Kähler and Almost-Kähler Geometric Flows

Doctoral Thesis

Author:
Julian Pook

Supervisor:
Joel Fine

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Contents

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1. Introduction

A guiding question in differential geometry is the following: Given a family of geometric objects, does there exist a best object in that family? It is often not obvious what qualities should distinguish an object as "best" and the precise definition is tacitly part of the question which, despite its vagueness, leads to numerous important notions and problems. Indeed, many facets of differential geometry can be interpreted in terms of the best object mantra. Examples include harmonic maps as best maps between Riemannian manifolds, encompassing geodesics and minimal submanifolds, Einstein metrics as best metrics on certain manifolds, extremal metrics as best metrics in a given Kähler class, including Kähler–Einstein metrics or Hermite–Einstein metrics as best Hermitian metrics on a holomorphic vector bundle over a Kähler manifold.

In many cases, including the above examples, the problem of finding best objects can be phrased in terms of an elliptic, typically nonlinear partial differential equation. A useful tool in the search for solutions is to consider a related parabolic problem leading to the notion of geometric flows[1]. While not a priori easier than the original elliptic problem, passing to the parabolic picture gives access to additional analytical techniques. Furthermore, one might hope to relate the eventual nonexistence of solutions to the elliptic equation to singularity formation along the parabolic flow.

A testimony to the utility of geometric flows is G. Perelman's work on Ricci flow entailing a proof of Thurston's geometrisation conjecture via Hamilton's programme [28, 29, 24]. Following Perelman's breakthrough in three-dimensional geometry, it has been proposed by G. Tian – J. Song that Kähler–Ricci flow implements a general Kähler version of the minimal model programme in algebraic geometry which aims to find the best representative in the birational equivalence class of a given algebraic variety. It is conjectured that on a general Kähler manifold the flow continues through singularities and converges to an analytical minimal model endowed with a possibly singular Kähler–Einstein structure [2]. Later, G. Tian – J. Streets proposed symplectic curvature flow as a generalisation of Kähler–Ricci flow to almost Kähler and almost Hermitian geometry with the hope of finding canonical structures in these cases [37]. Other prominent examples are S. Donaldson's existence proof of Hermite–Einstein metrics on stable holomorphic bundles over algebraic surfaces using Hermitian Yang–Mills flow [9] or P. Chruściel's use of Calabi flow in the construction of Robinson–Trautman solutions to Einstein's equations in general relativity [8].

The principal objects of study in this thesis are twisted Calabi flow and time-dependent Hermitian Yang–Mills flow which generalise their namesake flows in the sense that they

[1]Here, the term *geometric flow* is used in the broader sense and denotes any (parabolic) evolution equation for geometric quantities. It is not necessarily a gradient flow.

involve additional external data. In the case of twisted Calabi flow, the Kähler–Ricci form ρ is replaced by $\rho + \alpha$, where the *twist* α is a time-dependent family of two-forms. Time-dependent Hermitian Yang–Mills flow is obtained from Hermitian Yang–Mills flow by allowing the Kähler metric on the base to depend on the time parameter. Both arise as first order approximations to Calabi flow on two types of adiabatic fibrations considered by J. Fine [13] and Y.-J. Hong [22, 21] respectively.

It is shown that on a compact Riemann surface of genus at least one and for smooth initial data twisted Calabi flow exists for all positives times provided that the twist $\alpha(t)$ is negative semidefinite and stays within a fixed cohomology class. Moreover, if the twist converges to a limit α_∞ as $t \to \infty$, then the solution converges to the α_∞-twisted cscK metric considered by J. Fine [13, 14] and J. Song – G. Tian [35].

Similar results are obtained for time-dependent Hermitian Yang–Mills flow. If X is a compact Riemann surface with Kähler class κ and $E \to X$ a holomorphic vector bundle, then for any smooth family of Kähler forms $\omega(t)$ in κ and any Hermitian metric h_0 on E, Hermitian Yang–Mills flow with respect to $\omega(t)$ starting at h_0 exists for all times. If $E \to X$ is κ-slope stable and $\omega(t)$ converges to ω_∞ at an exponential rate, then the solution converges exponentially to a ω_∞-Hermite–Einstein metric.

In addition, the thesis presents several explicit solutions to symplectic curvature flow which can be grouped into two types: left invariant solutions on nilmanifolds and static non-Kähler solutions on twistor fibrations over hyperbolic space. The latter are the first compact examples of potential limit objects for symplectic curvature flow that do not admit a Kähler structure.

Remark. By a slight abuse of language, both, complex manifolds admitting Kähler metrics and complex manifolds with a given Kähler structure are referred to as Kähler manifolds. In addition, regarding the complex structure as fixed, the term “Kähler metric” can refer to a Kähler form or the corresponding Riemannian metric, understanding that one uniquely defines the other.

2. Twisted Calabi Flow

2.1. Introduction

One of the key features in the study of compact Kähler manifolds is that the Kähler metrics in a given Kähler class κ can be parametrised by functions. If (X, J, ω_0) is Kähler with $\omega_0 \in \kappa$, then owing to the $\bar{\partial}\partial$-lemma any other Kähler form in κ can be written as $\omega_0 + i\bar{\partial}\partial\varphi$ with the *Kähler potential* $\varphi \in C^\infty(X, \mathbb{R})$ being unique up to a constant. Conversely, the two-form $\omega_\varphi := \omega_0 + i\bar{\partial}\partial\varphi$ for $\varphi \in \mathcal{H} := \{\varphi \in C^\infty(X, \mathbb{R}) \,|\, \omega_0 + i\bar{\partial}\partial\varphi > 0\}$ defines a Kähler metric in κ, so the set of Kähler metrics in κ can be identified with $\mathcal{H}/\mathbb{R}$. It is, however, often more convenient to work with Kähler potentials than with the metrics themselves. As an open subset of $C^\infty(X, \mathbb{R})$, $\mathcal{H}$ carries the structure of a Fréchet manifold, on which

$$(f_1, f_2)_\varphi := \int_X f_1 \cdot f_2 \, \frac{\omega_\varphi^n}{n!} \,, \quad f_1, f_2 \in C^\infty(X, \mathbb{R}) = T_\varphi\mathcal{H} \,.$$

defines a Riemannian metric. This metric, independently due to Mabuchi, Semmes and Donaldson, formally turns $\mathcal{H}$ into nonpositively curved symmetric space. In addition, proving the existence of weak geodesics, Chen [6] showed that $\mathcal{H}$ is a genuine metric space. These properties play an important role in the search for canonical metrics in Kähler geometry.

The question if a given Kähler class contains a canonical representative has been attributed to Calabi who proposed to look for *extremal metrics*, i.e. critical points of the *Calabi energy*

$$Ca\colon \mathcal{H} \to \mathbb{R} \,, \quad \varphi \mapsto Ca(\varphi) = \int_X (S(\omega_\varphi) - \underline{S})^2 \, \frac{\omega_\varphi^n}{n!} \,. \tag{2.1}$$

Here $S(\omega_\varphi)$ denotes the scalar curvature of the metric ω_φ and $\underline{S}$ the average scalar curvature which is a cohomological constant and independent of $\varphi \in \mathcal{H}$. The variation of Ca at $\varphi \in \mathcal{H}$ in direction $\psi \in T_\varphi\mathcal{H} = C^\infty(X, \mathbb{R})$ is given by

$$(dCa)_\varphi\psi = 2\int_X (\mathcal{D}_\varphi^*\mathcal{D}_\varphi S(\omega_\varphi)) \cdot \psi \, \frac{\omega_\varphi^n}{n!} \,, \tag{2.2}$$

where the Lichnerowicz operator $\mathcal{D}_\varphi = \bar{\partial}\nabla$ acting on a function f is defined by applying the $\bar{\partial}$ operator on the holomorphic tangent bundle (TX, J) to ∇f. The $L^2(X, \omega_\varphi)$-adjoint of $\mathcal{D}_\varphi$ is denoted by $\mathcal{D}_\varphi^*$ and the subscript indicates that all operations are performed with respect to the metric g_φ defined by ω_φ. Integrating by parts, one sees that $\ker \mathcal{D}_\varphi^*\mathcal{D}_\varphi = \ker \mathcal{D}_\varphi$, so ω_φ is extremal precisely if the g_φ-gradient of the scalar curvature defines a holomorphic vector field. If X does not admit nontrivial holomorphic vector fields — or more generally the Futaki invariant of (X,κ) vanishes — this is equivalent to the scalar curvature being constant, i.e. $S(\omega_\varphi) = \underline{S}$.

Encompassing constant scalar curvature Kähler and Kähler–Einstein metrics, extremal Kähler metrics have been the object of extensive study (cf. e.g. [30] for an overview). A method Calabi himself proposed to find extremal metrics is to deform an initial metric via a fourth order parabolic evolution equation known as Calabi flow. The Calabi flow equation reads

$$\partial_t\varphi = -(S(\omega_\varphi) - \underline{S}) \text{ or } \partial_t\omega = -i\bar{\partial}\partial S(\omega) \tag{2.3}$$

on the level of Kähler potentials and Kähler forms respectively. Calabi flow bears resemblance to (normalised) Ricci flow. Both compare variations in the metric defining quantity — the Kähler potential in the case of Calabi flow and the metric itself in the case of Ricci flow — to the only natural curvature tensor living in the same space, the scalar and the Ricci curvature. This analogy, however, is only formal. Analytically, as a fourth order equation, Calabi flow requires a different set of tools than the second order Ricci flow. One does, for instance, not have a maximum principle and has to rely on monotone quantities such as the Calabi energy (which is manifestly nonincreasing under Calabi flow) to obtain a priori bounds. Calabi flow on compact manifolds has been conjectured to smoothly exist for all times and to converge to a constant scalar curvature Kähler (cscK) metric at infinity, provided such metrics exist in the given Kähler class (in [36] the author states that the long-time existence conjecture is due to Calabi and attributes the convergence conjecture to Donaldson). J. Streets showed in [36] that the long-time existence conjecture is true albeit in a very weak sense. A central motivation for long-time behaviour conjectures is the existence of functional $Ma\colon \mathcal{H} \to \mathbb{R}$, called the *Mabuchi energy* (or *K-energy*), whose defining property is

$$(dMa)_\varphi\psi = \int_X (S(\omega_\varphi) - \underline{S})\psi\, \frac{\omega_\varphi^n}{n!}\,. \tag{2.4}$$

The right hand side of (2.4) defines a closed one-form which by contractibility of $\mathcal{H}$ has to be exact. If φ evolves according to (2.3), then $\partial_t Ma(\varphi) = -Ca(\varphi)$, so Mabuchi energy is decreasing and convex along Calabi flow. Moreover, Chen–Tian [7] showed that Mabuchi energy is bounded from below if the Kähler class admits cscK metrics. When $\mathcal{H}$ is equipped with the Donaldson–Semmes–Mabuchi metric, Calabi flow becomes precisely the gradient flow of Mabuchi energy and in light of $\mathcal{H}$ being a nonpositively curved metric space one might hope the flow to be well behaved [27].

In the case of compact Riemann surfaces, Calabi flow is fairly well understood. Starting at an arbitrary initial Kähler metric, Calabi flow exists for all times and converges at an exponential rate to a cscK metric of the same volume (that metric is unique except on the Riemann sphere, where the Möbius group $\mathrm{PGl}(2,\mathbb{C})$ acts biholomorphically and generates a nontrivial family of Fubini–Study metrics, all of which are cscK). This was first proved by P. Chruściel [8]. Later, X.X. Chen [5] gave a new proof using a slightly different approach which has since been refined by M. Struwe [38]. All three authors exploit the fact that on a Riemann surface any two Hermitian (and thus Kähler) metrics are conformally equivalent to parametrise Kähler metrics by $e^u g_0$ for a reference Kähler metric g_0 of the same volume and rewrite the Calabi flow equation in terms of the logarithm u of the conformal factor. For analytical reasons, the authors choose g_0 to be a constant scalar curvature Kähler metric, thereby assuming a priori knowledge about the existence of a suitable limit object provided by the unformisation theorem. S.-C. Chang [4] later removed this assumption providing a new proof of the uniformisation theorem using Calabi flow. Another specificity to Riemann surfaces exploited by the authors is the existence of energy functionals decreasing along Calabi flow, for which it is unclear whether they generalise to higher dimensions. Chruściel used a physically motivated Bondi mass loss formula in addition to the fact that Calabi energy is decreasing in order to derive a sufficiently strong a priori bound on u to prove long-time existence. In Chen's and Struwe's proof, the analytic role of Bondi mass is fulfilled by the *Liouville energy*, another functional on $\mathcal{H}$ decreasing along Calabi flow. In additional to being mathematically more natural, the Liouville energy has been conjectured by Chen to admit a higher-dimensional generalisation which would be useful in the general study of Calabi flow.

The objective in this chapter is to study *twisted Calabi flow* on compact Riemann surfaces of positive genus. Instead of cscK metrics, twisted Calabi flow is designed to find so called *twisted cscK metrics* appearing in the work of Fine [13, 14] and of Song–Tian [35] which can roughly be understood as canonical metrics on the base of certain fibrations retaining information on the varying moduli of the fibres. More abstractly, given a closed real two-form α on a Kähler manifold (X, J), one can look within a given Kähler class κ for solutions to the equation

$$\Lambda_\omega(\rho(\omega) + \alpha) = \hat{S}, \tag{2.5}$$

where $\hat{S}$ is a cohomological constant, $\rho(\omega)$ the Kähler–Ricci form of the metric $\omega \in \kappa$ and Λ_ω denotes the (pointwise) adjoint of wedging with ω. Note that $\hat{S}$ is cohomological and only depends on the first Chern class of (X, J), the cohomology class of α and the volume of the Kähler class $[\omega]$. Also observe that $\Lambda_\omega \rho(\omega) = S(\omega)$ which justifies calling (2.5) the twisted cscK equation. On a Riemann surface of positive genus this equation can always be solved uniquely if $\int_X \alpha \leqslant 0$ and one can ask whether the solution can be found via the twisted Calabi flow

$$\partial_t \varphi = -(\Lambda_{\omega_\varphi}(\rho(\omega_\varphi) + \alpha) - \hat{S})\,. \tag{2.6}$$

For the application in mind (cf. Chapter 4), it is important to allow α to vary in the time parameter t. We show that on a compact Riemann surface of positive genus for negative semidefinite twists $\alpha(t)$ in a given cohomology class, the equation (2.6) admits a unique smooth long-time solution. Furthermore, if $\alpha(t)$ converges to a limiting twist α_∞ in a suitably strong sense, then the solution to twisted Calabi flow converges exponentially fast to the α_∞-twisted cscK metric.

2.2. Notation

2.2.1. Parametrisation of Metrics

Throughout this chapter (X, J, ω_0) denotes a compact Riemann surface of positive genus with fixed complex structure J and a smooth background Kähler metric ω_0. The background metric is used to define the space of Kähler potentials $\mathcal{H}$ and the metric corresponding to a potential $\varphi \in \mathcal{H}$ is denoted by $\omega_\varphi = \omega_0 + i\bar{\partial}\partial\varphi$. In a local holomorphic coordinate $z = x + iy$ a Riemannian metric g is locally determined by $g_{xx} := g(\partial_x, \partial_x)$, $g_{xy} = g(\partial_x, \partial_y)$ and $g_{yy} = g(\partial_y, \partial_y)$. If g is J-invariant, then $g_{xx} = g_{yy}$ and $g_{xy} = 0$, so any other J-invariant metric g' can be expressed as $g' = e^u g$ for $u = \log g'_{xx}/g_{xx}$. The locally defined u is independent of the chosen holomorphic coordinate and defines a real valued function on X. Since $H^2(X,\mathbb{R}) \cong \mathbb{R}$, a Kähler class is uniquely determined by its volume. These considerations imply that Kähler metrics in $[\omega_0]$ can also be parametrised as metrics of the same volume conformally equivalent to ω_0 via

$$\left\{ u \in C^\infty(X,\mathbb{R}) \,\middle|\, \int_X e^u\,\omega_0 = \int_X \omega_0 = \mathrm{vol}(X, [\omega_0]) \right\}.$$

2.2.2. Geometric Operators and Curvature

On a general Kähler manifold (X,ω), wedging with the Kähler form ω defines a map $\Omega^{p,q}(X) \to \Omega^{p+1,q+1}(X)$, $\alpha \mapsto \alpha\wedge\omega$. The (pointwise) adjoint of this map $\Lambda_\omega \colon \Omega^{p,q}(X) \to \Omega^{p-1,q-1}(X)$ is called *contraction with* ω. On a two-form α it can be computed as the factor of proportionality between $\alpha \wedge \omega^{n-1}/(n-1)!$ and the volume form $\omega^n/n!$. In the case of a Riemann surface (X,ω), any two-form is of type $(1,1)$ and hence a pointwise multiple of ω, i.e. given $\alpha \in \Omega^2(X,\mathbb{R})$, $\Lambda_\omega\alpha$ is the unique smooth function such that $\alpha = \Lambda_\omega\alpha\cdot\omega$. It follows from this description that if ω and ω' are related by $\omega' = e^u\omega$, then $\Lambda_{\omega'} = e^{-u}\Lambda_\omega$. For metrics parametrised by Kähler potentials $\varphi \in \mathcal{H}$, the contraction Λ_{ω_φ} is abbreviated by Λ_φ.

The ∂-*Laplacian* associated to a Kähler metric ω on functions is defined by $\Delta_\omega f = \Lambda_\omega i\bar{\partial}\partial f$. It follows from the Kähler identities that Δ_ω is one half of the full Riemannian Laplacian. As with the contraction one has $\Delta_{\omega'} = e^{-u}\Delta_\omega$ if $\omega' = e^u\omega$ and we abbreviate Δ_{ω_φ} by Δ_φ. Conformal factor and Kähler potential are related by $(1 + \Delta_0\varphi) = e^u$ if $\omega_\varphi = \omega_0 + i\bar{\partial}\partial\varphi = e^u\omega_0$.

