilibrium.

2.1 Functional symmetry relations

In this section, we shall consider the effect of time-reversal symmetry on quantum systems driven by a time-dependent perturbation.

We suppose that the system is described by a Hamiltonian operator $H(t; \mathcal{B})$ which depends on the time t and the magnetic field $\mathcal{B}$. The time-reversal operator Θ is an antilinear operator such that $\Theta^2 = I$ and which has the effect of changing the sign of all odd parameters such as magnetic fields:

$$\Theta H(t; \mathcal{B}) \Theta = H(t; -\mathcal{B}). \quad (2.1)$$

We first introduce the *forward process*. The system is initially in thermal equilibrium at the inverse temperature $\beta = 1/k_B T$. The initial state of the system is described by the canonical density matrix

$$\rho(0) = \frac{e^{-\beta H(0; \mathcal{B})}}{Z(0)}, \quad (2.2)$$

where the partition function is given in terms of the corresponding free energy $F(0)$ by $Z(0) = \text{tr} e^{-\beta H(0; \mathcal{B})} = e^{-\beta F(0)}$. Starting from this equilibrium situation at the initial time $t = 0$, the system evolves until some final time $t = T$ under the Hamiltonian dynamics. The corresponding forward time evolution is defined as

$$i\hbar \frac{\partial}{\partial t} U_F(t; \mathcal{B}) = H(t; \mathcal{B}) U_F(t; \mathcal{B}), \quad (2.3)$$

with the initial condition $U_F(0; \mathcal{B}) = I$. We can think of this time-dependence as arising from a control parameter X whose magnitude is changing in time according to a given schedule. In the Heisenberg representation, observables evolve according to

$$A_F(t) = U_F^\dagger(t) A U_F(t)$$

which also concerns the time-dependent Hamiltonian

$$H_F(t) = U_F^\dagger(t) H(t; \mathcal{B}) U_F(t).$$

The average of an observable is thus obtained from

$$\langle A_F(t) \rangle = \text{tr} \rho(0) A_F(t).$$

We note that the dependence on the magnetic field is implicit in these expressions.

The *backward process* is introduced similarly but in the magnetic field reversed. The system is perturbed according to the time-reversed protocol $H(T - t; -\mathcal{B})$, starting at the initial time $t = 0$ from the density matrix

$$\rho(T) = \frac{e^{-\beta H(T; -\mathcal{B})}}{Z(T)}, \quad (2.4)$$

where the free energy $F(T)$ is given in terms of the partition function according to $Z(T) = \text{tr} e^{-\beta H(T; -\mathcal{B})} = e^{-\beta F(T)}$. The system ends at time $t = T$ with the Hamiltonian $H(0; -\mathcal{B})$. The evolution operator of the backward process is defined as

$$i\hbar \frac{\partial}{\partial t} U_R(t; \mathcal{B}) = H(T - t; \mathcal{B}) U_R(t; \mathcal{B}), \quad (2.5)$$

with the initial condition $U_R(0; \mathcal{B}) = I$, and is related to the one of the forward process by the following

Lemma 1. *The forward and backward time evolution operators are related to each other according to*

$$\Theta U_F(T - t; \mathcal{B}) U_F^\dagger(T; \mathcal{B}) \Theta = U_R(t; -\mathcal{B}), \quad (2.6)$$

where t is an arbitrary time $0 \leq t \leq T$.

This lemma is proved by first substituting t to $T - t$ in Eq. (2.3) to get

$$-i\hbar \frac{\partial}{\partial t} U_F(T - t; \mathcal{B}) = H(T - t; \mathcal{B}) U_F(T - t; \mathcal{B}).$$

Multiplying this equation by $U_F^\dagger(T; \mathcal{B}) \Theta$ from the right and by Θ from the left, we find

$$i\hbar \frac{\partial}{\partial t} \Theta U_F(T - t; \mathcal{B}) U_F^\dagger(T; \mathcal{B}) \Theta = H(T - t; -\mathcal{B}) \Theta U_F(T - t; \mathcal{B}) U_F^\dagger(T; \mathcal{B}) \Theta,$$

where we used the antilinearity $\Theta i = -i\Theta$ of the time-reversal operator and its further property (2.1). This shows that the expression $\Theta U_F(T - t; \mathcal{B}) U_F^\dagger(T; \mathcal{B}) \Theta$ obeys the same evolution equation (2.5) as $U_R(t; -\mathcal{B})$. Since they also satisfy the same initial condition, $\Theta U_F(T; \mathcal{B}) U_F^\dagger(T; \mathcal{B}) \Theta = U_R(0; -\mathcal{B}) = I$, we have proven Eq. (2.6). QED.

With this lemma, we can now demonstrate the

Theorem 1. *Let us consider an arbitrary observable A with a definite parity under time reversal: $\Theta A \Theta = \epsilon_A A$, with $\epsilon_A = \pm 1$. It satisfies the following functional relation:*

$$\left\langle e^{\int_0^T dt \lambda(t) A_F(t)} e^{-\beta H_F(T)} e^{\beta H(0; \mathcal{B})} \right\rangle_F = e^{-\beta \Delta F} \left\langle e^{\epsilon_A \int_0^T dt \lambda(T-t) A_R(t)} \right\rangle_R, \quad (2.7)$$

where $\lambda(t)$ is an arbitrary function while the subscripts F and R stand for the forward or backward protocol, respectively. $\Delta F = F(T) - F(0)$ is the difference of the free energies of the initial equilibrium states (2.4) and (2.2) of the backward and forward processes.

In order to prove Eq. (2.7), we first consider the quantity $A_F(t)$ which can be written as

$$\begin{aligned}
A_F(t) &= U_F^\dagger(t) A U_F(t) \\
&= U_F^\dagger(T) U_F(T) U_F^\dagger(t) A U_F(t) U_F^\dagger(T) U_F(T) \\
&= \epsilon_A U_F^\dagger(T) \Theta A_R(T-t) \Theta U_F(T),
\end{aligned} \tag{2.8}$$

where we have inserted the identity $U_F^\dagger(T) U_F(T) = I$ to obtain the second equality. At the third equality, we inserted $\Theta^2 = I$ between the evolution operators and we used $\Theta A \Theta = \epsilon_A A$ along with Eq. (2.6). The connection is thus established with the backward process. Integrating over time with an arbitrary function $\lambda(t)$ and taking the exponential of both sides, the previous expression becomes

$$\exp \left(\int_0^T dt \lambda(t) A_F(t) \right) = U_F^\dagger(T) \Theta \exp \left(\epsilon_A \int_0^T dt \lambda(T-t) A_R(t) \right) \Theta U_F(T),$$

after the change of integration variables $t \rightarrow T-t$ in the right-hand side.

Therefore, the left-hand side of Eq. (2.7) reads

$$\begin{aligned}
&\text{tr} \rho(0) \exp \left(\int_0^T dt \lambda(t) A_F(t) \right) \exp[-\beta H_F(T)] \exp[\beta H(0; \mathcal{B})] \\
&= \frac{1}{Z(0)} \text{tr} \exp \left(\epsilon_A \int_0^T dt \lambda(T-t) A_R(t) \right) \Theta \exp[-\beta H(T)] \Theta \\
&= \frac{Z(T)}{Z(0)} \text{tr} \exp \left(\epsilon_A \int_0^T dt \lambda(T-t) A_R(t) \right) \rho(T) \\
&= e^{-\beta \Delta F} \left\langle \exp \left(\epsilon_A \int_0^T dt \lambda(T-t) A_R(t) \right) \right\rangle_R.
\end{aligned}$$

We used the invariance of the trace over cyclic permutations as well as the exponential of Eq. (2.8) at the first equality. In the second equality, we introduced the equilibrium density matrix (2.4) which is precisely the initial condition of the backward process. To obtain the last equality, we used that the partition functions have been expressed in terms of the corresponding free energies. This completes the proof of the theorem. QED.

We notice that related results have previously been considered in the restricted case where there is no change in free energy $\Delta F = 0$ [25]. The present theorem allows us to recover the quantum Jarzynski equality as a special case of Eq. (2.7) if $\lambda = 0$:

$$\left\langle e^{-\beta H_F(T)} e^{\beta H(0; \mathcal{B})} \right\rangle_F = e^{-\beta \Delta F}.$$

The factor inside the bracket can indeed be interpreted in the quantum setting in terms of the work performed on the system during the forward process [134, 210, 208] in spite of the non-commutativity of the energy operators $H_F(T)$ and $H(0; \mathcal{B})$ and thanks to the protocol with von Neumann quantum measurements

of the energy at the initial and final times. It is only in the classical limit that both energies commute and the classical work can be formed as $W_{cl} = [H_F(T) - H(0; B)]_{cl}$, in which case both exponentials in the left-hand side of the relation (2.7) becomes $\exp(-\beta W_{cl})$ which is the classical version of this relation.

Also, we can obtain various correlation functions by taking functional derivatives of relation (2.7). In particular, we show in the next section how to derive an expression of the linear response theory from the symmetry relation (2.7).

2.2 Linear response theory

In this section, we show how the generalized work relation (2.7) can be used to recover the linear response theory. For this purpose, we consider a perturbation of the form

$$H(t) = H(0; B) - X(t)B,$$

where the perturbation $X(t)$ is such that $X(t) = 0$ for $t \leq 0$ and $X(T) = 0$ for $T \leq t$. The observable B is arbitrary and should not be confused with the magnetic field. In order to obtain the linear response of an observable A with respect to the perturbation $-X(t)B$, we take the functional derivative of Eq. (2.7) with respect to $\lambda(T)$, around $\lambda = 0$. This yields

$$\left\langle A_F(T) e^{-\beta H_F(T)} e^{\beta H(0; B)} \right\rangle_F = \epsilon_A \langle A_R(0) \rangle_R = \epsilon_A \langle A \rangle_{eq, -B}, \quad (2.9)$$

where we used that $\Delta F = 0$ since $X(0) = X(T) = 0$. Since the reversed process also starts at equilibrium, the average in the right-hand side is an equilibrium average, albeit with a reversed magnetic field. Nevertheless, we have that $\epsilon_A \langle A \rangle_{eq, -B} = \langle A \rangle_{eq, B}$ by using time reversal. We now have to calculate the exponentials of the initial and final Hamiltonians. Since, in the Heisenberg representation, the total time derivative of the Hamiltonian equals its partial derivative, $dH_F/dt = (\partial H/\partial t)_F$, we can write

$$\exp[-\beta H_F(T)] = \exp[-\beta(H(0; B) + E)]$$

with

$$E = \int_0^T dt \left(\frac{\partial H}{\partial t} \right)_F = - \int_0^T dt \dot{X}(t) B_F(t) = \int_0^T dt X(t) \dot{B}_F(t),$$

where the last equality follows from an integration by parts. We now use the expression

$$\exp[\beta(P + Q)] \exp(-\beta P) = 1 + \int_0^\beta du \exp[u(P + Q)] Q \exp(-uP)$$

which can be proved by differentiating with respect to β . To first order in Q , we may neglect Q in the last exponential function, $\exp[u(P + Q)]$. Taking $P = -H(0; \mathcal{B})$ and $Q = -E$ and developing to first order in X , we get

$$\begin{aligned} e^{-\beta H_F(T)} e^{\beta H(0; \mathcal{B})} &= 1 - \int_0^T dt X(t) \int_0^\beta du e^{-uH_0} \dot{B}(t) e^{uH_0} + O(X^2) \\ &= 1 - \int_0^T dt X(t) \int_0^\beta du \dot{B}(t + i\hbar u) + O(X^2) \end{aligned}$$

where $B(t) = \exp(iH_0 t/\hbar) B \exp(-iH_0 t/\hbar)$ since, at first order in the driving force, the time evolution proceeds under the unperturbed Hamiltonian $H_0 \equiv H(0; \mathcal{B})$. Inserting this expansion into Eq. (2.9) and after some manipulations using the time invariance of correlation function as well as the KMS property $\rho A = A(i\hbar\beta)\rho$, we finally find

$$\langle A_F(T) \rangle = \langle A \rangle_{\text{eq}, \mathcal{B}} + \int_0^T dt X(T-t) \phi_{AB}(t) + O(X^2), \quad (2.10)$$

with the response function

$$\phi_{AB}(t) = \int_0^\beta du \langle \dot{B}(-i\hbar u) A(t) \rangle_{\text{eq}}. \quad (2.11)$$

Equations (2.10) and (2.11) are the well-known expressions of linear response theory in the canonical ensemble, also known as the Green-Kubo formula [105, 131]. The Casimir-Onsager reciprocity relations for the conductivities [175, 37] are obtained by taking $A = J_\mu/V$ and $\dot{B} = J_\nu$ in terms of the current $J_\mu = \sum_n e_n \dot{x}_{n\mu}$ and the volume V , in which case the time-reversal symmetry implies $\phi_{\mu\nu}(t; \mathcal{B}) = \phi_{\nu\mu}(t; -\mathcal{B})$ and $\sigma_{\mu\nu}(\omega; \mathcal{B}) = \sigma_{\nu\mu}(\omega; -\mathcal{B})$ for the tensor of conductivities $\sigma_{\mu\nu}(\omega; \mathcal{B}) = \int_0^\infty dt e^{i\omega t} \phi_{\mu\nu}(t; \mathcal{B})$. Higher-order terms in the expansion can be obtained as well.

2.3 Summary

In this chapter, we investigated the time-reversal symmetry in driven quantum systems. We have obtained a universal quantum work relation which involves arbitrary observables at arbitrary times. This result relates an average over the forward process ponderated by the quantum analogue of the work to an average over the reversed process. By taking functional derivatives, we can obtain relations for arbitrary correlation functions, which are the consequence of microreversibility. In the simplest case, it can be used to recover the well-known Jarzynski equality. On the other hand, we can also straightforwardly derive from the universal relation the linear response theory of an arbitrary observable. In this regard, this relation unifies in a common framework the work relations and the response theory, thereby opening the possibility to obtain further general relations which are valid not only close to equilibrium but also in the far-from-equilibrium regime. In the next chapter, we proceed to study open systems sustaining a nonequilibrium stationary state.

Fluctuation Theorem for Macroscopic Currents

Out-of-equilibrium systems are ubiquitous in nature and they play an essential role in many physical, chemical and biological phenomena [183, 168, 164, 151, 169, 171, 122, 80]. In an isolated system, a nonequilibrium state will spontaneously relax toward the state of thermodynamic equilibrium. However, a nonequilibrium steady state can be maintained in open systems by exchanging energy or matter with thermostats or chemistats. Such reservoirs may generate thermodynamic forces – also called affinities [51] – because they introduce inhomogeneities in the temperature, the pressure, or the chemical potentials of the different species of molecules. In turn, the thermodynamic forces or affinities generate fluxes of energy or matter across the system, which contribute to positive entropy production as described at the macroscopic level by nonequilibrium thermodynamics.

On the other hand, from the scale of the micrometer down to the nanometer, the molecular fluctuations manifest themselves and require a stochastic description. Thermodynamics concepts such as currents and affinities can be identified in the stochastic description thanks to a graphical representation of the process. In this representation, the cyclic trajectories play a crucial role. In this chapter, we will focus on the statistical behavior of the nonequilibrium fluxes crossing the system in a steady state. We shall prove that the fluctuations of the macroscopic currents obey a fluctuation relation, which is a consequence of an underlying fundamental symmetry of the dynamical evolution. In contrast with Eq. (1.5), the fluctuation symmetry is here expressed in terms of the macroscopic affinities driving the system out of equilibrium. Finally, we will show how this result can be applied to several systems of interest such as chemical reactions networks or molecular motors.

3.1 Mesoscopic description

3.1.1 Master equation

Many nonequilibrium phenomena are successfully described at the mesoscopic level in terms of Markovian random processes. Examples of such jump processes include birth-and-death processes in stochastic chemical kinetics and population dynamics [168, 190, 169], as well as kinetic processes in quantum field theory [229] and in quantum optics [147]. In some simple systems, such processes can be rigorously derived from the underlying deterministic or quantum dynamics by introducing an appropriate partition of the phase space [211, 84] or in some scaling limit [204].

Such continuous-time random processes are ruled by an evolution equation, called the *master equation*, for the probability to find the system in a coarse-grained state ω at time t :

$$\frac{dP_t(\omega)}{dt} = \sum_{\rho, \omega'} [W_\rho(\omega'|\omega)P_t(\omega') - W_\rho(\omega|\omega')P_t(\omega)] . \quad (3.1)$$

The quantities $W_\rho(\omega'|\omega)$ denote the rates of the transitions $\omega' \xrightarrow{\rho} \omega$ allowed by the elementary processes $\rho = 1, 2, \dots, r$. Each one of these elementary processes may independently contribute to the entropy production so that it is important to separate them in the master equation. This is for example the case in nonequilibrium reaction systems where the variable ρ corresponds to the different reactions [122, 85].

A trajectory picture of this stochastic process is obtained as follows. When the system is in state ω , it waits a random time exponentially distributed with an average $1/R(\omega)$, where $R(\omega) \equiv \sum_{\rho, \omega'} W_\rho(\omega|\omega')$. It then jumps onto another state ω' with probability $W_\rho(\omega|\omega')/R(\omega)$. The probability of a full trajectory is obtained by iterating these steps. This description is at the basis of the simulation algorithm by Gillespie [99]. Ergodicity of these stochastic processes entails the equivalence between time and ensemble averages.

The master equation (3.1) is known to obey a *H-theorem* for the entropy

$$S_t = \sum_{\omega} S(\omega)P_t(\omega) - \sum_{\omega} P_t(\omega) \ln P_t(\omega) \quad (3.2)$$

associated with the probability distribution $P_t(\omega)$ describing the state of the system at the time t [190, 122]. $S(\omega)$ denotes the entropy due to the statistical distribution of all the degrees of freedom which are not specified by the coarse-grained state ω [85]. For instance, if the coarse-grained state ω only specifies the numbers of the particles of the different species, $S(\omega)$ is the entropy of the statistical distribution of the positions and momenta of the particles. The second term is the contribution to the entropy due to the statistical distribution $P_t(\omega)$ of the coarse-grained states [85]. The entropy is here calculated in the units of Boltzmann's constant $k_B \simeq 1.38 \cdot 10^{-23}$ J/K. The time derivative of the entropy is given by

$$\frac{dS}{dt} = \frac{d_i S}{dt} + \frac{d_e S}{dt}, \quad (3.3)$$

in terms of the entropy flux $d_e S/dt$ and the non-negative entropy production

$$\frac{d_i S}{dt} = \frac{1}{2} \sum_{\rho, \omega, \omega'} J_\rho(\omega, \omega') A_\rho(\omega, \omega') \geq 0, \quad (3.4)$$

where

$$J_\rho(\omega, \omega') \equiv P_t(\omega) W_\rho(\omega|\omega') - P_t(\omega') W_\rho(\omega'|\omega) \quad (3.5)$$

is the current of the transition $\omega \xrightarrow{\rho} \omega'$ and

$$A_\rho(\omega, \omega') \equiv \ln \frac{P_t(\omega) W_\rho(\omega|\omega')}{P_t(\omega') W_\rho(\omega'|\omega)} \quad (3.6)$$

the corresponding affinity [190, 122]. The H -theorem asserts that the entropy production (3.4) is always non-negative in agreement with the second law of thermodynamics and thus characterizes the irreversibility of the process.

In a stationary state where $dP/dt = 0$, the master equation (3.1) can also be written in terms of the currents (3.5) as

$$\sum_{\rho, \omega'} J_\rho(\omega', \omega) = 0, \quad (3.7)$$

which is nothing else than the Kirchhoff current law [190].

In the equilibrium stationary state, the conditions of detailed balance

$$P_{eq}(\omega) W_\rho(\omega|\omega') = P_{eq}(\omega') W_\rho(\omega'|\omega) \quad (3.8)$$

are satisfied for all the possible forward and backward transitions $\omega \xrightleftharpoons{\rho} \omega'$. Thereby, the currents (3.5) and the affinities (3.6) as well as the entropy production (3.4) all vanish at equilibrium. An equivalent condition, known as *Kolmogorov's criterion* [129], can be expressed in terms of the sole transition rates. Let us consider a cyclic path $\omega_1 \rightarrow \omega_2 \rightarrow \dots \rightarrow \omega_n \rightarrow \omega_1$. Such a cycle will be denoted by $c \equiv (\omega_1 \omega_2 \dots \omega_n)$ and the corresponding reversed cycle by $c_- \equiv (\omega_1 \omega_n \dots \omega_2)$. Here, we define the quantities $w_c \equiv W(\omega_1|\omega_2) \dots W(\omega_{n-1}|\omega_n) W(\omega_n|\omega_1)$ as the product of the transition rates along the cycle c . The equilibrium conditions (3.8) are equivalent to

$$w_c = w_{c_-} \quad (3.9)$$

for every cycle c . If this condition is violated for a cycle, the system is in a nonequilibrium stationary state characterized by the presence of non vanishing probability currents and a positive entropy production [190, 121]. This illustrates the importance of the cyclic trajectories in the stochastic process. This observation can be further elaborated, as done in the next section.

3.1.2 Cycle representation

As we observed in the previous section, thermodynamic equilibrium can be rephrased in terms of the cyclic paths of the stochastic process. Our purpose in this section is to show that the cyclic trajectories link in a fundamental way the transition rates of the stochastic process to the thermodynamic description.

At the macroscopic level, nonequilibrium constraints are imposed to a system if the temperature, the pressure, or the chemical potentials differ between the reservoirs surrounding the system. These constraints are characterized by the global thermodynamic forces or affinities defined by the differences of the temperatures, pressures, or chemical potentials of the reservoirs. These global affinities do not directly appear in the transition rates $W_\rho(\omega|\omega')$ of the master equation (3.1). These rates depend on the temperature, the pressure, or the chemical potentials of the reservoir responsible for the transition $\omega \xrightarrow{\rho} \omega'$ and also on the states ω and ω' . In contrast, the global affinities $\{A_\alpha\}$ are defined by the *differences* of the temperatures, pressures, or chemical potentials of the reservoirs and are independent of the particular states ω or ω' . In this regard, the global affinities are macroscopic. We shall show below how a cyclic representation of the stochastic process can be used to recover these macroscopic affinities.

We now introduce the cycle analysis of the random process. For a system ruled by the master equation (3.1), a graph G is associated as follows: each state ω of the system corresponds to a vertex or node while the edges represent the different transitions $\omega \xrightarrow{\rho} \omega'$ allowed between the states. Accordingly, two states are connected by several edges if several elementary processes ρ allow transitions between them. Several examples will be detailed in Sec. 3.3.

An orientation is given to each edge of the graph G . The directed edges are thus defined by

$$e \equiv \omega \xrightarrow{\rho} \omega'. \quad (3.10)$$

Let f be a directed subgraph of G . The orientation of the subgraph f with respect to its edges $\{e\}$ is described by introducing the quantity

$$S_e(f) \equiv \begin{cases} +1 & \text{if } e \text{ and } f \text{ are parallel,} \\ -1 & \text{if } e \text{ and } f \text{ are antiparallel,} \\ 0 & \text{if } e \text{ is not in } f, \end{cases} \quad (3.11)$$

where e and f are said to be parallel (resp. antiparallel) if f contains the edge e in its reference (resp. opposite) orientation.

A graph G usually presents a huge number of cyclic paths c . However, all cyclic paths are not independent. They can be expressed by a linear combination of a smaller subset of cycles, called the *fundamental set*, which plays the role of a basis in the space of cycles. The fundamental set generalizes the concept of mesh currents introduced in the analysis of planar electrical circuits. A method has been provided by Schnakenberg to identify all the independent cycles of a

graph [190]. Formally, the method is based on the definition of a *maximal tree* $T(G)$, which is a subgraph of the graph G satisfying the following properties:

- T contains all the vertices of G ;
- T is connected;
- T contains no circuit, i.e., no cyclic sequence of edges.

In general a given graph G has several maximal trees.

The edges l of G which do not belong to T are called the *chords* of T . If we add to T one of its chords l , the resulting subgraph $T + l$ contains exactly one circuit, c_l , which is obtained from $T + l$ by removing all the edges which are not part of the circuit. Each chord l thus defines a unique cycle c_l called a *fundamental cycle*. Henceforth, we will use the convention that the orientation is such that $S_l(c_l) = 1$, i.e., the cycles are oriented as the chords l .

For a graph with N vertex and E edges, there exist $E - N + 1$ chords. However, the number of maximal trees cannot be easily calculated as it crucially depends on the topology of the graph.¹

We can now formulate many important thermodynamic concepts in terms of cycles. In a stationary state, the mean number of cycles per unit time $J_{c\pm}$ accomplished in the positive and negative orientations are related to each other by $J_{c+}/J_{c-} = \exp(w_{c+}/w_{c-})$ as shown in Refs. [113, 121]. It is therefore natural to define the affinity of a cycle as

$$A(c) = \ln \frac{w_{c+}}{w_{c-}}. \quad (3.12)$$

According to Kolmogorov's criterion (3.9), these affinities consistently vanish at equilibrium.

Each cycle can now be decomposed in terms of the fundamental cycles [190]. For instance, the affinity of an arbitrary cycle c can be expressed as

$$A(c) = \sum_l S_l(c) A(c_l), \quad (3.13)$$

where the sum extends over all the chords. This shows in particular that the affinity of an arbitrary cycle is a linear combination of the affinities of a fundamental set [190]. Accordingly, the maximal tree T can be chosen arbitrarily because each cycle c_l can be redefined by linear combinations of the fundamental cycles. Equation (3.13) also implies that the equilibrium conditions (3.9) are redundant. Detailed balance will hold if and only if conditions (3.9) are satisfied for the fundamental cycles only. Indeed, if the affinities $A(c_l) = 0$ of all fundamental cycles vanish, so does the affinity of any cycle c as shown by Eq. (3.13), fulfilling relations (3.9). This illustrates the fact that the fundamental cycles constitute a basis identifying the *independent* contributions to the stochastic process.

¹ Kirchhoff's matrix tree theorem relates the number of maximal trees to the product of the positive eigenvalues of the admittance matrix of the graph G .

Finally, the entropy production (3.4) can be expressed in terms of the fundamental cycles as

$$\frac{d_i \mathcal{S}}{dt} = \sum_l J_l A_l,$$

where $A_l = A(c_l)$ and J_l is the flux (3.5) corresponding to the chord l . The entropy production is thus written as the product of the fluxes and the affinities, in concordance with macroscopic thermodynamics [169].

These observations led Hill [113] and Schnakenberg [190] to identify the macroscopic nonequilibrium constraints of a system to the affinities of the fundamental cycles c_l of the graph. This identification is verified in a large class of processes including diffusion processes in lattice gases, nonequilibrium chemical reactions, and electronic transport in mesoscopic conductors [112, 115, 190, 139, 2, 3, 4], as will be shown on several examples in section 3.3. These conditions are weaker than in systems with external mechanical forces where the affinities can be directly identified at the level of the transition rates themselves.

We notice that there can still exist more cycles c_l than macroscopic processes α . The reason is that the graph describes all the possible states and transitions at the mesoscopic level while the currents α are typically macroscopic and fewer than the mesoscopic states. The affinities or thermodynamic forces $A(c_l)$ associated with the various cycles c_l of a graph G may thus take the same value for all cycles corresponding to the same current α : $A(c_l) = A_\alpha$ for all $c_l \in \alpha$.

3.2 Demonstration of the fluctuation theorem

In this section we shall derive a fluctuation theorem for the currents. Our construction is based on Schnakenberg network analysis [190] which is used to obtain the independent physical processes in the system. We use the graphical representation of the system where the nodes correspond to the states and the edges e correspond to transitions between the states of the system. The observables we are interested in are the independent currents. The instantaneous current on the chord l is defined by

$$j_l(t) \equiv \sum_{n=-\infty}^{+\infty} S_l(e_n) \delta(t - t_n), \quad (3.14)$$

where t_n is the time of the random transition e_n during a path of the stochastic process. We use the convention that j_l is oriented as the graph G since $S_l(e_n)$ is equal to (-1) if the transition e_n is (anti)parallel to the chord l . The current (3.14) is a fluctuating random variable. The so-called *Helfand moment* [111] associated with the current is defined by

$$G_l(t) \equiv \int_0^t dt' j_l(t'). \quad (3.15)$$

The quantity of interest is the generating function of the *independent* currents crossing the chords of the system

$$F(\lambda) \equiv \lim_{t \rightarrow \infty} -\frac{1}{t} \ln \left\langle e^{-\sum_l \lambda_l \int_0^t dt' j_l(t')} \right\rangle, \quad (3.16)$$

where the sum is taken over all the chords. We refer to Appendix A for a discussion on the properties of such a generating function. It here describes the long time behavior of the probability distribution of the currents in the system. We notice that the generating function (3.16) can be written as [93]

$$F(\lambda) \equiv \lim_{t \rightarrow \infty} -\frac{1}{t} \ln \left\langle e^{-\sum_l \lambda_l G_l(t)} \right\rangle$$

in terms of the Helfand moments (3.15). Our goal will be to prove the following symmetry relation

$$F(\lambda) = F(A - \lambda), \quad (3.17)$$

where we introduced the vector A regrouping the fundamental affinities A_l .