On a Kähler manifold (X,ω) of complex dimension n, one can conveniently compute the Kähler–Ricci form $\rho(\omega)$ as the i times the curvature of the Chern connection on the anti-canonical bundle $K_X^{-1} = (T^{1,0}X)^n$ endowed with the Hermitian metric $h_\omega := \omega^n$ which is locally given by $F_{h_\omega} = \bar{\partial}h_\omega^{-1}\partial h_\omega = \bar{\partial}\partial\log h_\omega$. The scalar curvature is obtained from the Kähler–Ricci form by contracting with ω, i.e. $S(\omega) = \Lambda_\omega\rho(\omega)$. In the case of a Riemann surface, the Kähler–Ricci forms of the background metric ω_0 and $e^u\omega_0$ are related by

$$\rho(e^u\omega_0) = iF_{e^u h_{\omega_0}} = i\bar{\partial}\partial\log(e^u h_{\omega_0}) = i\bar{\partial}\partial u + \rho(\omega_0)\,,$$

so for the scalar curvature one has

$$S(e^u\omega_0) = \Lambda_{e^u\omega_0}\rho(e^u\omega_0) = e^{-u}\Lambda_0(i\bar{\partial}\partial u + \rho(\omega_0)) = e^{-u}(\Delta_0 u + S(\omega_0))\,. \tag{2.7}$$

Scalar curvature can also be seen as a map

$$\mathrm{Sc}\colon \mathcal{H} \to C^\infty(X, \mathbb{R}), \quad \mathrm{Sc}(\varphi) = S(\omega_\varphi) = \Lambda_\varphi i\bar{\partial}\partial \log \omega_\varphi,$$

where in the last expression ω_φ is interpreted as a Hermitian product on the holomorphic line bundle $T^{1,0}X$ in a local trivialisation. From here (cf. also Appendix A.3) one obtains

$$(d\,\mathrm{Sc})_\varphi \cdot \psi = \Delta_\varphi^2 \psi - \mathrm{Sc}(\varphi)\Delta_\varphi \psi$$

for the derivative of the scalar curvature map, which using the relation $\mathcal{D}_\varphi^* \mathcal{D}_\varphi \psi = \Delta_\varphi^2 \psi - \mathrm{Sc}(\varphi)\Delta_\varphi \psi + \frac{1}{2} \cdot g_\varphi(d\,\mathrm{Sc}(\varphi), d\psi)$ can be reexpressed as

$$(d\,\mathrm{Sc})_\varphi \cdot \psi = \mathcal{D}_\varphi^* \mathcal{D}_\varphi \psi - \frac{1}{2} g_\varphi(d\,\mathrm{Sc}(\varphi), d\psi). \tag{2.8}$$

2.2.3. Analysis

For various analytic arguments, it is necessary to allow metrics that are not a priori smooth and to lower the regularity requirements on u. For our purposes the spaces of interest are $C^k(X, g)$ and the Sobolev spaces $L_k^p(X, g)$ which can be defined as the completion of $C^\infty(X)$ with respect to the norms

$$\begin{aligned} \|\varphi\|_{C^k(X,g)} &= \sum_{j=0}^{k} \sup_X |\nabla^j \varphi|, \\ \|\varphi\|_{L_k^p(X,g)} &= \left(\sum_{j=0}^{k} \int_X |\nabla^j \varphi|^p \,\omega \right)^{\frac{1}{p}}, \end{aligned}$$

where all gradients and pointwise norms are taken with respect to the inner products induced by the metric g on the respective tensors. To lighten the notation, explicit mention of the metric is omitted if norms are taken with respect to the background metric g_0. The *Sobolev embedding theorems* provide continuous linear embeddings $L_k^p \hookrightarrow L_l^q$ if

$$\frac{1}{p} \leqslant \frac{1}{q} + \frac{k-l}{2}$$

and $L_k^p \hookrightarrow C^l$ if

$$\frac{1}{p} < \frac{k-l}{2}$$

for integers $k, l \geqslant 0$ and reals $p, q \geqslant 1$. Moreover, the embeddings are compact whenever the inequalities are strict (the second always is).

Remark. There exists a stronger version of the second embedding into Hölder spaces. However, we only require the stated version.

As a boundary case of the Sobolev embeddings, L_1^2 does not quite embed into C^0, but a function in L_1^2 cannot have its L^p-norms grow too quickly in p. A precise statement is provided by the *Moser–Trudinger inequality.*

Proposition 2.2.1. *If $f \in L^2_1$, then $e^f \in L^1$. Moreover, there exist positive constants C, μ such that*

$$\int_X e^f \, \omega_0 \leqslant C e^{\mu \|f\|^2_{L^2_1}} .$$

The *Gagliardo–Nirenberg interpolation inequalities*, which in their general version permit finer interpolation between Sobolev spaces than the embedding theorems are an essential tool for the higher regularity analysis. Of interest here is only the following case.

Proposition 2.2.2. *Let $2 \leqslant p \leqslant \infty$ and $j, m \in \mathbb{N}$ with $j < m$. There exists a constant $C > 0$ such that for all $f \in L^2_m(X, g)$ with $\int_X f \, \omega_0 = 0$ the following inequality holds:*

$$\|\nabla^j f\|_{L^p} \leqslant C \|\nabla^m f\|^a_{L^2} \|f\|^{1-a}_{L^2} , \quad a = \frac{j + 1 - 2/p}{m} .$$

Corollary 2.2.3. *Let $2 \leqslant p \leqslant \infty$ and $k, l \in \mathbb{N}$ such that $2 \leqslant k < l + 2$. There exists a constant $C > 0$ such that for all $f \in L^2_{l+2}$ with $\int_X f \, \omega_0 = 0$ the following holds:*

$$\|\nabla^k f\|_{L^p} \leqslant C \left(\|\nabla^{l+2} f\|^a_{L^2} \|f\|^{1-a}_{L^2_2} + \|f\|_{L^2_2} \right) , \quad a = \frac{k - 1 - 2/p}{l} .$$

Morally, Corollary 2.2.3 is obtained from Proposition 2.2.2 by replacing f by $\nabla^2 f$ (which is no longer a function). A proof can be found in [8]. The Sobolev, Moser–Trudinger and Gagliardo–Nirenberg inequalities are discussed in a more general form in [1].

2.2.4. Twisted Calabi Flow

The goal of this chapter is to prove the following theorem:

Theorem 2.2.4. *Let (X, J) be a compact Riemann surface of positive genus and $\mathcal{H}$ the space of Kähler potentials for a Kähler class κ with respect to a background metric $\omega_0 \in \kappa$. Let $\alpha(t)$, $t \geqslant 0$ be a smooth one-parameter family of real closed negative semidefinite two-forms in a fixed cohomology class such that $\alpha(t)$ converges to a real closed negative-semidefinite two-form α_∞ and $\partial_t \alpha(t)$ to 0 at exponential rates in C^k for all $k \in \mathbb{N}_0$. Denote by $\underline{S}$ the average scalar curvature and by $\underline{\alpha}$ the integral of α and set $\hat{S} := \underline{S} + \underline{\alpha}$. Then the α-twisted Calabi flow equation*

$$\partial_t \varphi(t) = -(\mathrm{Sc}(\varphi(t)) + \Lambda_{\varphi(t)} \alpha(t) - \hat{S}) \tag{2.9}$$

starting at smooth initial data $\varphi(0)$ admits a unique long-time solution $\varphi(t)$. Moreover, if $\omega_{\varphi(t)} = e^{u(t)} \omega_0$, then $u(t)$ converges exponentially to the logarithm of the conformal factor of the α_∞-twisted cscK metric in C^k for all $k \in \mathbb{N}_0$.

Observe that in terms of u, equation (2.9) reads

$$\partial_t u = e^{-u}(\partial_t e^u) = e^{-u} \partial_t (1 + \Delta_0 \varphi) = \Delta_\varphi \partial_t \varphi = -\Delta_\varphi (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha) ,$$

which in light of (2.7) can be expressed as

$$\partial_t u = -e^{-u}\Delta_0[e^{-u}(\Delta_0 u + \mathrm{Sc}_0 + \Lambda_0\alpha)]\,. \tag{2.10}$$

This is a quasilinear fourth order parabolic PDE and standard theory guarantees existence and uniqueness of smooth solutions for small times $t \in [0, T[$. Unless explicit note is made to the contrary, the background metric g_0 is taken to be the *α_∞-twisted cscK metric* satisfying $\mathrm{Sc}_0 + \Lambda_0\alpha_\infty = \hat{S}$, whose existence and uniqueness proof is essentially that of the uniformisation theorem (cf. Appendix B.1). With this choice of g_0, the claim of Theorem 2.10 then states that $u(t)$ should exponentially converge to 0. Also note that if $e^{u(t)}\omega_0$ solves twisted Calabi flow with respect to $\alpha(t)$ in the class $[\omega_0]$, then for any $c > 0$ the path $ce^{u(t/c^2)}\omega_0$ is a solution of twisted Calabi flow with respect to $\alpha(t/c^2)$ in the class $c[\omega_0]$. We can hence assume without loss of generality that $[\omega_0]$ has unit volume.

The proof of Theorem 2.2.4 is organised as follows. Twisted versions of Mabuchi, Liouville and Calabi energy are defined and shown to be uniformly bounded in time along twisted Calabi flow, which is used to obtain an a priori L^2_2-bound on u. Chruściel's higher regularity arguments are then adopted to improve the uniform bounds on u from L^2_2 to any C^k implying long-time existence of twisted Calabi flow. Lastly, convergence and its exponentiality are established.

Remark. In our context, a function f on $X \times [0, T[$ for $T \in]0, \infty]$ is called *uniformly bounded* or *controlled* in C^k or L^p_k if there exists a constant independent of $t \in [0, T[$ such that the respective norm satisfies $\|f\| \leqslant C$. In general, C, C', etc. denote constants independent of t whose precise value is allowed to change from line to line.

2.3. Energy Functionals

In their proofs of the long-time existence and convergence of Calabi flow on compact Riemann surfaces, X.X. Chen [5] and M. Struwe [38] rely on the boundedness of the Liouville energy, Calabi energy and to a lesser extent of Mabuchi energy along the flow. Bounds on Liouville and Calabi energy imply an a priori bound on $\|u\|_{L^2_2}$, which turns out to be sufficient to extend short-time solutions to arbitrary positive times. X.X. Chen [5] uses the boundedness of Mabuchi energy to show that Calabi energy tends to zero as $t \to \infty$. Both, Mabuchi and Liouville energy, are easiest defined in terms of their variations, which on a Riemann surface are given by $(dMa)_\varphi \cdot \psi = \int_X (S(\omega_\varphi) - \underline{S})\psi\,\omega_\varphi$ and $(dF)_\varphi \cdot \psi = \int_X g_\varphi(dS(\omega_\varphi), d\psi)\,\omega_\varphi$. Choosing a reference metric g_0 in $\mathcal{H}$, the variational expressions can be integrated to

$$\begin{aligned} Ma(\varphi) &= \int_X u\,\omega_\varphi - \frac{1}{2}\underline{S}\varphi\Delta_0\varphi\,\omega_0 + (\mathrm{Sc}_0 - \underline{S})\varphi\,\omega_0\,, \\ F(\varphi) &= \int_X u[\Delta_0 u + 2\,\mathrm{Sc}_0]\,\omega_0\,. \end{aligned}$$

An explicit formula for the Mabuchi energy in arbitrary dimension can be found in e.g. [30]. The existence of a higher-dimensional version of the Liouville energy has been conjectured by X.X. Chen (cf. [5]).

As for regular Calabi flow, the key to obtaining critical a priori bounds for solutions to twisted Calabi flow lies in the boundedness of certain energy functionals, namely twisted versions of Mabuchi, Liouville and Calabi energy. Unfortunately, the arguments in [38, 5] cannot be adopted directly, as the twisted Liouville and twisted Calabi energy as defined below are not manifestly decreasing under twisted Calabi flow. However, their time-derivatives are decreasing in leading order and the lower order pieces can be shown to have uniformly bounded time integral by using a lower bound on twisted Mabuchi energy. The twist $\alpha(t)$ is assumed to satisfy the assumptions of Theorem 2.2.4 throughout.

2.3.1. Twisted Mabuchi energy

The definition of twisted Mabuchi energy requires a choice of a reference Kähler metric g_0. While not necessary at this point, we choose this to be the same α_∞-twisted cscK metric used to define $\mathcal{H}$.

Definition 2.3.1. *Twisted Mabuchi energy* is the functional on $\mathcal{H} \times \Omega^2(X, \mathbb{R})$ given by

$$Ma(\varphi, \alpha) := \int_X u\,\omega_\varphi - \frac{1}{2}\hat{S}\varphi\Delta_0\varphi\,\omega_0 + (\mathrm{Sc}_0 + \Lambda_0\alpha - \hat{S})\varphi\,\omega_0\,,$$

where u is understood to depend on φ via $u = \log(1+\Delta_0\varphi)$. In this definition, $\hat{S} = \underline{S} + \underline{\alpha}$ and $\underline{\alpha}$ depends on the cohomology class of α.

Proposition 2.3.2. *Twisted Mabuchi energy is uniformly bounded along twisted Calabi flow.*

Proof. For $\alpha = 0$, twisted Mabuchi energy reduces to regular Mabuchi energy which is known to be bounded below on Riemann surfaces [5]. Denote by Δ_0^{-1} the Green's operator seen as a homeomorphism of $C_0^\infty(X, g_0) = \{f \in C^\infty(X) \mid \int_X f\,\omega_0 = 0\}$. Since $\underline{\alpha}$ is nonpositive and $\Lambda_0\alpha - \underline{\alpha}$ has zero integral we can estimate

$$\begin{aligned}
Ma(\varphi, \alpha) &= \int_X u\,\omega_\varphi - \frac{1}{2}(\underline{S} + \underline{\alpha})\varphi\Delta_0\varphi\,\omega_0 + (\mathrm{Sc}_0 - \underline{S} + \Lambda_0\alpha - \underline{\alpha})\varphi\,\omega_0 \\
&= Ma(\varphi, 0) + \frac{|\underline{\alpha}|}{4}\int_X |d\varphi|^2_{g_0}\,\omega_0 + \int_X (\Lambda_0\alpha - \underline{\alpha})\varphi\,\omega_0 \\
&\geqslant -C - \left|\int_X (\Lambda_0\alpha - \underline{\alpha})\varphi\,\omega_0\right| \\
&= -C - \left|\int_X (1 + \Delta_0\varphi)\cdot\Delta_0^{-1}(\Lambda_0\alpha - \underline{\alpha})\,\omega_0\right| \\
&\geqslant -C - \sup_X |\Delta_0^{-1}(\Lambda_0\alpha - \underline{\alpha})| \cdot \int_X \omega_\varphi \\
&\geqslant -\left(C + \sup_{X, t\geqslant 0} |\Delta_0^{-1}(\Lambda_0\alpha - \underline{\alpha})|\right).
\end{aligned}$$

By the assumptions convergence assumptions on $\alpha(t)$, the second term is bounded, so twisted Mabuchi energy has a lower bound depending only on the path $\alpha(t)$.

It remains to find an upper bound. To this end, we individually consider the variations of Ma in φ and in α. One computes

$$\begin{aligned}
(\delta_\varphi Ma)_{(\varphi,\alpha)}\dot\varphi &= \int_X \Delta_\varphi\dot\varphi\,\omega_\varphi + u\Delta_\varphi\dot\varphi\,\omega_\varphi - \hat S\dot\varphi\Delta_0\varphi\,\omega_0 + \dot\varphi(\mathrm{Sc}_0 + \Lambda_0\alpha - \hat S)\,\omega_0 \\
&= \int_X \dot\varphi\Delta_0 u - \hat S\dot\varphi(1+\Delta_0\varphi) + \dot\varphi(\mathrm{Sc}_0 + \Lambda_0\alpha)\,\omega_0 \\
&= \int_X \dot\varphi[\Delta_0 u + \mathrm{Sc}_0 + \Lambda_0\alpha - e^u\hat S]\,\omega_0 \\
&= \int_X \dot\varphi[\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat S]\,\omega_\varphi\,,
\end{aligned}$$

which in the direction of twisted Calabi flow becomes

$$(\delta_\varphi Ma)_{(\varphi,\alpha)}(-(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat S)) = -\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat S)^2\,\omega_\varphi \leqslant 0\,.$$

We remark that the expression $Ca(\varphi,\alpha) := \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat S)^2\,\omega_\varphi$ is the twisted Calabi energy which will be examined later.

For the variation of Ma in direction of α one finds

$$(\delta_\alpha Ma)_{(\varphi,\alpha)}\dot\alpha = \int_X \varphi\Lambda_0\dot\alpha\,\omega_0\,,$$

so the total time-derivative of twisted Mabuchi energy is

$$\partial_t Ma(\varphi,\alpha) = -Ca(\varphi,\alpha) + \int_X \varphi\Lambda_0\dot\alpha\,\omega_0$$

and can be estimated by

$$\begin{aligned}
\partial_t Ma(\varphi,\alpha) &\leqslant \int_X \varphi\Lambda_0\dot\alpha\,\omega_0 \\
&\leqslant \int_X \Delta_0\varphi\,\Delta_0^{-1}\Lambda_0\dot\alpha\,\omega_0 \\
&= \int_X (1+\Delta_0\varphi)\,\Delta_0^{-1}\Lambda_0\dot\alpha\,\omega_0 \\
&\leqslant \sup_X |\Delta_0^{-1}\Lambda_0\dot\alpha| \cdot \int_X e^u\omega_0 \\
&= \sup_X |\Delta_0^{-1}\Lambda_0\dot\alpha|\,.
\end{aligned}$$

By the decay properties of α, the integral $\int_0^\infty \sup_X |\Delta_0^{-1}\Lambda_0\dot\alpha|\,dt$ (depending only on $\alpha(t)$ and the background metric) is finite and one gets

$$Ma(\varphi(t),\alpha(t)) \leqslant Ma(\varphi(0),\alpha(0)) + \int_0^\infty \sup_X |\Delta_0^{-1}\Lambda_0\dot\alpha|\,dt$$

which bounds the twisted Mabuchi energy from above along twisted Calabi flow. □

Remark. The upper bound on twisted Mabuchi energy is technically not required for Theorem 2.2.4, but the details of the proof are used to bound twisted Liouville and twisted Calabi energy.

2.3.2. Twisted Liouville energy

Like twisted Mabuchi energy, twisted Liouville energy depends on the choice of a background metric g_0. In this case the choice of g_0 being the α_∞-twisted cscK metric does matter.

Definition 2.3.3. *Twisted Liouville energy* is the functional on $\mathcal{H} \times \Omega^2(X, \mathbb{R})$ given by

$$F(\varphi, \alpha) := \int_X u[\Delta_0 u + 2\Lambda_0\alpha + 2\,\mathrm{Sc}_0]\,\omega_0 = \frac{1}{2}\int_X |du|_0^2\,\omega_0 + 2\int_X u(\Lambda_0\alpha + \mathrm{Sc}_0)\,\omega_0\,.$$

Proposition 2.3.4. *Twisted Liouville energy is uniformly bounded along twisted Calabi flow.*

Proof. To establish a lower bound we want to use the Poincaré inequality to estimate the second integral appearing in the above right hand side expression for the twisted Liouville energy in terms of $\|du\|_{L^2}$ and then complete the square with the first integral. Set $\widetilde{u} := \int_X u\,\omega_0$. Since we normalised to unit volume, Jensen's inequality implies that $\widetilde{u} = \int_X u\,\omega_0 \leqslant \log\int_X e^u\,\omega_0 = 0$. With the choice of background metric and the zero-integral property of $\Lambda_0(\alpha - \alpha_\infty)$ in mind we can estimate

$$\begin{aligned}
F(\varphi, \alpha) &= \frac{1}{2}\int_X |du|_0^2\,\omega_0 + 2\int_X u(\Lambda_0\alpha + \mathrm{Sc}_0)\,\omega_0 \\
&= \frac{1}{2}\int_X |du|_0^2\,\omega_0 + \underbrace{2\hat{S}\cdot\widetilde{u}}_{\geqslant 0} + 2\int_X u\Lambda_0(\alpha - \alpha_\infty)\,\omega_0 \\
&\geqslant \frac{1}{2}\int_X |du|_0^2\,\omega_0 + 2\int_X (u - \widetilde{u})\Lambda_0(\alpha - \alpha_\infty)\,\omega_0 \\
&\geqslant \frac{1}{2}\int_X |du|_0^2\,\omega_0 - 2\sup_X |\Lambda_0(\alpha - \alpha_\infty)| \left(\int_X (u - \widetilde{u})^2\,\omega_0\right)^{\frac{1}{2}} \\
&\geqslant \frac{1}{2}\int_X |du|_0^2\,\omega_0 - 2\sup_X |\Lambda_0(\alpha - \alpha_\infty)|\lambda^{-\frac{1}{2}} \left(\int_X |du|_0^2\,\omega_0\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left[\left(\int_X |du|_0^2\,\omega_0\right)^{\frac{1}{2}} - 2\lambda^{-\frac{1}{2}}\sup_X |\Lambda_0(\alpha - \alpha_\infty)|\right]^2 - 2\lambda^{-1}\sup_X |\Lambda_0(\alpha - \alpha_\infty)|^2 \\
&\geqslant -2\lambda^{-1}\sup_{X, t\geqslant 0} |\Lambda_0(\alpha - \alpha_\infty)|^2\,,
\end{aligned}$$

where λ is the first nonzero eigenvalue of the full g_0-Laplacian. Since $\alpha(t)$ is bounded in t, $F(\varphi, \alpha)$ can be bounded from below by a constant depending on the path $\alpha(t)$.