In order to prove Eq. (3.17), we define

$$F_t(\omega, \lambda) \equiv \left\langle e^{-\sum_l \lambda_l G_l(t)} \right\rangle_\omega$$

as the mean value of $\exp[-\sum_l \lambda_l G_l(t)]$ conditioned on the system being in state ω at time $t = 0$. The generating function (3.16) is then obtained as $F(\lambda) = \lim_{t \rightarrow \infty} (-1/t) \sum_\omega P_{st}(\omega) F_t(\omega, \lambda)$. We will first derive an evolution equation for the functions $F_t(\omega, \lambda)$. The quantity $F_{t+dt}(\omega, \lambda)$ is calculated by considering all the possible transitions occurring in an infinitesimal time dt starting from state ω at time $t = 0$, before evolving the ensemble average for a time t starting from the state ω' reached at time dt . All these possible transitions must be weighted with their respective occurrence probabilities multiplied by the contributions of the exponential factors $\exp[-\sum_l \lambda_l G_l(t)]$. For clarity purpose, we present the case where only one type of transition between two states is present, i.e., the variable ρ can be omitted. The demonstration can be extended to the general case as well [6]. We also introduce the quantities

$$z(\omega|\omega') \equiv \sum_l S_l(\omega \rightarrow \omega') \lambda_l,$$

which are expressed in terms of the orientation numbers (3.11) and where the sum is taken over all chords. They take the values $\pm \lambda_l$ when one of the transitions l occur in the direct or reversed orientation and 0 otherwise. The quantity $F_{t+dt}(\omega, \lambda)$ can then be written as

$$F_{t+dt}(\omega, \lambda) = \sum_{\omega'} P_{dt}(\omega|\omega') e^{-z(\omega|\omega')} F_t(\omega', \lambda). \quad (3.18)$$

In Eq. (3.18), the exponential factor comes from the contributions of the quantity $\exp[-\sum_l \lambda_l G_l(t)]$ during the transition $\omega \rightarrow \omega'$. $P_{dt}(\omega|\omega')$ corresponds to

the probability that, starting from state ω , the system ends in state ω' after a time dt . This conditional probability is given by

$$P_{dt}(\omega|\omega') = \begin{cases} W(\omega|\omega')dt + O(dt^2) & \text{if } \omega \neq \omega' \\ 1 - \sum_{\omega'} W(\omega|\omega')dt + O(dt^2) & \text{if } \omega = \omega'. \end{cases}$$

Inserting this formula into Eq. (3.18) and taking the limit $dt \rightarrow 0$ yields

$$\begin{aligned} \frac{d}{dt} F_t(\omega, \lambda) &= \sum_{\omega'} \left[W(\omega|\omega') e^{-z(\omega|\omega')} F_t(\omega', \lambda) - W(\omega|\omega') F_t(\omega, \lambda) \right] \\ &\equiv \hat{L}_\lambda F_t(\omega, \lambda) \end{aligned} \quad (3.19)$$

with the initial condition $F_{t=0}(\omega, \lambda) = 1$. Whereupon we find that

$$\left\langle e^{-\sum_i \lambda_i \int_0^t dt' j_i(t')} \right\rangle = \sum_{\omega} P_{st}(\omega) F_t(\omega, \lambda) = \sum_{\omega, \omega'} P_{st}(\omega) \left[e^{\hat{L}_\lambda t} \right]_{\omega, \omega'}.$$

With our conditions on the transition rates, the Perron-Frobenius theorem is of application and there is a unique maximal eigenvector V_λ (which is positive)

$$\hat{L}_\lambda V_\lambda = -F(\lambda) V_\lambda. \quad (3.20)$$

By virtue of this relation, the limit in Eq. (3.16) exists and the leading eigenvalue of the operator (3.19) gives the generating function of the currents.

We now prove the symmetry (3.17). This was first accomplished in Ref. [6] using the determinant expansion in terms of cycles. We will here present a more direct approach. Our strategy to prove relation (3.17) will be to show that the evolution operator (3.19) presents the corresponding symmetry. Indeed, an appropriate change of basis $\hat{U}$ will show that

$$\hat{L}'_\lambda = \hat{U} \hat{L}_\lambda \hat{U}^{-1} = \hat{L}'_{A-\lambda}, \quad (3.21)$$

where T denotes the transpose. As a consequence, all eigenvalues of the operator $\hat{L}$ will present the symmetry $\lambda \rightarrow A - \lambda$. Since the generating function (3.16) is given by the largest eigenvalue of $\hat{L}$ according to (3.20), this will conclude the proof of the result (3.17).

The construction goes as follows. We consider an operator $\hat{U}$ of the form

$$\hat{U}_{\omega\omega'} = u_{\omega} \delta_{\omega, \omega'}.$$

This operator is invertible if none of the coefficients u is equal to zero, in which case $\hat{U}_{\omega\omega'}^{-1} = (1/u_{\omega}) \delta_{\omega, \omega'}$. In this new basis, the operator (3.19) is transformed into $\hat{L}'_{\omega\omega'} = \hat{L}_{\omega\omega'} (u_{\omega}/u_{\omega'})$. We now proceed to choose the elements u_{ω} . For all transitions $\omega \rightarrow \omega'$ which do not correspond to a chord, we impose that the ratios $u_{\omega}/u_{\omega'}$ satisfy

$$\frac{u_{\omega}}{u_{\omega'}} = \left(\frac{W_{\omega'\omega}}{W_{\omega\omega'}} \right)^{\frac{1}{2}}. \quad (3.22)$$

Thereby, the elements of $\hat{L}'$ now take the values

$$\hat{L}'_{\omega\omega'} = \sqrt{W_{\omega\omega'}W_{\omega'\omega}} = \hat{L}'_{\omega'\omega} \quad (3.23)$$

if the transition $\omega \rightarrow \omega'$ is not a chord, because $z_{\omega\omega'} = 0$ and $L_{\omega\omega'} = W_{\omega\omega'}$ in this case. These elements are now symmetric.

At this point, we have fixed the value of $N - 1$ ratios of the form (3.22). Indeed, if there are E edges in the graph and N states, there exist $E - N + 1$ chords in the graph so that we have fixed $E - (E - N + 1) = N - 1$ ratios. This is precisely the number of independent variables we can specify. The operator $\hat{U}$ is now determined up to a multiplicative factor.

The remaining elements of $\hat{L}'$ corresponding to the chords are now obtained as follows. For a fundamental cycle $c_l = (\omega_1, \dots, \omega_l)$, we have the identity

$$\prod_{i=1}^l \frac{u_{\omega_i}}{u_{\omega_{i+1}}} = \frac{u_{\omega_l}}{u_{\omega_1}} \prod_{i=1}^{l-1} \frac{u_{\omega_i}}{u_{\omega_{i+1}}} = 1, \quad (3.24)$$

where $\omega_{l+1} \equiv \omega_1$ and the transition $\omega_l \rightarrow \omega_1$ corresponds to the chord l in the positive direction. By construction, a fundamental cycle c_l only contains its associated chord l . Hence, using Eq. (3.22) as well as Eq. (3.12), equation (3.24) becomes

$$\frac{u_{\omega_1}}{u_{\omega_l}} = \left(\frac{W_{\omega_l\omega_1}}{W_{\omega_1\omega_l}} \right)^{\frac{1}{2}} \prod_{i=1}^l \left(\frac{W_{\omega_{i+1}\omega_i}}{W_{\omega_i\omega_{i+1}}} \right)^{\frac{1}{2}} = \left(\frac{W_{\omega_l\omega_1}}{W_{\omega_1\omega_l}} \right)^{\frac{1}{2}} e^{-A_l/2}.$$

Accordingly, the operator elements associated with the chord l read

$$\hat{L}'_{\omega_l\omega_1}(\lambda_l) = \sqrt{W_{\omega_l\omega_1}W_{\omega_1\omega_l}} e^{A_l/2 - \lambda_l} = \hat{L}'_{\omega_1\omega_l}(A_l - \lambda_l)$$

because $z_{\omega_l\omega_1}(\lambda_l) = \lambda_l$ and $L_{\omega_l\omega_1}(\lambda_l) = \exp(-z_{\omega_l\omega_1})W_{\omega_l\omega_1}$. This result, along with Eq. (3.23), proves Eq. (3.21). Consequently, all the eigenvalues of $\hat{L}_\lambda$, including the generating function, will present the symmetry (3.17). QED.

The fluctuation theorem for the current is thus related to a fundamental symmetry property of the evolution operator. It is also remarkable that this relation is valid arbitrarily far from equilibrium. In addition, in this new basis the evolution operator $\hat{L}$ takes a more symmetric form that is more stable for numerical algorithms searching for eigenvalues. Eventually, one can check in the same way that this construction can be extended to the case where there exist several transitions ρ between two states [6].

As explained in section 3.1.2, we can now regroup the different microscopic currents corresponding to a given macroscopic process α :

$$j_\alpha(t) \equiv \sum_{l \in \alpha} \sum_{n=-\infty}^{+\infty} S_l(e_n) \delta(t - t_n). \quad (3.25)$$

On the other hand, we may set $\lambda_l = \lambda_\alpha$ for all $l \in \alpha$ and define $Q(\{\lambda_\alpha\}) = F(\{\lambda_l = \lambda_\alpha\})$. Since all fundamental cycles contributing to a given transport process α have the same affinities, we have demonstrated the

Theorem 2. *The generating function of the macroscopic currents*

$$Q(\lambda) \equiv \lim_{t \rightarrow \infty} -\frac{1}{t} \ln \left\langle e^{-\sum_\alpha \lambda_\alpha \int_0^t dt' j_\alpha(t')} \right\rangle \quad (3.26)$$

obeys the symmetry relation

$$Q(\lambda) = Q(A - \lambda). \quad (3.27)$$

We note that this symmetry property involves the ensemble of currents in the system. In Ref. [7], we derived the necessary and sufficient conditions under which a fluctuation theorem will hold for a *single* macroscopic current. These conditions are expressed in terms of geometrical and thermodynamical conditions on the graph. In such a situation, a given current j_α satisfies a fluctuation symmetry regardless of the other thermodynamic processes at stake [7].

We also notice that Theorem 2 can be extended to the more general class of semi-Markov processes. In this framework, the waiting-time distribution between the jumps is non-exponential, implying a non-Markovian behavior. The demonstration of the fluctuation theorem (3.27) for this class of stochastic processes is presented in Appendix B.

As shown in the Appendix A, the generating functions are closely related to the large fluctuations. Large deviation theory indeed shows that, asymptotically,

$$P(G_t/t = \zeta) \sim e^{-tI(\zeta)} \quad (t \rightarrow \infty),$$

where G_t regroups the Helfand moments and I is the Legendre transform of Q . Hence, the fluctuation symmetry (3.27) is reflected on the rate function $I(\zeta)$ as follows:

$$\begin{aligned} I(\zeta) &= \max_{\lambda} \{Q(\lambda) - \lambda\zeta\} \\ &= \max_{\lambda} \{Q(A - \lambda) - \lambda\zeta\} \\ &= \max_{\lambda} \{Q(\lambda) - (A - \lambda)\zeta\} \\ &= I(-\zeta) - A \cdot \zeta \end{aligned}$$

or, equivalently,

$$P(G_t/t = \zeta) \sim e^{K \cdot A} P(G_t/t = -\zeta) \quad (3.28)$$

in the limit $t \rightarrow \infty$. This relation shows that the probability to observe a given positive fluctuation is exponentially more likely than the probability to observe

the opposite one. This exponential dependence is proportional to time as well as to the affinities driving the systems out of equilibrium.

Relations (3.27) and (3.28) contain more information than the fluctuation theorem for the entropy production obtained by Lebowitz and Spohn [139]. Indeed, this last relation can be recovered from the present theorem by merging the different currents onto a single quantity, which is accomplished by the substitution $\lambda_\alpha \rightarrow \lambda A_\alpha$. In this case we find the symmetry $Q'(\lambda) = Q'(1 - \lambda)$ [139, 6], but going the other way around is not possible. This shows that the fluctuation theorem for currents is more detailed than the usual fluctuation theorem for the entropy production.

In Chapter 4, we shall explore the consequences of this symmetry for the nonlinear response theory. Here below, we will illustrate the results of this section on different mesoscopic systems.

3.3 Applications

In this section, we survey a number of chemical, physical, and biological systems where the fluctuation symmetry (3.27) or (3.28) can be applied.

3.3.1 Chemical reaction networks

Chemical reactions can be driven out of equilibrium by pumping reactants into a reactor and allowing the outflow of products. If the reactor is continuously stirred and maintained at constant temperature, the system is homogeneous and isothermal. Under such conditions, the reacting system can be described by the randomly fluctuating numbers of molecules of the intermediate species.

Our purpose in this section is to illustrate Schnakenberg's network theory with the example of the nonlinear chemical network



Particles enter into the system from three different reservoirs, A, B, and C. The particles of the species X are produced by three different reactions from so many reactants. The reaction constants are denoted by $k_{\pm\rho}$ with $\rho = 1, 2, 3$. Out of equilibrium, fluxes of matter will cross the system between the three reservoirs. In a stationary state only two such fluxes will be independent and they will constitute the relevant macroscopic observables. According to the mass-action law [168, 164, 151, 169, 171, 122], the transition rates of these reactions are proportional to the concentrations and given by

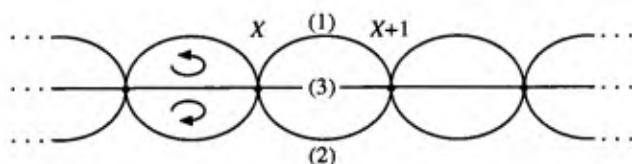


Fig. 3.1. Graph G associated with the reaction network (3.29).

$$\begin{aligned}
 W_1(X|X+1) &= k_1[A]\Omega, \\
 W_1(X|X-1) &= k_{-1}X, \\
 W_2(X|X+1) &= k_2[B]X, \\
 W_2(X|X-1) &= k_{-2}\Omega^{-1}X(X-1), \\
 W_3(X|X+1) &= k_3[C]\Omega^{-1}X(X-1), \\
 W_3(X|X-1) &= k_{-3}\Omega^{-2}X(X-1)(X-2),
 \end{aligned}$$

where X is the number of particles of the species X , $[\cdot]$ denotes the mean concentration in the reservoirs, and Ω an extensivity parameter such as the volume of the reactor. The transition rates are nonlinear functions of the internal state X . The state of the system is described in terms of the probability $P(X, t)$ that the system contains the number X of particles of the species X at the time t . The master equation ruling the time evolution takes the form:

$$\frac{d}{dt}P(X, t) = \sum_{\rho=1}^3 \sum_{\nu=\pm 1} \left[W_{\rho}(X-\nu|X)P(X-\nu, t) - W_{\rho}(X|X-\nu)P(X, t) \right].$$

We notice that neither the macroscopic currents nor the macroscopic affinities are apparent in the master equation contrary to the systems where the affinities are given in terms of mechanical forces. In order to identify them, we use the graph analysis of Schnakenberg [190]. The graph of this stochastic process is depicted in Fig. 3.1.

At equilibrium, the detailed balance conditions (3.8) hold which imply that

$$\frac{k_1[A]}{k_{-1}} = \frac{k_2[B]}{k_{-2}} = \frac{k_3[C]}{k_{-3}} = \frac{\langle X \rangle_{\text{eq}}}{\Omega}$$

and the equilibrium probability distribution $P_{\text{eq}}(X)$ is Poissonian of average $\langle X \rangle_{\text{eq}}$. Out of equilibrium, the detailed balance conditions are no longer satisfied and the probability distribution $P_{\text{st}}(X)$ of the nonequilibrium steady state is known to be non-Poissonian for such nonlinear chemical networks [168, 164, 151, 169, 171]. For the present model, the steady state is given by:

$$\begin{aligned}
 P_{\text{st}}(X) &= P_{\text{st}}(0) \prod_{x=1}^X \frac{\sum_{\rho} W_{\rho}(x-1|x)}{\sum_{\rho} W_{\rho}(x|x-1)} \\
 &= P_{\text{st}}(0) \prod_{x=1}^X \frac{k_1[A]\Omega + k_2[B](x-1) + k_3[C](x-1)(x-2)\Omega^{-1}}{k_{-1}x + k_{-2}x(x-1)\Omega^{-1} + k_{-3}x(x-1)(x-2)\Omega^{-2}}.
 \end{aligned}$$

An affinity can be associated with each reaction ($\rho = 1, 2, 3$) according to Eq. (3.6):

$$A_\rho(X) \equiv \ln \frac{P_{\text{st}}(X)W_\rho(X|X+1)}{P_{\text{st}}(X+1)W_\rho(X+1|X)}.$$

These mesoscopic affinities depend on the internal state X , and thus fluctuates in time. Therefore, they do not correspond to the time-independent macroscopic affinities. These latter are instead given by the Schnakenberg conditions. A possible maximal tree is given by choosing all edges corresponding to reaction $\rho = 3$. The chords correspond in this case to all edges associated with the reactions $\rho = 1, 2$. The corresponding cycles are depicted in Fig. 5.1. They start from the state X and go to the state $X+1$ by the edge $\rho = 3$ and return to the state X by the edges $\rho = 1$ or $\rho = 2$. The corresponding macroscopic affinities are given by

$$A_\rho \equiv \ln \frac{W_3(X|X+1)W_\rho(X+1|X)}{W_3(X+1|X)W_\rho(X|X+1)}$$

or

$$A_1 = \ln \frac{k_{-1}k_3[C]}{k_{-3}k_1[A]}, \quad A_2 = \ln \frac{k_{-2}k_3[C]}{k_{-3}k_2[B]}, \quad (3.30)$$

which are independent of the internal state X and thus constant in time as it should.

This example shows that the mesoscopic affinities (3.6) in general depend on the state of the system and will thus fluctuate in time together with the corresponding currents (3.5) along a stochastic trajectory of the process. This problem is overcome by introducing the affinities defined by the conditions (3.12) and corresponding to the macroscopic affinities (3.30). These latter no longer depend on the mesoscopic state and are thus independent of time as expected for nonequilibrium constraints from macroscopic reservoirs. The application of these concepts to arbitrary networks of reactions can be found in Ref. [2].

3.3.2 Electronic transport in mesoscopic junctions

The nature of the current flow at low temperatures through mesoscopic structures has received a lot of attention during recent years. After initial focus on the conductance, which measures the average number of electrons transmitted in time, there has been an increasing interest for the noise power, a measure for the variance of the transmitted charge [24]. The next logical step is to study the full distribution function of the charge transmitted through a mesoscopic conductor, which is now within experimental reach [76]. After the pioneering work of Ref. [142], several methods have been developed in order to obtain the full counting statistics in mesoscopic conductors. In the semiclassical limit, Nazarov and Bagretz derived a circuit theory based on Keldysh Green's

function formalism [163]. The statistics of the currents can also be described in terms of a stochastic path integral [181] or a cascade of Boltzmann-Langevin equations describing the fluctuations of the currents [161]. The purpose of this section is to apply the fluctuation theorem for currents in this recent context, based on a master equation description [3, 67]. The fluctuation symmetry has also been shown using Keldysh Green's function formalism [213].

We consider two mesoscopic tunnel junctions coupled in series as described in Ref. [1]. A schematic representation of the system is given in Fig. 3.2. When the charging energy $E_c = e^2/2C$, where $C = C_L + C_R$, is larger than the thermal energy $k_B T$, electron tunneling events across the junction become correlated and give rise to a variety of phenomena such as Coulomb blockade, leading to steps in the current-voltage characteristics [1]. The voltage in the central region between the two junctions V_M fluctuates depending on the number N of excess electrons in this region. The voltage drop across the left junction, $V_L - V_M(N)$, and the right junction, $V_M(N) - V_R$, are found using classical electrodynamics to be

$$\begin{aligned} V_L - V_M(N) &= \frac{C_R}{C} V + \frac{Ne}{C} + V_p, \\ V_M(N) - V_R &= \frac{C_L}{C} V - \frac{Ne}{C} - V_p. \end{aligned}$$

The additional voltage V_p has been included to account for any misalignment of the Fermi level in the middle region with respect to the Fermi levels of the left and right leads when V and N are zero [1]. In this semiclassical description, the state of the system is determined by the probability $P(N, t)$ to have a number of excess electrons N in the middle region, which obeys the master equation:

$$\frac{dP(N, t)}{dt} = \sum_{\rho=L, R} \sum_{\pm} [W_{\rho}(N \pm 1|N)P(N \pm 1, t) - W_{\rho}(N|N \pm 1)P(N, t)].$$

The system is controlled by four tunneling rates: the rate for electrons to tunnel into the central region from the left $W_L(N|N+1)$ and right $W_R(N|N+1)$, and the rate for electrons to tunnel out of the central region to the left $W_L(N|N-1)$ and right $W_R(N|N-1)$. These rates are computed via Fermi's golden rule and take the form

$$W_{\rho}(N|N \pm 1) = \frac{1}{e^2 R_{\rho}} \frac{\pm e[V_M(N) - V_{\rho}] - E_c}{1 - e^{\beta\{\mp e[V_M(N) - V_{\rho}] + E_c\}}}$$

with $\rho = L, R$ and the inverse temperature $\beta = 1/(k_B T)$. This model successfully reproduces the experimental data on the complicated structure of the I - V characteristics [1].

The graph of the system is depicted in Fig. 3.3. Calculating the quantity (3.12) along the cycle shown in Fig. 3.3, one finds

$$\frac{W_R(N|N+1)W_L(N+1|N)}{W_L(N|N+1)W_R(N+1|N)} = \exp \frac{eV}{k_B T}.$$

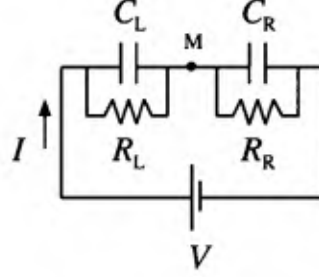


Fig. 3.2. Schematic representations of the two junctions which are denoted L for left and R for right. They have the resistances R_L and R_R , and capacitances C_L and C_R respectively. The two junctions are driven by an ideal constant voltage source V . $I = J$ is the current intensity.

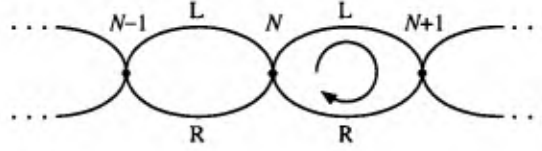


Fig. 3.3. Graph associated with the random process of the conductor of Fig. 3.2.

One can thus extract the potential difference applied to the double junction by considering the affinities of the cycles. The current fluctuation theorem can be applied and the distribution of the charge transmitted to the double junction therefore satisfies

$$\frac{P \left[(1/t) \int_0^t j_\rho(t') dt' = +\alpha \right]}{P \left[(1/t) \int_0^t j_\rho(t') dt' = -\alpha \right]} \simeq \exp \left(\alpha \frac{eV}{k_B T} t \right) \quad (t \rightarrow \infty),$$

where $\rho = L, R$ are the chords which can be chosen to correspond to the left or right junction. The fluctuation theorem for the currents can be extended to multi-terminal systems, i.e., systems with three or more junctions [3].

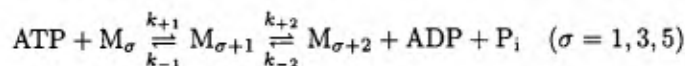
Since the conditions of application of the current fluctuation theorem holds, we infer that the response coefficients obey the Onsager reciprocity relations as well as their higher-order generalizations as derived in the next chapter. In this regard, we notice that the zero-frequency current noise $\Sigma_{\alpha\beta}$ can be computed from the generating function (3.26) by taking second derivatives. This quantity is of central interest to characterize the fluctuations of the currents in mesoscopic conductors [24]. Furthermore, the knowledge of the power spectrum in the vicinity of equilibrium gives access to the nonlinear response coefficients and their nonlinear symmetries, as will be shown in Chapter 4.

3.3.3 Molecular motors

In this section, we introduce a discrete-state model describing the motion of the F_1 motor studied by Kinosita and coworkers in Ref. [232]. The F_1 protein complex is composed of three large α - and β -subunits circularly arranged around a smaller γ subunit. The three β -subunits are the reactive sites for the hydrolysis of ATP, while the γ -subunit plays the role of rotation shaft to which a bead of 40 nm-diameter is glued. The mechanism of rotational catalysis was proposed by Boyer using a bi-site activation [27]. Nevertheless, experimental data cannot distinguish for the moment between the bi-site and three-site activations. The observation [232] clearly shows that the rotation takes place in six steps: ATP binding induces a rotation of about 90° followed by the release of ADP and P_i with a rotation of about 30° . Therefore, the hydrolysis of one ATP corresponds to a rotation by 120° and a revolution of 360° to three sequential ATP hydrolysis in the three β -subunits. The six successive states of the hydrolytic motor $M = F_1$ can thus be specified by the angle θ of the shaft and the occupancy of the sites of the three β -subunits as

$$\begin{aligned} M_1 &= [\theta = 0, (\text{ADP} + P_i, \emptyset, x)] & M_2 &= [\theta = \frac{\pi}{2}, (\text{ADP} + P_i, \text{ATP}, x)] \\ M_3 &= [\theta = \frac{2\pi}{3}, (\emptyset, \text{ADP} + P_i, x)] & M_4 &= [\theta = \frac{5\pi}{6}, (x, \text{ADP} + P_i, \text{ATP})] \\ M_5 &= [\theta = \frac{4\pi}{3}, (x, \emptyset, \text{ADP} + P_i)] & M_6 &= [\theta = \frac{11\pi}{6}, (\text{ATP}, x, \text{ADP} + P_i)] \end{aligned}$$

where x stands either for $\emptyset$ or ADP for the bi- or three-site mechanism. If the site is empty, the F_1 complex jumps to the following state with the rate $k_{+1}[\text{ATP}]$, and with the rate k_{+2} if the site is occupied. The backward transitions being possible, the complex can jump to the preceding state with the rate k_{-1} if the site is occupied and the rate $k_{-2}[\text{ADP}][P_i]$ if it is empty. This process can be summarized by the following reaction scheme



with a cyclic ordering $M_7 \equiv M_1$. This is a six-state model with the transition rates

$$\begin{aligned} w_{+1} &\equiv k_{+1}[\text{ATP}], \\ w_{-1} &\equiv k_{-1}, \\ w_{+2} &\equiv k_{+2}, \\ w_{-2} &\equiv k_{-2}[\text{ADP}][P_i]. \end{aligned}$$

The graph of this model is depicted in Fig. 3.4. The three-fold symmetry of the F_1 -ATPase is taken into account in the model by the symmetry of the transition rates. Similar models can be used to describe the dynamics of others molecular motors such as kinesin [74, 196, 138, 144].

The affinity of the cycle of Fig. 3.4 is given by

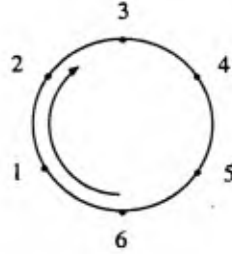


Fig. 3.4. Graph associated with the six-state model.

$$A \equiv 3 \ln \frac{k_{+1}k_{+2}[\text{ATP}]}{k_{-1}k_{-2}[\text{ADP}][\text{P}_i]}. \quad (3.31)$$

The maximum work which can be done per revolution by the F_1 motor is $3(\mu_{\text{ATP}} - \mu_{\text{ADP}} - \mu_{\text{P}_i}) = Ak_B T$. The detailed balance conditions (3.9) should be satisfied at the thermodynamic equilibrium, which implies the vanishing of the affinity (3.31). Accordingly, equilibrium is reached if $w_{+1}w_{+2} = w_{-1}w_{-2}$. Since the standard free enthalpy of hydrolysis is equal to

$$\Delta G^0 = G_{\text{ATP}}^0 - G_{\text{ADP}}^0 - G_{\text{P}_i}^0 = 50 \text{ pN nm}$$

and the temperature of the experiment of Ref. [232] is 23° Celsius, the equilibrium concentrations of the reactant and products obey

$$\frac{[\text{ATP}]_{\text{eq}}}{[\text{ADP}]_{\text{eq}}[\text{P}_i]_{\text{eq}}} = \frac{k_{-1}k_{-2}}{k_{+1}k_{+2}} = e^{-\Delta G^0/k_B T} = 4.89 \cdot 10^{-6} \text{ M}^{-1},$$

which is a constraint on the reaction constants from equilibrium thermodynamics. We notice that, under physiological conditions, the concentrations are about $[\text{ATP}] \simeq 10^{-3} \text{ M}$, $[\text{ADP}] \simeq 10^{-4} \text{ M}$, and $[\text{P}_i] \simeq 10^{-3} \text{ M}$, so that ATP is in large excess with respect to its equilibrium concentration $[\text{ATP}]_{\text{eq}} \simeq 4.89 \cdot 10^{-13} \text{ M}$, which shows that the system is typically very far from equilibrium.

The reaction constants $k_{\pm p}$ can be determined from the experimental data, yielding [4, 94]

$$\begin{aligned} k_{+1} &= (2.6 \pm 0.5) \cdot 10^7 \text{ M}^{-1} \text{ s}^{-1}, \\ k_{-1} &= (138 \pm 34) \cdot 10^{-6} \text{ s}^{-1}, \\ k_{+2} &= (387 \pm 27) \text{ s}^{-1}, \\ k_{-2} &= (3.5 \pm 0.8) \cdot 10^8 \text{ M}^{-2} \text{ s}^{-1}. \end{aligned}$$

We observe that these reaction constants range over about twelve orders of magnitude, which is characteristic of a stiff stochastic process.

The rotation of the motor can be described by the number of revolutions R_t accomplished during a time t . The corresponding generating function

$$Q(\lambda) = \lim_{t \rightarrow \infty} -\frac{1}{t} \ln \langle e^{-\lambda R_t} \rangle$$

is given by [4]

$$Q(\lambda) = \frac{1}{2} \left\{ w_{+1} + w_{+2} + w_{-1} + w_{-2} - \left[(w_{+1} + w_{-2} + w_{-1} + w_{+2})^2 + 4w_{+1}w_{+2} \left(e^{(\lambda-A)/3} + e^{-\lambda/3} - 1 - e^{-A/3} \right) \right]^{\frac{1}{2}} \right\}. \quad (3.32)$$

This generating function presents the symmetry

$$Q(\lambda) = Q(A - \lambda)$$

as it should. The rotation of the motor will thus satisfy the large-deviation relation

$$\frac{P(R_t/t = +\alpha)}{P(R_t/t = -\alpha)} \simeq \exp A\alpha t \quad (t \rightarrow \infty). \quad (3.33)$$

Another fluctuation relation can be derived for the cumulative number of steps S_t the motor performs during a time t . The generating function of this variable can be calculated at finite times [4] and shows the symmetry

$$P(S_t = s) \simeq P(S_t = -s) e^{sA/6}. \quad (3.34)$$

In Fig. 3.5 we show that this fluctuation relation is indeed satisfied. For a given affinity, the fluctuation theorem (3.34) for the displacement of the motor is easier to observe experimentally than the one (3.33) for the full revolutions. As seen in Fig. 3.5, the probability distribution of the displacement here takes a specific form where the odd displacements are almost never occurring. Indeed, for these values of the concentrations of the chemical species, the probability to be on odd sites is about 4 orders of magnitude lower than the probability to be on even sites. The system almost never stays on odd site and immediately jumps to the next or previous site.

The generating function (3.32) allows us to derive not only the mean current but also the higher-order moments by differentiation. The first derivative corresponds to the steady-state current or mean rotation velocity, which is given by

$$V = \frac{w_{+1}w_{+2} - w_{-1}w_{-2}}{3(w_{+1} + w_{+2} + w_{-1} + w_{-2})}.$$

This corresponds to a kinetics of Michaelis-Menten type in the absence of the products ADP and P_i of the reaction (i.e., in the limit $A = +\infty$) where the steady-state current is given by

$$V = \frac{k_{+1}k_{+2}[\text{ATP}]}{3(k_{+1}[\text{ATP}] + k_{+2} + k_{-1})} = \frac{V_{\max}[\text{ATP}]}{[\text{ATP}] + K_M},$$

with the maximum value $V_{\max} = k_{+2}/3$ and the Michaelis-Menten constant $K_M = (k_{+2} + k_{-1})/k_{+1}$.

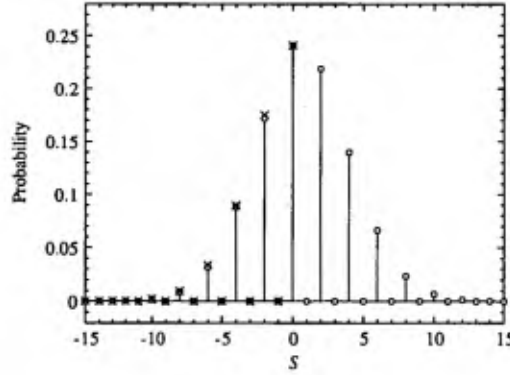


Fig. 3.5. Probability $P(S_t = s)$ (open circles) that the F_1 motor performs $s = S_t$ steps during the time interval $t = 10^4$ s compared with the expression $P(S_t = -s) e^{sA/6}$ (crosses) expected from the fluctuation theorem for $[ATP] = 6 \cdot 10^{-8}$ M and $[ADP][P_i] = 10^{-2}$ M².

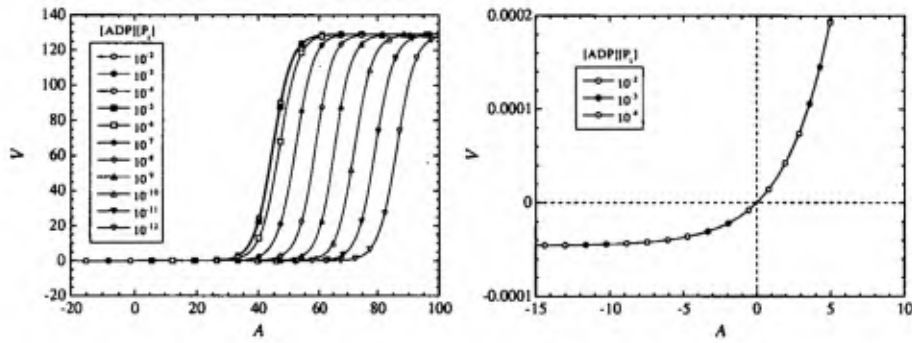


Fig. 3.6. (a) Mean rotation velocity V of the F_1 motor with a bead of 40 nm-diameter versus the affinity A for different concentrations $[ADP][P_i]$ of the products. (b) Zoom of (a) giving the mean rotation velocity V of the F_1 motor versus the affinity A around the equilibrium at $A = 0$ for different concentrations $[ADP][P_i]$ of the products.

The mean rotation velocity is depicted in Fig. 3.6a as a function of the affinity (3.31) for different concentrations of the products. We observe that the V - A curve is highly nonlinear as a consequence of the stiffness of the process. Even the vanishing of the mean velocity at the thermodynamic equilibrium $A = 0$ is not visible in Fig. 3.6a. A zoom is carried out in the vicinity of equilibrium in Fig. 3.6b where we observe that, indeed, the mean velocity vanishes linearly with the affinity as expected. This linear regime does not extend by more than one decade around the equilibrium concentration. Typically, the motor is very far from equilibrium and is functioning in the nonlinear regime. This shows the crucial importance of these nonlinear regimes of nonequilibrium thermodynamics for the understanding of biological molecular motors.

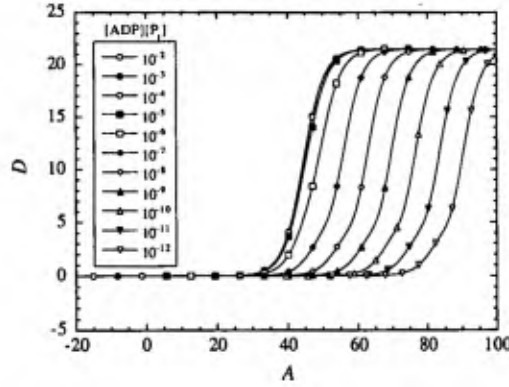


Fig. 3.7. Diffusion coefficient D of the F_1 motor with a bead of 40 nm-diameter versus the affinity A for different concentrations $[ADP][P_i]$ of the products.

A second derivative of the generating function selects out the diffusion coefficient

$$D = \frac{w_{+1}w_{+2} + w_{-1}w_{-2}}{18(w_{+1} + w_{+2} + w_{-1} + w_{-2})} - \frac{(w_{+1}w_{+2} - w_{-1}w_{-2})^2}{9(w_{+1} + w_{+2} + w_{-1} + w_{-2})^3}.$$

This diffusion coefficient characterizes the fluctuations in the rotation of the motor. Another consequence of the stiffness of the motor is that the diffusion coefficient depicted in Fig. 3.7 is small relative to the mean velocity. For most values of the affinity, the ratio of the mean velocity to the diffusion coefficient is about $V/D \simeq 6$, which is characteristic of a correlated rotation slightly affected by the fluctuations. The nonlinearities confer thus to these rotary motors a very robust dynamical behavior.

3.4 Summary

In this chapter, we derived a fluctuation relation for the macroscopic currents crossing the system in a nonequilibrium steady state. This result is expressed by the symmetry property (3.27) for the generating function of the currents.

At the mesoscopic scale, the discreteness of matter results in a random evolution of the trajectories of the system. A statistical description is obtained in terms of a master equation for the evolution of the probability distribution in phase space. This probability distribution will evolve towards a stationary distribution which can be an equilibrium or nonequilibrium state, depending on whether the detailed balance conditions are satisfied or not. A nonequilibrium steady state is characterized by the presence of probability currents, maintained by a difference of thermodynamic potentials between the reservoirs coupled to the system. In a nonequilibrium steady state, the violation of detailed balance along cyclic trajectories can be related to the macroscopic affinities. In this way, the connection with thermodynamics can be established.

However, thermodynamic quantities such as the nonequilibrium currents fluctuate along the stochastic trajectories. The analysis carried out at the level of a statistical ensemble of trajectories shows that the dynamics presents an underlying hidden symmetry (3.21). In turn, this symmetry property translates into the large-deviation relation (3.28) for the fluctuations of the macroscopic currents. This relation reveals that the probability to observe a positive value of the currents is exponentially greater than the probability to observe its opposite value. The exponential behavior depends on the thermodynamic affinities, which entails deep connections between fluctuations and thermodynamic forces.

This result can be applied to a large variety of systems, as shown here for chemical, physical, and biological systems. These systems typically function in the nonlinear regime, so that the fluctuation theorem is a favored tool to explore the nonequilibrium properties of mesoscopic systems.

Since the fluctuation relation for the currents is expressed in terms of the macroscopic affinities and is valid arbitrarily far from equilibrium, it is natural to explore the consequences of this relation on the response theory. This will be the purpose of the next chapter.

Nonlinear Response Theory

Onsager's classic work of 1931 [175] has shown that the linear response coefficients relating the currents to the thermodynamic forces obey reciprocity relations as a consequence of the reversibility of the underlying microscopic dynamics. Subsequently, Green [105] and Kubo [131] independently expressed the response coefficients in terms of the equilibrium correlation function of the currents. The fluctuation theorem is also closely related to microreversibility and is valid far from equilibrium. In particular, close enough to equilibrium where the response of the system is linear in the affinities, the Onsager reciprocity relations can be deduced from the fluctuation theorem [77, 139, 2].

On the other hand, it is known that far-from-equilibrium systems may present nonlinear responses to nonequilibrium constraints. The response is said to be nonlinear if the currents crossing the nonequilibrium system depend nonlinearly on the affinities. The coefficients characterizing such nonlinear responses are obtained by expanding the currents in the powers of the affinities. The terms linear in the affinities are the linear response coefficients obeying Onsager's reciprocity relations. The terms which are quadratic, cubic, quartic, etc... in the affinities are called the nonlinear response coefficients. We may wonder if the nonlinear response coefficients would obey relations beyond Onsager's ones as the consequence of the fundamental microreversibility.

In this chapter, we shall show that an affirmative answer can be given to this question thanks to the fluctuation theorem for the currents, proved in Chapter 3. In this framework, expressions for the response coefficients of the currents at arbitrary orders in the affinities are given in terms of the fluctuations of the cumulative currents. Remarkable relations are obtained which are the consequences of microreversibility beyond Onsager reciprocity relations [5].

4.1 Nonlinear expansion

The link between the mesoscopic and macroscopic scales is made thanks to the concept of ensemble or time averaging. The macroscopic currents J_α are thus obtained from the mesoscopic currents (3.25) or their associated Helfand moments G_α as follows:

$$J_\alpha = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \langle j_\alpha(t') \rangle dt' = \lim_{t \rightarrow \infty} \frac{\langle G_\alpha \rangle}{t}. \quad (4.1)$$

In general, the macroscopic currents can be expanded as power series of the macroscopic affinities:

$$J_\alpha = \sum_\beta L_{\alpha,\beta} A_\beta + \frac{1}{2} \sum_{\beta,\gamma} L_{\alpha,\beta\gamma} A_\beta A_\gamma + \frac{1}{6} \sum_{\beta,\gamma,\delta} L_{\alpha,\beta\gamma\delta} A_\beta A_\gamma A_\delta + \dots$$

The linear response of the currents J_α with respect to a small perturbation in the affinities A_β is characterized by the Onsager coefficients $L_{\alpha,\beta}$ and the nonlinear response by the higher-order coefficients $L_{\alpha,\beta\gamma}$, $L_{\alpha,\beta\gamma\delta}$, ...

On the other hand, the generating function (3.26) can be used to obtain the mean currents as well as higher-order cumulants by expanding in Taylor series

$$Q(\lambda, \mathbf{A}) \simeq \sum_\alpha J_\alpha \lambda_\alpha + O(\lambda^2),$$

as shown in Appendix A. We here emphasize the dependence of Q on the affinities. The symmetry of the fluctuation theorem (3.27) reads

$$Q(\lambda, \mathbf{A}) = Q(\mathbf{A} - \lambda, \mathbf{A}) \quad (4.2)$$

and explicitly involves the thermodynamic forces. Our goal will be to explore the consequences of this relation on the response coefficients.

A first observation is that the antisymmetric part of the generating function can be recovered from the symmetric part. We thus consider the symmetric and antisymmetric parts

$$Q_\pm(\lambda, \mathbf{A}) = \frac{1}{2} [Q(\lambda, \mathbf{A}) \pm Q(-\lambda, \mathbf{A})] \equiv \hat{P}_\pm Q(\lambda, \mathbf{A}),$$

where we introduced the projection operators $\hat{P}_\pm$. Equivalently, we have

$$Q(\lambda, \mathbf{A}) = Q_+(\lambda, \mathbf{A}) + Q_-(\lambda, \mathbf{A}), \quad (4.3a)$$

$$Q(-\lambda, \mathbf{A}) = Q_+(\lambda, \mathbf{A}) - Q_-(\lambda, \mathbf{A}). \quad (4.3b)$$

Expressing the symmetry (4.2) in terms of the translation operator

$$\exp(y \partial / \partial x) \phi(x) = \phi(x + y)$$

and substituting Eqs. (4.3) into the resulting expression gives

$$[\exp(-\mathbf{A} \cdot \partial/\partial\lambda) + 1]Q_-(\lambda, \mathbf{A}) = [\exp(-\mathbf{A} \cdot \partial/\partial\lambda) - 1]Q_+(\lambda, \mathbf{A}).$$

Operating on both sides of this equation by $[\exp(-\mathbf{A} \cdot \partial/\partial\lambda) + 1]^{-1}$, we have

$$Q_-(\lambda, \mathbf{A}) = -\tanh\left(\frac{1}{2}\mathbf{A} \cdot \frac{\partial}{\partial\lambda}\right)Q_+(\lambda, \mathbf{A}). \quad (4.4)$$

We see that the odd part Q_- is completely determined by the even part Q_+ . The fluctuation symmetry eliminates half of the degrees of freedom of the generating function. Now, substituting the projection operators $\hat{P}_\pm$ into (4.4) yields

$$\hat{P}_-Q(\lambda, \mathbf{A}) = -\tanh\left(\frac{1}{2}\mathbf{A} \cdot \frac{\partial}{\partial\lambda}\right)\hat{P}_+Q(\lambda, \mathbf{A}). \quad (4.5)$$

If we analyse the effect of the odd operator $\tanh(2^{-1}\mathbf{A} \cdot \partial/\partial\lambda)$ on $\hat{P}_+$, we can verify that

$$\tanh(2^{-1}\mathbf{A} \cdot \partial/\partial\lambda)\hat{P}_+ = \hat{P}_-\tanh(2^{-1}\mathbf{A} \cdot \partial/\partial\lambda). \quad (4.6)$$

By virtue of (4.6), equation (4.5) takes the form

$$\hat{P}_-\left[Q(\lambda, \mathbf{A}) + \tanh\left(\frac{1}{2}\mathbf{A} \cdot \frac{\partial}{\partial\lambda}\right)Q(\lambda, \mathbf{A})\right] = 0, \quad (4.7)$$

yielding another expression of the fundamental symmetry (4.2).

We now explore the consequences of this symmetry on the fluctuations and their response. A characterization of the fluctuations can be obtained from the Taylor series of the generating function of the currents

$$Q(\lambda, \mathbf{A}) = \sum_{m=1}^{\infty} \frac{1}{m!} K_{\alpha_1 \dots \alpha_m}(\mathbf{A}) \lambda_{\alpha_1} \dots \lambda_{\alpha_m}, \quad (4.8)$$

where we use the convention to sum over repeated indices. Indeed, the coefficients $K_{\alpha_1 \dots \alpha_m}$ are related to the cumulants of the Helfand moments (3.15) as

$$K_{\alpha_1 \dots \alpha_m}(\mathbf{A}) = (-1)^{m+1} \lim_{t \rightarrow \infty} \frac{1}{t} \langle \langle G_{\alpha_1}(t) \dots G_{\alpha_m}(t) \rangle \rangle_{\mathbf{A}}. \quad (4.9)$$

The coefficients $K_{\alpha_1 \dots \alpha_m}$ can in turn be expanded in powers of the affinities,

$$K_{\alpha_1 \dots \alpha_m}(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!} L_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} A_{\beta_1} \dots A_{\beta_n},$$

so that we can write the generating function (4.8) as

$$Q = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} L_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} \lambda_{\alpha_1} \dots \lambda_{\alpha_m} A_{\beta_1} \dots A_{\beta_n}. \quad (4.10)$$

The coefficients $L_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n}$ characterize the response of the quantities $K_{\alpha_1 \dots \alpha_m}$ with respect to the affinities $A_{\beta_1} \dots A_{\beta_n}$. These coefficients are obtained by differentiating the function Q with respect to the parameters λ and the affinities A . In particular, the response coefficients of the macroscopic currents are given by $L_{\alpha, \beta_1 \dots \beta_n}$. We notice that the action of the operators $\hat{P}_{\pm}$ on the series (4.10) is to multiply it by $1/2$ and to select out the terms for which $(-1)^m = \pm 1$.

We now consider the action of the operator $A \cdot \partial/\partial \lambda$ on the series (4.10) in detail. We have

$$A_{\gamma} \frac{\partial Q}{\partial \lambda_{\gamma}} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(m-1)!n!} L_{\alpha_1 \dots \alpha_{m-1} \gamma, \beta_1 \dots \beta_n} \lambda_{\alpha_1} \dots \lambda_{\alpha_{m-1}} A_{\beta_1} \dots A_{\beta_n} A_{\gamma}.$$

We may write the last series in the form

$$\sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \bar{L}_{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_l} \lambda_{\alpha_1} \dots \lambda_{\alpha_k} A_{\beta_1} \dots A_{\beta_l},$$

where

$$\bar{L}_{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_l} = \sum_{i=1}^l L_{\alpha_1 \dots \alpha_k \beta_i, \beta_1 \dots \beta_{i-1} \beta_{i+1} \dots \beta_l}. \quad (4.11)$$

We will treat (4.11) as the result of the action of an operator σ which transposes $L_{\alpha_1 \dots \alpha_{k+1}, \beta_1 \dots \beta_{l-1}}$ into $\bar{L}_{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_l}$:

$$\bar{L}_{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_l} = (\sigma L)_{\alpha_1 \dots \alpha_{k+1}, \beta_1 \dots \beta_{l-1}}.$$

Then, as a consequence of (4.11), we have

$$(\sigma L)_{\alpha_1 \dots \alpha_{k+1}, \beta_1 \dots \beta_l} = \sum_{i=1}^l L_{\alpha_1 \dots \alpha_k \beta_i, \beta_1 \dots \beta_{i-1} \beta_{i+1} \dots \beta_l}. \quad (4.12)$$

This formula defines the action of the operator σ . Thus, we see that the action of the operator $A \cdot \partial/\partial \lambda$ on Q is equivalent to the action of the operator σ on the coefficients of the series. We now use this equivalence after inserting (4.10) into (4.7). Taking into account that $\hat{P}_{-}$ selects out the terms with property $(-1)^m = -1$, we obtain

$$L_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} = -[\tanh(\sigma/2)L]_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} \quad (4.13)$$

for $(-1)^m = -1$. The function $\tanh(z/2)$ can be represented as

$$\tanh(z/2) = \sum_{j=0}^{\infty} c_{2j+1} z^{2j+1}, \quad (4.14)$$

where

$$c_{2j+1} = 2 \frac{4^{j+1} - 1}{(2j+2)!} B_{2j+2}. \quad (4.15)$$

Here B_{2j+2} are the Bernoulli numbers. The first coefficients of (4.14) are

$$c_1 = \frac{1}{2}, \quad c_3 = -\frac{1}{24}, \quad c_5 = \frac{1}{240}, \quad \dots$$

By virtue of (4.14), equation (4.13) takes the form

$$L_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} = - \sum_{j \geq 0} c_{2j+1} (\sigma^{2j+1} L)_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} \quad (4.16)$$

at $(-1)^{m+1} = 1$. The number of terms on the right-hand side of (4.16) is finite. Indeed, we see from (4.12) that the action of the operator σ decreases the number of subscripts appearing to the right of the comma. This number cannot be negative. For this reason $(\sigma^{2j+1} L)_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} = 0$ for $2j+1 > n$. The number of term in (4.16) is thus equal to the integer part of $(n+1)/2$. Using Eqs. (4.12), (4.15), and (4.16), we have for $m = 1, 3, 5, \dots$

$$L_{\alpha_1 \dots \alpha_m} = 0, \quad (4.17)$$

$$-2 L_{\alpha_1 \dots \alpha_m, \beta} = L_{\alpha_1 \dots \alpha_m, \beta}, \quad (4.18)$$

$$-2 L_{\alpha_1 \dots \alpha_m, \beta \gamma} = L_{\alpha_1 \dots \alpha_m, \beta, \gamma} + L_{\alpha_1 \dots \alpha_m, \gamma, \beta}. \quad (4.19)$$

These relations correspond to the values $n = 0, n = 1, n = 2$ respectively. All these relations and the others derived from (4.16) are independent. We now consider relations (4.13) or (4.16) at successive orders in detail.

4.1.1 Linear response

We here consider all independent two-subscript relations. They are given by Eq. (4.18) with $m = 1$, i.e.

$$L_{\alpha, \beta} = -\frac{1}{2} L_{\alpha \beta}.$$

This expression shows that the response coefficients are symmetric:

$$L_{\alpha, \beta} = L_{\beta, \alpha}. \quad (4.20)$$

These are the Onsager reciprocity relations [175].

By using Eq. (4.9), we obtain the Onsager coefficients as

$$L_{\alpha, \beta} = \lim_{t \rightarrow \infty} \frac{1}{2t} \langle G_{\alpha}(t) G_{\beta}(t) \rangle_{\text{eq}} = \frac{1}{2} \int_{-\infty}^{+\infty} \langle j_{\alpha}(t) j_{\beta}(0) \rangle_{\text{eq}} dt, \quad (4.21)$$

in terms of the Helfand moments or in terms of the time correlation functions of the instantaneous currents. Here, the statistical average is carried out with respect to the state of thermodynamic equilibrium. In Eq. (4.21), the formulas giving the coefficients in terms of the Helfand moments or cumulative currents are known as the Einstein-Helfand formulas [65, 111]. The other formulas giving the coefficients in terms of the time correlation functions are known as the Green-Kubo formulas [105, 131] or the Yamamoto-Zwanzig formulas in the context of chemical reactions [231, 233].

4.1.2 Second-order response

We here consider all three-subscript relations. From Eq. (4.17) with $m = 3$, we find

$$L_{\alpha\beta\gamma} = 0, \quad (4.22)$$

showing that the third cumulant vanishes at equilibrium. This is true for all odd cumulants at equilibrium as can be seen from Eq. (4.17).

Another relation comes from Eq. (4.19) with $m = 1$, i.e.,

$$L_{\alpha,\beta\gamma} = -\frac{1}{2} (L_{\alpha\beta,\gamma} + L_{\alpha\gamma,\beta}). \quad (4.23)$$

The right-hand terms of this equation are related to the spectral function of the nonequilibrium current fluctuations defined by

$$\Sigma_{\alpha\beta}(\omega) \equiv \int_{-\infty}^{+\infty} e^{i\omega t} \langle [j_{\alpha}(t) - \langle j_{\alpha} \rangle] [j_{\beta}(0) - \langle j_{\beta} \rangle] \rangle dt, \quad (4.24)$$

where the statistical average is here taken with respect to the nonequilibrium steady state. Here, the quantities $L_{\alpha\beta,\gamma}$ are expressed as

$$\begin{aligned} L_{\alpha\beta,\gamma} &\equiv \frac{\partial}{\partial A_{\gamma}} \int_{-\infty}^{+\infty} \langle [j_{\alpha}(t) - \langle j_{\alpha} \rangle] [j_{\beta}(0) - \langle j_{\beta} \rangle] \rangle dt \Big|_{A=0} \\ &= \frac{\partial}{\partial A_{\gamma}} \Sigma_{\alpha\beta}(\omega = 0) \Big|_{A=0}, \end{aligned} \quad (4.25)$$

which characterize the sensitivity of the current fluctuations out of equilibrium.

Thanks to the fluctuation theorem for the currents, we can therefore relate the second-order nonlinear response coefficients to quantities characterizing the nonequilibrium fluctuations such as the spectral functions of the currents in the nonequilibrium steady state. We notice that the number of derivatives with respect to the affinities has indeed been reduced.

4.1.3 Third-order response

We here consider all four-subscript relations. Equation (4.18) with $m = 3$ shows that

$$L_{\alpha\beta\gamma,\delta} = -\frac{1}{2} L_{\alpha\beta\gamma\delta}, \quad (4.26)$$

which relates the response of the third cumulant to the equilibrium fourth cumulant. In particular, we find that the tensor $L_{\alpha\beta\gamma,\delta}$ is totally symmetric, which is another non-trivial consequence of the fluctuation theorem. Relations (4.17)-(4.19) do not lead to any other four-subscripts relations.

On the other hand, relation (4.16) with $n = 3$ and $m = 1$ reads

$$L_{\alpha,\beta\gamma\delta} = -\frac{1}{2} (L_{\alpha\beta,\gamma\delta} + L_{\alpha\gamma,\beta\delta} + L_{\alpha\delta,\beta\gamma}) + \frac{1}{4} L_{\alpha\beta\gamma\delta}, \quad (4.27)$$

which expresses the third-order response in terms of the second-order response of the power spectra and of the equilibrium fourth cumulant.