An upper bound on twisted Liouville energy can be obtained by estimating its time-derivative. We again compute the variation in φ and in α separately. It is

$$\begin{aligned}(\delta_\varphi F)_{(\varphi,\alpha)}\dot\varphi &= 2\int_X \Delta_\varphi\dot\varphi[\Delta_0 u + \mathrm{Sc}_0 + \Lambda_0\alpha]\,\omega_0\\ &= 2\int_X \Delta_\varphi\dot\varphi[e^u(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)]\,\omega_0\\ &= \int_X g_\varphi(d\dot\varphi, d(\mathrm{Sc}_\varphi + \Lambda_\varphi\alpha))\,\omega_\varphi\,,\end{aligned}$$

which under Calabi flow becomes

$$(\delta_\varphi F)_{(\varphi,\alpha)}(-(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat S)) = -\int_X |d(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)|^2_\varphi\,\omega_\varphi \leqslant 0\,.$$

The variation in α-direction is

$$(\delta_\alpha F)_{(\varphi,\alpha)}\dot\alpha = 2\int_X u\Lambda_0\dot\alpha\,\omega_0\,,$$

so for the total time-derivative of twisted Liouville energy one has

$$\partial_t F(\varphi,\alpha) = -\int_X |d(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)|^2_\varphi\,\omega_\varphi + 2\int_X u\Lambda_0\dot\alpha\omega_0\,,$$

which can be estimated by

$$\begin{aligned}\frac{1}{2}\partial_t F(\varphi,\alpha) &\leqslant \int_X u\Lambda_0\dot\alpha\,\omega_0\\ &= \int_X \Delta_0 u\,\underbrace{\Delta_0^{-1}\Lambda_0\dot\alpha}_{=:\eta}\,\omega_0\\ &= \int_X \mathrm{Sc}(\varphi)\eta\,\omega_\varphi - \int_X \mathrm{Sc}_0\,\eta\,\omega_0\\ &= \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat S)\eta\,\omega_\varphi - \int_X \Lambda_\varphi\alpha\,\eta\,\omega_\varphi + \hat S\int_X \eta\,\omega_\varphi - \int_X \mathrm{Sc}_0\,\eta\,\omega_0\,.\end{aligned}$$

In light of the volume constraint $\int_X \omega_0 = \int_X \omega_\varphi = 1$ the last three terms satisfy

$$-\int_X \Lambda_\varphi\alpha\,\eta\,\omega_\varphi + \hat S\int_X \eta\,\omega_\varphi - \int_X \mathrm{Sc}_0\,\eta\,\omega_0 \leqslant \sup_X|\eta|\cdot\underbrace{\left[\sup_{X,t\geqslant 0}|\Lambda_0\alpha| + |\hat S| + \int_X|\mathrm{Sc}_0|\,\omega_0\right]}_{K},$$

whereas using Cauchy–Schwarz on the first term gives

$$\begin{aligned}\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat S)\eta\,\omega_\varphi &\leqslant \sup_X|\eta|\,(Ca(\varphi,\alpha))^{\frac{1}{2}}\\ &= \sup_X|\eta|\left(-\partial_t Ma(\varphi,\alpha) + \int_X \varphi\Lambda_0\dot\alpha\,\omega_0\right)^{\frac{1}{2}}\,.\end{aligned}$$

In the last step we have used the expression for twisted Calabi energy found in the proof of boundedness of twisted Mabuchi energy (Proposition 2.3.2). Integrating the expression for $\partial_t F$ from 0 to $\tau \geqslant 0$, these estimates give

$$
\begin{aligned}
\frac{1}{2}F(\varphi(\tau),\alpha(\tau)) &= \frac{1}{2}F(\varphi(0),\alpha(0)) + \frac{1}{2}\int_0^\tau \partial_t F(\varphi,\alpha)\,dt \\
&\leqslant C + K\int_0^\tau \sup_X |\eta|\,dt + \int_0^\tau \sup_X |\eta| \left(-\partial_t Ma + \int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\right)^{\frac{1}{2}} dt \\
&\leqslant C + K\int_0^\infty \sup_X |\eta|\,dt + \left(\int_0^\tau \sup_X |\eta|^2\,dt\right)^{\frac{1}{2}} \left(\int_0^\tau -\partial_t Ma + \int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\,dt\right)^{\frac{1}{2}} \\
&\leqslant C' + \left(\int_0^\infty \sup_X |\eta|^2\,dt\right)^{\frac{1}{2}} \left(-Ma(\tau) + Ma(0) + \int_0^\tau\int_X (1+\Delta_0\varphi)\eta\,\omega_0\,dt\right)^{\frac{1}{2}} \\
&\leqslant C' + C''\left(-Ma(\tau) + Ma(0) + \int_0^\tau\int_X \eta\,\omega_\varphi\,dt\right)^{\frac{1}{2}} \\
&\leqslant C' + C''\left(-Ma(\tau) + C'''\right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used Cauchy–Scharz and the convergence assumptions on α. Consequently, the lower bound on twisted Mabuchi energy gives the desired upper bound for twisted Liouville energy.

□

The boundedness of the twisted Liouville energy along twisted Calabi flow has an important implication:

Corollary 2.3.5. *Along twisted Calabi flow, $\|u\|_{L^2_1}$ is uniformly bounded in t.*

Proof. From the computation used to establish a lower bound on twisted Liouville energy in the proof of Proposition 2.3.4 we recall

$$
F(\varphi,\alpha) \geqslant \frac{1}{2}\left[\left(\left(\int_X |du|_0^2\,\omega_0\right)^{\frac{1}{2}} - 2\lambda^{-\frac{1}{2}}\sup_X |\Lambda_0(\alpha-\alpha_\infty)|\right]^2 - 2\lambda^{-1}\sup_X |\Lambda_0(\alpha-\alpha_\infty)|^2 .
$$

Uniform boundedness of $\sup_X |\Lambda_0(\alpha-\alpha_\infty)|$ and the upper bound on F imply the uniform boundedness of $\|du\|_{L^2}^2 = \int_X |du|_0^2\,\omega_0$. Via the Poincaré inequality, this controls $\|u-\widetilde{u}\|_{L^2}$ and hence $\|u-\widetilde{u}\|_{L^2_1}$, so in order to obtain a genuine bound on $\|u\|_{L^2_1}$ it remains to find an estimate on $\widetilde{u}$. By Jensen's inequality we already established $\widetilde{u} \leqslant 0$. To find a lower bound we use the volume constraint and the Moser–Trudinger inequality (Proposition 2.2.1). It is

$$
e^{-\widetilde{u}} = e^{-\widetilde{u}}\int_X e^u\,\omega_0 = \int_X e^{u-\widetilde{u}}\,\omega_0 \leqslant Ce^{\mu\|u-\widetilde{u}\|_{L^2_1}^2},
$$

which implies $\widetilde{u} \geqslant -\log C - \mu\|u-\widetilde{u}\|_{L^2_1}^2$ as desired. □

Unfortunately, control over $\|u\|_{L^2_1}$ is not quite sufficient to imply uniform boundedness of u in L^∞. However, the Moser–Trudinger inequality allows us to uniformly bound $\int_X e^{su}\omega_0$ along twisted Calabi flow for any $s \in \mathbb{R}$ by a constant $C(s)$. Indeed, one has

$$\int_X e^{su}\omega_0 \leqslant Ce^{\mu s^2 \|u\|^2_{L^2_1}}\,. \tag{2.11}$$

This will be useful in establishing uniform bounds on twisted Calabi energy.

2.3.3. Twisted Calabi energy

Definition 2.3.6. *Twisted Calabi energy* is the functional on $\mathcal{H} \times \Omega^2(X, \mathbb{R})$ given by

$$Ca(\varphi, \alpha) := \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S})^2\,\omega_\varphi$$

Proposition 2.3.7. *Twisted Calabi energy is uniformly bounded along twisted Calabi flow.*

Proof. Twisted Calabi energy is manifestly nonnegative, so it suffices to find an upper bound. This is again done by estimating the time-derivative of twisted Calabi energy and then integrating. Using (2.8), the variation in direction of φ can be expressed as

$$\begin{aligned}
(\delta_\varphi Ca)_{(\varphi,\alpha)}\dot{\varphi} &= 2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S})\mathcal{D}^*_\varphi \mathcal{D}_\varphi \dot{\varphi}\,\omega_\varphi \\
&- \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S}) g_\varphi(d\,\mathrm{Sc}(\varphi), d\dot{\varphi})\,\omega_\varphi \\
&- 2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S})\Lambda_\varphi \alpha \Delta_\varphi \dot{\varphi}\,\omega_\varphi \\
&+ \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S})^2 \Delta_\varphi \dot{\varphi}\,\omega_\varphi\,.
\end{aligned}$$

Integrating the third term by parts gives

$$\begin{aligned}
&-2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S})\Lambda_\varphi \alpha \Delta_\varphi \dot{\varphi}\,\omega_\varphi \\
&= -\int_X g_\varphi(d(\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha), d\dot{\varphi})\,\alpha - \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S}) g_\varphi(d\Lambda_\varphi \alpha, d\dot{\varphi})\,\omega_\varphi\,.
\end{aligned}$$

and we observe that the second term on the right hand side combines with the second term of the above expression for $(\delta_\varphi Ca)_{(\varphi,\alpha)}\dot{\varphi}$ to cancel out the fourth term. What remains is

$$\begin{aligned}
(\delta_\varphi Ca)_{(\varphi,\alpha)}(\dot{\varphi}) &= 2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S})\mathcal{D}^*_\varphi \mathcal{D}_\varphi \dot{\varphi}\,\omega_\varphi \\
&- \int_X g_\varphi(d(\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha), d\dot{\varphi})\,\alpha\,.
\end{aligned}$$

Under twisted Calabi flow this becomes

$$(\delta_\varphi Ca)_{(\varphi,\alpha)}(-(\mathrm{Sc}(\varphi)+\Lambda_\varphi\alpha-\hat{S})) = -2\|\mathcal{D}_\varphi(\mathrm{Sc}(\varphi)+\Lambda_\varphi\alpha)\|^2_{L^2(X,g_\varphi)}+\int_X |d(\mathrm{Sc}(\varphi)+\Lambda_\varphi\alpha)|^2_{g_\varphi}\alpha\,.$$

Due to the nonpositivity of α, both terms are nonpositive.

Remark. This is the only instance where pointwise nonpositivity of α is used. The fact that the elliptic problem of finding twisted cscK metrics only requires α be integrally nonpositive (cf. Theorem B.1.1 in Appendix B.1) suggest one should also be able to relax the pointwise condition on α to an integral one in the parabolic case.

The variation in the α-direction is

$$(\delta_\alpha Ca)_{\varphi,\alpha}\dot{\alpha} = 2\int_X (\mathrm{Sc}(\varphi)+\Lambda_\varphi\alpha-\hat{S})\Lambda_\varphi\dot{\alpha}\,\omega_\varphi$$

and can be estimated by

$$\begin{aligned}
\int_X (\mathrm{Sc}(\varphi)+\Lambda_\varphi\alpha-\hat{S})\Lambda_\varphi\dot{\alpha}\,\omega_\varphi &\leqslant Ca^{\frac{1}{2}}\cdot\left(\int_X (\Lambda_\varphi\dot{\alpha})^2\,\omega_\varphi\right)^{\frac{1}{2}} \\
&\leqslant \left(-\partial_t Ma+\int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\right)^{\frac{1}{2}}\cdot\sup_X|\Lambda_0\dot{\alpha}|\cdot\left(\int_X e^{-u}\,\omega_0\right)^{\frac{1}{2}} \\
&\leqslant C\sup_X|\Lambda_0\dot{\alpha}|\left(-\partial_t Ma+\int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\right)^{\frac{1}{2}}\,,
\end{aligned}$$

where in the last step we used that $\int_X e^{-u}\,\omega_0$ is uniformly bounded along Calabi flow by (2.11). We can thus estimate the total time-derivative of twisted Calabi energy by

$$\partial_t Ca \leqslant C\sup_X|\Lambda_0\dot{\alpha}|\left(-\partial_t Ma+\int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\right)^{\frac{1}{2}}$$

and integration from 0 to $\tau\geqslant 0$ gives

$$\begin{aligned}
Ca(\varphi(\tau),\alpha(\tau)) &= Ca(\varphi(0),\alpha(0))+\int_0^\tau \partial_t Ca\,dt \\
&\leqslant C'+C\int_0^\tau \sup_X|\Lambda_0\dot{\alpha}|\left(-\partial_t Ma+\int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\right)^{\frac{1}{2}}\,dt \\
&\leqslant C'+C\left(\int_0^\tau \sup_X|\Lambda_0\dot{\alpha}|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^\tau -\partial_t Ma+\int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\,dt\right)^{\frac{1}{2}}\,.
\end{aligned}$$

By the convergence properties of α, the integral $\int_0^\infty \sup_X|\Lambda_0\dot{\alpha}|^2\,dt$ is finite and the same is true for $\int_0^\infty\int_X \varphi\Lambda_0\dot{\alpha}\,\omega_0\,dt$ as was shown in the proof of the boundedness of twisted Mabuchi energy (Proposition 2.3.2). Hence

$$Ca(\tau)\leqslant C'+C''\left(-Ma(\tau)+Ma(0)+C'''\right)^{\frac{1}{2}}$$

and twisted Calabi energy is uniformly bounded along twisted Calabi flow. □

Before the boundedness of twisted Calabi energy can be used to control $\|u\|_{L^2_2}$ an intermediate step is required.

Corollary 2.3.8. *The logarithm u of the conformal factor is uniformly bounded in C^0 along twisted Calabi flow.*

Proof. The idea is to use twisted Calabi energy to obtain a bound on $\|u\|_{L^{3/2}_2}$. The Sobolev embedding $L^{3/2}_2 \hookrightarrow C^0$ then proves the claim. Using Hölder's inequality with $p = 4$ and $q = 4/3$ and the Moser–Trudinger inequality in the form (2.11) with $s = 3$ we estimate

$$\begin{aligned}
\int_X |\Delta_0 u|^{\frac{3}{2}}\,\omega_0 &= \int_X e^{\frac{1}{2}u}|\Delta_\varphi u|^{\frac{3}{2}}\,\omega_\varphi \\
&\leqslant \left(\int_X e^{2u}\,\omega_\varphi\right)^{\frac{1}{4}}\left(\int_X (\Delta_\varphi u)^2\,\omega_\varphi\right)^{\frac{3}{4}} \\
&= \left(\int_X e^{3u}\,\omega_0\right)^{\frac{1}{4}}\left(\int_X (\mathrm{Sc}(\varphi) - e^{-u}\,\mathrm{Sc}_0)^2\,\omega_\varphi\right)^{\frac{3}{4}} \\
&\leqslant C\left(\int_X 2(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})^2 + 2(\hat{S} - \Lambda_\varphi\alpha - e^{-u}\,\mathrm{Sc}_0)^2\,\omega_\varphi\right)^{\frac{3}{4}} \\
&\leqslant C\left(2Ca + C'\right)^{\frac{3}{4}}\ .
\end{aligned}$$

The desired uniform bound on $\|u\|_{L^{3/2}_2}$ then follows from elliptic estimates. □

Corollary 2.3.9. *The logarithm u of the conformal factor is uniformly bounded in L^2_2 along twisted Calabi flow.*

Proof. The proof is very similar to that of Corollary 2.3.8. With the uniform C^0-boundedness of u in mind one estimates

$$\begin{aligned}
\int_X (\Delta_0 u)^2\,\omega_0 &= \int_X e^u(\Delta_\varphi u)^2\,\omega_\varphi \\
&\leqslant \sup_X e^u \cdot \int_X (\Delta_\varphi u)^2\,\omega_\varphi \\
&\leqslant C\int_X (\mathrm{Sc}(\varphi) - e^{-u}\,\mathrm{Sc}_0)^2\,\omega_\varphi \\
&\leqslant C\int_X 2(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})^2 + 2(\hat{S} - \Lambda_\varphi\alpha - e^{-u}\,\mathrm{Sc}_0)^2\,\omega_\varphi \\
&\leqslant C(2Ca + C')\,.
\end{aligned}$$

Again, the desired bound on $\|u\|_{L^2_2}$ follows from elliptic estimates. □

2.4. Higher a priori Bounds

The presentation largely follows [8], though some adjustments are necessary to deal with additional terms appearing in our case. For integers $k_1, ..., k_s$ and a tensor S we use the notation

$$S = \text{``}\nabla^{k_1} u \bowtie \cdots \bowtie \nabla^{k_s} u\text{''}$$

to indicate that S is an algebraic expression (possibly involving the metric g_0) multilinearly and nontrivially depending on the ${k_i}^{\text{th}}$ covariant derivatives of u. We also allow the expressions to contain a factor of a bounded smooth function on $X \times [0, \infty[$, such as $\Lambda_0 \alpha$ or derivatives thereof. One has

$$\text{``}\nabla^{k_1} u \bowtie \cdots \bowtie \nabla^{k_s} u\text{''} \leqslant C|\nabla^{k_1} u| \cdots |\nabla^{k_s} u| \,,$$

where $|\cdot|$ is the pointwise norm on tensors induced by g_0. We also recall that the choice of background metric g_0 was such that $\int_X \omega_0 = 1$ and $\mathrm{Sc}_0 + \Lambda_0 \alpha_\infty = \hat{S}$. All covariant derivatives are taken with respect to g_0. We start with a few preparatory lemmas.

Lemma 2.4.1. *Let $l \in \mathbb{N}_0$. The time-derivative of $\nabla^l u$ can be written as*

$$\partial_t \nabla^l u = e^{-2u}[-\Delta_0^2 \nabla^l u - \nabla^l \Delta_0 \Lambda_0(\alpha - \alpha_\infty) + R_l^1 + R_l^2 + R_l^3 + R_l^4] \,,$$

where R_l^i are expressions of the form

$$\begin{aligned} R_l^1 &= \text{``}\nabla u \bowtie \nabla^{l+3} u\text{''} \\ R_l^2 &= \text{``}(\nabla u + (\nabla u)^2 + \nabla^2 u) \bowtie \nabla^{l+2} u\text{''} \\ R_l^3 &= \sum_{\substack{s \geqslant 2 \\ 1 \leqslant i_j \leqslant 1+l \\ i_1 + \cdots + i_s + \leqslant l+4}} \text{``}\nabla^{i_1} u \bowtie \cdots \bowtie \nabla^{i_s} u\text{''} \\ R_l^4 &= \sum_{1 \leqslant i \leqslant l+2} \text{``}\nabla^i u\text{''} \,. \end{aligned}$$

Remark. The ∂-Laplacian appearing in (2.10) is only half the Hodge-Laplacian acting on functions, which normally would lead to confusing prefactors in the analysis. In favour of a cleaner presentation, we redefine ∇ to be $1/\sqrt{2}$ times the metric covariant derivative and denote by Δ_0 acting on tensors $1/2$ times the rough Laplacian for the remainder of the higher regularity analysis.