4.1.4 Time-odd parameters

The symmetry relation (4.2) has been derived from a mesoscopic description with even variables under time-reversal symmetry. On the other hand, as already indicated by Eq. (2.1), the time-reversal operation acts differently in the presence of odd variables such as magnetic fields. This situation cannot strictly be coped with a master equation description. In regard of the results of Chapter 2, the stationary fluctuation theorem for the currents presumably generalizes to

$$Q(\lambda, A, B) = Q(A - \lambda, A, -B). \quad (4.28)$$

This fluctuation relation has been verified in a specific model of electronic transport in Ref. [188]. Following the reasoning of Sect. 4.1, the symmetry (4.28) would lead to

$$L_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n}(B) = (-1)^m \sum_{j \geq 0} \frac{1}{j!} (\sigma^j L)_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n}(-B) \quad (4.29)$$

at the level of the response theory. In contrast with the zero-magnetic field case, the index m can here take any positive integer. We show below that this relation indeed leads to the Onsager-Casimir relations.

Relation (4.29) with $m = 2$ and $n = 0$ yields

$$L_{\alpha\beta}(B) = L_{\alpha\beta}(-B),$$

so that the power spectrum is even in the magnetic field at equilibrium. When $m = n = 1$, we obtain

$$L_{\alpha,\beta}(B) = -L_{\alpha,\beta}(-B) - L_{\alpha\beta}(-B) \quad (4.30)$$

while the case $m = 0$ and $n = 2$ leads to

$$0 = L_{\alpha,\beta}(-B) + L_{\beta,\alpha}(-B) + L_{\alpha\beta}(-B), \quad (4.31)$$

where we used that $L_{\alpha\beta} = 0$. Finally, combining relations (4.30) and (4.31) gives rise to the Onsager-Casimir relations

$$L_{\alpha,\beta}(B) = L_{\beta,\alpha}(-B),$$

which generalize Onsager symmetry (4.20) in presence of a magnetic field [37].

4.2 Summary

In this chapter, we have shown that the fluctuation theorem for the currents (4.2) implies not only Onsager's reciprocity relations along with the Green-Kubo and Einstein-Helfand formulas for the linear response coefficients, but also further remarkable relations for the nonlinear response coefficients at arbitrarily high orders. These results find their origin in the validity of the fluctuation

theorem for the currents far from equilibrium in stochastic rate processes. The obtained relations are thus the consequences of the microreversibility.

The response coefficients are defined by expanding the currents crossing the nonequilibrium system in powers of the affinities. Therefore, the response coefficients are defined with respect to the equilibrium state where the affinities vanish. Nevertheless, we can estimate the currents further away from equilibrium if we use an expansion up to high powers of the affinities. This explains that we need a general property valid far from equilibrium, such as the fluctuation theorem for the currents, in order to obtain relations on the nonlinear response coefficients at arbitrary orders.

Typically, the relations described in the present chapter connect quantities measuring the statistical correlations among m fluctuating cumulative currents to corresponding quantities among $m - n$ of them with n extra derivatives with respect to an affinity. The former characterizes the fluctuations at m^{th} order and the latter the n^{th} order sensitivity of the fluctuations at the lower $(m - n)^{\text{th}}$ order with respect to the nonequilibrium constraints. This is the case for instance for the equality (4.26) between the sensitivity of the nonequilibrium correlations among three cumulative currents under changes of an affinity and the fluctuations between four cumulative currents in the equilibrium thermodynamic state. This is the feature which is found in the relations here described. In particular, and in contrast with linear response, the nonlinear response coefficients are not always determined by equilibrium properties.

In conclusion, the theory developed in the present chapter provides a general framework to formulate the nonlinear response theory in nonequilibrium processes. These results have important applications for the nonlinear response properties of many nonequilibrium systems such as the chemical and biochemical reactions, the full counting statistics in mesoscopic conductors, the effusion of ideal gases [42], or Van den Broeck's demons [216]. The present results could be especially important in nonequilibrium systems at the micro- and nanoscales, where the nonlinear response properties turn out to be dominant, as seen in the previous chapter.

Time Asymmetry and Entropy Production

According to the second law of thermodynamics, nonequilibrium systems produce entropy, which is a macroscopic concept, in a time asymmetric way. The lack of understanding of this time asymmetry in terms of concepts closer to the microscopic description of the motion of particles has always been a difficulty. Only recently, general relationships have been discovered which allows us to express the thermodynamic time asymmetry at the mesoscopic level of description in terms of the probabilities ruling the molecular or thermal fluctuations in nonequilibrium systems.

Recently, the concept of time-reversed entropy per unit time was introduced in analogy with the standard Kolmogorov-Sinai entropy per unit time [86, 87, 140, 162, 182]. These entropies per unit time characterize the exponential decay rate of the typical paths and their time-reversed counterparts. Therefore, these entropies per unit time characterize dynamical randomness, alias temporal disorder, in time series of the nonequilibrium fluctuations.

Under nonequilibrium conditions, the probabilities of the typical paths and those of the corresponding time-reversed paths break the time-reversal symmetry. The fact is that the decay rate of the typical time-reversed paths is larger than the decay rate of the typical paths. Their difference is positive and gives the well-known thermodynamic entropy production [86], so that the entropy production turns out to be directly related to the breaking of the time-reversal symmetry in the dynamical randomness of the nonequilibrium fluctuations. The entropy production thus finds its origin in the time asymmetry of dynamical randomness, alias temporal disorder, in systems driven out of equilibrium.

The purpose of this chapter is to introduce these concepts and to provide experimental evidence for this remarkable connection. We investigate the time-reversal symmetry breaking in two systems, namely a driven Brownian particle and a driven *RC* circuit. We show that the asymmetry in the time series of the nonequilibrium fluctuations is directly linked to the thermodynamic entropy production [9, 10].

5.1 Dynamical entropies and entropy production

Newton's equations are symmetric under time-reversal. This means that there exists an involution Θ that transforms a phase-space curve C solution of Newton's equations into another curve ΘC also solution of Newton's equations. However, an important observation is that the time-reversed curve ΘC does not need to be identical to the original curve C :

$$C \neq \Theta C. \quad (5.1)$$

This is the well-known phenomenon of *spontaneous symmetry breaking*, according to which the solutions of an equation have a lower symmetry than the equation itself. This is not always so: the phase-space trajectories of an harmonic oscillator are identical to their time-reversed counterparts. Yet, the breaking of time-reversal can already be observed for the motion of a free particle. In this case, the solution of Newton's equations reads $C = \{x_0 + v_0 t, t \in \mathbb{R}\}$ while the time-reversed curve $\Theta C = \{x_0 - v_0 t, t \in \mathbb{R}\}$ is distinct from C in phase space if $v_0 \neq 0$.

Accordingly, we can now understand the origin of time asymmetry: while the time-reversed trajectories are always possible, a selection of initial conditions will break the time-reversal symmetry. The symmetry breaking (5.1) opens the possibility of giving a different statistical weight to the forward and backward trajectories. Specifically, suppose we have a partition $\mathcal{P}$ of phase space into cells $\{\omega_i\}$. We observe the system at fixed time intervals τ and we denote by $\mu(\omega_0 \omega_1 \cdots \omega_{n-1})$ the probability to observe the path $\omega_0 \omega_1 \cdots \omega_{n-1}$. The principle of microscopic reversibility states that

$$\mu(\omega_0 \omega_1 \cdots \omega_{n-1}) = \mu(\omega_{n-1} \cdots \omega_1 \omega_0) \quad (5.2)$$

at equilibrium, i.e., the occurrence of a path or trajectory is as probable as the time-reversed one. In contrast, out of equilibrium a statistical selection of trajectories occurs, which results in a violation of this equality.

An important problem is precisely to understand by which mechanisms this selection process occurs in nonequilibrium situations. The fact is that fluxes of matter or energy imposed at the boundaries of an open system explicitly break the time-reversal symmetry. Indeed, in nonequilibrium steady states (NESS), the particles incoming at boundaries typically have a smooth probability distribution, whereas the outgoing particles have a probability distribution which depends on their interaction inside the system and are therefore finely correlated. The time-reversed steady state is in principle possible but it would require the incoming particles to present a probability distribution exactly correlated according to the interaction they will undergo inside the system. Since, in general, the environment from where the particles come does not know about the interaction taking place inside the system, the occurrence of the time-reversed steady state is highly improbable [84]. Accordingly, in a NESS, the probability distribution of the microscopic trajectories inside the system differs from the probability distribution of the time-reversed trajectories so that the system is in an irreversible state. The irreversibility of a NESS thus finds its

origin in the selection of the initial conditions for the trajectories incoming the open system [211, 84]. The breaking of time-reversal symmetry may manifest itself for instance in some time correlation functions and we may wonder if the time-reversal symmetry breaking would not also concern other properties closely related to both the dynamics and the thermodynamics. In particular, we expect that the degree of violation or, equivalently, the importance of the time-symmetry breaking would be a measure of the nonequilibrium conditions. Such a connection can be elaborated at the level of dynamical randomness.

Dynamical randomness is the fundamental property of temporal disorder in the time series. The quantity measuring the rate of production of information and the dynamical randomness is the so-called *entropy per unit time*. The standard entropy per unit time is defined as the mean decay rate of the multiple-time probability as [48]

$$h \equiv \lim_{n \rightarrow \infty} -\frac{1}{n\tau} \sum_{\omega_0 \omega_1 \dots \omega_{n-1}} \mu(\omega_0 \omega_1 \dots \omega_{n-1}) \ln \mu(\omega_0 \omega_1 \dots \omega_{n-1}). \quad (5.3)$$

According to the Shannon-McMillan-Breiman theorem [23], the multiple-time probability indeed decays according to

$$\mu(\omega_0 \omega_1 \dots \omega_{n-1}) \sim e^{-n\tau h}$$

for almost all the trajectories if the process is ergodic. This result shows that the more rapid the spreading of the probabilities along all possible paths, the higher will be the entropy per unit time. Therefore, the entropy per unit time characterizes the dynamical randomness of the time evolution observed with the measuring device. The entropy per unit time (5.3) is the rate of production of information by the random process, i.e., the minimum number of nats required to reliably record the time series during one time unit. The entropy per unit time (5.3) is thus the maximal compression factor one can achieve when compressing data drawn from the distribution μ [49]. The supremum of the dynamical entropy (5.3) over all the possible partitions $\mathcal{P}$ defines the Kolmogorov-Sinai (KS) entropy per unit time of dynamical systems theory [48]. According to Pesin's theorem (1.3), the KS entropy is given by the sum of positive Lyapunov exponents [61]. For stochastic processes, the KS entropy depends on the sampling time as well as on the coarse-grained partition. Stochastic processes are usually such that randomness can be found at all spatial and time scales, which entails that the entropy per unit time will diverge for small time and spatial resolution [97]. This divergence will saturate at the microscopic scale where the deterministic feature of the underlying dynamics becomes apparent.

Analogously, a quantity characterizing the disorder in time for the reversed paths can be introduced. In order to compare with the properties of the time-reversed trajectories, we consider the time-reversed path $\omega_{n-1} \dots \omega_1 \omega_0$. As observed in Chapter 2, the dissipation is linked to the backward process with reversed nonequilibrium constraints. The backward process is characterized by an steady measure $\bar{\mu}$ where all the odd driving forces have been reversed. We are thus interested in the probability

$$\bar{\mu}(\omega_{n-1} \cdots \omega_1 \omega_0) \sim e^{-n\tau h^R}$$

of occurrence of the time-reversed path taking place in the backward process. The decay rate of this multiple-time probability defines the time-reversed entropy per unit time

$$h^R \equiv \lim_{n \rightarrow \infty} -\frac{1}{n\tau} \sum_{\omega_0 \omega_1 \cdots \omega_{n-1}} \mu(\omega_0 \omega_1 \cdots \omega_{n-1}) \ln \bar{\mu}(\omega_{n-1} \cdots \omega_1 \omega_0). \quad (5.4)$$

We emphasize that the average is taken with respect to the probability of the forward path. We can say that the time-reversed entropy per unit time characterizes the dynamical randomness of the time-reversed paths in the forward process of the NESS.

We now proceed to the link between these dynamical entropies and irreversibility. The entropy production $d_i S/dt$ in the stationary state of a random process is related to the difference between the time-reversed and standard dynamical entropies (5.4) and (5.3) according to the central result:

$$\frac{1}{k_B} \frac{d_i S}{dt} = h^R - h \geq 0 \quad (5.5)$$

for small enough sampling time τ . The non-negativity of the entropy production is an immediate consequence of the fact that the difference $h^R - h$ between Eqs. (5.4) and (5.3) is a relative entropy per unit time which is known to be non-negative [228].¹ At equilibrium, the absence of any driving force and the principle of microscopic reversibility (5.2) result in a vanishing difference between h^R and h , as it should.

This remarkable result (5.5) has been shown for Markovian processes where the entropy production is given by Eq. (3.4) and for thermostatted Hamiltonian systems where the phase-space contraction rate plays the role of the entropy production [86].² In the next chapter, section 6.1, we shall argue that expression (5.5) corresponds to the entropy production of non-Markovian stochastic processes as well.

The expression of the entropy production as a relative entropy is interesting in itself. Relative entropies have deep information theoretic meanings. For instance, Stein's lemma [49] relates the relative entropy between two distributions to the statistical errors made when trying to discriminate the origin of data between the two distributions. The relative entropy is also related to the

¹ A more stringent bound can be derived for the entropy production. The following inequality for the relative entropy $D(P_1 \| P_2)$ between two probability distributions can be found, e.g., in Ref. [49]:

$$D(P_1 \| P_2) \geq \frac{1}{2} \| P_1 - P_2 \|^2,$$

where we introduced the distance $\| P_1 - P_2 \| \equiv \sum_x |P_1(x) - P_2(x)|$.

² This interpretation is not always valid. See for example Refs. [60, 225]

extra cost of compressing a sequence drawn from a probability distribution by using an encoding scheme based on the incorrect one [49]. The entropy production thus measures the information cost (in units of $k_B T$) of describing the temporal evolution in the time-reversed frame of reference. In this sense, the entropy production can be understood as expressing the difficulty to distinguish the arrow of time.

The non-negativity of the entropy production leads to the principle of *temporal ordering* according to which, in nonequilibrium steady states, the typical paths are more ordered in time than their corresponding time reversals [89]. Indeed, the probabilities of the forward paths decay more slowly in time than their time-reversed path since $h^R > h$ in nonequilibrium conditions. This changes our perspective on the second law of thermodynamic: nonequilibrium conditions perform a selection of trajectories. This time ordering of the trajectories occur as soon as the system is driven out of equilibrium, in contrast with Glansdorff-Prigogine threshold for the apparition of dissipative structures. This time-ordering is possible at the expense of the spatial disorder of the total system composed of the subsystem plus its environment. The consequences of these observations will be developed in Chapter 6.

The relationship which is here described belongs to the same family of large-deviation properties as the escape-rate and chaos-transport formulas [95, 55, 92], the steady-state or transient fluctuation theorem [69, 79], and the nonequilibrium work fluctuation theorems [118, 50]. All these relationships express an irreversible property as the difference between two decay rates of mesoscopic or microscopic properties [87, 88]. However, relation (5.5) is here expressed in terms of two quantities characterizing the fine-grained dynamics. Indeed, dynamical entropies characterize the intrinsic time scales of the system and are expressed in terms of n -time correlation functions, which are at the highest level of the hierarchy. We therefore probe the very fine scales of the dynamics. This will be further discussed in section 5.2.5.

5.2 Thermodynamic time asymmetry in nonequilibrium fluctuations

We here present the analysis of experiments on driven Brownian motion and electric noise in a RC circuit, showing that thermodynamic entropy production can be related to the breaking of time-reversal symmetry in the statistical description of these nonequilibrium systems [9, 10]. The experiments were performed by the group of S. Ciliberto at the Ecole Normale Supérieure de Lyon, France.

5.2.1 Stochastic description

We consider a Brownian particle in a fixed optical trap and surrounded by a fluid moving at the speed u . In a viscous fluid such as water solution at room temperature and pressure, the motion of a dragged micrometric particle is

overdamped. In this case, its Brownian motion can be modeled by the following Langevin equation [218]:

$$\alpha \frac{dz}{dt} = F(z) + \alpha u + \xi_t, \quad (5.6)$$

where α is the viscous friction coefficient, $F = -\partial_z V$ the force exerted by the potential V of the laser trap, αu is the drag force of the fluid moving at speed u , and ξ_t a Gaussian white noise with its average and correlation function given by

$$\begin{aligned} \langle \xi_t \rangle &= 0, \\ \langle \xi_t \xi_{t'} \rangle &= 2 k_B T \alpha \delta(t - t'). \end{aligned}$$

In the special case where the potential is harmonic of constant k , $V = kz^2/2$, the stationary probability density is Gaussian

$$p_{st}(z) = \sqrt{\frac{\beta k}{2\pi}} \exp \left[-\frac{\beta k}{2} (z - u \tau_R)^2 \right], \quad (5.7)$$

with the relaxation time

$$\tau_R = \frac{\alpha}{k}$$

and the inverse temperature $\beta = (k_B T)^{-1}$. The maximum of this Gaussian distribution is located at the distance $u \tau_R$ of the minimum of the confining potential. This shift is due to dragging and corresponds to the position where there is a balance between the frictional and harmonic forces.

The work W_t done on the system by the moving fluid during the time interval t is given by [197, 219, 218]

$$W_t = - \int_0^t u F(z_{t'}) dt' \quad (5.8)$$

while the heat Q_t generated by dissipation is

$$Q_t = \int_0^t (\dot{z}_{t'} - u) F(z_{t'}) dt'. \quad (5.9)$$

Both quantities are related by the change in potential energy $\Delta V_t \equiv V(z_t) - V(z_0)$ so that

$$Q_t = W_t - \Delta V_t.$$

In a stationary state, the mean value of the dissipation rate is equal to the mean power done by the moving fluid since $\lim_{t \rightarrow \infty} (1/t) \langle \Delta V_t \rangle = 0$. The thermodynamic entropy production is thus given by

$$\frac{d_i S}{dt} = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\langle Q_t \rangle}{T} = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\langle W_t \rangle}{T} = \frac{\alpha u^2}{T} \quad (5.10)$$

Brownian particle	RC circuit
z_t	$q_t - It$
$\dot{z}_t$	$\dot{q}_t - I$
ξ_t	$-\delta V_t$
α	R
k	$1/C$
u	$-I$

Table 5.1. The analogy between the Brownian particle and the electric RC circuit. For the Brownian particle, z_t is its position, $\dot{z}_t$ its velocity, ξ_t the Langevin fluctuating force, α the viscous friction coefficient, k the harmonic strength or stiffness of the optical trap, and u the fluid speed. For the electric circuit, q_t is the electric charge passing through the resistor during time t , $\dot{q}_t = i_t$ the corresponding current, δV_t the fluctuating electric potential of the Nyquist noise, R the resistance, C the capacitance, and I the mean current source.

in the stationary state.³

An equivalent system is an RC electric circuit driven out of equilibrium by a current source which imposes the mean current I [218, 81]. The current fluctuates in the circuit because of the intrinsic Nyquist thermal noise. This electric circuit and the dragged Brownian particle, although physically different, are known to be formally equivalent by the correspondence shown in Table 5.1 [218].

Our aim is to show that one can extract the heat dissipated along a fluctuating path by comparing the probability of this path with the probability of the corresponding time-reversed path having also reversed the external driving, i.e., $u \rightarrow -u$ for the dragged Brownian particle (respectively, $I \rightarrow -I$ for the RC circuit).

For this purpose, we use a path integral formulation. A stochastic trajectory is uniquely defined by specifying the noise history of the system ξ_t . Indeed, the solution of the stochastic equation (5.6), i.e.,

$$\begin{aligned} z_t &= z_0 + \int_0^t dt' \dot{z}_{t'} \\ &= z_0 + \frac{1}{\alpha} \int_0^t dt' \left[F(z_{t'}) + \alpha u + \xi_{t'} \right] \end{aligned}$$

is uniquely specified if the noise history is known. Since we consider a Gaussian white noise, the probability to have the noise history ξ_t is given by [176]

$$P[\xi_t] \propto \exp \left[-\frac{1}{4k_B T \alpha} \int_0^t dt' (\xi_{t'})^2 \right].$$

According to Eq. (5.6), the probability of a trajectory z_t starting from the fixed initial point z_0 is thus written as

³ Although the mean values of the work W and heat Q are equal, the extra term ΔV between them does change their large fluctuations properties, as pointed out in Ref. [219].

$$P[z_t|z_0] \propto \exp \left[-\frac{1}{4k_B T \alpha} \int_0^t dt' (\alpha \dot{z} - F(z) - \alpha u)^2 \right]. \quad (5.11)$$

We remark that the corresponding joint probability is obtained by multiplying the conditional probability (5.11) with the stationary probability density (5.7) of the initial position as

$$P[z_t] \propto P[z_t|z_0] p_{\text{st}}(z_0).$$

To extract the heat dissipated along a trajectory, we consider the probability of a given path over the probability to observe the reversed path having also reversed the sign of the driving u . The reversed path is thus defined by $z_t^R = z_{t-t'}$ which implies $\dot{z}_t^R = -\dot{z}_{t-t'}$. Therefore, we find that

$$\begin{aligned} \ln \frac{P_+[z_t|z_0]}{P_-[z_t^R|z_0^R]} &= \frac{1}{4k_B T \alpha} \int_0^t dt' \left[(\alpha \dot{z} + F(z) - \alpha u)^2 - (\alpha \dot{z} - F(z) - \alpha u)^2 \right] \\ &= \frac{1}{k_B T} \int_0^t dt' (\dot{z} - u) F(z) \\ &= \frac{1}{k_B T} \left[V(z_0) - V(z_t) - u \int_0^t dt' F(z) \right] \\ &= \frac{Q_t}{k_B T}, \end{aligned} \quad (5.12)$$

which is precisely the heat Q_t dissipated along the path expressed in the thermal unit $k_B T$. The dissipation can thus be related to time-symmetry breaking already at the level of mesoscopic paths. Relations similar to Eq. (5.12) are known for Boltzmann's entropy production [149], for time-dependent systems [50, 195, 125], and in the context of Onsager-Machlup theory [209]. We emphasize that the reversal of u is essential to get the dissipated heat from the way the path probabilities P_+ and P_- differ. This result can be generalized to the underdamped case and in the presence of a magnetic field, as shown in Appendix C. In addition to the driving u , the magnetic field must then be reversed as well, although neither the work nor the heat explicitly depend on this latter. This stems from the time-reversal symmetry, which has the effect of changing the sign of all odd parameters.

We notice that the main difference between the path probabilities P_+ and P_- comes from the shift between the mean values of the fluctuations under forward or backward driving. Indeed, the average position is equal to $\langle z \rangle_+ = u\tau_R$ under forward driving at speed $+u$, and $\langle z \rangle_- = -u\tau_R$ under backward driving at speed $-u$. The shift $2u\tau_R$ in the average positions implies that a typical path of the forward time series falls, after its time reversal, in the tail of the probability distribution P_- of the backward time series. Therefore, the probabilities P_- of the time-reversed forward paths in the *backward* time series are typically lower than the probabilities P_+ of the corresponding forward paths. The above derivation (5.12) shows that the dissipation can be obtained in terms of their ratio P_+/P_- . We emphasize that this derivation holds for anharmonic potentials as well as harmonic ones, so that the result is general in this respect.

In the stationary state, the mean entropy production (5.10) is given by averaging the dissipated heat (5.12) over all possible trajectories:

$$\begin{aligned}
 \frac{d_i S}{dt} &= \lim_{t \rightarrow \infty} \frac{k_B}{t} \left\langle \ln \frac{P_+[z_t|z_0]}{P_-[z_t^R|z_0^R]} \right\rangle \\
 &= \lim_{t \rightarrow \infty} \frac{k_B}{t} \left\langle \ln \frac{P_+[z_t]}{P_-[z_t^R]} \right\rangle \\
 &= \lim_{t \rightarrow \infty} \frac{k_B}{t} \int \mathcal{D}z_t P_+[z_t] \ln \frac{P_+[z_t]}{P_-[z_t^R]} \geq 0
 \end{aligned} \tag{5.13}$$

which results from the fact that the terms at the boundaries of the time interval are vanishing for the statistical average in the long-time limit. The mean entropy production (5.13) is thus given by a so-called Kullback-Leibler distance [132] or relative entropy [228], which is known to be always non negative. Therefore, the mean entropy production satisfies the second law of thermodynamics, as it should.

We point out that the heat dissipated along an individual path given by Eq. (5.12) is a fluctuating quantity and may be either positive or negative. We here face the paradox raised by Maxwell that the dissipation is non-negative on average but has an undetermined sign at the level of the individual stochastic paths. The second law of thermodynamics holds for entropy production defined after statistical averaging with the probability distribution. We remain with fluctuating mechanical quantities at the level of individual mesoscopic paths or microscopic trajectories.

5.2.2 Dynamical randomness and entropy production

The aim of this section is to present a method to characterize the property of dynamical randomness in the time series and to show how this property is related to the thermodynamic entropy production when the paths are compared with their time reversals.

(ε, τ) -entropies per unit time

For random processes which are continuous in time and in their variable, the trajectories should be sampled with a resolution ε and with a sampling time τ . Therefore, the entropy per unit time depends *a priori* on each one of them and we talk about the (ε, τ) -entropy per unit time. Such a quantity has been introduced by Shannon as the rate of information generation by continuous sources [202]. The theory of this quantity was developed under the names of ε -entropy [130] and rate distortion function [19]. More recently, the problem of characterizing dynamical randomness has reappeared in the study of chaotic dynamical systems. A numerical algorithm was proposed by Grassberger, Procaccia and coworkers [104, 44] in order to estimate the Kolmogorov-Sinai entropy per unit time. Thereafter, it was shown that the same algorithm also applies to stochastic processes, allowing us to compare the property of dynamical randomness

of different random processes [97, 84]. Moreover, these dynamic entropies were measured for Brownian motion at equilibrium [90, 29]. We here present the extension of this method to out-of-equilibrium fluctuating systems.

Since we are interested in the probability of a given succession of states obtained by sampling the signal $Z(t)$ at small time intervals τ , a multi-time random variable is defined according to $\mathbf{Z} = [Z(t_0), Z(t_0 + \tau), \dots, Z(t_0 + n\tau - \tau)]$, which represents the signal during the time period $t - t_0 = n\tau$. For a stationary process, the probability distribution does not depend on the initial time t_0 . From the point of view of probability theory, the process is defined by the n -time joint probabilities

$$P_s(\mathbf{z}; d\mathbf{z}, \tau, n) = \Pr\{\mathbf{z} < \mathbf{Z} < \mathbf{z} + d\mathbf{z}; s\} = p_s(\mathbf{z})d\mathbf{z},$$

where $p_s(\mathbf{z})$ denotes the probability density for $\mathbf{Z}$ to take the values $\mathbf{z} = (z_0, z_1, \dots, z_{n-1})$ at times $t_0 + i\tau$ ($i = 0, 1, 2, \dots, n-1$) for some nonequilibrium driving $s = u/|u| = \pm 1$. Now, due to the continuous nature in time and in space of the process, we will consider the probability $P_+(\mathbf{Z}_m; \varepsilon, \tau, n)$ for the trajectory to remain within a distance ε of some reference trajectory $\mathbf{Z}_m$, made of n successive positions of the Brownian particle observed at time intervals τ during the forward process. This reference trajectory belongs to an ensemble of M reference trajectories $\{\mathbf{Z}_m\}_{m=1}^M$, allowing us to take statistical averages. These reference trajectories define the patterns, i.e., the recurrences of which are searched for in the time series.

On the other hand, we can introduce the quantity $P_-(\mathbf{Z}_m^R; \varepsilon, \tau, n)$ which is the probability for a reversed trajectory of the reversed process to remain within a distance ε of the reference trajectory $\mathbf{Z}_m$ (of the forward process) for n successive positions.

Suppose we have two realizations over a very long time interval $L\tau \gg n\tau$ given by the time series $\{z_{\pm}(k\tau)\}_{k=1}^L$, respectively for the forward (+) and backward (-) processes. Within these long time series, sequences of length n are compared with each other. We thus consider an ensemble set of $1 \ll M \ll L$ reference sequences, which are all of length n :

$$\mathbf{Z}_m = [Z(m\tau), \dots, Z(m\tau + n\tau - \tau)], \quad m \in \{m_1, \dots, m_M\}.$$

These reference sequences are taken at equal time intervals in order to sample the forward process according to its probability distribution P_+ . The distance between a reference sequence and another sequence of length n is defined by

$$\text{dist}_n(\mathbf{Z}_m, \mathbf{Z}_j) = \max\{|Z(m\tau) - Z(j\tau)|, \dots, |Z(m\tau + n\tau - \tau) - Z(j\tau + n\tau - \tau)|\}$$

for $j = 1, 2, \dots, L' = L - n + 1$. The probability for this distance to be smaller than ε is then evaluated by

$$P_+(\mathbf{Z}_m; \varepsilon, \tau, n) = \frac{1}{L'} \text{Number}\{\mathbf{Z}_j : \text{dist}_n(\mathbf{Z}_m, \mathbf{Z}_j) \leq \varepsilon\}. \quad (5.14)$$

The average of the logarithm of these probabilities over the different reference sequences gives the block entropy

$$H(\varepsilon, \tau, n) = -\frac{1}{M} \sum_{m=1}^M \ln P_+(Z_m; \varepsilon, \tau, n), \quad (5.15)$$

also called the mean pattern entropy. The (ε, τ) -entropy per unit time is then defined as the rate of the linear growth of the block entropy as the length n of the reference sequences increases [104, 44, 97]:

$$h(\varepsilon, \tau) = \lim_{n \rightarrow \infty} \lim_{L', M \rightarrow \infty} \frac{1}{\tau} [H(\varepsilon, \tau, n+1) - H(\varepsilon, \tau, n)]. \quad (5.16)$$

Similarly, the probability of a reversed trajectory in the reversed process can be evaluated by

$$P_-(Z_m^R; \varepsilon, \tau, n) = \frac{1}{L'} \text{Number}\{\tilde{Z}_j : \text{dist}_n(Z_m^R, \tilde{Z}_j) \leq \varepsilon\}, \quad (5.17)$$

where $Z_m^R = [Z(m\tau + n\tau - \tau), \dots, Z(m\tau)]$ is the time reversal of the reference path Z_m of the forward process, while $\{\tilde{Z}_j\}_{j=1}^{L'}$ are the paths of the reversed process (with the opposite driving $-u$). In similitude with Eqs. (5.15) and (5.16), we may introduce the time-reversed block entropy

$$H^R(\varepsilon, \tau, n) = -\frac{1}{M} \sum_{m=1}^M \ln P_-(Z_m^R; \varepsilon, \tau, n) \quad (5.18)$$

and the time-reversed (ε, τ) -entropy per unit time

$$h^R(\varepsilon, \tau) = \lim_{n \rightarrow \infty} \lim_{L', M \rightarrow \infty} \frac{1}{\tau} [H^R(\varepsilon, \tau, n+1) - H^R(\varepsilon, \tau, n)]. \quad (5.19)$$

We notice that the dynamical entropy (5.16) gives the decay rate of the probabilities to find paths within a distance ε from a typical path $Z = (Z_0, Z_1, Z_2, \dots, Z_{n-1})$ with $Z_i = Z(t_0 + i\tau)$:

$$P_+(Z; \varepsilon, \tau, n) \sim \exp[-n\tau h(\varepsilon, \tau)] \quad (n \rightarrow \infty)$$

as the number n of time intervals increases. In the case of ergodic random processes, this property is known as the Shannon-McMillan-Breiman theorem [23]. The decay rate h characterizes the temporal disorder, i.e., dynamical randomness, in both deterministic dynamical systems and stochastic processes [104, 44, 61, 97, 84]. On the other hand, the time-reversed dynamical entropy (5.19) is the decay rate of the probabilities of the time-reversed paths in the reversed process:

$$P_-(Z^R; \varepsilon, \tau, n) \sim \exp[-n\tau h^R(\varepsilon, \tau)] \quad (n \rightarrow \infty).$$

Since h^R is the decay rate of the probability to find, in the backward process, the time-reversed path corresponding to some typical path of the forward process, the exponential $\exp(-h^R \Delta t)$ evaluates the amount of time-reversed paths among the typical paths (of duration Δt). The time-reversed entropy per

unit time h^R thus characterizes the rareness of the time-reversed paths in the forward process.

The dynamical randomness of the stochastic process ruled by the Langevin equation (5.6) can be characterized in terms of its (ε, τ) -entropy per unit time. This latter is calculated for the case of a harmonic potential in Appendix D. For small values of the spatial resolution ε , we find that

$$h(\varepsilon, \tau) = \frac{1}{\tau} \ln \sqrt{\frac{\pi e D \tau_R}{2 \varepsilon^2}} (1 - e^{-2\tau/\tau_R}) + O(\varepsilon^2), \quad (5.20)$$

with the diffusion coefficient of the Brownian particle $D = k_B T / \alpha$. The (ε, τ) -entropy per unit time increases as the resolution ε decreases, meaning that randomness is found on smaller and smaller scales in typical trajectories of the Brownian particle. After having obtained the main features of the (ε, τ) -entropy per unit time, we go on in the next subsection by comparing it with the time-reversed (ε, τ) -entropy per unit time, establishing the connection with thermodynamics.

Thermodynamic entropy production

Under nonequilibrium conditions, detailed balance does not hold so that the probabilities of the paths and their time reversals are different. Similarly, the decay rates h and h^R also differ. Their difference can be calculated by evaluating the path integral (5.13) by discretizing the paths with the sampling time τ and resolution ε :

$$\begin{aligned} \frac{d_i S}{dt} &= \lim_{t \rightarrow \infty} \frac{k_B}{t} \int \mathcal{D}z_t P_+[z_t] \ln \frac{P_+[z_t]}{P_-[z_t^R]} \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{\tau \rightarrow 0} \lim_{n \rightarrow \infty} \lim_{L', M \rightarrow \infty} \frac{k_B}{n\tau} \frac{1}{M} \sum_{m=1}^M \ln \frac{P_+(Z_m; \varepsilon, \tau, n)}{P_-(Z_m^R; \varepsilon, \tau, n)}. \end{aligned}$$

The statistical average is carried out over M paths of the forward process and thus corresponds to the average with the probability $P_+[z_t]$. The logarithm of the ratio of probabilities can be splitted into the difference between the logarithms of the probabilities, leading to the difference of the block entropies (5.18) and (5.15). The limit $n \rightarrow \infty$ of the block entropies divided by n can be evaluated from the differences between the block entropy at $n+1$ and the one at n , whereupon the (ε, τ) -entropies per unit time (5.16) and (5.19) appear. Finally, the mean entropy production in the nonequilibrium stationary state is given by the difference between the time-reversed and direct (ε, τ) -entropies per unit time:

$$\frac{d_i S}{dt} = \lim_{\varepsilon \rightarrow 0} \lim_{\tau \rightarrow 0} k_B [h^R(\varepsilon, \tau) - h(\varepsilon, \tau)]. \quad (5.21)$$

The difference between h^R and h characterizes the time asymmetry of the ensemble of typical paths effectively realized during the forward process. Equation (5.21) shows that this time asymmetry is related to the thermodynamic

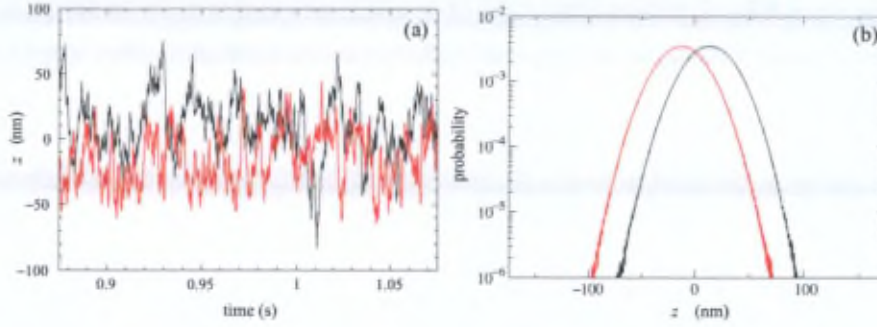


Fig. 5.1. (a) The time series of a typical path z_t for the trapped Brownian particle in the fluid moving at the speed u for the forward process (upper curve) and $-u$ for the reversed process (lower curve) with $u = 4.24 \times 10^{-6}$ m/s. (b) Gaussian probability distributions of the forward and backward experiments. The mean value is located at $\pm u\tau_R = \pm 12.9$ nm.

entropy production. The entropy production is thus expressed as the difference of two usually very large quantities which increase for ε going to zero [97, 84]. Nevertheless, the difference remains finite and gives the entropy production.

5.2.3 Driven Brownian motion

The first experimental system we have investigated is a Brownian particle trapped by an optical tweezer, which is composed by a large numerical aperture microscope objective ($\times 63$, 1.3) and by an infrared laser beam with a wavelength of 980 nm and a power of 20 mW on the focal plane. The trapped polystyrene particle has a diameter of 2 μm and is suspended in a 20% glycerol-water solution. The particle is trapped at 20 μm from the bottom plate of the cell which is 200 μm thick. The detection of the particle position x_t is done using a He-Ne laser and an interferometric technique [191]. This technique allows us to have a resolution on the position of the particle of 10^{-11} m. In order to apply a shear to the trapped particle, the cell is moved with a feedback-controlled piezo actuator which insures a perfect linearity of displacement.

The potential is harmonic: $V = kz^2/2$. The stiffness of the potential is $k = 9.62 \cdot 10^{-6}$ kg s $^{-2}$. The relaxation time is $\tau_R = \alpha/k = 3.05 \cdot 10^{-3}$ s, which has been determined by measuring the decay rate of the autocorrelation of x_t . The variable x_t is acquired at the sampling frequency $f = 8192$ Hz. The temperature is $T = 298$ K.

The mean square displacement of the Brownian particle in the optical trap is $\sigma = \sqrt{k_B T/k} = 20.7$ nm, while the diffusion coefficient is $D = \sigma^2/\tau_R = 1.4 \times 10^{-13}$ m 2 /s. We notice that the relaxation time is longer than the sampling time since their ratio is $f\tau_R = 25$.

In order to test experimentally that entropy production is related to the time asymmetry of dynamical randomness according to Eq. (5.21), time series have been recorded for several values of $|u|$. For each value, a pair of time

series is generated, one corresponding to the forward process and the other to the reversed process, having first discarded the transient evolution. The time series contain up to 2×10^7 points each. Figure 5.1a depicts examples of paths z_t for the trapped Brownian particle in the moving fluid. Figure 5.1b shows the corresponding stationary distributions for the two time series. They are Gaussian distributions shifted according to Eq. (5.7).

The analysis of these time series is performed by calculating the block entropy (5.15) versus the path duration $n\tau$, and this for different values of ε . Figure 5.2a shows that the block entropy increases linearly with the path duration $n\tau$ up to a maximum value fixed by the total length of the time series. The forward entropy per unit time $h(\varepsilon, \tau)$ is thus evaluated from the linear growth of the block entropy (5.15) with the time $n\tau$.

Similarly, the time-reversed block entropy (5.18) is computed using the same reference sequences as for the forward block entropy, reversing each one of them, and getting their probability of occurrence in the backward time series. The resulting time-reversed block entropy is depicted in Fig. 5.2b versus $n\tau$ for different values of ε . Here also, we observe that H^R grows linearly with the time $n\tau$ up to some maximum value due to the lack of statistics over long sequences because the time series is limited. Nevertheless, the linear growth is sufficiently extended that the backward entropy per unit time $h^R(\varepsilon, \tau)$ can be obtained from the slopes in Fig. 5.2b.

Figure 5.2c depicts the difference between the backward and forward block entropies H^R and H versus the time $n\tau$, showing the time asymmetry due to the nonequilibrium constraint. We notice that the differences $H^R - H$ are small compared with the block entropies themselves, meaning that dynamical randomness is large although the time asymmetry is small. Accordingly, the values $H^R - H$ are more affected by the experimental limitations than the block entropies themselves. In particular, the saturation due to the total length of the time series affects the linearity of $H^R - H$ versus $n\tau$. However, we observe the expected independence of the differences $H^R - H$ on ε . Indeed, the slope which can be obtained from the differences $H^R - H$ versus $n\tau$ cluster around a common value (contrary to what happens for H and H^R). According to Eq. (5.21), the slope of $H^R - H$ versus $n\tau$ gives the thermodynamic entropy production.

This prediction is indeed verified. Figure 5.3 compares the difference $h^R(\varepsilon, \tau) - h(\varepsilon, \tau)$ with the thermodynamic entropy production given by the rate of dissipation (5.10) as a function of the speed u of the fluid. We see the good agreement between both, which is the experimental evidence that the thermodynamic entropy production is indeed related to the time asymmetry of dynamical randomness.

The dynamical randomness of the Langevin stochastic process can be further analyzed by plotting the scaled entropy per unit time $\tau h(\varepsilon, \tau)$ versus the scaled resolution $\delta \equiv \varepsilon/\sqrt{1 - \exp(-2\tau/\tau_R)}$ for different values of the time interval τ , as depicted in Fig. 5.4a. According to Eq. (5.20), the scaled entropy per unit time should behave as $\tau h(\varepsilon, \tau) \simeq \ln(1/\delta) + C$ with some constant C in the limit

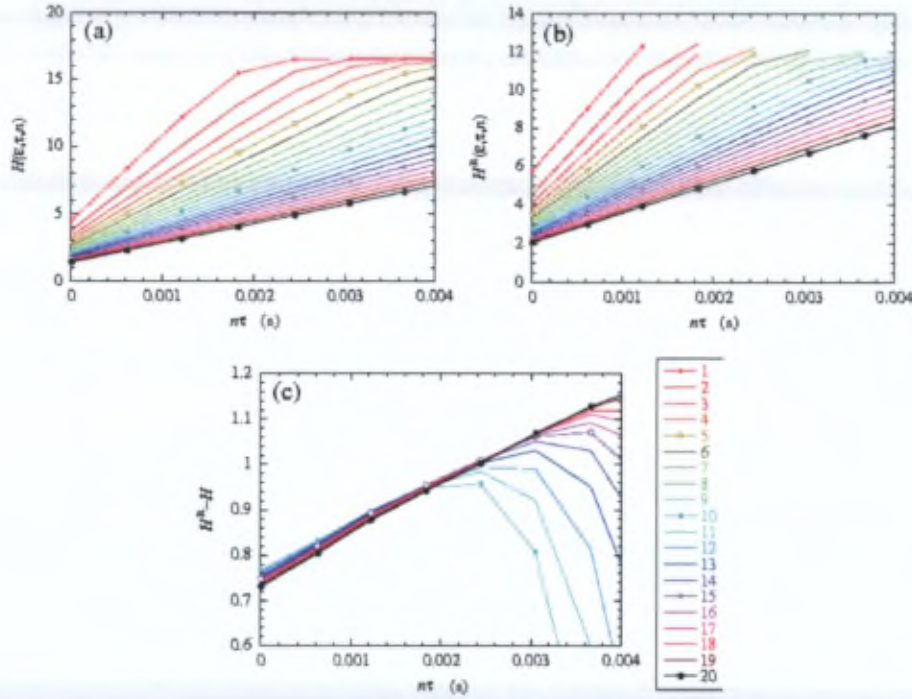


Fig. 5.2. (a) The block entropy (or mean pattern entropy) as a function of time $n\tau$ for the trapped Brownian particle in the fluid moving at the speed $u = 4.24 \times 10^{-6}$ m/s. The different curves correspond to different values of $\varepsilon = k \times 0.558$ nm with $k = 1, \dots, 20$ given in the right-hand column. The distance used in this calculation is defined by taking the maximum among the distances $|Z(t) - Z_m(t)|$ for the times $t = 0, \tau, \dots, (n-1)\tau$. The larger slopes correspond to the smaller value of ε . The linear growth persists up to a maximal value of the mean pattern entropy given by the total length of the time series: $H_{\max} = \ln(1.4 \times 10^7)$. (b) The mean reversed block entropy as a function of time corresponding to (a). After a linear growth, H^R saturates and falls down to zero (not shown) because of the finiteness of the time series which limits the statistics. (c) Differences between the backward and forward (ε, τ) dynamical entropies in (b) and (a) for ε between 5.6–11.2 nm for the Brownian particle. Straight lines are fitted to the first part of the curves and their slopes give the entropy production according to Eq. (5.21).

$\delta \rightarrow 0$. Indeed, we verify in Fig. 5.4a that, in the limit $\delta \rightarrow 0$, the scaled curves only depend on the variable δ with the expected dependence $\ln(1/\delta)$.

For large values of δ , the experimental curves deviate from the logarithmic approximation (5.20), since this latter is only valid for $\delta \rightarrow 0$. The calculation in Appendix D shows that we should expect corrections in powers of ε^2 to be added to the approximation as $\ln(1/\delta)$.

In Fig. 5.4b, we depict the scaled direct and reversed (ε, τ) -entropies per unit time. We compare the behavior of h^R with the behavior $\tau h^R \simeq \tau(h + k_B^{-1} d_i S/dt)$ expected from the formula (5.21). This figure shows that

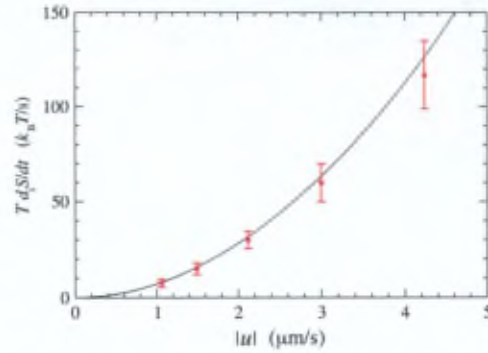


Fig. 5.3. Entropy production of the Brownian particle versus the driving speed u . The solid line is the well-known rate of dissipation given by Eq. (5.10). The dots depict the results of Eq. (5.21) calculated with the differences between the (ε, τ) -entropies per unit time.

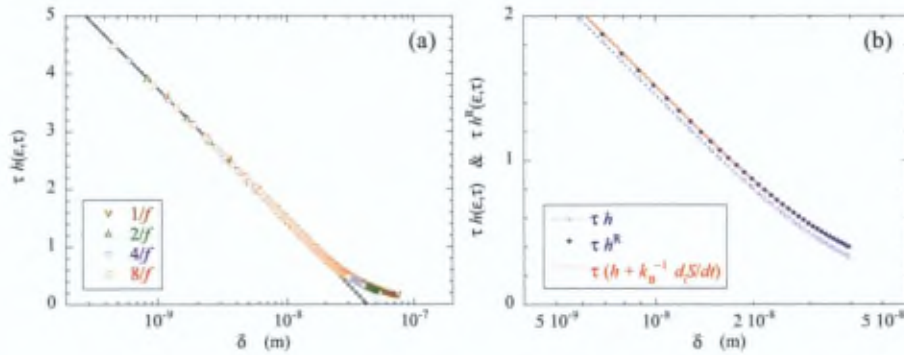


Fig. 5.4. (a) (ε, τ) -entropy per unit time of the Brownian particle scaled by the time interval τ as a function of $\delta = \varepsilon / \sqrt{1 - \exp(-2\tau/\tau_R)}$, for different values of the time interval $\tau = 1/f, 2/f, 4/f, 8/f$, with the sampling time $1/f = 1/8192$ s. The dots are the results of the computation from the time series for the speed $u = 4.24 \times 10^{-6}$ m/s. The solid line depicts the expected behavior according to Eqs. (5.20) and (5.21). (b) Scaled reversed and direct (ε, τ) -entropies per unit time for $\tau = 4/f$. The solid line is the result expected from Eq. (5.21).

the direct and reversed (ε, τ) -entropies per unit time are quantities which are large with respect to their difference due to the nonequilibrium constraint. This means that the positive entropy production is a small effect on the top of a substantial dynamical randomness.

Maxwell's demon vividly illustrates the paradox that the dissipated heat is always positive at the macroscopic level although it would take both signs if considered at the microscopic level of individual trajectories. The resolution of Maxwell's paradox can be remarkably demonstrated with the experimental data. Indeed, the heat dissipated along an individual stochastic trajectory is

given by Eq. (5.9) and can be obtained by searching for recurrences in the time series according to Eq. (5.12). The conditional probabilities entering Eq. (5.12) are evaluated in terms of the joint probabilities (5.14) and (5.17) according to

$$P_+(Z; \varepsilon, \tau, n | Z_0) = P_+(Z; \varepsilon, \tau, n) / P_+(Z_0; \varepsilon, \tau, 1), \quad (5.22)$$

$$P_-(Z^R; \varepsilon, \tau, n | Z_{n-1}) = P_-(Z^R; \varepsilon, \tau, n) / P_-(Z_{n-1}; \varepsilon, \tau, 1), \quad (5.23)$$

where we notice that the probabilities with $n = 1$ are approximately equal to the corresponding stationary probability density (5.7) multiplied by the range $dz = 2\varepsilon$. The heat dissipated along two randomly selected paths are plotted in Fig. 5.5. We see the very good agreement between the values computed with Eq. (5.9) using each path and Eq. (5.12) using the probabilities of recurrences in the time series. We observe that, at the level of individual trajectories, the heat exchanged between the particle and the surrounding fluid can be positive or negative because of the molecular fluctuations. It is only by averaging over the forward process that the dissipated heat takes the positive value depicted in Fig. 5.3. Indeed, Fig. 5.3 is obtained after averaging over many reference paths as those of Fig. 5.5. The positivity of the thermodynamic entropy production results from this averaging, which solves Maxwell's paradox.

5.2.4 Electric noise in RC circuits

The second system we have investigated is an RC electric circuit driven out of equilibrium by a current source which imposes the mean current I [81]. The current fluctuates in the resistor because of the intrinsic Nyquist thermal noise [218]. The RC electric circuit and the dragged Brownian particle, although physically different, are known to be formally equivalent by the correspondence given in Table 5.1.

The electric circuit is composed of a capacitor with capacitance $C = 278$ pF in parallel with a resistor of resistance $R = 9.22$ M Ω . The relaxation time of the circuit is $\tau_R = RC = 2.56 \times 10^{-3}$ s. The charge q_t going through the resistor during the time interval t is acquired at the sampling frequency $f = 8192$ Hz. The temperature is here also equal to $T = 298$ K.

The mean square charge of the Nyquist thermal fluctuations is $\sigma = \sqrt{k_B T C} = 6.7 \times 10^3 e$ where $e = 1.602 \times 10^{-19}$ C is the electron charge. The diffusion coefficient is $D = \sigma^2 / \tau_R = 1.75 \times 10^{10} e^2 / s$. The ratio of the relaxation time to the sampling time is here equal to $f \tau_R = 21$.

As for the first system, pairs of time series for opposite drivings $\pm I$ are recorded. Their length are 2×10^7 points each. Figure 5.6 depicts an example of a pair of such paths with the corresponding probability distribution of the charge fluctuations.

The block entropies are here also calculated using Eqs. (5.15) and (5.18) and the (ε, τ) -entropies per unit time are obtained from their linear growth as a function of the time $n\tau$. The scaled entropies per unit time τh are depicted versus δ in Fig. 5.7a. Here again, the scaled entropy per unit time is verified to depend only on δ for $\delta \rightarrow 0$, as expected from the analytical calculation in Appendix D. In Fig. 5.7b, we compare the scaled reversed (ε, τ) -entropy per

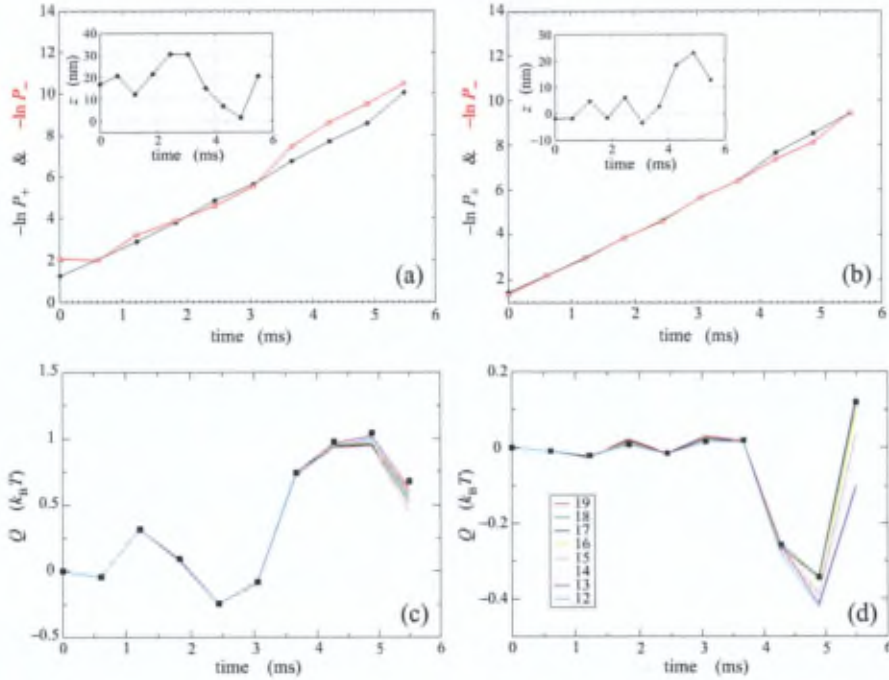


Fig. 5.5. Measure of the heat dissipated by the Brownian particle along two randomly selected trajectories in the time series of Fig. 5.1. The fluid speeds are $\pm u$ with $u = 4.24 \times 10^{-6}$ m/s. (a)-(b) Inset: The two randomly selected trajectories. The conditional probabilities P_+ and P_- of the corresponding forward (filled circles) and the backward (open circles) paths for $\varepsilon = 8.4$ nm, as evaluated by Eqs. (5.22) and (5.23). These probabilities present an exponential decrease modulated by the fluctuations. At time zero, the conditional probabilities are not defined and, instead, we plotted at this time the stationary probabilities for indication. (c)-(d) The dissipated heat given by the logarithm of the ratio of the forward and backward probabilities according to Eq. (5.12) for different values of $\varepsilon = k \times 0.558$ nm with $k = 12, \dots, 19$ in the range 6.7-10.6 nm. They are compared with the values (squares) directly calculated from Eq. (5.9). For small values of ε , the agreement is quite good for short time and are within experimental errors for larger time.

unit time to the behavior $\tau h^R \simeq \tau (h + k_B^{-1} d_i S/dt)$, expected by our central result (5.21).

The difference between the time-reversed and direct (ε, τ) -entropies per unit time is then compared with the dissipation rate expected with Joule's law. We observe the nice agreement between both in Fig. 5.8, which confirms the validity of Eq. (5.21).

5.2.5 Discussion

In this section, we discuss about the comparison between the present results and other nonequilibrium relations.

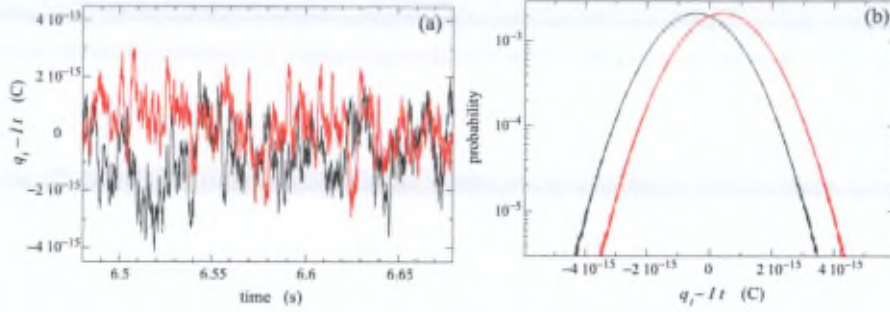


Fig. 5.6. (a) The time series of a typical path $q_t - It$ for the Nyquist noise in the RC electric circuit driven by the current I (upper curve) and opposite current $-I$ (lower curve) with $I = 1.67 \times 10^{-13}$ A. (b) Gaussian probability distributions of the forward and backward experiments. The unit of the electric charge $q_t - It$ is the Coulomb (C).