Proof. This is shown by induction on l. Using $\Delta_0(f_1 f_2) = (\Delta_0 f_1) f_2 + f_1(\Delta_0 f_2) - g(df_1, df_2)$, $\Delta_0 e^f = e^f(\Delta_0 f - \frac{1}{2}|df|_0^2)$ as well as $\mathrm{Sc}_0 + \Lambda_0 \alpha = \hat{S} + \Lambda_0(\alpha - \alpha_\infty)$ one computes for $l = 0$:

$$\begin{aligned} \partial_t u &= -e^{-u} \Delta_0(e^{-u}(\Delta_0 u + \mathrm{Sc}_0 + \Lambda_0 \alpha)) \\ &= e^{-2u}[-\Delta_0^2 u - \Delta_0(\mathrm{Sc}_0 + \Lambda_0 \alpha) - g_0(du, d\Delta_0 u) \\ &+ (\Delta_0 u)^2 + \frac{1}{2}|du|_0^2 \Delta_0 u + (\mathrm{Sc}_0 + \Lambda_0 \alpha) \Delta_0 u - g_0(du, d(\mathrm{Sc}_0 + \Lambda_0 \alpha))] \\ &= e^{-2u}[-\Delta_0^2 u - \Delta_0 \Lambda_0(\alpha - \alpha_\infty) + R_0^1 + R_0^2 + R_0^3 + R_0^4] \,. \end{aligned}$$

Now suppose the claim is true for l. Then

$$
\begin{aligned}
\partial_t \nabla^{l+1} u &= \nabla(\partial_t \nabla^l u) \\
&= \nabla(e^{-2u}[-\Delta_0^2 \nabla^l u - \nabla^l \Delta_0 \Lambda_0(\alpha - \alpha_\infty) + R_l^1 + R_l^2 + R_l^3 + R_l^4]) \\
&= e^{-2u}(-2\nabla u)[-\Delta_0^2 \nabla^l u - \nabla^l \Delta_0 \Lambda_0(\alpha - \alpha_\infty) + R_l^1 + R_l^2 + R_l^3 + R_l^4] \quad (2.12) \\
&+ e^{-2u}[-\nabla \Delta_0^2 \nabla^l u - \nabla^{l+1} \Delta_0 \Lambda_0(\alpha - \alpha_\infty)] \quad (2.13) \\
&+ e^{-2u}[\nabla(R_l^1 + R_l^2 + R_l^3 + R_l^4)] \quad (2.14)
\end{aligned}
$$

The first term in (2.12) is $(\nabla u) \cdot \Delta_0^2 \nabla^l u = $ “$\nabla u \bowtie \nabla^{4+l} u$” and constitutes a part of R_{l+1}^1. The second term is $(\nabla u) \cdot \nabla^l \Delta_0 \Lambda_0(\alpha - \alpha_\infty) = $ “∇u” and subsumed in R_{l+1}^4. In a similar fashion $(\nabla u) R_l^1 = $ “$(\nabla u)^2 \bowtie \nabla^{l+3} u$” contribues to R_{l+1}^2, $(\nabla u) R_l^2$ to R_{l+1}^3, $(\nabla u) R_l^3$ to R_{l+1}^3 and $(\nabla u) R_l^4$ to R_{l+1}^3 and R_{l+1}^2.

Commuting Δ^2 and ∇ (see e.g. Appendix A.1) one obtains $-\nabla \Delta_0^2 \nabla^l u = -\Delta_0^2 \nabla^{l+1} u + \sum_{j=1}^{l+3}$ “$\nabla^j u$” (the curvature terms are all bounded), so (2.13) accounts for the two special terms and a contribution to R_{l+1}^4.

It remains to examine the contributions from (2.14). The first is

$$\nabla R_l^1 = \text{“}\nabla u \bowtie \nabla^{l+3} u\text{”} + \text{“}\nabla^2 u \bowtie \nabla^{l+3} u\text{”} + \text{“}\nabla u \bowtie \nabla^{l+4} u\text{”}\,,$$

where the first term can arise since the notation “S” allowed the appearance of bounded smooth functions. The first two terms contribute to R_{l+1}^2, and the third to R_{l+1}^1. The second contribution is

$$
\begin{aligned}
\nabla R_l^2 &= \text{“}(\nabla u + (\nabla u)^2 + \nabla^2 u) \bowtie \nabla^{l+2} u\text{”} \\
&+ \text{“}(\nabla^2 u + (\nabla u)\nabla^2 u + \nabla^3 u) \bowtie \nabla^{l+2} u\text{”} \\
&+ \text{“}(\nabla u + (\nabla u)^2 + \nabla^2 u) \bowtie \nabla^{l+3} u\text{”}\,.
\end{aligned}
$$

Here, the first term adds to R_{l+1}^3 and the third to R_{l+1}^2, whereas the second term contributes to R_{l+1}^3 and in the case of $l = 0$ also to R_{l+1}^2. Next one has

$$\nabla R_l^3 = \sum_{\substack{s \geqslant 2 \\ 1 \leqslant i_j \leqslant 2+l \\ i_1 + \cdots + i_s \leqslant l+5}} \text{“}\nabla^{i_1} u \bowtie \cdots \bowtie \nabla^{i_s} u\text{”}$$

which by definition is subsumed under R_{l+1}^3. Lastly,

$$\nabla R_l^4 = \sum_{1 \leqslant i \leqslant l+3} \text{“}\nabla^i u\text{”}$$

which contributes R_{l+1}^4. □

The purpose of the previous lemma is to more precisely describe the lower order terms of the time-derivative of the functionals

$$E_l := \int_X e^{2u} |\nabla^l u|^2 \, \omega_0 \tag{2.15}$$

indexed by $l \in \mathbb{N}$.

Lemma 2.4.2. *For the time-derivative of E_l under twisted Calabi flow (2.10) the following estimate holds:*

$$\partial_t E_l \leqslant -2\|\nabla^{l+2}u\|_{L^2}^2 + C(\Phi_l^1 + \Phi_l^2 + \Phi_l^3) + Ce^{-\delta t}\|u - \widetilde{u}\|_{L_2^2},$$

where

$$\begin{aligned}
\Phi_l^1 &= \int_X |\nabla u||\nabla^{l+1}u||\nabla^{l+2}u|\,\omega_0\,,\\
\Phi_l^2 &= \sum_{\substack{s\geqslant 3\\ 1\leqslant i_j\leqslant l+1\\ i_1+\cdots+i_s\leqslant 2l+4}} \int_X |\nabla^{i_1}u|\cdots|\nabla^{i_s}u|\,\omega_0\,,\\
\Phi_l^3 &= \sum_{\substack{1\leqslant i_1,i_2\leqslant l+2\\ i_1+i_2\leqslant 2l+2}} \int_X |\nabla^{i_1}u||\nabla^{i_2}u|\,\omega_0
\end{aligned}$$

and δ is a positive constant. Recall that $\widetilde{u}$ was defined to be the average of u with respect to the background metric.

Proof. It follows from Lemma 2.4.1 that

$$\begin{aligned}
\partial_t E_l &= \int_X \partial_t(e^{2u})|\nabla^l u|^2\,\omega_0 + 2\int_X e^{2u}(\partial_t\nabla^l u, \nabla^l u)\,\omega_0\\
&= \underbrace{2\int_X e^{2u}(\partial_t u)|\nabla^l u|^2\,\omega_0}_{I_l} - \underbrace{2\int_X (\Delta_0\nabla^l u, \Delta_0\nabla^l u)\,\omega_0}_{II_l^0} + \sum_{r=1}^4 \underbrace{2\int_X (R_l^r, \nabla^l u)\,\omega_0}_{II_l^r}\\
&- \underbrace{2\int_X (\nabla^l\Delta_0\Lambda_0(\alpha-\alpha_\infty), \nabla^l u)}_{III_l}
\end{aligned}$$

We estimate each term individually.

The first term

$$I_l = -2\int_X \Delta_0^2 u\cdot|\nabla^l u|^2\,\omega_0 - 2\int_X \Delta_0\Lambda_0(\alpha-\alpha_\infty)\cdot|\nabla^l u|^2\,\omega_0 + 2\int_X (R_0^1+R_0^2+R_0^3+R_0^4)|\nabla^l u|^2\,\omega_0$$

needs to be examined separately for $l = 1, l = 2$ and the general case $l \geqslant 3$ to account for the leading order contributions potentially not stemming from the $|\nabla^l u|^2$ factor. For $l = 1$, one can integrate $\Delta_0^2 u \cdot |\nabla u|^2$ by parts once to obtain an integrand which can be estimated by $|\nabla^3 u||\nabla^2 u||\nabla u|$, so the integral is of type Φ_1^1. The integrand in the second summand can be estimated by $|\nabla u|^2$, so the integral is of type Φ_1^3. After integrating $R_0^1|\nabla u|^2$ by parts once to get rid of the third derivative, all contributions of the third integral in I_1 can be estimated in terms of Φ_1^2. If $l = 2$, after integrating the fourth derivative in Δ^4 by parts, the first and the third term in I_2 can be estimated by Φ_2^2 and the second by Φ_2^3. For $l \geqslant 3$ the first and third summand can directly be estimated by Φ_l^2 and the second again by Φ_l^2.

The term II_l^0 contains the highest order contribution to $\partial_t E_l$ and — in light of (A.1) in Appendix A.1 — can be expressed as

$$II_l^0 = -2\int_X |\Delta_0 \nabla^l u|^2 \,\omega_0 = -2\int_X |\nabla^{l+2} u|^2 \,\omega_0 + \int_X \text{“}|\nabla^{l+1} u|^2\text{”}\,\omega_0 \,.$$

It follows that $II_l^0 \leqslant -2\|\nabla^{l+2} u\|_{L^2}^2 + C\Phi_l^3$.

The term II_l^1 is of the form

$$II_l^1 = \int_X \text{“}\nabla u \bowtie \nabla^{l+3} u \bowtie \nabla^l u\text{”}\,\omega_0$$

which after integration by parts on the highest order derivative becomes

$$\int_X \text{“}\nabla^2 u \bowtie \nabla^{l+2} u \bowtie \nabla^l u\text{”} + \text{“}\nabla u \bowtie \nabla^{l+2} u \bowtie \nabla^{l+1} u\text{”} + \text{“}\nabla u \bowtie \nabla^{l+2} u \bowtie \nabla^l u\text{”}\,\omega_0 \,.$$

The middle summand is estimated by Φ_l^1 and after another integration by parts to get rid of the $(l+2)^{\text{nd}}$ derivative, the third summand by Φ_l^2. The first summand is estimated by Φ_l^1 if $l = 1$ and after an integration by parts by Φ_l^2 for $l \geqslant 2$.

For II_l^2 one has

$$II_l^2 = \int \text{“}(\nabla u + (\nabla u)^2 + \nabla^2 u) \bowtie \nabla^{l+2} u \bowtie \nabla^l u\text{”}\,\omega_0 \,.$$

In the case of $l = 1$ this becomes

$$\int_X \text{“}(\nabla u)^2 \bowtie \nabla^3 u\text{”} + \text{“}(\nabla u)^3 \bowtie \nabla^3 u\text{”} + \text{“}\nabla^2 u \bowtie \nabla^3 u \bowtie \nabla u\text{”}\,\omega_0 \,.$$

The last summand is dominated by Φ_1^1 and after integrating by parts the third derivative, the first and second summand are less than $C\Phi_1^2$. For $l \geqslant 2$ one can estimate II_l^2 by Φ_l^2 after integrating by parts to get rid of the $(l+2)^{\text{nd}}$ derivative.

It follows from the definition of R_l^3 that

$$II_l^3 = \sum_{\substack{s\geqslant 2\\ 1\leqslant i_j\leqslant l+1\\ i_1+\cdots+i_s\leqslant l+4}} \int \text{“}\nabla^{i_1} u \bowtie \cdots \bowtie \nabla^{i_s} u \bowtie \nabla^l u\text{”}\,\omega_0 \,,$$

which can directly be estimated by Φ_l^2.

The term II_l^4 is of the form

$$II_l^4 = \sum_{1\leqslant i\leqslant l+2} \int_X \text{“}\nabla^i u \bowtie \nabla^l u\text{”}\,\omega_0 \,,$$

which is controlled by Φ_l^3.

Lastly, consider the remaining term

$$III_l = \int_X (\nabla^l \Delta_0 \Lambda_0(\alpha - \alpha_\infty), \nabla^l u)\, \omega_0 \,.$$

Denoting by ∇^* the $L^2(X, g_0)$-adjoint of ∇, define D_l to be the differential operator $D_l = \nabla^{*l-2}\nabla^l$ for $l \geqslant 2$ and $D_1 = \mathrm{id}$. Integrating by parts and applying Cauchy–Schwarz yields

$$|III_l| \leqslant \|D_l \Delta_0 \Lambda_0(\alpha - \alpha_\infty)\|_{L^2} \|\nabla^2 u\|_{L^2} \leqslant Ce^{-\delta t}\|u - \widetilde{u}\|_{L^2_2}\,,$$

where the exponentially decaying factor on the right hand side is owing to the assumed exponential convergence of α to α_∞ in C^k for any $k \in \mathbb{N}$. □

Remark. For our purposes it would have been sufficient to estimate the last term as $III_l \leqslant \|\nabla^{*l}\nabla^l \Delta_0 \Lambda_0(\alpha - \alpha_\infty)\|_{C^0}\|u\|_{L^1} \leqslant C\|u\|_{L^1}$, but the above estimates in terms of $\|u - \widetilde{u}\|_{L^2_2}$ fit in more naturally with the treatment of the other terms. Also, the exponential decay factor, while nice and a byproduct of the choice of background metric, is not strictly necessary and will be left out from here on.

Lemma 2.4.3. *For $l \in \mathbb{N}$, there exists a finite index set B and constants $\alpha_\beta, \gamma_\beta$ for each $\beta \in B$ satisfying $0 < \alpha_\beta < 2$ and $\gamma_\beta \geqslant 2 - \alpha_\beta$ such that*

$$\partial_t E_l \leqslant -2\|\nabla^{l+2}u\|^2_{L^2} + C\sum_{\beta \in B} \|\nabla^{l+2}u\|^{\alpha_\beta}_{L^2}\|u - \widetilde{u}\|^{\gamma_\beta}_{L^2_2} + C\|u - \widetilde{u}\|_{L^2_2}\,, \tag{2.16}$$

where $\widetilde{u} = \int_X u\, \omega_0$.

Proof. We only need to estimate the three terms Φ^r_s for $r = 1, 2, 3$ on the right hand side of the expression for $\partial_t E_l$ in Lemma 2.4. Starting with Φ^1_l one can estimate

$$\Phi^1_l = \int_X |\nabla u||\nabla^{l+1}u||\nabla^{l+2}u|\, \omega_0 \leqslant \|\nabla u\|_{L^4}\|\nabla^{l+1}u\|_{L^4}\|\nabla^{l+2}u\|_{L^2}\,.$$

For the first factor one has $\|\nabla u\|_{L^4} \leqslant C\|u - \widetilde{u}\|_{L^2_2}$ by the Sobolev embeddings. For the second factor one can use the Gagliardo–Nirenberg inequality to obtain $\|\nabla^{l+1}u\|_{L^4} \leqslant C\|\nabla^{l+2}u\|^{(l+2-1/2)/(l+2)}_{L^2}\|u - \widetilde{u}\|^{1-(l+2-1/2)/(l+2)}_{L^2}$. With the obvious inequality $\|u - \widetilde{u}\|_{L^2} \leqslant \|u - \widetilde{u}\|_{L^2_2}$ this combines to

$$\Phi^1_l \leqslant C\|\nabla^{l+2}u\|^{1+\frac{l+2-1/2}{l+2}}_{L^2}\|u - \widetilde{u}\|^{2-\frac{l+2-1/2}{l+2}}_{L^2_2}$$

which is of the required form.

Next we estimate

$$\Phi^3_l = \sum_{\substack{1 \leqslant i_1 \leqslant i_2 \leqslant l+2 \\ i_1 + i_2 \leqslant 2l+2}} \int_X |\nabla^{i_1}u||\nabla^{i_2}u|\, \omega_0 \leqslant \sum_{\substack{1 \leqslant i_1 \leqslant i_2 \leqslant l+2 \\ i_1 + i_2 \leqslant 2l+2}} \|\nabla^{i_1}u\|_{L^2}\|\nabla^{i_2}u\|_{L^2}\,.$$

If $i_2 = l+2$, then $i_1 \leqslant l$ and one can apply Gagliardo–Nirenberg to the i_1-term and estimate $\|\nabla^{i_1} u\|_{L^2} \|\nabla^{i_2} u\|_{L^2} \leqslant C\|\nabla^{l+2}u\|_{L^2}^{1+i_1/(l+2)} \|u-\tilde{u}\|_{L^2}^{1-i_1/(l+2)}$ which is of the form (2.16) after again estimating $\|u-\tilde{u}\|_{L^2} \leqslant \|u-\tilde{u}\|_{L^2_2}$. If $i_2 < l+2$, then $\|\nabla^{i_1} u\|_{L^2} \nabla^{i_2} u\|_{L^2} \leqslant C\|\nabla^{l+2}u\|_{L^2}^{(i_1+i_2)/(l+2)} \|u-\tilde{u}\|_{L^2}^{2-(i_1+i_2)/(l+2)}$ by application of the Gagliardo–Nirenberg inequality to both factors. This also fits (2.16) since $i_1 + i_2 \leqslant 2l+2$.