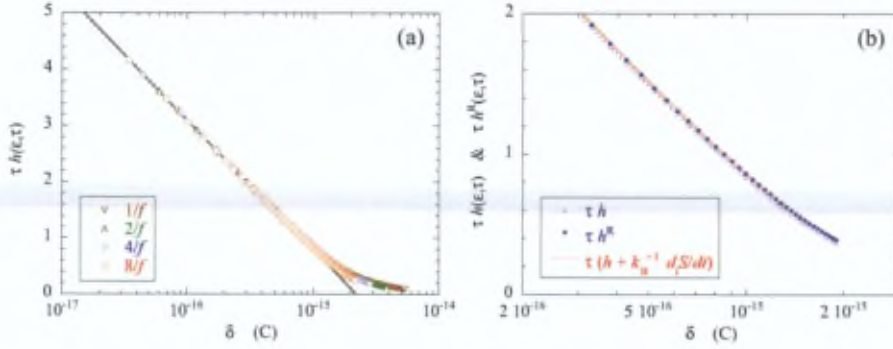


Fig. 5.7. (a) (ε, τ) -entropy per unit time of the RC electric circuit scaled by the time interval τ as a function of $\delta = \varepsilon/\sqrt{1 - \exp(-2\tau/\tau_R)}$, for different values of the time interval $\tau = 1/f, 2/f, 4/f, 8/f$, with the sampling time $1/f = 1/8192$ s. The dots are the results of the computation from the time series for the current $I = 1.67 \times 10^{-13}$ A. The solid line depicts the expected behavior according to Eqs. (5.20) and (5.21). (b) Scaled reversed and direct (ε, τ) -entropies per unit time for $\tau = 4/f$. The solid line is the result expected from Eq. (5.21). The unit of δ is the Coulomb (C).

The relation (5.21) expresses the entropy production as the difference between the backward and forward (ε, τ) -entropies per unit time. The backward process is obtained by reversing the driving constraints, which is also a characteristic feature of Crooks relation [50]. However, Crooks relation is concerned with systems driven by time-dependent external controls starting at equilibrium. In contrast, our results apply to nonequilibrium steady states. Another point is that Crooks relation deals with the fluctuations of the work performed on the system, while the present relation (5.21) gives the *mean* value of the entropy production and, this, in terms of path probabilities. In this respect, the relation (5.21) is closer to a formula recently obtained for the mean value of the dissipated work in systems driven from equilibrium by time-dependent

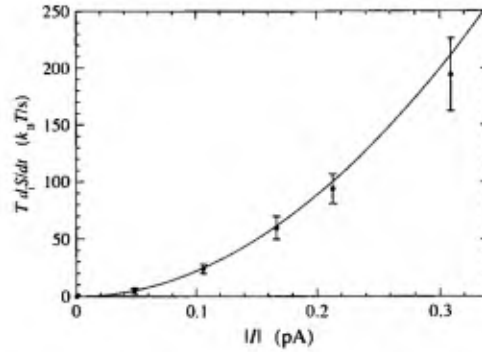


Fig. 5.8. Entropy production of the RC electric circuit versus the injected current I . The solid line is the Joule law, $T d_i S/dt = RI^2$. The dots are the results of Eq. (5.21).

external controls [125]. This formula relates the mean value of the dissipated work to the logarithm of the ratio between two phase-space probability densities associated with the forward and backward processes, respectively. These phase-space probability densities could in principle be expressed as path probabilities. Nevertheless, these latter would be defined for systems driven over a finite time interval starting from the equilibrium state, although the present equation (5.21) applies to nonequilibrium steady states reached in the long-time limit.

We now compare our results to the extended fluctuation theorem, which concerns nonequilibrium steady states [219, 218, 81]. The extended fluctuation theorem is a symmetry relation of the large-deviation properties of the fluctuating heat dissipation (5.9) during a time interval t . The probability that this fluctuating quantity takes the value

$$\zeta \simeq \frac{1}{t} \frac{Q_t}{T}$$

decays exponentially with the rate

$$J(\zeta) \equiv \lim_{t \rightarrow \infty} -\frac{1}{t} \ln P \left\{ \zeta < \frac{Q_t}{tT} < \zeta + d\zeta \right\}. \quad (5.24)$$

The extended fluctuation theorem states that the ratio of the probability of a positive fluctuation to the probability of a negative one increases exponentially as $\exp(\zeta t)$ in the long-time limit $t \rightarrow \infty$ and over a range of values of ζ , which is limited by its average $\langle \zeta \rangle$ [219, 218, 81]. Taking the logarithm of the ratio and the long-time limit, the extended fluctuation theorem can therefore be expressed as the following symmetry relation for the decay rate (5.24):

$$\zeta = k_B [J(-\zeta) - J(\zeta)] \quad \text{for} \quad -\langle \zeta \rangle \leq \zeta \leq \langle \zeta \rangle.$$

In this form, we notice the analogy with Eq. (5.21). *A priori*, the decay rate (4.25) can be compared with the (ε, τ) -entropy per unit time, which is also a

decay rate. However, the decay rate (5.24) concerns the probability of all the paths with dissipation ζ while the (ε, τ) -entropy per unit time concerns the probability of the paths within a distance ε of some reference typical paths. The (ε, τ) -entropy per unit time is therefore probing more deeply into the fluctuations down to the microscopic dynamics. In principle, this latter should be reached by zooming to the limit $\varepsilon \rightarrow 0$.

A closer comparison can be performed by considering the mean value of the fluctuating quantity ζ which gives the thermodynamic entropy production:

$$\langle \zeta \rangle = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\langle Q_t \rangle}{T} = \frac{d_i \mathcal{S}}{dt}.$$

Since the decay rate (5.24) vanishes at the mean value, $J(\langle \zeta \rangle) = 0$, we obtain the formula

$$\langle \zeta \rangle = k_B J(-\langle \zeta \rangle),$$

which can be quantitatively compared with Eq. (5.21) since both give the thermodynamic entropy production. Although the time-reversed entropy per unit time $h^R(\varepsilon, \tau)$ is *a priori* comparable with the decay rate $J(-\zeta)$, it turns out that they are different and satisfy the inequality $h^R(\varepsilon, \tau) \geq J(-\langle \zeta \rangle)$ since the entropy per unit time is always non negative $h(\varepsilon, \tau) \geq 0$. Moreover, $h(\varepsilon, \tau)$ is typically a large positive quantity. The greater the dynamical randomness, the larger the entropy per unit time $h(\varepsilon, \tau)$, as expected in the limit where ε goes to zero. This shows that the (ε, τ) -entropy per unit time probes finer scales in the path space where the time asymmetry is tested.

5.3 Summary

We have here presented detailed experimental results giving evidence that the thermodynamic entropy production finds its origin in the time asymmetry of dynamical randomness in the nonequilibrium fluctuations of two experimental systems. The first is a Brownian particle driven by an optical tweezer moving a constant speed $0 \leq |u| < 4.3 \mu\text{m/s}$. The second is the electric noise in an *RC* circuit driven by a constant source of current $0 \leq |I| < 0.3 \text{ pA}$. In both systems, long time series are recorded, allowing us to carry out the statistical analysis of their properties of dynamical randomness.

The dynamical randomness of the fluctuations is characterized in terms of (ε, τ) -entropies per unit time, one for the forward process and the other for the reversed process with opposite driving. These entropies per unit time measure the temporal disorder in the time series. The fact that the stochastic processes is continuous implies that the entropies per unit time depend on the resolution ε and the sampling time τ . The temporal disorder of the forward process is thus characterized by the entropy per unit time $h(\varepsilon, \tau)$, which is the mean decay rate of the path probabilities. On the other hand, the time asymmetry of the process can be tested by evaluating the amount of time-reversed paths of the forward process among the paths of the reversed process. This amount is evaluated by the probabilities of the time-reversed forward

paths in the reversed process and its mean decay rate, which defines the time-reversed entropy per unit time $h^R(\varepsilon, \tau)$. The time asymmetry in the process can be measured by the difference $h^R(\varepsilon, \tau) - h(\varepsilon, \tau)$. At equilibrium where detailed balance holds, we expect that the probability distribution ruling the time evolution is symmetric under time reversal so that this difference should vanish. In contrast, out of equilibrium, detailed balance is no longer satisfied and we expect that the breaking of the time-reversal symmetry by the invariant probability distribution of the nonequilibrium steady state. In this case, a non-vanishing difference is expected.

The analysis of the experimental data shows that the difference of (ε, τ) -entropies per unit time is indeed non vanishing. Moreover, we have the remarkable result that the difference gives the thermodynamic entropy production. The agreement between the difference and the thermodynamic entropy production is obtained for the driven Brownian motion up to an entropy production of nearly $120 k_B T/s$. For electric noise in the RC circuit, the agreement is obtained up to an entropy production of nearly $200 k_B T/s$. These results provide strong evidence that the thermodynamic entropy production arises from the breaking of time-reversal symmetry of the dynamical randomness in out-of-equilibrium systems.

Dynamical Randomness and the Physics of Information

Information is something that can be encoded in the state of a physical system, and a computation is a task that can be performed with a physically realizable device. Thus the quest for better ways to acquire, store, transmit, and process information leads us to seek more powerful methods for understanding and controlling the physical world. Information and algorithmic theories are concerned with theoretical concepts and formulations while the physics of information studies the various physical constraints governing information processing and computational tasks. In this respect, we show in this chapter that the temporal ordering principle introduced in Chapter 5 provides novel perspectives on these problems.

In this direction, the concepts introduced in the previous chapter are used to show that Landauer's principle can be understood in terms of time asymmetry in the dynamical randomness generated by the physical process of the erasure of digital information. In this way, Landauer's principle is generalized, showing that the dissipation associated with the erasure of a sequence of bits produces entropy at the rate $k_B D$ per erased bit, where D is Shannon's information per bit [8].

In addition, we survey general fluctuating copolymerization processes, which present the ability to store dynamical fluctuations in the organization of its constitutive elements. The dissipation associated to these nonequilibrium processes turns out to be closely related to the information generated. This shows in particular how information acquisition stems from the interplay between stored patterns and dynamical evolution in nonequilibrium environments. We apply these results to the process of DNA replication, showing that information acquisition can be nearly maximal by purely entropic effect, that is, in the low dissipation regime.

6.1 Landauer's principle

Landauer's principle asserts that the minimal dissipation accompanying the erasure of one bit of information produces an entropy equal to $k_B \ln 2$, where k_B is Boltzmann's constant [135]. This important result is based on the observation that the processing of digital information is a physical process among others and should thus obey the laws of thermodynamics. Landauer's result has therefore supported the idea that information processing devices working at temperature T should dissipate at least $k_B T \ln 2$ of energy during an elementary act of information [31, 223]. Later, Bennett's work clarified the point that Landauer's dissipation is the feature of logically irreversible operations on data, i.e., operations whose inverse is not unique, such as data erasure [15]. Moreover, Bennett used Landauer's result to resolve Maxwell's demon paradox [17], as discussed in the Introduction. Since then, Landauer's principle has been explicitly verified in specific cases, for instance, in bistable potentials with white noise [203], for a bit in contact with a thermal reservoir [180], or from coarse graining in phase space [125].

Landauer's result has been verified using case studies and it is only recently that nonequilibrium statistical mechanics has been sufficiently advanced in order to reach its understanding in a general framework. Indeed, it has been established in Chapter 5 that the second law of thermodynamics finds its origin in the time asymmetry of the property of dynamical randomness, i.e., the temporal disorder that a physical process develops during its time evolution. Dynamical randomness is characterized by the decay rates

$$h = \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{\omega_1 \omega_2 \dots \omega_n} \mu(\omega_1 \omega_2 \dots \omega_n) \ln \mu(\omega_1 \omega_2 \dots \omega_n) \quad (6.1)$$

and

$$h^R = \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{\omega_1 \omega_2 \dots \omega_n} \mu(\omega_1 \omega_2 \dots \omega_n) \ln \bar{\mu}(\omega_n \dots \omega_2 \omega_1) \quad (6.2)$$

for the forward and time-reversed process, respectively. The paths are observed every time interval τ (considered as the unit time) and, in contrast with Eqs. (5.3)-(5.4), the entropies (6.1) and (6.2) are here defined as entropies per step and are dimensionless. These dynamical entropies may be considered as rates of production of information generated by the fluctuations of the process during its time evolution. In this sense, they characterize the dynamical randomness of the process. The difference between these quantities gives the thermodynamic entropy production

$$\Delta_i S = k_B (h^R - h), \quad (6.3)$$

where $\Delta_i S$ is the entropy produced during the unit time τ . Equation (6.3) explicitly shows that the thermodynamic entropy production comes from the time asymmetry in the more microscopic property of dynamical randomness.

In this section, we shall show that the aforementioned formula (6.3) allows us to relate the thermodynamic entropy production to the information as it is physically recorded *in space* inside some information processing device, generalizing in this way Landauer's principle. We here consider the process of erasure of *statistically correlated* random bits, that is a sequence of bits $\sigma_1\sigma_2\dots\sigma_m\dots$ (with $\sigma_i = 0$ or 1). We assume that this sequence is initially recorded on some spatially extended support such as a recording tape inside the device. The recorded data are described by some probability distribution $p(\sigma_1\sigma_2\dots\sigma_m)$ giving the occurrence frequencies of the sequences $\sigma_1\sigma_2\dots\sigma_m$ in the memory of the device. This probability distribution is general with possible spatial correlations among the bits σ_i . The randomness of the sequence is characterized by the quantity

$$D = \lim_{m \rightarrow \infty} -\frac{1}{m} \sum_{\sigma_1\sigma_2\dots\sigma_m} p(\sigma_1\sigma_2\dots\sigma_m) \ln p(\sigma_1\sigma_2\dots\sigma_m). \quad (6.4)$$

This is an entropy per bit of information in the sense of Shannon, which we call the information or disorder D per bit of the sequence. In the case the bits are randomly distributed with equal probability and independently of each other, the information per bit is equal to $D = \ln 2$. In general, we have the inequality $D \leq \ln 2$.

The thermodynamic entropy produced during the process of erasure of the aforementioned sequence can be obtained using the formula (6.3) according to the following reasoning. We suppose that one bit is erased every unit time τ . Let us associate a state ω_i with the sequence of bits $(\dots\sigma_m\dots\sigma_{i+1}\sigma_i.000\dots)$, as done in Fig. 6.1. When viewed forward in time, the erasure process transforms the state ω_i into ω_{i+1} . Therefore, the process of erasure does not generate dynamical randomness since the outcome is unique every time a bit is erased. Accordingly, the dynamical entropy per unit time (6.1) vanishes, $h = 0$, since the probability measure μ takes the unit value for the unique path followed during erasure and vanishes for all the other paths. On the other hand, the backward process corresponds to the generation of a sequence of bits. This reversed process is not unique and generates dynamical randomness at the rate $h^R = D$ given by the information contained in the sequence of bits (see Fig. 6.1). Indeed, the sequence of bits $(\sigma_1\sigma_2\dots\sigma_m)$ now appears at random with the probability $p(\sigma_1\sigma_2\dots\sigma_m)$ and the probability measure $\bar{\mu}$ of this reversed process is distributed among several possible time-reversed paths. This observation can be understood by the following example. Let us consider a particle in a bistable potential, where the left and right wells correspond to the bits 0 and 1. We can slowly deform the potential in order to force the particle to end in the left well, which is equivalent to the erasure of the initial bit. Since we want to minimize the dissipation, any such deformation must pass through a potential with a single minimum. Undoing the latter transformation will thus result in the particle being in the left or right well with equal probability. This is the analog of our backward process, where the sequences of bits are now generated according to their respective probabilities. In this way, we can understand how dissipation is closely related to logical irreversibility [135]. Finally, we infer from

Eq. (6.3) that the thermodynamic entropy production of the erasure is given by

$$\Delta_i S = k_B(h^R - h) = k_B D \text{ per bit.} \quad (6.5)$$

Landauer's principle is recovered in the particular case of statistically independent random bits of equal probability for which $\Delta_i S = k_B D = k_B \ln 2$ [85]. This shows that Landauer's principle can be understood from concepts uniquely based on dynamical randomness. Therefore, it is completely model independent and $k_B D$ is the *minimal* dissipation one can achieve for correlated random bits. Another way to obtain this result is to note that there exist universal coding schemes [49] that will asymptotically compress any ergodic sequence of length m to its maximal possible value mD_2 (the subscript 2 indicates that the information is calculated with logarithms in base 2). It is essential that this compression step can be made without any dissipation, which results from Bennett's work on reversible computation [15]. Once compressed, the sequence is composed of uncorrelated random bits with relative probability one half.¹ The usual Landauer principle can then be applied to this compressed sequence which leads to a dissipation of $k_B D_2 \ln 2 = k_B D$ per bit.

To avoid this computational step and in the case where the exact probability distribution p of the bits remains unknown, we can still obtain the minimal entropy production (6.5) as follows. The extra cost of compressing data using an *a priori* distribution q instead of the correct distribution p is the relative entropy $D(p||q)$, given by [49]

$$D(p||q) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{\sigma_1 \sigma_2 \dots \sigma_m} p(\sigma_1 \sigma_2 \dots \sigma_m) \ln \frac{p(\sigma_1 \sigma_2 \dots \sigma_m)}{q(\sigma_1 \sigma_2 \dots \sigma_m)}.$$

It has been proven in Ref. [230] that there exists a stationary ergodic process $\bar{q}$ with the property to have a vanishing relative entropy, $D(p||\bar{q}) = 0$, with respect to any other stationary ergodic process with probability distribution p . Therefore, the erasure process can be done assuming this particular distribution, which results in no extra cost with respect to the minimal entropy production (6.5).

We notice that the disorder D is positive if the probabilities $p(\sigma_1 \sigma_2 \dots \sigma_m)$ characterizing the sequence of bits decay exponentially. Nevertheless, information can be sporadically distributed along the sequence of bits, in which case the probabilities $p(\sigma_1 \sigma_2 \dots \sigma_m)$ decay as stretched exponentials so that the information per bit vanishes $D \rightarrow 0$ in the long-sequence limit $m \rightarrow \infty$. The fact is that such sporadic sequences are not uncommon in complex systems [96, 58, 59, 165, 167]. Accordingly, the above considerations lead to the interesting result that dissipation can be arbitrarily small during the erasure of sequences of sporadically distributed information.

¹ Otherwise it would be possible to further compress the sequence, leading to a compression smaller than D_2 in contradiction with Shannon's bound on data compression.

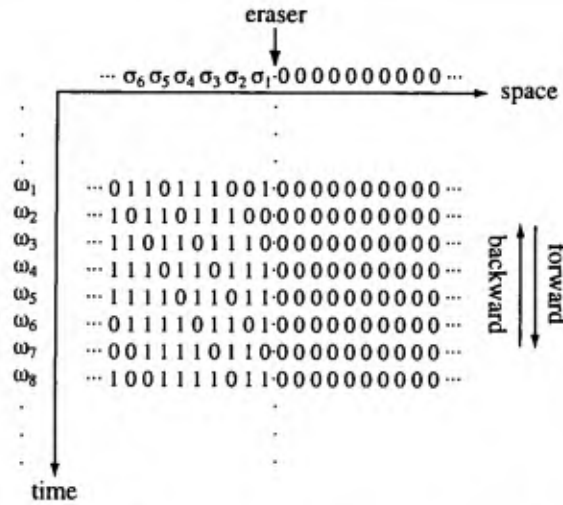


Fig. 6.1. Space-time plot of the physical process of erasure of a sequence of bits $\sigma_1 \sigma_2 \dots \sigma_m \dots$ of information. The bits are distributed along the space axis on the recording tape of the information processing device. The eraser is located somewhere along the recording tape and transforms each bit into a zero in this illustrative example. At every instant of time, the state of the system is given by the current sequence of bits: $\omega_j = \dots \sigma_{j+2} \sigma_{j+1} \sigma_j . 00000 \dots$ where the dot denotes the location of the eraser.

Finally, we point out that, as long as Landauer's principle is equivalent to the second law of thermodynamics [18], the agreement between the argument based on dynamical randomness and the universal compression procedure shows that Eq. (6.3) is the appropriate measure of dissipation for general ergodic stationary stochastic processes.

6.2 Nonequilibrium generation of information in copolymerization processes

The origin of biological information is one of the major challenges for our understanding of the living organisms. Since the discovery of DNA, the biochemical support for the storage of genetic information is known. DNA is a copolymer which keeps the memory of information on the living organism in its structure. This molecular structure is stable at ambient temperatures by the binding energy between the nucleotides, allowing the heredity of genetic information across generations. As observed *in vitro* in evolution experiments on RNA and viruses [155, 22], the processing of biological information can be discussed in terms of the dynamics of populations associated with the different possible genetic sequences, the populations evolving by replications and mutations into quasi-species [63, 62]. Such population dynamics are nonequilibrium processes where dissipation is compensated by energy supply and the entropy produced by dis-

sipation is evacuated to the environment of these open systems. However, this view relies on macroscopic concepts such as the population sizes, which are largely separated from the nanoscale of the genetic sequences. Moreover, observations reveal that biological systems have structures and functions at every scale down to the molecular level and the understanding of their origin is a challenge.

Actually, the information in DNA copolymers is processed and replicated by mechanisms taking place at the molecular level in the presence of thermal fluctuations. These fluctuations are due to the random motion of the atoms and molecules composing DNA, the transcription or replication machinery, and their environment. In this regard, biological information processing is ruled by the statistical laws of motion and thermodynamics. At the thermodynamic equilibrium, the principle of detailed balance implies that no information can be spontaneously processed or generated because each random motion is statistically balanced by the corresponding reversed motion. Therefore, equilibrium is the stage of erratic motion where information generation is highly improbable.

On the other hand, as shown in Chapter 5, nonequilibrium fluctuating systems present a time asymmetry in which the typical random paths followed by the system during its time evolution turn out to be more probable than their time reversal [86, 89, 9, 10]. The remarkable result is that this temporal ordering of nonequilibrium fluctuations is the consequence of the second law of thermodynamics. This phenomenon explains that dynamical order can be naturally generated in molecular motion under nonequilibrium conditions.

The goal of this section is thus to show that the implication of this nonequilibrium temporal ordering in copolymerization processes is the generation of information. Indeed, copolymerization can store in space the dynamical order which is generated in time by the nonequilibrium processes, establishing a coupling from time back to the spatial support of information. The possibility of such transfers of information has already been envisaged at the macroscopic level [170] and is here shown to arise at the molecular level. We study copolymerization processes with or without the presence of an underlying template. Such processes have the ability to store information in the copolymer pattern, which in turn influences back the dynamics of the system. Information generation can thus be understood as a natural consequence of driving out of equilibrium systems with the ability to store and retrieve molecular information.

We first explain the tradeoff between dissipation and information in copolymerization processes in the framework of nonequilibrium statistical thermodynamics. We then proceed with the illustration of these results in the case of DNA replication.

6.2.1 Dissipation-information tradeoff

Copolymerization processes can be described at the macroscopic level in terms of the population sizes, i.e., the chemical concentrations of the monomers, dimers, trimers, etc... In this macroscopic description, the rate of dissipation, in particular, due to metabolism can be estimated by the thermodynamic entropy production given in terms of the chemical concentrations [101]. Since we

are interested in the molecular structure generated during copolymerization, we adopt a mesoscopic description at the scale of a single copolymer chain.

The stochastic growth process will be modelled by continuous-time random processes ruled by an evolution equation of the form (3.1). For many processes at fixed pressure and temperature T , the ratio of forward to backward transition rates can be expressed in terms of the free enthalpies $G(\omega)$ of the coarse-grained states ω according to

$$\frac{W(\omega, \omega')}{W(\omega', \omega)} = \exp \frac{G(\omega) - G(\omega')}{k_B T},$$

where k_B is Boltzmann's constant [114]. In this framework, the thermodynamic entropy of the system is given by

$$S_t = \sum_{\omega} P_t(\omega) S(\omega) - k_B \sum_{\omega} P_t(\omega) \ln P_t(\omega), \quad (6.6)$$

where $S(\omega)$ is the entropy of the coarse-grained state ω [85]. The entropy (6.6) varies in time with the probability distribution ruled by the evolution equation (3.30). It is known that these variations dS/dt are due to the exchange of entropy with the environment $d_e S/dt$ and the internal production of entropy

$$\frac{d_i S}{dt} = \frac{dS}{dt} - \frac{d_e S}{dt} \geq 0 \quad (6.7)$$

which is always non negative in agreement with the second law of thermodynamics [190, 169, 122]. The exchange of entropy is given by

$$\frac{d_e S}{dt} = \frac{1}{T} \frac{d\langle H \rangle_t}{dt} \quad (6.8)$$

in terms of the time variation of the mean enthalpy of the system: $\langle H \rangle_t = \sum_{\omega} P_t(\omega) H(\omega)$ [122, 85]. The enthalpy $H(\omega)$ of the coarse-grained state ω is related to the corresponding free enthalpy and entropy by Gibbs' relation: $G(\omega) = H(\omega) - TS(\omega)$. In a stationary state, the entropy does not change in time, $dS/dt = 0$, so that the entropy produced inside the system is evacuated to its environment: $d_i S/dt = -d_e S/dt$.

We first consider a copolymerization process without template and with M different monomers (see Fig. 5.1a). An arbitrary chain ω can grow by adding a monomer m or shorten by removing the last monomer m of the actual chain: $\omega + m \rightleftharpoons \omega m$. The length l of the chain ω is supposed to range from zero to infinity so that the growth process can go on without limit and possibly reach a regime where the synthesized copolymer has stationary statistical properties [45, 46]. It is assumed that, in the stationary regime, the probability to have a certain chain ω at time t becomes

$$P_t(\omega) = p_t(l) \mu_l(\omega),$$

where $p_t(l)$ is the probability distribution of the lengths l at time t and $\mu_l(\omega)$ is the probability to have a chain ω provided that its length is l . The distribution

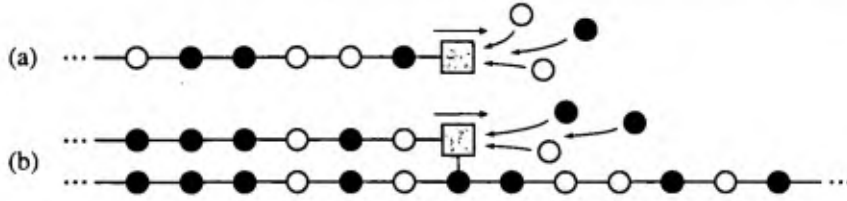


Fig. 6.2. Schematic representations of: (a) a copolymerization process without a template; (b) a copolymerization process with a template. The circles depict the monomers and the square the catalyst of polymerization.

$\mu_l(\omega)$ gives the composition of the copolymer in the stationary regime and is normalized as $\sum_{\omega} \mu_l(\omega) = 1$, where the sum is restricted to all the chains ω of length l . The entropy (6.6) can thus be written as

$$S_t = \sum_{l, \omega} p_t(l) \mu_l(\omega) S(\omega) - k_B \sum_{l, \omega} p_t(l) \mu_l(\omega) \ln \mu_l(\omega) - k_B \sum_l p_t(l) \ln p_t(l). \quad (6.9)$$

The first term is the contribution from the thermodynamic entropy $S(\omega)$ of the copolymer ω in its environment. In the stationary regime, we may introduce the mean entropy per monomer as

$$s = \lim_{l \rightarrow \infty} \frac{1}{l} \sum_{\omega} \mu_l(\omega) S(\omega), \quad (6.10)$$

so that the first term becomes $\langle l \rangle_t s$ with the mean length $\langle l \rangle_t = \sum_l p_t(l) l$. The growth velocity is thus given by

$$v = \frac{d \langle l \rangle_t}{dt}. \quad (6.11)$$

The second term in Eq. (6.9) can be expressed in terms of the disorder in the composition of the copolymer. This disorder can be characterized by the Shannon entropy per monomer defined as

$$D(\text{polymer}) = \lim_{l \rightarrow \infty} -\frac{1}{l} \sum_{\omega} \mu_l(\omega) \ln \mu_l(\omega). \quad (6.12)$$

The disorder takes a value in the range $0 \leq D \leq \ln M$ limited by the number of different monomers composing the copolymer. In the stationary regime, the second term is controlled by the disorder in the chain since

$$\begin{aligned} -\sum_{l, \omega} p_t(l) \mu_l(\omega) \ln \mu_l(\omega) &= \sum_l p_t(l) l \left[-\frac{1}{l} \sum_{\omega} \mu_l(\omega) \ln \mu_l(\omega) \right] \\ &\simeq \langle l \rangle_t D(\text{polymer}) \end{aligned}$$

at long times.