It remains to estimate Φ_l^2. Let

$$\Psi = \int_X |\nabla^{i_1} u| \cdots |\nabla^{i_s} u| \, \omega_0$$

be a summand of Φ_l^2, i.e. $s \geqslant 3$, $1 \leqslant i_j \leqslant l+1$ and $i_1 + \cdots + i_s \leqslant 2l+4$. Let $k \geqslant 0$ be the number of i_js for which $i_j = 1$ and reorder the indices such that $i_j \geqslant 2$ for $1 \leqslant j \leqslant s-k$. Applying Hölder's inequality to Ψ yields

$$\Psi = \int_X |\nabla u|^k |\nabla^{i_1} u| \cdots |\nabla^{i_{s-k}} u| \leqslant C\|\nabla u\|_{L^{kp_0}}^k \|\nabla^{i_1} u\|_{L^{p_1}} \|\nabla^{i_{s-k}} u\|_{L^{p_{s-k}}} ,$$

for positive numbers $p_0, ..., p_{s-k}$ satisfying $1/p_0 + 1/p_1 + \cdots 1/p_{s-k} = 1$. We emphasise that the indices i_j, $1 \leqslant j \leqslant s-k$ satisfy $2 \leqslant i_j \leqslant l+1$ and $i_1 + \cdots + i_{s-k} \leqslant 2l+4-k$. Corollary 2.2.3 of the Gagliardo–Nirenberg inequalities implies

$$\Psi \leqslant \|u-\tilde{u}\|_{L^2_2}^k \prod_{j=1}^{s-k} \left(\|\nabla^{l+2} u\|_{L^2}^{a_j} \|u-\tilde{u}\|_{L^2_2}^{1-a_j} + \|u-\tilde{u}\|_{L^2_2} \right) , \quad a_j = \frac{i_j - 1 - 2/p_j}{l}$$

and evaluating the product gives

$$\Psi \leqslant C \sum_{\beta \in B'} \|\nabla^{l+2} u\|_{L^2}^{\alpha_\beta} \|u-\tilde{u}\|_{L^2_2}^{\gamma_\beta} ,$$

where the total exponent satisfies $\alpha_\beta + \gamma_\beta = s \geqslant 3$. In order for this to be of the form (2.16), we require $\alpha_\beta < 2$. The largest exponent of $\|\nabla^{l+2} u\|$ satisfies

$$\alpha_{\max} = \sum_{j=1}^{s-k} a_j = \frac{1}{l} \sum_{j=1}^{s-k} (i_j - 1 - 2/p_j) \leqslant \frac{2l+4-k}{l} - \frac{s-k}{l} - \frac{2(1-1/p_0)}{l} \leqslant \frac{2l-1+2/p_0}{l} ,$$

so choosing $p_0 > 2$ implies $\alpha_\beta \leqslant \alpha_{\max} < 2$ as needed. □

We can now combine the already established uniform bound on u in L^2_2 with the preceding lemma to show that norms involving higher derivatives of u cannot exhibit finite time singularities.

Lemma 2.4.4. *For $k \in \mathbb{N}$ there exists a positive constant C such that*

$$\partial_t E_k \leqslant -\|\nabla^{k+2} u\|_{L^2}^2 + C\|u-\tilde{u}\|_{L^2_2} . \tag{2.17}$$

Remark. The term on the rightmost side of (2.17) can of course be estimated by a constant, but the present form turns out to be useful to establish exponential convergence of $u \to 0$ in higher Sobolev norms.

Proof. The analysis of section 2.3 shows that u is bounded in $L^2_2(X, g_0)$ for as long as the flow exists. By Lemma 2.4.3, the time-derivative of E_k is of the form

$$\partial_t E_k \leqslant -2\|\nabla^{k+2}u\|^2_{L^2} + C\sum_{\beta\in B}\|\nabla^{k+2}u\|^{\alpha_\beta}_{L^2}\|u-\widetilde{u}\|^{\gamma_\beta}_{L^2_2} + C\|u-\widetilde{u}\|_{L^2_2} \tag{2.18}$$

for a finite index set B and constants $\alpha_\beta, \gamma_\beta$ satisfying $0 < \alpha_\beta < 2$ and $\gamma_\beta \geqslant 2-\alpha_\beta$. An application of Young's inequality with ε

$$ab \leqslant \varepsilon a^p + \frac{1}{q}(p\varepsilon)^{-q/p}b^q\,, \quad \frac{1}{q}+\frac{1}{p} = 1$$

with $p = 2/\alpha_\beta$ and $q = 2/(2-\alpha_\beta)$ yields

$$\|\nabla^{k+2}u\|^{\alpha_\beta}_{L^2}\|u-\widetilde{u}\|^{\gamma_\beta}_{L^2_2} \leqslant \varepsilon\|\nabla^{k+2}u\|^2_{L^2} + C(\varepsilon, l, \beta)\|u-\widetilde{u}\|^{2\gamma_\beta/(2-\alpha_\beta)}_{L^2_2}\,.$$

Since $2\gamma_\beta/(2-\alpha_\beta) \geqslant 2$, one further estimates

$$\|u-\widetilde{u}\|^{2\gamma_\beta/(2-\alpha_\beta)}_{L^2_2} = \|u-\widetilde{u}\|^{2\gamma_\beta/(2-\alpha_\beta)-1}_{L^2_2}\|u-\widetilde{u}\|_{L^2_2} \leqslant C'\|u-\widetilde{u}\|_{L^2_2}\,.$$

Choosing ε sufficiently small compared to the constant appearing in (2.17) one then has

$$C\sum_{\beta\in B}\|\nabla^{k+2}u\|^{\alpha_\beta}_{L^2}\|u-\widetilde{u}\|^{\gamma_\beta}_{L^2_2} \leqslant \|\nabla^{k+2}u\|^2_{L^2} + C''\|u-\widetilde{u}\|_{L^2_2}\,,$$

so $\partial_t E_k \leqslant -\|\nabla^{k+2}u\|^2_{L^2} + C''\|u-\widetilde{u}\|_{L^2_2}$ as claimed. □

Proposition 2.4.5. *Let $0 < T \leqslant \infty$ and $u \in C([0,T[, L^2_2(X, g_0))$ be a solution to twisted Calabi flow (2.10) with smooth initial data $u(0) = u_0$. Then*

$$\forall t \in [0, T[\colon\ \|u\|_{L^2_k} \leqslant C(1+t)^{\frac{1}{2}}\,,$$

where the constant C depends on k and the initial data. In particular, if $T < \infty$, then

$$\sup_{0\leqslant t<T}\|u\|_{L^2_k} < \infty$$

for any k.

Proof. Since uniform boundedness of u in L^2_2 implies uniform boundedness in $C^0(X)$, the estimate

$$\|u\|^2_{L^2_k} \leqslant C(|\widetilde{u}|^2 + \|\nabla^k u\|^2_{L^2}) \leqslant C\left(1 + \sup_X e^{-2u}\int_X e^{2u}|\nabla^k u|^2\,\omega_0\right) \leqslant C(1+E_k)$$

shows that is suffices to establish $E_k \leqslant C(1+t)$. But this follows from integrating (2.17) and the uniform boundedness of $\|u-\widetilde{u}\|_{L^2_2}$. □

Corollary 2.4.6. *Under the assumptions stated in Theorem 2.2.4, twisted Calabi flow admits a unique long-time solution* $u \in C^\infty(X \times [0, \infty[)$.

Proof. For given smooth initial data, let u be the smooth unique short-time solution to (2.10) provided by standard theory of parabolic equations. Suppose u only exists on $[0, T[$ for a maximal existence time $T \in]0, \infty[$. For a sequence t_i in $[0, T[$ converging to T consider the sequence $u(t_i)$. By Proposition 2.4.5 the sequence $u(t_i)$ is bounded in L^2_k for any k and by the compactness of the embeddings $L^2_{k+2} \hookrightarrow C^k$ has convergent subsequences in each C^k. By passing to such a subsequence, we assume that $u(t_i)$ converges in C^0 and denote by $u(T)$ its limit. Since u is uniformly bounded in C^4, so is $\partial_t u$ in C^0 and one has

$$\begin{aligned}
\|u(t) - u(T)\|_{C^0} &\leqslant \inf_j \left(\|u(t) - u(t_j)\|_{C^0} + \|u(t_j) - u(T)\|_{C^0} \right) \\
&= \inf_j \|u(t) - u(t_j)\|_{C^0} \\
&\leqslant \inf_j \left| \int_t^{t_j} (\partial_s u)(s)\, ds \right| \\
&\leqslant \sup_{0 \leqslant t < T} \|\partial_t u\|_{C_0} \cdot \inf_j |t - t_j| \\
&= C \cdot \inf_j |t - t_j| \,,
\end{aligned}$$

which implies that $u(t) \to u(T)$ in C^0. In addition, $u(T)$ has to coincide with limits of subsequences converging in C^k, so $u(T)$ is in fact smooth (in fact the convergence $u(t) \to u(T)$ is in C^∞, which can be shown using e.g. Lemma A.2.2 in Appendix A.2). In particular $u(T)$ can be taken as smooth initial data and the flow can be continued contradicting the maximality of T. □

2.5. Convergence and Exponentiality

With long-time existence of twisted Calabi flow established, one can ask how a solution $u(t)$ behaves as $t \to \infty$. We already know that $\|u - \tilde{u}\|_{L^2_2}$ is uniformly bounded in $t \in [0, \infty[$ and proceed to show that it in fact converges to zero at an exponential rate. Lemma 2.4.4 is then used to show that $u(t)$ converges to zero in each C^k. To lighten the presentation we refer to the solution of $\alpha(t)$-twisted Calabi flow for fixed initial metric and twist $\alpha(t)$ by either $\omega(t), \varphi(t)$ or $u(t)$, where the three are related by $\omega(t) = (1 + \Delta_0 \varphi)\omega_0 = e^{u(t)}\omega_0$. In addition, we write $Ca(t)$ instead of $Ca(\varphi(t), \alpha(t))$ for twisted Calabi energy evaluated at $\varphi(t)$ and $\alpha(t)$ and extend this notation to other energy functionals.

2.5.1. Convergence in $L^2_2(X, g_0)$

We start out by showing that twisted Calabi energy has the right limiting behaviour.

Lemma 2.5.1. *Twisted Calabi energy tends to zero along Calabi flow.*

Proof. Recall that twisted Mabuchi energy is bounded along Calabi flow and satisfies

$$\partial_t Ma = -Ca + \int_X \varphi \Lambda_0 \dot{\alpha}\, \omega_0\,.$$

The proof of the boundedness of twisted Mabuchi energy (Proposition 2.3.2) also shows that $\int_0^\infty \int_X \varphi \Lambda_0 \dot{\alpha}\, \omega_0 \, dt < \infty$, so

$$\int_0^\infty Ca(t)\, dt < \infty\,.$$

Also recall from the proof of the boundedness of twisted Calabi energy (Proposition 2.3.7) that

$$\partial_t Ca \leqslant 2 \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S}) \Lambda_0 \dot{\alpha}\, \omega_0$$

and applying Cauchy–Schwarz gives the estimate $\partial_t Ca \leqslant 2Ca^{1/2} \sup_{X, t \geqslant 0} |\Lambda_0 \dot{\alpha}|$ which is bounded from above by a positive constant K. Now suppose that $Ca(t)$ does not tend to zero as $t \to \infty$. Then there exists a monotone and unbounded sequence (t_i) in $[0, \infty[$ and $\varepsilon > 0$ such that $Ca(t_i) \geqslant \varepsilon$ for all $i \in \mathbb{N}$. By passing to a subsequence one can assume that the intervals $I_i :=]t_i - \varepsilon/(2K), t_i[$ are disjoint and contained in $[0, \infty[$. For $s \in I_i$ we estimate

$$Ca(t_i) - Ca(s) = \int_s^{t_i} (\partial_t Ca)(t)\, dt \leqslant K \cdot (t_i - s)$$

which, in light of $t_i - s < \varepsilon/(2K)$, implies

$$Ca(s) \geqslant Ca(t_i) - K \cdot (t_i - s) > Ca(t_i) - \varepsilon/2 \geqslant \varepsilon - \varepsilon/2 = \varepsilon/2\,.$$

This construction yields infinitely many disjoint intervals $I_i \subset [0, \infty[$ such that

$$\int_{I_i} Ca(t)\, dt \geqslant \frac{\varepsilon^2}{4K}\,,$$

which contradicts $\int_0^\infty Ca(t) < \infty$. □

Corollary 2.5.2. *Along twisted Calabi flow one has* $\lim_{t \to \infty} \|\Delta_0 u(t)\|_{L^2(X, g_0)} = 0$.

Proof. We recall that $\mathrm{Sc}_0 + \Lambda_0\alpha_\infty = \hat{S}$ with $\hat{S} = \mathrm{Sc}_0 + \Lambda_0\alpha_\infty = \underline{S} + \underline{\alpha} \leqslant 0$ and estimate

$$\begin{aligned}
& \|\Delta_0 u\|_{L^2}^2 + \int_X \Delta_0 u \cdot \Lambda_0(\alpha - \alpha_\infty)\,\omega_0 \\
\leqslant\ & \|\Delta_0 u\|_{L^2}^2 + \int_X \Delta_0 u \cdot \Lambda_0(\alpha - \alpha_\infty)\,\omega_0 + \frac{1}{2}|\hat{S}| \int_X e^u |du|_0^2\,\omega_0 \\
=\ & \|\Delta_0 u\|_{L^2}^2 + \int_X \Delta_0 u \cdot \Lambda_0(\alpha - \alpha_\infty)\,\omega_0 - \hat{S}\int_X (e^u - 1)\Delta_0 u\,\omega_0 \\
=\ & \int_X (\Delta_0 u)\left[\Delta_0 u + \Lambda_0(\alpha - \alpha_\infty) - \hat{S}(e^u - 1)\right]\omega_0 \\
=\ & \int_X (\Delta_0 u)\left[\Delta_0 u + \mathrm{Sc}_0 + \Lambda_0\alpha - \hat{S}e^u\right]\omega_0 \\
=\ & \int_X (\Delta_0 u)e^u\left[\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S}\right]\omega_0 \\
\leqslant\ & \left(\int_X (\Delta_0 u)^2\,\omega_0\right)^{1/2}\left(\int_X e^{2u}(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})^2\,\omega_0\right)^{1/2} \\
\leqslant\ & \|\Delta_0 u\|_{L^2} \cdot \sup_X e^{u/2} \cdot Ca^{1/2}\,.
\end{aligned}$$

Bringing $\int_X \Delta_0 u \cdot \Lambda_0(\alpha - \alpha_\infty)\,\omega_0$ to the right hand side, applying Cauchy–Schwarz and dividing by $\|\Delta_0 u\|_{L^2}$ gives

$$\|\Delta_0 u\|_{L^2} \leqslant C\left(Ca^{1/2} + \|\Lambda_0(\alpha - \alpha_\infty)\|_{L^2}\right).$$

The right hand side tends to 0 as $t \to \infty$ proving the claim. □

Corollary 2.5.3. *Along twisted Calabi flow,* $\lim_{t\to\infty}\|u(t) - \widetilde{u}(t)\|_{L^2_2(X,g_0)} = 0$ *as well as* $\lim_{t\to\infty}\|u(t)\|_{L^2_2(X,g_0)} = 0$*.*

Proof. The first part of the claim, $\lim_{t\to\infty}\|u(t) - \widetilde{u}(t)\|_{L^2_2} = 0$, follows immediately from Corollary 2.5.2 and the estimate $\|u(t) - \widetilde{u}(t)\|_{L^2_2} \leqslant C\|\Delta_0 u\|_{L^2}$. For the second part, observe that

$$f \mapsto \log\int_X e^f\,\omega_0$$

defines a continuous map $L^2_2 \to \mathbb{R}$ which vanishes at 0. Applying this to $f = u(t) - \widetilde{u}(t)$ gives

$$0 = -\lim_{t\to\infty}\log\int_X e^{u(t)-\widetilde{u}(t)}\,\omega_0 = -\lim_{t\to\infty}(-\widetilde{u}(t) + \log\int_X e^{u(t)}\,\omega_0 = \lim_{t\to\infty}\widetilde{u}(t)\,.$$

Since $\|u\|_{L^2_2}^2 = \|u - \widetilde{u}\|_{L^2_2}^2 + \widetilde{u}^2$, this implies the desired convergence $\lim_{t\to\infty}\|u(t)\|_{L^2_2} \to 0$. □

2.5.2. Exponentiality and Convergence in $C^k(X, g_0)$

We now proceed to show that the convergence of $\|u\|_{L^2_2}$ and $\|u - \widetilde{u}\|_{L^2_2} \to 0$ occurs at an exponential rate. The following technical lemma in conjunction with Lemma 2.4.4 then implies exponential convergence of u to zero in C^k for any $k \in \mathbb{N}_0$.

Lemma 2.5.4. *Let $f\colon [0,\infty[\to [0,\infty[$ be a differentiable function satisfying the differential inequality*

$$\partial_t f \leqslant -af + Be^{-\gamma t}$$

for constants $a, \gamma > 0$ and $B \in \mathbb{R}$. Then for any $0 < \delta < \min\{a, \gamma\}$ there exists a C such that $f(t) \leqslant Ce^{-\delta t}$.

Proof. For $B \leqslant 0$ the nonnegativity of f implies that if $f(t_0) = 0$, then $f(t) = 0$ for all $t \geqslant t_0$. For the open connected subset of $[0,\infty[$ where $f(t) \neq 0$ one has $\partial_t \log f \leqslant -a$ and integration yields the claim for $\delta = a$. If $B > 0$, then observe that the differential inequality remains true for any $0 < \gamma' < \gamma$, so we can assume that $\gamma \leqslant a$. Now the estimate

$$\partial_t(e^{\gamma t} f) \leqslant \gamma e^{\gamma t} f - a e^{\gamma t} f + B \leqslant B$$

yields $e^{\gamma t} f(t) \leqslant f(0) + B \cdot t$, so for any $0 < \varepsilon < \gamma$ we have

$$f(t) \leqslant (f(0) + B \cdot t)e^{-\varepsilon t} \cdot e^{-(\gamma-\varepsilon)t}\,.$$

Since $(f(0) + B \cdot t)e^{-\varepsilon t}$ is bounded uniformly in t, setting $\delta := \gamma - \varepsilon$ concludes the proof. □

Proposition 2.5.5. *Let $\varphi(t)$ be a solution to twisted Calabi flow. Then there exist positive constants C, δ such that $Ca(t) \leqslant Ce^{-\delta t}$.*

Proof. We will show that there exist positive constants C, C' such that for sufficiently large times one has the estimate $\partial_t Ca \leqslant -C' Ca + C\|\Lambda_0 \dot{\alpha}\|_{L^2}$. The exponential decay of $\|\Lambda_0 \dot{\alpha}\|_{L^2}$ and Lemma 2.5.4 then prove the claim.

Denote by $o(1)$ for any smooth function of t that tends to 0 as $t \to \infty$. Since $u(t) \to 0$ in C^0 as $t \to \infty$, we have $\sup_X e^{\pm u} = 1 + o(1)$. Observe that for any function f and metric g, the constant a that minimises $\|f - a\|_{L^2(X,g)}$ is given by the average of f with respect to the metric g. Denoting by $\widetilde{S}$ to be the average of $\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha$ with respect to the background metric g_0, one thus have the estimate

$$Ca \leqslant \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi - \widetilde{S})^2\, \omega_\varphi \leqslant (1 + o(1))\|\,\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \widetilde{S}\|^2_{L^2}\,.$$

Combining the Poincaré inequality and the Cauchy–Schwarz inequality one can estimate $\|f - \widetilde{f}\|_{L^2} \leqslant \|\Delta f\|_{L^2}$, which in the case at hand yields

$$Ca \leqslant C(1 + o(1))\|\Delta_0(\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha)\|_{L^2}\,. \tag{2.19}$$

Combining this estimate with $Ca = o(1)$ and the Sobolev embedding $L^2_2 \hookrightarrow L^4_1$ we now have

$$\begin{aligned}
\partial_t Ca &= 2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi \alpha - \hat{S})(\Delta^2_\varphi \partial_t \varphi - \mathrm{Sc}(\varphi)\Delta_\varphi \partial_t\varphi - \Lambda_\varphi\alpha\Delta_\varphi\partial_t\varphi)\,\omega_\varphi \\
&\quad + \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})^2 \Delta_\varphi \partial_t\varphi\,\omega_\varphi + 2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})\Lambda_\varphi\dot{\alpha}\,\omega_\varphi \\
&= -2\int_X (\Delta_\varphi(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S}))^2\,\omega_\varphi - \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})^2\Delta_\varphi\partial_t\varphi\,\omega_\varphi \\
&\quad -2\hat{S}\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})\Delta_\varphi\partial_t\varphi\,\omega_\varphi + 2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})\Lambda_\varphi\dot{\alpha}\,\omega_\varphi \\
&= -2\int_X (\Delta_\varphi(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha))^2\,\omega_\varphi + \int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})|d(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)|^2_0\,\omega_0 \\
&\quad -|\hat{S}|\int_X |d(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})|^2_\varphi\,\omega_\varphi + 2\int_X (\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S})\Lambda_\varphi\dot{\alpha}\,\omega_\varphi \\
&\leqslant -2\inf_X e^{-u}\|\Delta_0(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)\|^2_{L^2} + \|\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha - \hat{S}\|_{L^2}\cdot\|d(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)\|^2_{L^4} \\
&\quad +2Ca^{1/2}\|e^{-u}\Lambda_0\dot{\alpha}\|_{L^2} \\
&\leqslant -2\inf_X e^{-u}\|\Delta_0(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)\|^2_{L^2} + \sup_X e^{-u}Ca^{1/2}\|d(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)\|^2_{L^4} \\
&\quad +C\|\Lambda_0\dot{\alpha}\|_{L^2} \\
&\leqslant (-2\inf_X e^{-u} + C\sup_X e^{-u}Ca^{1/2})\|\Delta_0(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)\|^2_{L^2} + C\|\Lambda_0\dot{\alpha}\|_{L^2} \\
&\leqslant -2(1 + o(1))\|\Delta_0(\mathrm{Sc}(\varphi) + \Lambda_\varphi\alpha)\|^2_{L^2} + C\|\Lambda_0\dot{\alpha}\|_{L^2} \\
&\leqslant -(2-1)C'Ca + C\|\Lambda_0\dot{\alpha}\|_{L^2}
\end{aligned}$$

for t sufficiently large such that all instances of $o(1)$ (including that in (2.19)) are small compared to $1/2$. ☐

Corollary 2.5.6. *Let $u(t)$ solve twisted Calabi flow. Then there exist positive constants C, δ such that $\|u - \widetilde{u}\|_{L^2_2} \leqslant Ce^{-\delta t}$ and $\|u\|_{L^2_2} \leqslant Ce^{-\delta t}$.*

Proof. Recall from the proof of Corollary 2.5.2 that

$$\|\Delta_0 u\|_{L^2} \leqslant C\left(Ca^{1/2} + \|\Lambda_0(\alpha - \alpha_\infty)\|_{L^2}\right).$$

Proposition 2.5.5 along with the exponential convergence of $\alpha \to \alpha_\infty$ implies that $\|\Delta_0 u\|_{L^2} \to 0$ at an exponential rate and hence that

$$\|u - \widetilde{u}\|_{L^2_2} \leqslant Ce^{-\delta t}$$

for suitable $C, \delta < 0$. To establish the second claim it remains to show that $\widetilde{u}(t)$ tends to zero exponentially. We write

$$|\widetilde{u}|(1 + o(1)) = |\widetilde{u}|\sum_{k=0}^{\infty}\frac{\widetilde{u}^k}{(k+1)!} = (1 - e^{\widetilde{u}}) = \int_X (e^u - e^{\widetilde{u}})\,\omega_0$$

and observe that $e^x - e^y = (x-y)b(x,y)$ where $b(x,y)$ is bounded if x, y vary within a bounded set. If t is sufficiently large, then $o(1)$ lies in, say $]-1/2, 1/2[$ and one has

$$|\tilde{u}| \leqslant 2\int_X (e^u - e^{\tilde{u}})\,\omega_0 \leqslant C\int_X |u-\tilde{u}|\,\omega_0 \leqslant C\|u-\tilde{u}\|_{L^2} \leqslant C' e^{-\delta t}\,.$$

□

Proposition 2.5.7. *Let $u(t)$ solve twisted Calabi flow. Then for any $k \in \mathbb{N}_0$ there exist positive constants C_k, δ_k such that $\|u\|_{L^2_k} < C_k e^{-\delta_k t}$. The same is true for the C^k-norms of u.*

Proof. From Lemma 2.4.4 we know that the time-derivative of $E_l = \|e^u \nabla^l u\|^2_{L^2}$, $l \in \mathbb{N}$ is given by

$$\partial_t E_l \leqslant -\|\nabla^{l+2} u\|^2_{L^2} + C\|u-\tilde{u}\|_{L^2_2}\,.$$

By the Gagliardo–Nirenberg inequality (2.2.2) one has

$$\|\nabla^l u\|^2_{L^2} \leqslant C\|\nabla^{l+2}u\|^{2a}_{L^2}\|u-\tilde{u}\|^{2(1-a)}_{L^2}\,, \quad a = l/(l+2)$$

and applying Young's inequality with ε and $p = 1/a$ to the right hand side gives

$$\|\nabla^l u\|^2_{L^2} \leqslant C\varepsilon\|\nabla^{l+2}u\|^2_{L^2} + C'(\varepsilon)\|u-\tilde{u}\|^2_{L^2}\,.$$

Estimating $E_l \leqslant \sup_X e^u \|\nabla^l u\|^2_{L^2}$ and choosing ε sufficiently small, this can be used to further estimate $\partial_t E_l$ by

$$\partial_t E_l \leqslant -E_l + C\|u-\tilde{u}\|^2_{L^2_2} + C\|u-\tilde{u}\|_{L^2_2} \leqslant -E_l + C'\|u-\tilde{u}\|_{L^2_2}\,.$$

The rightmost term on the right hand side decays exponentially by Corollary 2.5.6, so Lemma 2.5.4 can be applied to the above differential inequality. Since $\|\nabla^l u\|^2_{L^2} \leqslant \sup_X e^{-2u} E_l$, this shows that

$$\|\nabla^l u\|^2_{L^2} < Ce^{-\delta t}\,.$$

By the exponential convergence $u \to 0$ in L^2_2, this is also true for $l = 0$, so summing from $l = 1$ to k and taking the smallest occurring δ on the right hand side gives $\|u\|^2_{L^2_k} \leqslant Ce^{-\delta t}$ as claimed. The exponential decay of u in C^k-norms follows from the Sobolev embeddings $L^2_{k+2}(X) \hookrightarrow C^k(X)$. □

3. Time-Dependent Hermitian Yang–Mills Flow

3.1. Introduction

Hermitian Yang–Mills flow is a semilinear parabolic PDE for a path of Hermitian metrics $h(t)$ on a holomorphic vector bundle E over a Kähler manifold (X, ω) which tries to deform an arbitrary initial metric h_0 into a Hermite–Einstein metric with respect to ω_X, i.e. a metric h whose curvature F_h satisfies $\Lambda_\omega i F_h = \lambda \operatorname{id}_E$. Hermite–Einstein metrics can be seen as a best Hermitian metric compatible with a given Kähler metric ω_X, their Chern connections are instances of Yang–Mills connections which play an important role in four-manifold geometry (cf. [10]) and gauge theory (cf. C.1 for more details). In [9] S. Donaldson used Hermitian Yang–Mills flow to give a proof of the Kobayashi–Hitchin correspondence on projective algebraic surfaces and compact complex curves, relating the solvability of the Hermite–Einstein equation — an analytic problem — to the algebro-geometric condition of Takemoto–Mumford (or slope) stability of the holomorphic bundle $E \to X$. The analysis in [9] shows that the flow exists for all times over holomorphic bundles on compact Kähler manifolds, independently of stability. Donaldson related convergence to the existence of a lower bound for a functional M, which we refer to as the *Donaldson functional*. Hermitian Yang–Mills flow can be seen as the gradient flow of M and in some sense, the Donaldson functional plays a role similar to that of Mabuchi energy for Calabi flow. Hermitian Yang–Mills flow was also used in the study of Higgs–Bundles by Simpson in [34], wherein the author also relates convergence of Hermitian Yang–Mills flow at infinity to stability of the bundle and the properties of the Donaldson functional for bundles of base manifolds with arbitrary dimension.

We are interested in the case where the Kähler metric on the base X changes in time within its cohomology class. The resulting *time-dependent Hermitian Yang–Mills flow* (which we sometimes simply refer to as Hermitian Yang–Mills flow or abbreviate as HYMF) arises naturally in the construction of adiabatic approximations to Calabi flow on ruled manifolds (Chapter 4 contains a detailed account). For technical reasons we restrict our attention to the case of complex dimension one. The goal of this section is to prove the following theorem.

Theorem 3.1.1. *Let X a compact Riemann surface with a fixed Kähler class $\kappa \in H^{1,1}(X, \mathbb{R})$ and $\omega(t), t \in [0, \infty[$ a smooth one-parameter family in κ converging to a limit Kähler form ω_∞. The convergence is assumed to be exponential in the sense that the logarithm $u(t)$ of the conformal factor relating $\omega(t)$ and ω_∞ via $\omega(t) = e^{u(t)}\omega_\infty$ as well as its time-derivative tend to 0 at exponential rates in $C^k(X, g_\infty)$ for all $k \in \mathbb{N}_0$. Let furthermore $E \to X$ be a holomorphic rank r vector bundle assumed to be slope-stable with* *respect to the class κ. Denote by F_h the curvature of the Chern connection associated to a Hermitian metric h on E and by $\lambda = 2\pi\langle[c_1(E)], [X]\rangle / r \operatorname{Vol}(X, \Omega)$ the Hermite–Einstein constant. Then for any smooth initial Hermitian metric h_0 the time-dependent Hermitian Yang–Mills flow given by*

$$h^{-1}(t)(\partial_t h)(t) = -[\Lambda_{\omega(t)} i F_{h(t)} - \lambda \operatorname{id}_E], \tag{3.1}$$

admits a unique smooth long-time solution $h(t)$ with $h(0) = h_0$. Moreover, $h(t)$ converges exponentially fast in each C^k for $k \in \mathbb{N}_0$ to a ω_∞-Hermite–Einstein metric h_∞ characterised by $\Lambda_{\omega_\infty} i F_{h_\infty} = \lambda \operatorname{id}_E$ up to a constant factor.

After recalling some general facts, we present the proof in four steps: short-time existence, long-time existence, convergence and exponentiality of the convergence.

3.2. Preliminaries

3.2.1. Chern Connections on E and $\operatorname{End}(E)$

We recall some fundamental facts about Chern connections on Kähler manifolds for future reference. Let h, k be two Hermitian metrics on E. Let $\eta \in \operatorname{End}(E)$ be the endomorphism relating h and k via $k(\eta\cdot, \cdot) = h(\cdot, \cdot)$ (write $h = k\eta$). In a local trivialisation h, k are represented by Hermitian matrices which we also denote by h, k. In that trivialisation one has $\eta = k^{-1}h$.

The Chern connection d_h is the unique connection making h parallel and satisfying $d_h^{0,1} = \overline{\partial}$. It is locally represented by $d_h = d + A_h$, where the connection one-form is purely of type $(1,0)$ and given by $A_h = h^{-1}\partial h$. Splitting d_h into its type components, we write $d_h = \overline{\partial} + \partial_h$ with $\partial_h = \partial + A_h$. The curvature of (E, d_h) is given by $F_h = \overline{\partial} A_h = \overline{\partial}(h^{-1}\partial h)$. If k is another Hermitian metric on E related to h by η as above, then the Chern connections of h and k differ by a global $(1,0)$-form with values in $\operatorname{End}(E)$ given by $d_h - d_k = \eta^{-1}\partial_k \eta$. Since this is locally just $A_h - A_k$, one finds

$$F_h - F_k = \overline{\partial}(\eta^{-1}\partial_k \eta) \tag{3.2}$$

for the difference of the curvatures.

We will frequently deal with associated data on the bundle $\operatorname{End}(E)$, so we recall some basic relations here. The Hermitian product h on E induces the Hermitian product h' on $\operatorname{End}(E)$ which is given by $(\phi, \psi)_h := h'(\phi, \psi) = \operatorname{tr}(\phi \psi^h)$, where ϕ, ψ are sections of $\operatorname{End}(E)$ and ψ^h denotes the h-adjoint of ψ. In addition, $\operatorname{End}(E)$ inherits a holomorphic structure from E which we denote by $\overline{\partial}'$. The Chern connection $d_{h'}$ on $(\operatorname{End}(E), h', \overline{\partial}')$ coincides with the connection d'_h on $\operatorname{End}(E)$ induced by the Chern connection on E. As before one has $d'_h = \overline{\partial}' + \partial'_h$, where locally $\partial'_h = \partial + A'_h$. The connection one-form is given by $A'_h = \operatorname{ad}_{A_h} = [A_h, \cdot]$, defining $[\cdot, \cdot]$ on $\bigoplus_k \Lambda^k T^*_{\mathbb{C}} X \otimes \operatorname{End}(E)$ to be given by the usual commutator on the endomorphism part and the wedge product on the form part, i.e. if $\alpha \in \Lambda^k T^*_{\mathbb{C}} X$, $\beta \in \Lambda^l T^*_{\mathbb{C}} X$, $\phi, \psi \in \operatorname{End}(E)$ (all over the same base point), then $[\alpha \otimes \phi, \beta \otimes \psi] = \alpha \wedge \beta \otimes [\phi, \psi] = (-1)^{kl+1}[\beta \otimes \psi, \alpha \otimes \phi]$. For the curvatures one obtains

$F'_h = \mathrm{ad}_{F_h}$ and hence $F'_h - F'_k = \mathrm{ad}_{\overline{\partial}(\eta^{-1}\partial_k\eta)}$. We also remark that for $\phi \in \mathrm{End}(E)$ the h'-adjoint of ad_ϕ is given by ad_{ϕ^h}.

To lighten the notation a bit, we shall drop primes if the context makes it clear whether we are working on E or $\mathrm{End}(E)$.

3.2.2. Kähler Identities on Vector Bundles

The Riemannian metric g associated to the Kähler form ω on X endows the (complexified) tangent bundle $T_{\mathbb{C}}X$ and associated bundles — in particular $\Lambda^k T^*_{\mathbb{C}}X$ — with Hermitian products $(\cdot,\cdot)_\omega$. These can be expressed by $(\alpha,\beta)_\omega = *(\alpha \wedge *\bar{\beta})$, where $*$ is the Hodge-star operator associated to g. We can combine the Hermitian structures $(\cdot,\cdot)_\omega$ on $\Lambda^k T^*_{\mathbb{C}}X$ and $(\cdot,\cdot)_h$ on $\mathrm{End}(E)$ to a Hermitian structure $(\cdot,\cdot)_{h,\omega}$ on $\bigoplus_k \Lambda^k T^*_{\mathbb{C}}X \otimes \mathrm{End}(E)$ given by linearly extending

$$(\alpha \otimes \phi, \beta \otimes \psi)_{h,\omega} = (\phi,\psi)_h \cdot (\alpha,\beta)_\omega\,.$$

The pointwise Hermitian structure gives rise to an L^2-product on $\bigoplus_k \Omega^k(X, \mathrm{End}(E))$:

$$\langle a, b\rangle_{h,\omega} = \int_X (a,b)_{h,\omega}\, d\mathrm{vol}_\omega\,.$$

The Kähler (or Nakano) identities relate the operators $\partial_h, \overline{\partial}$ and their formal L^2-adjoints $\partial^*, \overline{\partial}^*_h$ to the metric data on the base. They read

$$\partial^* = i[\Lambda_\omega, \overline{\partial}], \quad \overline{\partial}^*_h = -i[\Lambda_\omega, \partial_h]\,. \tag{3.3}$$

Indeed, ∂^* is independent of h, justifying the omission of the subscript. A proof of the Kähler identities can be found in e.g. [23].

3.2.3. Relating Different Laplacians

If (E,h) is a Hermitian holomorphic vector bundle, one can define various Laplacians on the space of form-valued sections $\Omega^k(X,E)$:

$$\Delta_{\partial,h,\omega} = \partial^*\partial_h + \partial_h\partial^*, \quad \Delta_{\overline{\partial},h,\omega} = \overline{\partial}^*_h\overline{\partial} + \overline{\partial}\overline{\partial}^*_h, \quad \Delta_{d,h,\omega} = d^*_h d_h + d_h d^*_h\,.$$

With the help of the Kähler identities (3.3) and the relation $F_h = d_h^2 = (\overline{\partial}\partial_h + \partial_h\overline{\partial})$ one finds

$$\begin{aligned} \Delta_{\partial,h,\omega} + \Delta_{\overline{\partial},h,\omega} &= \Delta_{d,h,\omega}\,, \\ \Delta_{\partial,h,\omega} - \Delta_{\overline{\partial},h,\omega} &= [\Lambda_\omega, iF_h]\,. \end{aligned}$$

Defining the corresponding Laplacians on $\Omega^k(X, \mathrm{End}(E))$ in the same manner, the second identity becomes $\Delta_{\partial,h,\omega} - \Delta_{\overline{\partial},h,\omega} = [\Lambda_\omega, \mathrm{ad}_{iF_h}]$ which reduces to $\mathrm{ad}_{\Lambda_\omega iF_h}$ on $\Omega^0(X, \mathrm{End}(E))$.

It will be of importance to know how Laplacians on $\Omega^0(X, \mathrm{End}(E))$ defined by different Hermitian metrics h, k are related. For $\varphi \in \Omega^0(X, \mathrm{End}(E))$ compute

$$\begin{aligned}
(\Delta_{\partial,h,\omega} - \Delta_{\partial,k,\omega})\varphi &= i\Lambda_\omega \overline{\partial}(\partial_h - \partial_k)\varphi \\
&= i\Lambda_\omega \overline{\partial}[\eta^{-1}\partial_k \eta, \varphi] \\
&= i\Lambda_\omega [\overline{\partial}(\eta^{-1}\partial_k\eta), \varphi] - i\Lambda_\omega[\eta^{-1}\partial_k\eta, \overline{\partial}\varphi] \\
&= i\Lambda_\omega[F_h - F_k, \varphi] - i\Lambda_\omega[\eta^{-1}\partial_k\eta, \overline{\partial}\varphi] \\
&= \mathrm{ad}_{\Lambda_\omega i(F_h - F_k)}\,\varphi - i\Lambda_\omega[\eta^{-1}\partial_k\eta, \overline{\partial}\varphi]\,.
\end{aligned}$$

Performing a similar computation or simply using the relations relating Δ_∂ and $\Delta_{\overline{\partial}}$ one finds

$$\begin{aligned}
(\Delta_{\overline{\partial},h,\omega} - \Delta_{\overline{\partial},k,\omega})\varphi &= -i\Lambda_\omega[\eta^{-1}\partial_k\eta, \overline{\partial}\varphi]\,, \\
(\Delta_{d,h,\omega} - \Delta_{d,k,\omega})\varphi &= \mathrm{ad}_{\Lambda_\omega i(F_h - F_k)} - 2i\Lambda_\omega[\eta^{-1}\partial_k\eta, \overline{\partial}\varphi]\,.
\end{aligned}$$

Finally, all of the three Laplacians acting on $\Omega^0(X, E)$ or $\Omega^0(X, \mathrm{End}(E))$ depend on the metric via $\Delta_\omega = e^{-u}\Delta_{\omega'}$ if $\omega = e^u \omega'$.