The third term in Eq. (6.9) is the Shannon entropy of the distribution of lengths. Since this distribution broadens in time and may extend from $l = 0$ to $l = w_{\max}t$ where $w_{\max}$ is the largest among the growth rates $W(\omega, \omega m)$, this term typically increases logarithmically in time and does not contribute to the variation rate of the entropy because $\lim_{t \rightarrow \infty} (-1/t) \sum_l p_t(l) \ln p_t(l) = 0$.

Hence, we find that the time variation of the entropy is given in the stationary regime by

$$\frac{dS}{dt} = \lim_{t \rightarrow \infty} \frac{S(t)}{t} = v s + v D(\text{polymer}),$$

where we have chosen physical units where $k_B = 1$ for the simplicity of notations. On the other hand, the entropy exchange rate (6.8) is equal to

$$\frac{d_e S}{dt} = v \frac{h}{T} \quad (6.13)$$

in terms of the mean enthalpy per monomer which has a definition similar as Eq. (6.10). Finally, the entropy production (6.7) reads

$$\frac{d_i S}{dt} = v A \geq 0$$

with the growth velocity (6.11) and the thermodynamic force or affinity per monomer

$$A = -\frac{g}{T} + D(\text{polymer}) = \varepsilon + D(\text{polymer}), \quad (6.14)$$

where $g = h - Ts$ is the free enthalpy per monomer and $\varepsilon = -g/T$ is the mean driving force. In the case the growth velocity is positive, the affinity (6.14) can be interpreted as the entropy production per added monomer. The affinity has two contributions: the first from the driving force ε given in terms of the mean free enthalpy of copolymerization and the second due to the disorder (6.12) of the monomers in the chain. This shows in particular that the copolymer can grow by an entropic effect of disorder even when the driving forces are slightly negative. Indeed, the randomness incorporated in the chain has a non-negative contribution in the form of the disorder (6.12) to the thermodynamic force or affinity per monomer (6.14). This already shows how the stored pattern can influence back the dynamics of the system. The growth stops when each monomer is included with its equilibrium distribution, or equivalently when the mean driving force is equal to minus the equilibrium disorder.

In general, the transition rates $W(\omega, \omega')$ depend on the specific structure and composition of the whole chain ω so that nonlocal or cooperative effects can generate long-range correlations along the chain, thereby reducing the disorder (6.12). In many cases, the transition rates only depend on the few monomers at the end of the chain, in particular, if the copolymerization process is controlled by the free enthalpy changes of the reactions $\omega + m \rightleftharpoons \omega m$. In the most local process, the transition rates only depend on the last monomer m which is added to or removed from the chain. In this case, the transition rates are denoted as $W(\omega, \omega m) = k_{+m}$ and $W(\omega m, \omega) = k_{-m}$ and the quantities

$$\epsilon_m = \ln \frac{k_{+m}}{k_{-m}} \quad (6.15)$$

represent the driving forces favoring the chain growth. They are given by the free enthalpy changes of the chemical reactions and are measured in units of the thermal energy: $\epsilon_m = [G(\omega) - G(\omega m)] / (k_B T)$. We notice that these driving forces depend in general on control parameters such as the chemical concentrations of the different monomers in solution around the copolymer chain. Under the assumption of Eq. (6.15), the mean value of the driving forces can be calculated as $\epsilon = \sum_m \mu_1(m) \epsilon_m$ by averaging them over the normalized distribution $\mu_1(m)$ of monomers. If we further assume that there is no free enthalpy difference between the chains ωm , the rates are equal to each other and $\epsilon_m = \epsilon$ for all m , whereupon equilibrium occurs when the affinity (6.14) vanishes $A_{eq} = 0$ which implies that the driving force takes the value $\epsilon_{eq} = -\ln M$ and the disorder reaches its maximum value $D_{eq} = \ln M$.² On the other hand, we notice that, for fixed positive values of the thermodynamic entropy production and the growth velocity, the disorder can be reduced at the expense of the driving force ϵ according to Eq. (6.14). Therefore, the nonequilibrium drive may contribute to some extent to the ordering of the copolymer.

We next consider a copolymerization process under the influence of a template, which determines the probabilities to add or remove monomers (see Fig. 5.1b). This is typically the case during the DNA transcription or replication processes where the DNA sequence influences the probabilities to add the monomers, favoring the Watson-Crick pairing rule A-T and C-G. The template is thus composed of a chain $\alpha = \alpha_1 \alpha_2 \dots \alpha_l \dots$ and the transition rates now depend on this underlying substrate: $W(\omega, \omega' | \alpha)$. In this case, the previous considerations are modified in the following way. The entropy (6.6), the velocity (6.11), and the mean driving force are now averaged not only over the distribution of monomers but also over the template. We suppose that the template is described by a stationary distribution $\nu_l(\alpha) = \nu_l(\alpha_1 \alpha_2 \dots \alpha_l)$ which is normalized to unity $\sum_\alpha \nu_l(\alpha) = 1$ for all the chains of length l . In the stationary regime, the probability to have a chain ω at time t here takes the form

$$P_t(\omega) = p_t(l) \mu_l(\omega | \alpha)$$

at long times, where $\mu_l(\omega | \alpha)$ is the distribution of chains ω of length l grown on the template α . We expect that the template ensemble average corresponds to the average over a typical realization of the template sequence. We can here

² Precisely, at $\epsilon_{eq} = -\ln M$ we have a semi-infinite random-walk type of dynamics where no stationary distribution is reached. Detailed balance conditions hold when $\epsilon < -\ln M$, in which case the probability distribution of a polymer of length l is proportional to $\exp(l\epsilon)$, independently of its specific composition. In the more general case where a free enthalpy difference exists between the chains, detailed balance conditions are satisfied when $\sum_m \exp(\epsilon_m) < 1$. The equilibrium probability distribution of the polymers is then proportional to $\exp(-\sum_m n_m \epsilon_m)$, where n_m is the number of monomer of type m .

also introduce the mean entropy (6.10) and enthalpy per monomer by averaging over the template. The growth speed (6.11) is now obtained with a long typical sequence of the template.

On the other hand, the disorder (6.12) now becomes the Shannon conditional entropy of the copolymer with respect to the template:

$$D(\text{polymer}|\text{template}) = \lim_{l \rightarrow \infty} -\frac{1}{l} \sum_{\alpha} \nu_l(\alpha) \sum_{\omega} \mu_l(\omega|\alpha) \ln \mu_l(\omega|\alpha). \quad (6.16)$$

The conditional entropy between two random variables can also be expressed in terms of the mutual information³

$$I(X, Y) = D(X) - D(X|Y)$$

between the two variables [38]. The mutual information is bounded according to $0 \leq I(X, Y) \leq \min \{D(X), D(Y)\}$ and measures how much the knowledge on one of these variables reduces the uncertainty about the other. It is zero when the two variables are independent so that we recover the previous results (6.12) and (6.14) when the substrate does not influence the copolymerization process. Accordingly, the affinity here becomes

$$\begin{aligned} A &= \varepsilon + D(\text{polymer}|\text{template}) \\ &= \varepsilon + D(\text{polymer}) - I(\text{polymer}, \text{template}) \end{aligned} \quad (6.17)$$

and is thus directly expressed in terms of the mutual information that the copolymer acquires from its template or more generally from its environment. The mean driving force is given by $\varepsilon = -g/T$ as in Eq. (6.14).

To be specific, we can take transition rates of the form $W(\omega, \omega m|\alpha) = k_{+mn}$ and $W(\omega m, \omega|\alpha) = k_{-mn}$ if $n = \alpha_{l+1}$ is the template at the position $l+1$ and ω is a chain of length l . The corresponding driving forces

$$\varepsilon_{mn} = \ln \frac{k_{+mn}}{k_{-mn}} \quad (6.18)$$

now depend on the substrate as well. Accordingly, in the case (6.18), the mean driving force is given by $\varepsilon = \sum_{m,n} \nu_1(n) \mu_1(m|n) \varepsilon_{mn}$, where $\nu_1(n)$ is the distribution of monomers on the template.

If the copolymer grows and $v > 0$, the affinity is the entropy production per added monomer so that Eq. (6.17) shows that substantial information can be generated if the mean driving force ε is large enough. We illustrate this fundamental result with DNA replication in the next section.

6.2.2 DNA replication

In this section, we consider the process of DNA replication. In this case, the four nucleotides NTP, with $N = A, T, C$ or G , constitute the monomer units

³ As shown by Shannon [202], the maximal rate at which information can be sent with an arbitrarily small probability of error over a noisy transmission channel is given by the supremum of the mutual information over input data distributions.

as well as the template. The DNA polymerase copies the DNA with a fairly small error rate. In the actual DNA replication, an exonuclease will act as a proofreading mechanism in order to correct possible errors. Proofreading allows one to decrease the error rate up to the discrimination squared [116, 172, 16]. This mechanism can be modeled by considering several reactions μ during the copolymerization, namely $W_\mu(\omega, \omega'|\alpha)$. These additional pathways will change the average driving force and speed, but the disorder (6.16) will remain unchanged for a given distribution of nucleotides.

To analyze purely kinetic effects, we will assume in the following that no free enthalpy difference exists between correct and incorrect chains, implying $\varepsilon_{mn} = \varepsilon$. We will thus consider the different effects as a function of this parameter. The transition rates and the driving forces (6.18) incorporate external conditions such as the chemical concentrations of the monomers. Accordingly, the driving force ε will typically be a function of such control parameters.

For concreteness, we consider the case of the DNA polymerase Pol γ , which replicates the human mitochondrial DNA. Forward kinetic constants for the incorporation of both correct and incorrect nucleotides are available [141] and we used these for our simulations. The human mitochondrial DNA is 16.5 kb long and can be obtained from GenBank.⁴ To have a good statistical estimation of the disorder (6.16), we used longer DNA sequences generated from the same triplet distribution as the original mitochondrial DNA. Random trajectories corresponding to the master equation (3.30) are obtained with Gillespie's algorithm [99].

The minimal error rate η_{mn} one can achieve for a base m instead of n is given by $\eta_{mn} = k_{+mn} / \sum_m k_{+mn}$ and occurs in the infinite dissipation limit. In this limit, the maximal speed is given by $v_{\max}^{-1} = \sum_n \nu_1(n) / (\sum_m k_{+mn})$ which is here of about 34 bases per second. The maximal mutual information is given by $\sum_{m,n} \eta_{mn} \nu_1(n) \ln[\eta_{nm} / \sum_n \eta_{mn} \nu_1(n)]$ and takes the value $I_{\max} \simeq 1.337$ nats. Since we assumed no free enthalpy difference between the chains, equilibrium occurs when $\varepsilon = -\ln 4$, where each nucleotide is inserted with equal probability. Therefore, at equilibrium, the velocity and the mutual information vanish, while the error rate is 0.75.

In Fig. 6.3, we depict the fraction of errors as a function of the driving force and, in Fig. 6.4, the velocity of the replication process. These two observables present a steep decrease (respectively, increase) as the driving force ε becomes positive.

In Fig. 6.5, we represent the affinity (6.14) per copied nucleotide, which gives the corresponding entropy production since the growth velocity is here positive. This curve is non monotonic and presents a minima around $\varepsilon \simeq 0.015$. On the left-hand side of it, the external driving force is weak or negative and dissipation occurs chiefly due to the positive driving caused by the incorporation of errors whereas, on the right-hand side, dissipation is mainly due to the external driving force. This transition also corresponds to the sharp behavior observed in the

⁴ Homo sapiens DNA mitochondrion, 16569 bp, locus AC 000021, version GI:115315570, <http://www.ncbi.nlm.nih.gov>.

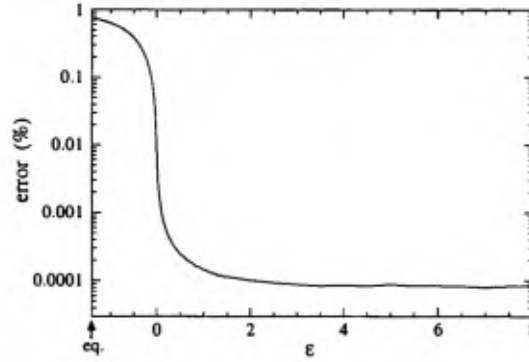


Fig. 6.3. Percentage of misincorporations as a function of the driving force ϵ . The minimal error rate is given by $1 - \nu_A \eta_{TA} - \nu_T \eta_{AT} - \nu_C \eta_{GC} - \nu_G \eta_{CG} \simeq 8.3 \cdot 10^{-5}$. The maximal error rate occurs at equilibrium where detailed balance conditions are satisfied, according to which each nucleotide insertion is balanced by its removal from the chain. Under this condition and if $\epsilon_{mn} = \epsilon$ for all m and n , every nucleotide is included with equal probability and the error rate is 0.75. The kinetic constants of Watson-Crick pairing are taken to be $k_{+TA} = 25 \text{ s}^{-1}$, $k_{+AT} = 45 \text{ s}^{-1}$, $k_{+CG} = 37 \text{ s}^{-1}$, and $k_{+GC} = 43 \text{ s}^{-1}$ [141]. The discrimination $d_{mn} = k_{+mn}/k_{+nn}$ between nucleotide m and template n is given by $d_{AA} = d_{AG} = d_{GA} = 1/280000$, $d_{CA} = 1/210000$, $d_{TT} = 1/250000$, $d_{CT} = 1/570000$, $d_{GT} = 1/3600$, $d_{AC} = 1/71000$, $d_{TC} = 1/640000$, $d_{CC} = 1/2300000$, $d_{TG} = 1/59000$, and $d_{GG} = 1/110000$. The reversed kinetic constants are taken as $k_{-mn} = k_{+mn} e^{-\epsilon}$ according to Eq. (6.18) with $\epsilon_{mn} = \epsilon$ (see text).

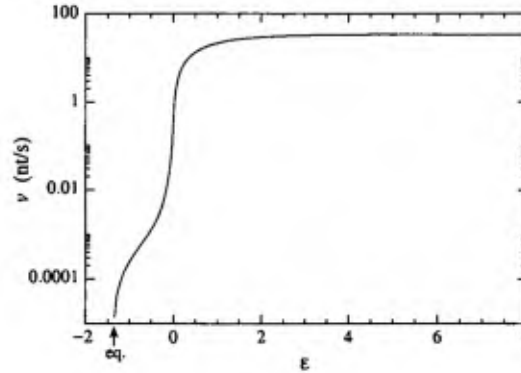


Fig. 6.4. Velocity of the replication process as a function of the driving force ϵ . The maximal speed is given by about 34 nucleotides per second as explained in the text.

number of errors and in the velocity of the process. The same behavior can also be observed in the mutual information I between the original and the copied DNA strands, as depicted in Fig. 6.6. It increases with the driving force, reaching nearly its maximal value after the threshold point. Afterwards, nearly all the original information has been acquired. The result is that the increase of mutual information corresponds to a reduction of disorder and allows fidelity

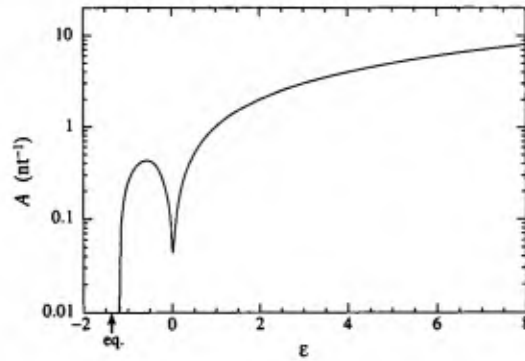


Fig. 6.5. Affinity per copied nucleotide as a function of the driving force ε . The local minimum is around $\varepsilon \simeq 0.015$ and indicates the transition between the error driven regime and the externally driven regime. Since the growth speed is positive, this affinity is the entropy production per copied nucleotide.

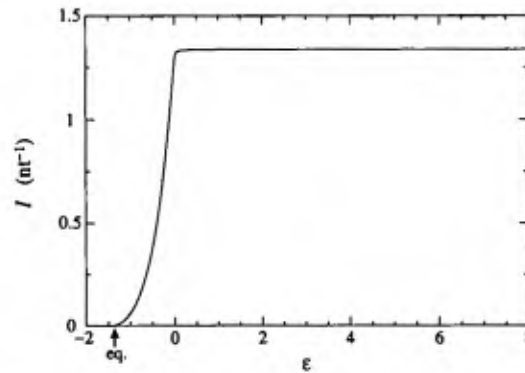


Fig. 6.6. Mutual information between the copied DNA strand and the original strand, as a function of the driving force ε . The saturation occurs after the transition between the error driven and the externally driven regimes. The value of the mutual information at saturation is $I_{\max} \simeq 1.337$ nats.

in the transfer of information. Contrary to the situation at equilibrium where this transfer is not possible, the nonequilibrium conditions turn out to induce the generation of mutual information.

6.3 Summary

In this chapter, we have investigated two different problems in the field of the physics of information. The second law of thermodynamics has fundamental implications on the way information is processed in physical systems and the principle of temporal ordering enlightens this problem with new understandings.

We have here obtained a generalization of Landauer's principle on the basis of recent advances in nonequilibrium statistical mechanics showing that the second law of thermodynamics can be understood in terms of time asymmetry in the property of dynamical randomness [85, 87, 89, 9]. These advances allow us to relate the thermodynamic entropy production during the erasure of a sequence of bits to the information contained in this sequence, taking into account the possible statistical correlations among the bits [8].

Furthermore, we have developed the nonequilibrium statistical thermodynamics of copolymerization processes, delineating the role of disorder in copolymer chains. In particular, we have shown how information can be generated out of equilibrium in these fluctuating processes. We have observed a transition between two regimes. At small or negative external driving forces, the growth process is essentially driven by the incorporation of errors in the copolymer chain while, beyond a threshold, it is driven by free enthalpy sources. After this transition point, it turns out that the extra dissipation does not lead to a substantial increase in the fidelity of the replicated strands, as measured by the error rate or the mutual information between the original and the copied DNA strand. Information acquisition thus appears to have an energy cost analogous to information erasure [135, 17, 8]. Similar or enhanced effects are expected in the presence of autocatalysis [101], possibly, with different dependences on control parameters.

Albeit the results apply to every copolymerization process, they are fundamental for DNA replication and transcription and the understanding of information acquisition by biological entities. Biological systems are characterized by several properties such as metabolism and self-reproduction. Each of them can be said to be encoded in the genetic code. While metabolism refers to nonequilibrium thermodynamics, self-reproduction involves information processing. The present work establishes an essential connection between both properties on the basis of nonequilibrium statistical thermodynamics. In this perspective, dissipation allows the emergence of temporal ordering and the spontaneous generation of information, explaining in this way a key feature of complex systems [167]. Information generation can thus be understood as resulting from the reduction of disorder in copolymers growing out of equilibrium and the ability to store and retrieve information in nonequilibrium environments. If furthermore this information can be copied and processed back by the nonequilibrium dynamics of the system, the self-reproduction of biological systems could be explained. From this viewpoint, we could say that biological systems are physico-chemical systems with a built-in thermodynamic arrow of time.

Conclusions and Perspectives

The present thesis has been devoted to the study of the thermodynamics of nanosystems, in the light of recent results in nonequilibrium statistical mechanics.

7.1 Overview

In the past fifteen years, new relationships have been discovered in nonequilibrium statistical mechanics. These relationships characterize the large fluctuations of several dynamical quantities in far-from-equilibrium situations. Today, these findings have significantly changed our understanding of thermodynamics by showing, for instance, how to formulate entropy production in fluctuating systems at the nanoscale [33]. In this work, we essentially developed two different aspects of this emerging field.

The first part aimed at presenting a unifying framework relating fluctuations and response in nonequilibrium systems. We first considered quantum systems driven by a time-dependent perturbation, in a setup similar to Crooks [50]. In this framework, we use the time-reversal symmetry operator to express the average of an arbitrary observable as an average, over the time-reversed process, weighted by a quantum version of the work. The quantum Jarzynski relation appears as a specific case of this functional relation. Besides, we can expand this relation in powers of the perturbation in order to connect the response of an observable to its fluctuations. In this way we can recover, in a unified framework, both response relations and fluctuation relations.

In this direction, we also considered nonequilibrium steady states in a stochastic framework. This approach allows us to connect thermodynamical properties such as currents or affinities to transition rates in the stochastic description. On this basis, we can derive a fluctuation relation for the macroscopic currents crossing the system in nonequilibrium situation. This symmetry property of the dynamics is expressed in terms of the affinities, connecting in a fundamental way the fluctuations to the nonequilibrium drives. In particular, the consequences of such a connection have been derived at all the orders of perturbation.

In the second part of this study, we gave experimental evidence showing that, at the level of both the individual trajectories and the statistical ensembles, entropy production can be understood as a symmetry breaking of time reversal under nonequilibrium conditions. This symmetry breaking is measured on mesoscopic quantities that characterize dynamical randomness in the forward or time-reversed direction, connecting in this way dissipation to temporal disorder. These developments led us to the field of the physics of information. Landauer's principle can indeed be understood as arising from a time-asymmetry in dynamical randomness. Moreover, we illustrated on a model system of copolymerization the following nonequilibrium organizing scheme. In nonequilibrium systems, the principle of temporal ordering shows that the random fluctuations should be biased, allowing the emergence of a dynamical order. If this dynamical order can be memorized by the system (here in the organization of the constitutive units), we can understand that information can be generated. If furthermore this information can be restored back to the nonequilibrium dynamics of the system, we have achieved a sustained nonequilibrium organization.

In conclusion, thermodynamic concepts can be consistently formulated at the nanoscale, in the form of large-deviation relations. In this respect, this work gives new insights into the thermodynamics of small systems, in particular on the problem of nonlinear transport as well as on the origin of irreversibility and its constructive role in nonequilibrium processes. We comment in the next section on several topics that could be further investigated.

7.2 Open questions

Many questions remain open at this point. Large-deviation relations offer new promising approaches that might help establish the foundation of a sound nonequilibrium statistical mechanics. Also, the applications of the fluctuation relations to the vast existing range of nanosystems, especially electronic and biological systems, is expected in the next few years, both at the theoretical and experimental levels. In regard of these general problems, we here detail some more specific issues and possible future research directions.

The fluctuation relations and their consequences offer many new perspectives. It is thus essential to explore at the fundamental level their range of validity and their repercussions on energetics and transport properties.

At the quantum level, virtually all fluctuation relations have been derived for system starting at equilibrium. Indeed, the construction of nonequilibrium stationary states in open quantum systems is still problematic. Accordingly, the analogue of the steady-state fluctuation theorems that would be expressed only in terms of the system observables (i.e., without the bath) is still lacking.

Also, a generalization of the fluctuation theorem for the currents in a purely Hamiltonian framework would be of fundamental interest. Such a generalization can be envisaged in the following circumstances. Let us suppose we have an open system in contact with reservoirs of particles at different concentrations.

If we assume that the reservoirs are at equilibrium at each time step and independent of the state of the system, the effect of a reservoir is to inject particles in the system with a Poissonian distribution, the average of which is equal to the reservoir concentration. The net flux of particles can be measured at any boundary and is given in terms of the outgoing and incoming particles. Accordingly, the distribution of particles at the boundary is given by the difference between two Poissonian distributions with density ρ_{out} and ρ_{in} , respectively. Now, the fact is that the difference of two Poissonian distributions is a Skellam distribution, which has the property to satisfy a symmetry relation with an exponential factor given by $\rho_{\text{out}}/\rho_{\text{in}}$. In this way we can directly obtain a fluctuation theorem for the currents. A more general statement might be obtained along these lines.

Another question concerns the statistical behavior of the nonequilibrium fluxes under time-dependent drivings. To address this issue, it appears necessary to include additional “inductive” elements in the graph formulation of the stochastic process in order to account for the conservation of the currents. This problem is relevant for many applications, for example in the study of noise and electronic transport in quantum dots [205].

Eventually, the nonlinear relations presented in Chapter 4 might also provide new insights into the mechanisms used by nanosystems in order to take advantage of the molecular fluctuations. In this respect, nonlinear effects are expected to be important in the understanding of the transport properties at the nanoscale. The nonlinear relations and their symmetries might give rise to new transport mechanisms or peculiar modes and regimes of functioning. Brownian motors with several nonequilibrium drivings offer model systems to further explore these possibilities.

On the other hand, statistical physics is nowadays challenged by the rapid development of nanosciences, and in particular of the biological science. In relation with our work, a fundamental problem pertains to the impact of noise on information transmission in biology. Indeed, biological systems have to solve many problems of computation and signal processing at the biochemical level. Surprisingly, many such crucial biological processes operate with small numbers of molecules and it is essential to assess the impact of noise associated with these small numbers [20, 102]. Remarkably, recent experiments suggest that these intracellular signaling processes operate with a precision close to the fundamental physical limits. This is the case for example in *Escherichia Coli*, where regulation of gene expression by transcription factors and control of the flagellar motor both appear close to a “noise floor” independent of all the details of the chemical kinetics [21]. These observations on functional performance suggest that, given the structural constraints coming from evolutionary and developmental compromises, various dynamical networks governing crucial tasks could result from a compromise between specificity and flexibility. For instance, noise in the flagellar motor of *E. Coli* appears to optimize the exploration of the environment while allowing a reliable response to external perturbations such as gradients of nutrients [66]. This view is in

contrast with the idea that bacteria would have evolved toward a deterministic motion always selecting the largest gradients of nutrients.¹ In this example, these aspects can be linked to the architecture and nonlinear behaviors of the underlying genetic and chemical networks [66], showing that some of the intricate machinery of the cell can be understood as a principled solution to these problems of noise and computation. In this direction, we believe that the link between fluctuation, dissipation, and information generation put forward in Chapter 6 could be further developed to address these questions. For instance, the inclusion of nonequilibrium thermodynamics to rate-distortion theory could provide us with new understandings on the physical limits to physical signaling in biological systems.

Finally, the time-ordering principle and the nonequilibrium self-organization scheme presented in Chapters 5 and 6 open new perspectives in various directions.

First, nonequilibrium systems present a generic temporal and spatial order, characterized for instance by the presence of long tails in the correlation functions [166, 128]. In this context, nonequilibrium constraints have already been identified as the source of the long-range correlations in some specific models [166, 128]. This observation seems closely related to the time-ordering principle and to the link between entropy production and dynamical randomness. Thus, general relationships could potentially be established and would be of great interest to further understand the origin, behavior, and constructive role of these phenomena.

Second, even though reversible computation is possible, the presence of noise or intrinsic errors results in a finite amount of dissipation if a device is to function properly. A quantitative formulation of these problems would be highly desirable, both at the fundamental level and for potential applications. In this direction, we mention the work of von Neumann [222], who had already envisaged this problem in a different context, although he did not incorporate energy dissipation in his analysis. He showed that reliable computation could be achieved from unreliable components under the condition that the noise level was below a certain threshold, but not otherwise. This is reminiscent of the already fairly low intrinsic error rates found in the functioning of the genetic apparatus. More generally, the existence of error-correcting schemes in DNA transcription and replication is an evidence that strategies optimizing the combination dissipation-reliability can be achieved.

Third, we believe that the new self-organizing scheme illustrated in section 6.2.2 on DNA replication provides a first step in the understanding of the problem of information acquisition by biological organisms, and especially on the

¹ Putting aside the problem of developing the (necessarily more complex) machinery associated with such a behavior, randomness here appears favorable due to the competition between bacteria. The remarkable result that optimal strategies - independently of any physical implementation - are probabilistic rather than deterministic or reflex-like finds its origin in the seminal works of von Neumann and Nash in economy and game theory.

problem of self-reproducing entities. Indeed, self-reproduction would be a particular case of a sustained non-equilibrium organization where the information about the structure and function of the entity is copied and processed back by the nonequilibrium dynamics. In this regard, we could envisage more complex polymerization dynamics that could lead to more evolved self-reproducing entities. For instance, nonlinear and autocatalytic reaction pathways, multiple reactive states or cooperative effects could already lead to complex patterns. Another possible direction would be to combine this organizing principle to the cellular automata approach. Self-reproduction in this context was first envisaged in the classical work of von Neumann [223]. He understood that machines could easily achieve trivial forms of self-reproduction based on template-replication or crystal-like growth. In contrast, he speculated that, unlike machines, biological organisms have the ability to increase their complexity without limit via self-replication (open-ended evolution). His insight that open-ended evolution requires the separation of a universal constructor from its own description (which needs to be copied separately) is all the more remarkable because it preceded the discovery of the structure of the DNA molecule as a genetic information storage in biological systems. The ability to achieve open-ended evolution lies in the fact that errors in the copying of the description can lead to viable variants of the automaton, which can then evolve via natural selection. von Neumann's original model was actually fairly complicated since he required its automaton to act as a Turing machine in addition of a universal constructor.² On the other hand, it has been pointed out that far simpler machines achieve self-replication [137], although none of these simpler machines are capable of open-ended evolution.³ Anyhow, all models developed so far are purely deterministic, whereas the fluctuations and the energetics aspects should play an important role in metabolism and evolution. The inclusion of these aspects might provide new insights into this fascinating problem. The understanding of the physical principles constraining self-reproducing entities could also deeply transform our views on the origin of life.