3.3. Short-Time Existence

We merely show that the evolution equation (3.1) defines a semilinear parabolic system. The existence of a unique smooth short-time solution is then guaranteed by standard theory. Setting $\eta(t) = h_0^{-1}h(t)$, we express (3.1) as

$$\begin{aligned}
\partial_t \eta &= -\eta[\Lambda_\omega i\overline{\partial}(\eta^{-1}\partial_{h_0}\eta) + \Lambda_\omega i F_{h_0} - \lambda\,\mathrm{id}_E] \\
&= -[i\Lambda_\omega \overline{\partial}\partial_{h_0}\eta - i\Lambda_\omega(\overline{\partial}\eta \wedge \eta^{-1}\partial_{h_0}\eta) + \eta\Lambda_\omega i F_{h_0} - \eta\lambda\,\mathrm{id}_E]\,,
\end{aligned}$$

which using the Kähler identities can be rearranged to

$$(\partial_t + \Delta_{h_0,\omega,\partial})\eta = i\Lambda_\omega\overline{\partial}\eta \wedge \eta^{-1}\partial_{h_0}\eta - \eta[\Lambda_\omega i F_{h_0} - \lambda\,\mathrm{id}_E]\,. \tag{3.4}$$

From this we see that time-dependent Hermitian Yang–Mills flow is a semilinear parabolic PDE (the highest order piece is linear).

3.4. Long-Time Existence

With the exception of higher regularity estimates, most techniques used in this section are an adaptation of Donaldson's to the case of a moving base metric. Working over a Riemann surface grants us the luxury that the contracted curvature $\Lambda_\omega F_h$ already controls the full curvature F_h.

To establish long-time existence we assume that the flow exists only on a maximal time interval $[0, T[, T < \infty$ and show that $h(t)$ converges to a smooth Hermitian metric as $t \to T$, permitting the flow to be extended beyond the maximal existence time T, hence contradicting its maximality. In the case at hand, the proof can be split into three steps. The first consists in showing that $h(t)$ converges in C^0 to a continuous metric $h(T)$. Using a maximum principle argument we then establish an a priori bound on $\Lambda_\omega iF_h$ uniform in $[0,T[$, which in conjunction with the C^0-boundedness also provides a C^1-bound via a blowup argument. Finally, we use a parabolic bootstrapping argument to establish that $\Lambda_\omega F_h$ remains uniformly bounded in C^k and infer that this also uniformly bounds $h(t)$ in C^k for any $k \in \mathbb{N}_0$. The C^0-limit $h(T)$ then has to be smooth and the convergence of $h(t)$ to $h(T)$ occurs in C^∞.

Remark. By maximum principle we refer to the parabolic maximum principle for heat equations on manifolds with varying metric which can be found in e.g. [40].

Remark. The precise argument for the boundedness in each C^k in conjunction with C^0-convergence implying C^∞ convergence is given by Lemma A.2.2 in Appendix A.2 (set $E = C^{k+1}$, $F = C^k$ and $G = C^0$ with S, T being the obvious inclusions).

3.4.1. Convergence in C^0 for finite time

The first step is to establish the C^0-convergence of $h(t)$ for $t \to T$. There is a natural notion of C^0-distance between two Hermitian metrics on a complex vector bundle induced by the *symmetric distance* of Hermitian inner products on a finite-dimensional complex vector space which we recall below.

Denote by $\mathcal{H}$ the set of Hermitian inner products on $\mathbb{C}^r$. The (right) action of $\mathrm{Gl}(r,\mathbb{C})$ on $\mathcal{H}$ given by $(h \cdot g)(\cdot,\cdot) = h(g\cdot, g\cdot)$ is transitive and the stabiliser of a point $h_0 \in \mathcal{H}$ (we can take h_0 to be the standard Hermitian product on $\mathbb{C}^r$) consists of those $g \in \mathrm{Gl}(r,\mathbb{C})$ that are unitary with respect to h_0, i.e. $\mathcal{H} \cong \mathrm{U}(r)\backslash\mathrm{Gl}(r,\mathbb{C})$. Writing $h_0(g\cdot, g\cdot) = h_0(g^*g\cdot,\cdot)$ leads to another description of $\mathcal{H}$. One shows that the map $\mathrm{U}(r)\cdot g \mapsto g^*g$ is a bijection between $\mathrm{U}(r)\backslash\mathrm{Gl}(r,\mathbb{C})$ to the positive h_0-self-adjoint endomorphisms $\mathrm{End}^+_{h_0}(\mathbb{C}^r)$. We can endow the homogeneous space $\mathrm{U}(r)\backslash\mathrm{Gl}(r,\mathbb{C})$ with a symmetric space structure by observing that the fixed point set $\mathrm{Gl}(r,\mathbb{C})^\sigma$ of the involutive automorphism $\sigma(g) := g^{-1*}$ of $\mathrm{Gl}(r,\mathbb{C})$ is precisely $\mathrm{U}(r)$. At the identity, this induces the splitting $\mathfrak{gl}(r,\mathbb{C}) = \mathfrak{u}(r) \oplus i\mathfrak{u}(r)$ as the ± 1-eigenspaces of $d\sigma$. We now identify $T_{\mathrm{U}(r)\cdot e}\,\mathrm{U}(r)\backslash\mathrm{Gl}(r,\mathbb{C})$ with $i\mathfrak{u}(r)$ and define an $\mathrm{Ad}_{\mathrm{U}(r)}$-invariant inner product on $i\mathfrak{u}(r)$, say $(a,b) := \mathrm{tr}(ab^*) = \mathrm{tr}(ab)$ and extend this right-invariantly to a Riemannian metric on $\mathcal{H}$ endowing the set of Hermitian inner products with a symmetric space structure.

We are interested in the geodesic distance in $\mathcal{H}$ in the $\mathrm{End}^+_{h_0}(\mathbb{C}^r)$-picture. For $H \in \mathrm{End}^+_{h_0}(\mathbb{C}^r)$ one has $T_H\,\mathrm{End}^+_{h_0}(\mathbb{C}^r) = \mathrm{End}_{h_0}(\mathbb{C}^r)$ with the inner product given by

$$\langle A, B\rangle_H = \mathrm{tr}(AH^{-1}BH^{-1})\,.$$

To see this, write $A = (a^*a)^{\cdot}$, $B = (b^*b)^{\cdot}$ and $H = g^*g$ for paths a, b in $\mathrm{Gl}(r,\mathbb{C})$ through g. Since the tangent map of right multiplication by g^{-1} on $\mathrm{U}(r)\backslash\mathrm{Gl}(r,\mathbb{C})$ is given by

$\dot{a} \mapsto (ag^{-1})^{\cdot}$ and we demanded right multiplication to be an isometry, we get

$$\begin{aligned}\langle A, B\rangle_H &= \langle (a^*a)^{\cdot}, (b^*b)^{\cdot}\rangle_{g^*g}\\ &= \langle [(ag^{-1})^*(ag^{-1})]^{\cdot}, [(bg^{-1})^*(bg^{-1})]^{\cdot}\rangle_{e^*e}\\ &= \mathrm{tr}(g^{-1*}a^*ag^{-1}g^{-1*}b^*bg^{-1})\\ &= \mathrm{tr}(AH^{-1}BH^{-1})\end{aligned}$$

as claimed. To compute the geodesic distances consider

$$d(\mathbf{1}, \exp(A)) = \int_0^1 \langle A\exp(tA), A\exp(tA)\rangle^{\frac{1}{2}}_{\exp(tA)}\, dt = \mathrm{tr}(A^2)^{\frac{1}{2}}\,.$$

The right invariance of the Riemannian metric implies that for $f, g \in \mathrm{Gl}(r, \mathbb{C})$ one has $d(g^*g, f^*f) = d(\mathbf{1}, (fg^{-1})^*fg^{-1})$. In the $\mathrm{End}^+_{h_0}$-picture this means that

$$d(H, K) = d(H^{\frac{1}{2}*}H^{\frac{1}{2}}, K^{\frac{1}{2}*}K^{\frac{1}{2}}) = d(\mathbf{1}, (K^{\frac{1}{2}}H^{-\frac{1}{2}})^*(K^{\frac{1}{2}}H^{-\frac{1}{2}}))\,.$$

Taking the logarithm of the positive self-adjoint endomorphism $(K^{\frac{1}{2}}H^{-\frac{1}{2}})^*(K^{\frac{1}{2}}H^{-\frac{1}{2}})$ and using the formula for $d(\mathbf{1}, \exp(A))$ one then obtains

$$d(H, K) = \mathrm{tr}(\log((K^{\frac{1}{2}}H^{-\frac{1}{2}})^*(K^{\frac{1}{2}}H^{-\frac{1}{2}}))^2)^{\frac{1}{2}}\,.$$

In other words, if λ_i for $i = 1, ..., r$ are the (positive) eigenvalues of $(K^{\frac{1}{2}}H^{-\frac{1}{2}})^*(K^{\frac{1}{2}}H^{-\frac{1}{2}})$, then

$$d(H, K) = d(\lambda_1, ..., \lambda_r) = \left(\sum_i (\log \lambda_i)^2\right)^{\frac{1}{2}}\,.$$

To reassure ourselves of this computation we check by hand that d is in fact symmetric. For convenience we define the map $\Phi\colon \mathcal{H}\times\mathcal{H} \to \mathcal{H}$ by $\Phi(H, K) = (K^{\frac{1}{2}}H^{-\frac{1}{2}})^*(K^{\frac{1}{2}}H^{-\frac{1}{2}})$ and note that $\Phi(H, K)^{-1} = \Phi(K, H)$. Now

$$d(K, H) = d(\lambda_1^{-1}, ..., \lambda_r^{-1}) = \left(\sum_i (\log \lambda_i^{-1})^2\right)^{\frac{1}{2}} = d(\lambda_1, ..., \lambda_r) = d(H, K)\,.$$

We could alternatively observe that $d = \sqrt{}\circ \mathrm{tr} \circ^2 \circ \log \circ \Phi = \sqrt{} \circ \mathrm{tr} \circ^2 \circ \log \circ \Phi^{-1}$.

An important feature of the metric d on $\mathcal{H}$ is its completeness. However, for the purpose of obtaining a C^0-bound on $h(t)$ under HYMF, the metric d is somewhat unwieldy and it is convenient to consider an alternative distance measure σ on $\mathcal{H}$. We define

$$\sigma(H, K) := \mathrm{tr}(HK^{-1}) + \mathrm{tr}(K^{-1}H) - 2n$$

and observe that much like d, σ also factors through Φ and Φ^{-1} in the sense that $\sigma = \mathrm{tr} \circ \Phi + \mathrm{tr} \circ \Phi^{-1} - 2n$. In terms of eigenvalues of $\Phi(H, K)$, we have

$$\sigma(H, K) = \sigma(\lambda_1, ..., \lambda_r) = \sum_i (\lambda_i + \lambda_i^{-1} - 2)\,.$$

Compared to d, σ has the disadvantage that it fails to satisfy the triangle inequality and hence does not define a genuine metric. However, d and σ are equivalent in the sense that they define the same topology on $\mathcal{H}$. In particular, σ can be used to determine whether a given sequence in $\mathcal{H}$ is Cauchy. We establish this equivalence by finding homeomorphisms Ξ, Θ of $[0,\infty[$ such that $d \leqslant \Xi \circ \sigma$ and $\sigma \leqslant \Theta \circ d$. First observe that we can assume all eigenvalues λ_i of $\Phi(H,K)$ to satisfy $\lambda_i \geqslant 1$ since both σ and d are invariant under $\lambda_j \mapsto \lambda_j^{-1}$ for any $1 \leqslant j \leqslant r$. Let λ_k be the biggest of the $\lambda_i \geqslant 1$. Define homeomorphisms $\phi, \psi \colon [1,\infty[\to [0,\infty[$ via $\psi(x) = \log x$ and $\phi(x) = x + x^{-1} - 2$. We estimate

$$\begin{aligned}
\sigma(\lambda_1, ..., \lambda_r) &\leqslant r(\lambda_k + \lambda_k^{-1} - 2) \\
&= r(\phi \circ \psi^{-1})(\log \lambda_k) \\
&= r(\phi \circ \psi^{-1})(\max_i |\log \lambda_i|) \\
&\leqslant r(\phi \circ \psi^{-1})((\textstyle\sum_i (\log \lambda_i)^2)^{\frac{1}{2}}) \\
&= r(\phi \circ \psi^{-1})(d(\lambda_1, ..., \lambda_r))
\end{aligned}$$

and similarly

$$\begin{aligned}
d(\lambda_1, ..., \lambda_r) &\leqslant r \max_i |\log \lambda_i| \\
&= r \log \lambda_k \\
&= r(\psi \circ \phi^{-1})(\lambda_k + \lambda_k^{-1} - 2) \\
&= r(\psi \circ \phi^{-1})(\max_i(\lambda_i + \lambda_i^{-1} - 2)) \\
&\leqslant r(\psi \circ \phi^{-1})(\textstyle\sum_i(\lambda_i + \lambda_i^{-1} - 2)) \\
&= r(\psi \circ \phi^{-1})(\sigma(\lambda_1, ..., \lambda_r))\,.
\end{aligned}$$

Setting $\Xi = r \cdot \psi \circ \phi^{-1}$ and $\Theta = r \cdot \phi \circ \psi^{-1}$ yields the desired homeomorphisms of $[0,\infty[$.

Now let $h(t), k(t)$ be two one-parameter families of Hermitian metrics on E for $t \in [0,T[$ solving equation (3.1). For $\eta = k^{-1}h$ it was $F_h - F_k = \overline{\partial}(\eta^{-1}\partial_k \eta)$. We compute

$$\begin{aligned}
\partial_t \operatorname{tr}(\eta) &= -\operatorname{tr}(k^{-1}(\partial_t k)k^{-1}h) + \operatorname{tr}(k^{-1}(\partial_t h)) \\
&= \operatorname{tr}([\Lambda_\omega i F_k - \lambda \operatorname{id}_E]\,\eta) - \operatorname{tr}(\eta\,[\Lambda_\omega i F_h - \lambda \operatorname{id}_E]) \\
&= -\operatorname{tr}(\eta\,\Lambda_\omega i(F_h - F_k)) \\
&= -\operatorname{tr}(\eta\, i\Lambda_\omega \overline{\partial}(\eta^{-1}\partial_k \eta)) \\
&= \operatorname{tr}(\eta\, i\Lambda_\omega \eta^{-1}(\overline{\partial}\eta)\eta^{-1}\partial_k \eta)) - \operatorname{tr}(\eta\, i\Lambda_\omega \eta^{-1}\overline{\partial}\partial_k \eta)) \\
&= i\Lambda_{\omega(t)} \operatorname{tr}((\overline{\partial}\eta)\eta^{-1}(\partial_k \eta)) - \operatorname{tr}(\Lambda_\omega i \overline{\partial}\partial_k \eta)\,.
\end{aligned}$$

The first term is nonpositive, since it can be written as $-\operatorname{tr}(\eta^{-1} i\Lambda_\omega(\partial_k \eta)(\overline{\partial}\eta))$, where η^{-1} is k-self-adjoint and positive definite while $i\Lambda_\omega(\partial_k\eta)(\overline{\partial}\eta) = i\Lambda_\omega(dz \wedge d\overline{z})\eta_{\partial_k}\eta_{\overline{\partial}} = |dz|^2_\omega \eta_{\partial_k}\eta_{\overline{\partial}}$ is k-self-adjoint and positive semidefinite (we wrote $d_k\eta = \eta_{\partial_k}dz + \eta_{\overline{\partial}}d\overline{z}$, where η_{∂_k} and $\eta_{\overline{\partial}}$ are k-adjoint to one another). The composition of two self-adjoint positive semidefinite endomorphisms has nonnegative trace: if A, B are positive semidefinite and self-adjoint and U diagonalises A, then $\operatorname{tr} AB = \operatorname{tr}(U^*UAU^*UB^{\frac{1}{2}}B^{\frac{1}{2}}) = \operatorname{tr}(A_{\text{diag}}UB^{\frac{1}{2}}U^*UB^{\frac{1}{2}}U^*)$ which can be seen to be nonnegative in a straightforward computation. The second term is $-\operatorname{tr}(\Delta_{\partial,k,\omega}\eta)$, where $\Delta_{\partial,k,\omega} = \partial^*\partial_h$ is the ∂-bundle Laplacian defined by $\omega(t)$ and h acting on endomorphisms of E. A quick computation shows that the trace intertwines the bundle Laplacian and the Laplacian on functions, i.e.

$$\begin{aligned}
\operatorname{tr}\Delta_{\partial,k,\omega}\eta &= \operatorname{tr}(\Lambda_\omega i\bar\partial\partial_k\eta) \\
&= \Lambda_\omega i\bar\partial \operatorname{tr}(\partial\eta + [A_k,\eta]) \\
&= i\Lambda_\omega\bar\partial\partial \operatorname{tr}(\eta) \\
&= \Delta_{\partial,\omega}\operatorname{tr}(\eta)
\end{aligned}$$

owing to the antisymmetry of the commutator. Here $\Delta_{\partial,\omega}$ is the ∂-Laplacian defined by the $\omega(t)$ acting on functions. This permits to estimate

$$(\partial_t + \Delta_{\omega(t)})\operatorname{tr}(\eta) \leqslant 0\,.$$

Reversing the roles of h and k shows that the same estimate holds for $\operatorname{tr}(\eta^{-1})$ and hence also for $\sigma(h,k)$. By the maximum principle we then know that $\sup_X \sigma(h,k)$ is nonincreasing in t. As a byproduct this also reproves that solutions to (3.1) are unique: Let $h(0) = k(0)$, then $\sup_X \sigma(h,k) = 0$ and hence $h(t) = k(t)$ for all $t \in [0,T[$.

We now have what we need to prove

Proposition 3.4.1. *If $h(t)$ is a smooth solution to (3.1) on $[0,T[$, $T < \infty$, then $h(t)$ converges to a continuous Hermitian metric $h(T)$ as $t \to T$.*

Proof. Let t_i be a sequence in $[0,T[$ converging to T in $[0,T]$. The claim is that $h(t_i)$ is Cauchy in the space of Hermitian metrics on E with the complete metric $\sup_X d$. Given $\varepsilon > 0$, find a $\delta > 0$ such that $\sigma(h(t),h(t')) < \varepsilon$ for any $0 \leqslant t,t' < \delta$. Then $\sigma(h(T-t'),h(T-t)) < \varepsilon$ for any $0 < t,t' \leqslant \delta$ since $\sup_X \sigma$ is nonincreasing. Now find N such that $i > N$ implies $T - t_i \leqslant \delta$. □

3.4.2. An a priori C^0-bound on $|\Lambda_\omega iF_h|_h$

The first step in controlling $h(t)$ in higher order is to bounding the contracted curvature $|\Lambda_{\omega(t)}iF_{h(t)}|_{h(t)}$ uniformly in C^0. This is done using the maximum principle. In order to compute $\partial_t|\Lambda_{\omega(t)}iF_{h(t)}|^2_{h(t)}$ we need some preparation.