² Primordial biological entities probably do not satisfy such strong requirements.

³ We refer the interested reader to the reference book by Freitas and Merkle [75].

A

Large deviation theory

Large deviation theory aims at describing the occurrence of rare events. It has therefore found many applications in the fields of geology, economy, statistical physics, and time series analysis, among others. In this appendix, we briefly summarize some important results of this field.

We first introduce some standard tools in probability theory. A continuous random variable Y is characterized by its probability density function $p(y)$. It is normalized to unity and we denote by $\langle \cdot \rangle$ the average with respect to this probability distribution:

$$\langle f(Y) \rangle = \int_{-\infty}^{+\infty} f(y)p(y)dy.$$

The generalization of this appendix to discrete and vector-valued random variables is straightforward.

Another representation of the probability distribution is obtained as follows. We introduce the *moment-generating function* of the random variable Y as

$$\psi(\lambda) = \langle e^{\lambda Y} \rangle.$$

It can be calculated from the moments of the random variable as

$$\psi(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \langle Y^n \rangle$$

and, conversely, the moments can be obtained from the moment-generating function by a series expansion in λ . This operation is reversible: the probability distribution function can be obtained as

$$p(y) = \langle \delta(Y - y) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda e^{-iy\lambda} \psi(i\lambda), \quad (\text{A.1})$$

where we used the Fourier transform of the Dirac delta distribution $\delta(y)$. The information content of the two representations is thus identical.

The moment-generating function also provides a useful bound. Applying Markov's inequality to the positive random variable $\exp(\lambda Y)$, we get that, for all $\lambda \geq 0$,

$$P(Y \geq a) \leq e^{-\lambda a} \psi(\lambda). \quad (\text{A.2})$$

Since this inequality holds for all positive λ , we can optimize over λ to get $P(Y \geq a) \leq \min_{\lambda \geq 0} \{e^{-\lambda a} \psi(\lambda)\}$.

The moment generating function can be written as the exponential of the *cumulant-generating function*:

$$F(\lambda) = \ln \langle e^{\lambda Y} \rangle$$

with

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \kappa_n.$$

The first cumulants are given in terms of the central moments $\mu_k = \langle (Y - \langle Y \rangle)^k \rangle$ as follows:¹

$$\begin{aligned} \kappa_1 &= \mu_1, \\ \kappa_2 &= \mu_2, \\ \kappa_3 &= \mu_3, \\ \kappa_4 &= \mu_4 - 3\mu_2^2, \\ \kappa_5 &= \mu_5 - 10\mu_2\mu_3. \end{aligned}$$

Similarly to the moment-generating function ψ , the cumulants are obtained by derivation of the function F . In order to underline the random variable Y , we will also use the notation $\langle\langle Y^k \rangle\rangle$ to denote the k^{th} cumulant.

The cumulants have no direct interpretation. However, we note a theorem by Marcinkewitz [152] which shows that the probability distribution function will violate its positiveness if its cumulant generating function is a polynomial of degree greater than two. Thus, two possibilities occur for the cumulants. Either only the two first cumulants are non-zero and Eq. (A.1) proves that p is a Gaussian, or one cumulant of order above two is nonzero and then an infinite number of cumulants are non-zero.

We now consider a sequence $\{S_n\}$ of random variables. We would like to describe the asymptotic or large n behavior of the sequence. In many applications, n corresponds to a time parameter while S_n is typically a sum over several random variables: $S_n = (1/n) \sum_k Y_k$. In this case, the fluctuations around the

¹ The cumulants can also be expressed in terms of the moments as follows. Let p be a integer between 0 and $k-1$. Let us write down k times the character Y . Then let us divide this writing in $p+1$ subsets with brackets: $\langle YY \rangle \langle Y \rangle \dots \langle YYY \rangle$. Let us call C_p the sum of all these kinds of terms. Then, the cumulant of order k is given by the sum $\langle\langle Y^k \rangle\rangle = \sum_{p=0}^{k-1} (-1)^p p! C_p$.

average are described by the central limit theorem. We now want to show how to characterize the large deviations further away from the mean.

Assume that the sequence of functions

$$F_n(\lambda) = \frac{1}{n} \ln \langle e^{n\lambda S_n} \rangle$$

exists for all λ and converges to some differentiable function

$$F(\lambda) = \lim_{n \rightarrow \infty} F_n(\lambda).$$

Then, the Gärtner-Ellis theorem [220] asserts that the sequence $\{S_n\}$ satisfies the large-deviation principle

$$P(S_n = \zeta) \sim e^{-nI(\zeta)} \quad \text{for } n \rightarrow \infty$$

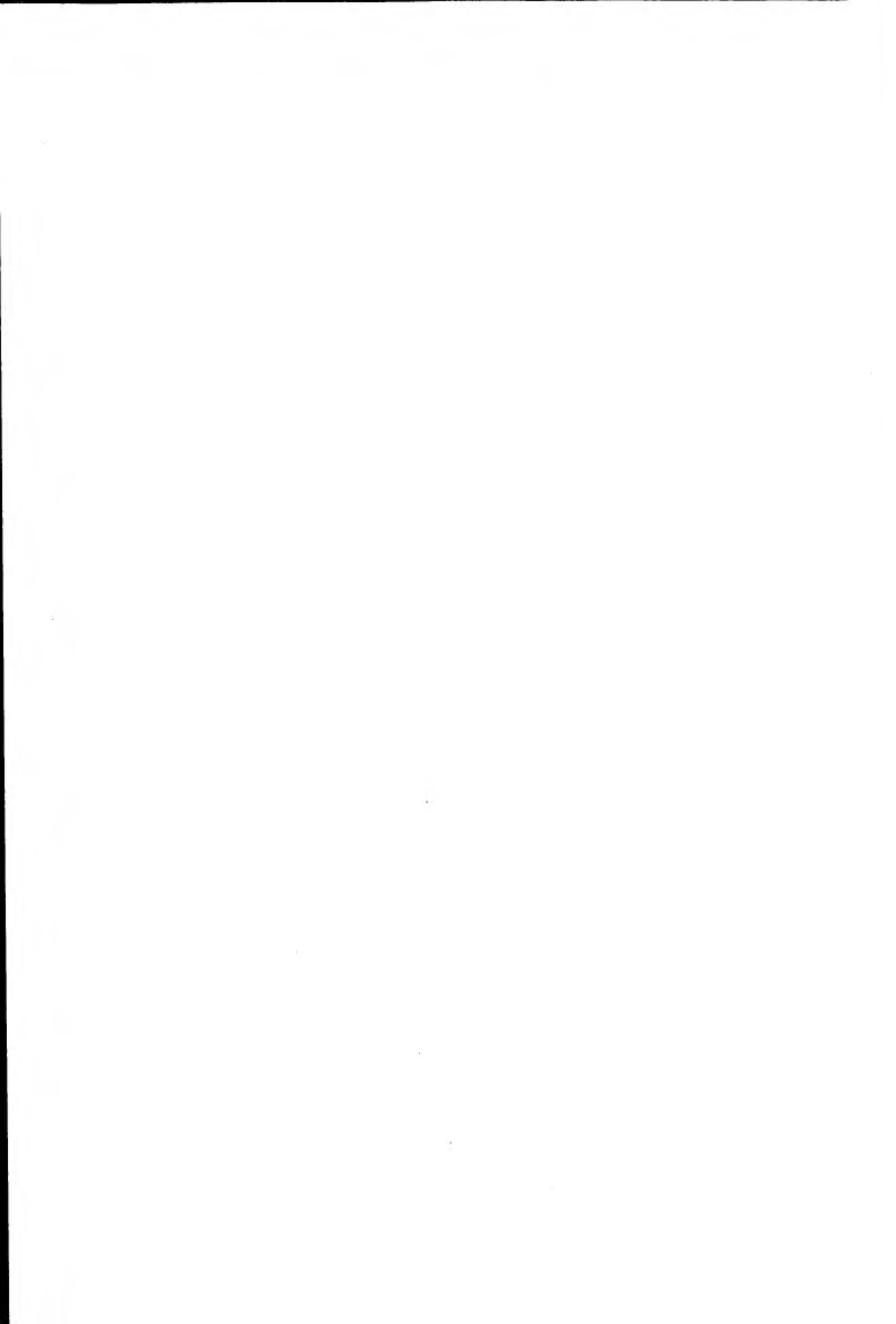
or, equivalently,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \ln P(S_n = \zeta) = I(\zeta),$$

with the *rate function* given by

$$I(\zeta) = \sup_{\lambda} \{\lambda\zeta - F(\lambda)\}.$$

The rate function $I(\zeta)$ characterizes the fluctuations or deviations from the average. It is positive and vanishes only at the mean value $\zeta_0 = \lim_{n \rightarrow \infty} \langle S_n \rangle$. By convexity, the rate function is quadratic around ζ_0 , which corresponds to the Gaussian fluctuations as described by the central limit theorem. The higher-order terms correspond to deviations further away from the mean. In this sense, the large fluctuations are characterized by the rate function I or, equivalently, by the generating function F .



B

Fluctuation theorem for semi-Markov processes

Our goal in this Appendix is to extend the fluctuation theorem for the currents to semi-Markov stochastic processes. Such an extension has been discussed in Ref. [184] for the heat dissipated. We here present the case of the fluctuation theorem for the macroscopic currents. Memory effects are known to occur in cases such as the transport of ions in membranes [14], enzyme kinetics, electronic transport in single-level quantum dots or metallic single-electron transistors which are strongly interacting systems [28], or dynamical mode transitions in noisy quantum dynamics. Such memory effects could also play a role in some recently observed molecular motors.

A semi-Markov process is a discrete non-Markovian renewal process which is fully described by the set of probability densities $\psi_{\omega\omega'}(\tau)$ for making random transitions within the time interval $[\tau, \tau + d\tau]$ from the state ω to the state ω' . These probability densities are necessarily positive and obey the normalization conditions

$$\sum_{\omega'} \int_0^\infty \psi_{\omega\omega'}(\tau) d\tau = 1, \quad (\text{B.1})$$

for all $\omega = 1, 2, \dots, N$. The subsequent residence-time intervals between the jumps are assumed to be mutually uncorrelated. The residence-time distribution $\psi_\omega(\tau)$ in the state ω reads

$$\psi_\omega(\tau) = \sum_{\omega'} \psi_{\omega\omega'}(\tau) = -\frac{d\phi_\omega(\tau)}{d\tau},$$

where $\phi_\omega(\tau)$ is the survival probability of the state ω obtained as

$$\phi_\omega(\tau) = \int_\tau^\infty \psi_\omega(t) dt.$$

The probability of a trajectory of the form

$$\omega_0 \xrightarrow{t_1} \omega_1 \xrightarrow{t_2} \dots \xrightarrow{t_n} \omega_n \quad (\text{B.2})$$

and ending at time t is thus given by

$$\phi_{\omega_n}(t - t_n) \psi_{\omega_n \omega_{n-1}}(t_n - t_{n-1}) \cdots \psi_{\omega_2 \omega_1}(t_2 - t_1) \psi_{\omega_1 \omega_0}(t_1 - t_0). \quad (\text{B.3})$$

This constitutes a general scheme for continuous-time random-walk theory [158].

Several descriptions for such non-Markovian processes have been developed [217, 40, 103]. We will here focus on the description introduced by Goychuk who also discussed the equivalence between all these theoretical schemes [103]. In this framework, the probability densities ψ are given by

$$\psi_{\omega\omega'}(\tau) = \psi_{\omega}(\tau) p_{\omega\omega'}$$

with *time-independent* transition probabilities $p_{\omega\omega'}$. In this case the normalization conditions (B.1) become

$$\sum_{\omega'} p_{\omega\omega'} = 1, \quad \int_0^{\infty} \psi_{\omega}(\tau) d\tau = 1.$$

The Markovian case is recovered when the survival probabilities $\phi_{\omega}(\tau)$ are exponential functions of time. We then have

$$\psi_{\omega}(\tau) = \left(\sum_{\omega'} W_{\omega\omega'} \right) \exp \left(-\tau \sum_{\omega'} W_{\omega\omega'} \right), \quad p_{\omega\omega'} = \frac{W_{\omega\omega'}}{\sum_{\omega'} W_{\omega\omega'}}.$$

The residence-time distribution $\psi_{\omega}(\tau)$ and the transition probabilities $p_{\omega\omega'}$ can be deduced from sampled trajectories. The study of the statistics of the residence-time intervals allows us to obtain the corresponding probability densities $\psi_{\omega}(\tau)$ hence the survival probabilities $\phi_{\omega}(\tau)$. Furthermore, the statistics of the transitions from one state to all the other states allows us to derive the corresponding conditional probabilities $p_{\omega\omega'}$.

We now derive the fluctuation theorem for the currents in close analogy to the demonstration of Chapter 3. The generating function of the currents (3.26) is now averaged with respect to the probabilities (B.3). Therefore, the quantity $F_{\omega\omega'}(t) \equiv \langle \exp -\lambda \cdot \mathbf{G}_t \rangle_{\omega\omega'}$, where the system starts in state ω at $t_0 = 0$ and ends in state ω' after a time t reads

$$F_{\omega\omega'}(\lambda, t) = \delta_{\omega\omega'} \phi_{\omega}(t) + \sum_{n=1}^{\infty} \int_0^t dt_1 \cdots dt_n \quad (\text{B.4})$$

$$\sum_{\omega_1, \dots, \omega_n} z_{\omega\omega_1}(t_1) z_{\omega_1\omega_2}(t_2 - t_1) \cdots z_{\omega_{n-1}\omega_n}(t_n - t_{n-1}) \phi_{\omega_n}(t - t_n) \delta_{\omega'\omega_n},$$

where the integration is subject to the condition $t_1 < t_2 < \cdots < t_n$. The quantities $z_{\omega\omega'}(t)$ are defined as

$$z_{\omega\omega'}(\lambda, t) \equiv \psi_{\omega\omega'}(t) \exp \left[\sum_l S_l(\omega \rightarrow \omega') \lambda_l \right].$$

These functions incorporate the contribution of the transitions to the Helfand moments G_l . To further study this process, we introduce the Laplace transform of the survival probabilities:

$$\hat{\phi}_\omega(s) \equiv \int_0^\infty dt e^{-st} \phi_\omega(t)$$

and similarly for $\hat{\psi}_{\omega\omega'}(s)$ and $\hat{w}_{\omega\omega'}(s)$. These quantities can be regrouped in matrix form as

$$\begin{aligned}\hat{\Phi}_{\omega\omega'}(s) &\equiv \delta_{\omega\omega'} \hat{\phi}_\omega(s) \\ \hat{\Psi}_{\omega\omega'}(s) &\equiv \hat{\psi}_{\omega\omega'}(s) \\ \hat{Z}_{\omega\omega'}(s) &\equiv \hat{z}_{\omega\omega'}(s).\end{aligned}$$

The Laplace transform of Eq. (B.4) is given by

$$\hat{F}_{\omega\omega'}(s) = \hat{\Phi}_{\omega\omega'}(s) + \sum_{n=1}^{\infty} \left[\hat{\Phi}(s) \hat{Z}^n(s) \right]_{\omega\omega'} = \left\{ \hat{\Phi}(s) [1 - \hat{Z}(s)]^{-1} \right\}_{\omega\omega'}.$$

The long-time exponential behavior of $F_{\omega\omega'}$, and thus of the generating function $Q = \sum_{\omega\omega'} P_{st}(\omega) F_{\omega\omega'}$, is given by the largest $s_0 = -Q$ such that the matrix

$$\hat{M}_{\omega\omega'}(s) \equiv [1 - \hat{Z}(s)]_{\omega\omega'}$$

presents a vanishing determinant: $\det \hat{M}(s_0) = \det[1 - \hat{Z}(s_0)] = 0$.

We can now proceed analogously to the demonstration of Sec. 3.2. Under the linear transform

$$\hat{U}_{\omega\omega'}(s) = \delta_{\omega\omega'} u_\omega(s),$$

where the functions u_ω satisfy the conditions

$$\frac{u_\omega}{u_{\omega'}} = \left(\frac{p_{\omega\omega'} \hat{\psi}_{\omega'}}{p_{\omega\omega'} \hat{\psi}_\omega} \right)^{\frac{1}{2}},$$

the matrix $\hat{M}'$ becomes

$$\hat{M}'_{\omega\omega'}(s) = \sqrt{\hat{\psi}_\omega \hat{\psi}_{\omega'} p_{\omega\omega'} p_{\omega'\omega}} \hat{M}_{\omega\omega'}(s)$$

if the transition $\omega \rightarrow \omega'$ is not a chord and for all s . For a chord l , we obtain

$$\hat{M}'_{\omega_l \omega_1}(s, \lambda_l) = \sqrt{\hat{\psi}_{\omega_l} \hat{\psi}_{\omega_1} p_{\omega_l \omega_1} p_{\omega_1 \omega_l}} e^{A_l/2 - \lambda_l} = \hat{M}'_{\omega_1 \omega_l}(s, A_l - \lambda_l).$$

With this change of basis, we arrive at the fundamental symmetry

$$\hat{M}'_\lambda = \hat{U} \hat{M}_\lambda \hat{U}^{-1} = \hat{M}'_{A-\lambda},$$

whereupon we find that all the eigenvalues of $\hat{M}$ as well as the generating function Q satisfy the same symmetry: $s_0 = -Q(\lambda) = -Q(A - \lambda)$. QED.

C

Dissipation and time reversal of Kramers' process

In this appendix, we generalize the calculations of Chapter 5 to the case of an underdamped charged Brownian particle in the presence of a magnetic field. For this purpose, we consider a Brownian particle in a two-dimensional (x - y) plane and immersed in a viscous fluid. The particle has a charge g and is subjected to the Lorentz force $g(\mathbf{v} \times \mathbf{B})$ due to a magnetic field aligned with the normal axis of the plane: $\mathbf{B} = B\mathbf{1}_z$. The particle is trapped in a potential well which is driven at speed $u \equiv \sqrt{2}v$ in the diagonal direction $\mathbf{1}_x + \mathbf{1}_y$. The fluctuating equations of motion read

$$\begin{aligned} m\ddot{x} + \gamma\dot{x} - gB\dot{y} - F_x(x - vt, y - vt) &= \xi_x(t) \\ m\ddot{y} + \gamma\dot{y} + gB\dot{x} - F_y(x - vt, y - vt) &= \xi_y(t), \end{aligned}$$

where $F_\alpha(x, y) = -\partial V/\partial\alpha$ is the force acting on the particle in the direction $\alpha = x$ or y . γ is the viscous friction and the random noise is correlated as

$$\langle \xi_\alpha(t) \xi_\beta(t') \rangle = 2k_B T \gamma \delta_{\alpha\beta} \delta(t - t'). \quad (\text{C.1})$$

In the following, we will work in the comoving frame

$$\begin{cases} q \equiv x - vt \\ r \equiv y - vt \end{cases}$$

so that the equations of motion become

$$m\ddot{q} + \gamma\dot{q} + \gamma v - gB(\dot{r} + v) - F_q(q, r) = \xi_q(t) \quad (\text{C.2a})$$

$$m\ddot{r} + \gamma\dot{r} + \gamma v + gB(\dot{q} + v) - F_r(q, r) = \xi_r(t), \quad (\text{C.2b})$$

The dissipated work over a time interval t is given by

$$W = v \int_0^t dt' [F_q(q_{t'}, r_{t'}) + F_r(q_{t'}, r_{t'})]$$

as the magnetic field does not perform any work (although the work does not explicitly depend on the magnetic field, the average work does).

Since we consider a Gaussian white noise with statistical properties (C.1), the probability to have the noise history $(\xi_q(t), \xi_r(t))$ is given by

$$P[\xi_q(t), \xi_r(t)] \propto \exp \left[-\frac{1}{4k_B T \gamma} \int_0^t dt' [\xi_q(t')]^2 + [\xi_r(t')]^2 \right]. \quad (\text{C.3})$$

To extract the heat dissipated along a trajectory, we consider the probability of a given path over the probability to observe the reversed path having also reversed the sign of the driving u as well as the magnetic field $\mathbf{B}$. The reversed path is thus defined by $z_{t'}^R = z_{t-t'}$ which implies $\dot{z}_{t'}^R = -\dot{z}_{t-t'}$ and $\ddot{z}_{t'}^R = \ddot{z}_{t-t'}$, with $z = q$ or r . According to Eqs. (C.3) and (C.2), we find after some algebra

$$\ln \frac{P_+[q_t, r_t | q_0, r_0]}{P_-[q_t^R, r_t^R | q_0^R, r_0^R]} = \frac{1}{k_B T} [W - \Delta K - \Delta V] \equiv Q.$$

Here,

$$\Delta K = \frac{m}{2} [(\dot{q}_t + v)^2 - (\dot{q}_0 + v)^2 + (\dot{r}_t + v)^2 - (\dot{r}_0 + v)^2]$$

is the difference of kinetic energy between final and initial states expressed in the comoving frame while

$$\Delta V = V(q_t, r_t) - V(q_0, r_0)$$

is the corresponding difference of potential energy. The heat Q dissipated along a random path can thus be obtained from the probabilities of the mesoscopic trajectories in the underdamped case as well, provided that we reverse the sign of *both* the driving constraint u and the magnetic field $\mathbf{B}$. We emphasize that, although the work and the heat do not depend explicitly on the magnetic field, it is only when reversing this latter that we obtain the heat dissipated from the underlying path probabilities.

D

Dynamical randomness of the Langevin process

In this appendix, we evaluate the (ε, τ) -entropy per unit time of the Grassberger-Procaccia algorithm for the Langevin stochastic process ruled by Eq. (5.6) in the case of a harmonic trap potential. In this case, the Langevin process is an Ornstein-Uhlenbeck stochastic process for the new variable

$$y \equiv z - u\tau_R.$$

The probability density that the continuous random variable $Y(t)$ takes the values $\mathbf{y} = (y_0, y_1, \dots, y_{n-1})$ at the successive times $0, \tau, 2\tau, \dots, (n-1)\tau$ factorizes since the random process is Markovian:

$$p(y_0, \dots, y_{n-1}) = p_{\text{st}}(y_0)G(y_0, y_1; \tau) \cdots G(y_{n-2}, y_{n-1}; \tau), \quad (\text{D.1})$$

where the Green function

$$G(y_0, y; t) = \frac{1}{\sqrt{2\pi\sigma^2(1 - e^{-2t/\tau_R})}} \exp \left[-\frac{(y - e^{-t/\tau_R}y_0)^2}{2\sigma^2(1 - e^{-2t/\tau_R})} \right]$$

is the conditional probability that the particle moves to the position y during the time interval t given that its initial position was y_0 [39]. The stationary probability density is given by the Gaussian distribution

$$p_{\text{st}}(y) = \lim_{t \rightarrow \infty} G(y_0, y; t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{y^2}{2\sigma^2} \right),$$

with the variance

$$\sigma^2 = \frac{k_B T}{k}.$$

Denoting by $\mathbf{y}^T$ the transpose of the vector $\mathbf{y}$, the joint probability density (D.1) can be written as the multivariate Gaussian distribution

$$p(y_0, \dots, y_{n-1}) = \frac{\exp \left(-\frac{1}{2} \mathbf{y}^T \cdot \mathbf{C}_n^{-1} \cdot \mathbf{y} \right)}{(2\pi)^{\frac{n}{2}} (\det \mathbf{C}_n)^{\frac{1}{2}}}$$

in terms of the correlation matrix

$$\mathbf{C}_n = \sigma^2 \begin{pmatrix} 1 & r & r^2 & \dots & r^{n-1} \\ r & 1 & r & \dots & r^{n-2} \\ r^2 & r & 1 & \dots & r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & 1 \end{pmatrix}$$

with

$$r = \exp(-\tau/\tau_R).$$

The inverse of the correlation matrix is given by

$$\mathbf{C}_n^{-1} = \frac{\sigma^{-2}}{(1-r^2)} \begin{pmatrix} 1 & -r & 0 & \dots & 0 & 0 \\ -r & 1+r^2 & -r & \dots & 0 & 0 \\ 0 & -r & 1+r^2 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+r^2 & -r \\ 0 & 0 & 0 & \dots & -r & 1 \end{pmatrix}$$

and its determinant by

$$\det \mathbf{C}_n = \sigma^{2n} (1-r^2)^{n-1}. \quad (\text{D.2})$$

The (ε, τ) -entropy per unit time is defined by

$$h(\varepsilon, \tau) = \lim_{n \rightarrow \infty} -\frac{1}{n\tau} \int_{-\infty}^{+\infty} dy_0 \dots \int_{-\infty}^{+\infty} dy_{n-1} p(y_0, \dots, y_{n-1}) \\ \times \ln \int_{-\varepsilon}^{+\varepsilon} d\eta_0 \dots \int_{-\varepsilon}^{+\varepsilon} d\eta_{n-1} p(y_0 + \eta_0, \dots, y_{n-1} + \eta_{n-1}),$$

where $\{y'_j = y_j + \eta_j\}_{j=0}^{n-1}$ represents the tube of trajectories satisfying the conditions $|y'_j - y_j| < \varepsilon$ with $j = 0, 1, \dots, n-1$, and centered around the reference trajectory sampled at the successive positions $\{y_j\}_{j=0}^{n-1}$. After expanding in powers of the variables η_j and evaluating the integrals over $-\varepsilon < \eta_j < +\varepsilon$, the logarithm is obtained as

$$\ln \int_{-\varepsilon}^{+\varepsilon} d\eta_0 \dots \int_{-\varepsilon}^{+\varepsilon} d\eta_{n-1} p(y_0 + \eta_0, \dots, y_{n-1} + \eta_{n-1}) \\ = \ln \frac{(2\varepsilon)^n}{(2\pi)^{\frac{n}{2}} (\det \mathbf{C}_n)^{\frac{1}{2}}} - \frac{1}{2} \mathbf{y}^T \cdot \mathbf{C}_n^{-1} \cdot \mathbf{y} + O(\varepsilon^2).$$

The integrals over $-\infty < y_j < +\infty$ can now be calculated to get the result (5.20) by using Eq. (D.2). We find

$$h(\varepsilon, \tau) = \frac{1}{\tau} \ln \sqrt{\frac{\pi e \sigma^2}{2\varepsilon^2} (1-r^2)} + O(\varepsilon^2). \quad (\text{D.3})$$

Since the relaxation time is given by $\tau_R = \alpha/k$ and the diffusion coefficient by $D = k_B T/\alpha$, the variance of the fluctuations can be rewritten as $\sigma^2 = k_B T/k =$

$D\tau_R$. Substituting in Eq. (D.3), we obtain the (ε, τ) -entropy per unit time given by Eq. (5.20). The above calculation shows that the (ε, τ) -entropy per unit time of the Ornstein-Uhlenbeck process is of the form

$$h(\varepsilon, \tau) = \frac{1}{\tau} \phi \left(\frac{D\tau_R}{\varepsilon^2}, e^{-\tau/\tau_R} \right)$$

with some function $\phi(s, r)$ of $s = D\tau_R/\varepsilon^2 = \sigma^2/\varepsilon^2$ and $r = \exp(-\tau/\tau_R)$.

In the limit where the time interval τ is much smaller than the relaxation time τ_R , the only dimensionless variable is the combination $D\tau/\varepsilon^2$. In this case, we recover the result that the (ε, τ) -entropy per unit time is given by

$$h(\varepsilon, \tau) = \frac{1}{\tau} \varphi \left(\frac{2D\tau}{\varepsilon^2} \right) \quad \text{for } \tau \ll \tau_R.$$

This (ε, τ) -entropy per unit time is characteristic of pure diffusion without trap potential, as previously shown [97, 84, 90].



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