First we remark that since $\Lambda_\omega iF_h$ is h-self-adjoint, one has $|\Lambda_\omega iF_h|^2_h = \operatorname{tr}((\Lambda_\omega iF_h)^2)$. Less conveniently, the base metric depends on the time parameter and we need to examine how contraction with the base metric behaves under taking derivates. On a Riemann surface this is not too complicated since if one writes $\omega(t) = e^{u(t)}\omega_\infty$, then $\Lambda_{\omega(t)} = e^{-u(t)}\Lambda_{\omega_\infty}$. For a $(1,1)$-form α, the time-derivative of $\Lambda_\omega\alpha$ then simply is $\partial_t\Lambda_\omega\alpha = -\dot u\Lambda_\omega\alpha$.

Remark. On a general Kähler manifold this formula becomes $\partial_t\Lambda_\omega\alpha = \frac{1}{2}\Lambda^2_\omega\alpha \wedge \dot\omega - \Lambda_\omega\dot\omega \cdot \Lambda_\omega\alpha$. If $\alpha = iF_h$, then the first term could potentially see the full curvature (and not only the contracted part) preventing us from applying the maximum principle to $|\Lambda_\omega F_h|^2$. On a Riemann surface this difficulty cannot arise.

Next we compute how the curvature behaves under the flow:

$$\begin{aligned}
\partial_t F_h &= \partial_t \overline{\partial}(h^{-1}\partial h) \\
&= -\overline{\partial}((h^{-1}\partial_t h)(h^{-1}\partial h)) + \overline{\partial}(h^{-1}\partial(\partial_t h)) \\
&= -\overline{\partial}((h^{-1}\partial_t h)(h^{-1}\partial h)) + \overline{\partial}((h^{-1}\partial h)(h^{-1}\partial_t h)) + \overline{\partial}\partial(h^{-1}\partial_t h) \\
&= \overline{\partial}([A_h, h^{-1}\partial_t h] + \partial(h^{-1}\partial_t h)) \\
&= \overline{\partial}\partial_h(h^{-1}\partial_t h) \\
&= -\overline{\partial}\partial_h \Lambda_\omega i F_h \\
&= -\overline{\partial}\overline{\partial}_h^* F_h \\
&= -\Delta_{\overline{\partial},h,\omega} F_h \,,
\end{aligned}$$

where in the last two steps we have used that $d_h F_h = 0$ (for degree reasons on a Riemann surface and by the Bianchi identity in general). The identities relating the different Laplacians derived in 3.2.3 imply $2\Delta_{\overline{\partial},h,\omega}F_h = 2\Delta_{\partial,h,\omega}F_h = \Delta_{d,h,\omega}F_h$, so the curvature F_h solves the bundle heat equation

$$(\partial_t + \Delta_{\overline{\partial},h,\omega})F_h = (\partial_t + \Delta_{\partial,h,\omega})F_h = (\partial_t + 1/2 \cdot \Delta_{d,h,\omega})F_h = 0\,.$$

In order to examine the behaviour of the contracted curvature under Hermitian Yang–Mills flow, we use the Kähler identities to observe

$$\begin{aligned}
\Delta_{\partial,h,\omega}\Lambda_\omega \xi &= \Lambda_\omega \Delta_{\overline{\partial},h,\omega}\xi \,, \\
\Delta_{\overline{\partial},h,\omega}\Lambda_\omega \xi &= \Lambda_\omega \Delta_{\partial,h,\omega}\xi \,, \\
\Delta_{d,h,\omega}\Lambda_\omega \xi &= \Lambda_\omega \Delta_{d,h,\omega}\xi
\end{aligned}$$

for $\xi \in \Omega^2(X, \mathrm{End}(E))$. For the contracted curvature one then obtains

$$(\partial_t + \Delta_{\partial,h,\omega})\Lambda_\omega i F_h = (\partial_t + \Delta_{\overline{\partial},h,\omega})\Lambda_\omega i F_h = (\partial_t + \frac{1}{2}\Delta_{d,h,\omega})\Lambda_\omega i F_h = -\dot{u}\Lambda_\omega i F_h\,.$$

As a last preparation, we observe that for $\varphi \in \Omega^0(X, \mathrm{End}(E))$ there holds

$$\begin{aligned}
\Delta_{\partial,h,\omega}(\varphi^2) &= \varphi \cdot \Delta_{\partial,h,\omega}\varphi + \Delta_{\partial,h,\omega}\varphi \cdot \varphi + i\Lambda_\omega(\overline{\partial}\varphi \wedge \partial_h\varphi - \partial_h\varphi \wedge \overline{\partial}\varphi)\,, \\
\Delta_{\overline{\partial},h,\omega}(\varphi^2) &= \varphi \cdot \Delta_{\overline{\partial},h,\omega}\varphi + \Delta_{\overline{\partial},h,\omega}\varphi \cdot \varphi + i\Lambda_\omega(\overline{\partial}\varphi \wedge \partial_h\varphi - \partial_h\varphi \wedge \overline{\partial}\varphi)\,.
\end{aligned}$$

We abbreviate $\kappa := \Lambda_\omega i F_h$ (this in not the Kähler class containing the metrics $\omega(t)$ which we also denoted by κ) and compute

$$\begin{aligned}
(\partial_t + \Delta_{\partial,\omega})|\kappa|_h^2 &= \mathrm{tr}[(\partial_t + \Delta_{\partial,h,\omega})\kappa^2] \\
&= 2\,\mathrm{tr}[\kappa(\partial_t + \Delta_{\partial,h,\omega})\kappa + i\Lambda_\omega(\overline{\partial}\kappa \wedge \partial_h\kappa)] \\
&= -2\dot{u}\,\mathrm{tr}[\kappa^2] + 2\,\mathrm{tr}[i\Lambda_\omega(\overline{\partial}\kappa \wedge \partial_h\kappa)] \\
&= -2\dot{u}|\kappa|_h^2 - |d_h\kappa|_{h,\omega}^2\,,
\end{aligned}$$

where we used the identity $2\,\mathrm{tr}[i\Lambda_\omega(\partial_h\kappa \wedge \overline{\partial}\kappa)] = |d_h\kappa|^2_{h,\omega}$ in the last step. Since the sign is crucial in our case, we take some space to go through the computation in detail. We prove the more general relation for self-adjoint $\varphi, \psi \in \Omega^0(X, \mathrm{End}(E))$:

$$(d_h\varphi, d_h\psi)_{h,\omega} = i\,\mathrm{tr}[\Lambda_\omega(\partial_h\varphi \wedge \overline{\partial}\psi + \partial_h\psi \wedge \overline{\partial}\varphi)]\,.$$

Write $d_h\varphi = \varphi_{\overline{\partial}}d\bar{z} + \varphi_\partial dz$. The endomorphisms $\varphi_{\overline{\partial}}$ and φ_∂ are then h-adjoint to each other and the same holds when replacing φ by ψ. Recalling $(\alpha, \beta)_\omega = *(\alpha \wedge *\bar{\beta})$ we compute

$$(d_h\varphi, d_h\psi)_{h,\omega} = (\varphi_\partial, \psi_\partial)_h|dz|^2_\omega + (\varphi_{\overline{\partial}}, \psi_{\overline{\partial}})_h|d\bar{z}|^2_\omega = \mathrm{tr}[\varphi_\partial\psi_{\overline{\partial}} + \psi_\partial\varphi_{\overline{\partial}}] * i(dz \wedge d\bar{z}) \quad (3.5)$$

on the one hand and

$$\mathrm{tr}[i\Lambda_\omega(\partial_h\varphi \wedge \overline{\partial}\psi + \partial_h\psi \wedge \overline{\partial}\varphi)] = \mathrm{tr}[\varphi_\partial\psi_{\overline{\partial}} + \psi_\partial\varphi_{\overline{\partial}}]\Lambda_\omega i(dz \wedge d\bar{z}) \quad (3.6)$$

on the other. Here, Λ_ω and $*$ coincide on the volume form ω and hence on all two-forms, so combining (3.5) and (3.6) yields the result.

The convergence assumptions on $u(t)$ imply that $\int_0^\infty \sup_X |\dot{u}| < \infty$, allowing the use of the maximum principle to obtain a uniform bound on $|\kappa|^2_h$. Indeed, the estimate

$$(\partial_t + \Delta_{\partial,\omega})|\kappa|^2_h \leqslant 2\sup_X |\dot{u}| \cdot |\kappa|^2_h$$

implies that the solution to $\partial_t\Phi(t) = \sup_X |\dot{u}|\Phi(t)$ satisfying the initial condition $\Phi(0) = \sup_X |\Lambda_{\omega_0} iF_{h_0}|^2$ dominates $|\kappa|^2_h$, so

$$|\kappa|^2_h \leqslant \Phi(t) = \Phi(0)\exp\left(\int_0^t \sup_X |\dot{u}|(s)\,ds\right)\,.$$

By assumption, the right hand side is uniformly bounded in t. We now know $|iF_{h(t)}|_h$ and hence the curvature $iF_{h(t)}$ to be uniformly bounded in $[0, T[$, even if $T = \infty$. Since $h(t)$ converges to a continuous metric $h(T)$ for $t \to T$, κ also remains bounded with respect to any reference metric, e.g. h_0. It follows then from the blowup argument (cf. Appendix C.3) that $h(t)$ is in fact bounded uniformly on $[0, T[$ in C^1 with respect to some fixed C^1-structure.

3.4.3. Higher a priori bounds on $\Lambda_\omega iF_h$

The principal ingredient in our proof for higher regularity is the fact that the contracted curvature $\kappa = \Lambda_\omega iF_h$ satisfies the linear parabolic PDE

$$\left(\partial_t + \frac{1}{2}\Delta_{d,h,\omega}\right)\kappa = -\dot{u}\kappa\,, \quad (3.7)$$

where we consider the right hand side as an inhomogeneity which is a bounded continuous section of $\mathrm{End}(E)$ over $X \times [0, T[$. The nonautonomous generator $2L := \Delta_{d,h,\omega}$ can be expressed as

$$\Delta_{d,h,\omega} = \Delta_{d,h_0,\omega} + \mathrm{ad}_{\Lambda_\omega i(F_h - F_{h_0})} - 2i\Lambda_\omega[\eta^{-1}\partial_{h_0}\eta, \overline{\partial}\,\cdot\,]$$

for $\eta = h_0^{-1}h$. The a priori C^0-bound on F_h and the C^1-bound on h imply that the coefficients of L are bounded in C^0 on $X \times [0, T[$. In addition, the highest order coefficient only depends on t via $\Delta_{d,h_0,\omega} = e^{-u}\Delta_{d,h_0,\omega_\infty}$, so a constant of ellipticity of L can be found independently of t. The idea is to use a classical inner L^p and Schauder estimates to infer boundedness of the equation's solution, κ, in higher parabolic Sobolev and Hölder spaces which in turn will increase control over the coefficients as well as the inhomogeneity. We will see that one can in fact bootstrap to uniformly bound κ in C^k for any $k \in \mathbb{N}_0$.

We adopt the notation and regularity result of Appendix A.4 (with $\mathrm{End}(E)$ taking the role of the vector bundle E in the appendix). The initial regularity results for the inhomogeneity $-\dot{u}\kappa$ and the coefficients of L establish that the hypotheses of Proposition A.4.3 are satisfied and we obtain $\|\kappa\|_{L^p_{1,2}(X_T,\omega_\infty,h_0)} < \infty$ (technically, Proposition A.4.3 only bounds $\|\kappa\|_{L^p_{1,2}(X'_T,\omega_\infty,h_0)}$ for $X'_T = X \times [\varepsilon, T[$, but for small times, say $0 \leqslant t \leqslant \varepsilon$, smoothness of κ already implies boundedness in C^k for any $k \in \mathbb{N}_0$). Consequently, the coefficients of L and the inhomogeneity are also bounded in $L^p_{1,2}$ (for any p) and hence in $C^{0,0,\alpha}$ for a sufficiently small $\alpha > 0$ by the parabolic Sobolev embedding A.4.5. The Schauder regularity theory in Proposition A.4.4 then implies that κ is bounded in $C^{1,2,\alpha}(X, \omega_\infty, h_0)$, again implying the same regularity on coefficients and inhomogeneity. Repeatedly using A.4.4, the argument can be iterated to show that κ is bounded in $C^{k,2k,\alpha}$ for any k (the bounds may of course grow in k). In particular, κ is for any $k \in \mathbb{N}_0$ bounded in $C^k(X, \omega_\infty, h_0)$ uniformly in $t \in [0, T[$.

It remains to examine how the uniform boundedness of κ in C^k implies that of h or equivalently $\eta = h_0^{-1}h$, provided η is already bounded in C^1. The key lies in using (3.2) and the Kähler identities to write

$$\Delta_{\partial,h_0,\omega}\eta = \eta\Lambda_\omega iF_h + i\Lambda_\omega(\bar{\partial}\eta)\eta^{-1}(\partial_{h_0}\eta) - \eta\Lambda_\omega iF_{h_0} \tag{3.8}$$

and using elliptic regularity theory in a similar fashion to the parabolic theory used above to bound $\kappa = \Lambda_\omega iF_h$ in C^k, $k \in \mathbb{N}_0$. First observe that the right hand side is bounded in C^0 which gives a bound in L^p_2 of η. For sufficiently high p, L^p_2 embeds into $C^{1+\alpha}$, so the right hand side is in fact uniformly bounded in $C^{0,\alpha}$. Elliptic Schauder theory then bounds η in $C^{2,\alpha}$ and the right hand side in $C^{1,\alpha}$. A bootstrapping argument then gives the desired uniform bound on η in C^k, $k \in \mathbb{N}_0$.

Remark. As an alternative to the parabolic regularity theory, one can also follow Donaldson's approach and apply the maximum principle to $|\nabla^k\kappa|_{h,\omega}$ and use induction on k to show that $\nabla^k\kappa$ is exponentially bounded. The advantage of our method is that it also works to establish uniform C^k-bounds on κ uniform in $t \in [0, \infty[$.

3.5. Convergence for $t \to \infty$

To show convergence of $h(t)$ as $t \to \infty$ in C^∞ we proceed in three steps. The first one is to show that $h(t)$ remains bounded in C^0 which is achieved by solving Hermitian Yang–Mills flow for very special initial conditions (which require slope stability of E) yielding a convergent one-parameter family of metrics $k(t)$ and using the fact that $\sigma(h, k)$

is nonincreasing. In the second step we use the Donaldson functional M to construct a sequence of times $(t_i) \to \infty$ for which $\kappa(t_i)$ converges to $\lambda\,\mathrm{id}_E$ in L^2 and conclude that $h(t)$ converges in C^1 to a ω_∞-Hermite–Einstein h_∞ metric on E. Finally, a variation of the parabolic regularity argument used to show long-time existence provides uniform bounds on κ in C^k, $k \in \mathbb{N}_0$ implying that the convergence of $h(t)$ to h_∞ occurs in C^∞.

3.5.1. Uniform C^0-boundedness of $h(t)$

We exploit that we are working over a Riemann surface by writing $\omega(t) = e^{u(t)}\omega_\infty$ (as before) and the fact that $[\omega]$-slope stability implies the existence of a ω_∞-Hermite–Einstein metric h_∞. They key observation is that the conformal class of Hermitian metrics containing h_∞ — that is the set of metrics $e^f h_\infty$ for a smooth real valued function f on X — is stable under Hermitian Yang–Mills flow. More specifically, if $k_0 = e^{f_0}h_\infty$, then solving the original flow equation

$$k^{-1}\partial_t k = -[e^{-u}\Lambda_{\omega_\infty} iF_k - \lambda\,\mathrm{id}_E]$$

is equivalent to finding an $f(t)$ satisfying

$$(\partial_t + \Delta_{\partial,\omega(t)})f = (1 - e^{-u})\lambda\,, \tag{3.9}$$

with initial condition $f(0) = f_0$. Too see this, observe that $F_{e^f h} = F_h + \bar{\partial}\partial f\,\mathrm{id}_E$. If f solves (3.9), set $k = e^f h_\infty$ and compute

$$\begin{aligned}
k^{-1}\partial_t k &= \partial_t f\,\mathrm{id}_E \\
&= -\Delta_{\omega(t)} f\,\mathrm{id}_E + (1 - e^{-u})\lambda\,\mathrm{id}_E \\
&= -\Delta_{\omega(t)} f\,\mathrm{id}_E - e^{-u}\Lambda_{\omega_\infty} iF_{h_\infty} + \lambda\,\mathrm{id}_E \\
&= -[\Lambda_{\omega(t)}(iF_{h_\infty} + i\bar{\partial}\partial f\,\mathrm{id}_E) - \lambda\,\mathrm{id}_E] \\
&= -[\Lambda_{\omega(t)} iF_{k(t)} - \lambda\,\mathrm{id}_E]\,.
\end{aligned}$$

The evolution equation (3.9) for f is just heat flow on functions with an inhomogeneity $(1-e^{-u})\lambda$ decaying exponentially to 0 (in C^∞), so f converges exponentially to a constant function and $k(t)$ to a multiple of h_∞ in C^∞. Since adding a constant to a solution of (3.9) gives a new solution, we can arrange for this multiple to be one.

Now if $h(t)$ with arbitrary initial condition h_0 is a long-time solution to HYMF, then we can consider the C^0-distance measure $\sigma(h(t), k(t))$, where $k(t)$ is a solution to HYMF starting in the conformal class as h_∞. We already know that $\sigma(h(t), k(t))$ is decreasing, so we can compute

$$d(h(t), h_\infty) \leqslant d(h(t), k(t)) + d(k(t), h_\infty) \leqslant \Xi(\sigma(h(t), k(t))) + \Xi(\sigma(k(t), h_\infty))\,,$$

where $\Xi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a orientation preserving diffeomorphism comparing d to σ. The first term on the right hand side is decreasing and the second tends to 0, hence $h(t)$ is bounded in C^0 uniformly for $t \in [0, \infty[$.

3.5.2. Convergence of $h(t)$ to h_∞ in C^1

For the Kähler metric ω_∞ we consider the *Donaldson functional* M defined on pairs of Hermitian metrics which satisfies the following properties:

- for Hermitian metrics h, k, l one has
$$M(h,l) + M(l,k) = M(h,k)\,,$$
- the variational property
$$\delta_h M(h_0,h)\cdot\eta = \int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)h^{-1}\eta]\omega_\infty$$
- and boundedness of $M(h_0,h)$ from below for a fixed reference metric h_0 and h varying, i.e.
$$\inf_h M(h_0,h) \geqslant -C(h_0)\,.$$

The last property uses stability of the bundle $E \to X$. For a construction of M, see Appendix C.2.

The variation of $M(h_0,\cdot)$ in direction of Hermitian Yang–Mills flow is

$$\begin{aligned}\partial_t M(h_0,h(t)) &= -\int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)(\Lambda_{\omega(t)} iF_h - \lambda\,\mathrm{id}_E)]\,\omega_\infty \\ &= -\int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)]\,\omega_\infty \\ &\quad -\int_X \operatorname{tr}[(\Lambda_{\omega(t)} - \Lambda_{\omega_\infty}) iF_h\,(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)]\,\omega_\infty\,.\end{aligned} \tag{3.10}$$

Reexpressing $(\Lambda_{\omega(t)} - \Lambda_{\omega_\infty}) = (e^{u(t)}-1)\Lambda_{\omega_\infty}$ and using the Cauchy–Schwarz inequality on the trace in the second integral, we estimate

$$\begin{aligned}\partial_t M(h_0,h(t)) &\leqslant -\int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)^2]\,\omega_\infty \\ &\quad +\sup_X(e^{u(t)}-1)\cdot\sup_{X,t}(\operatorname{tr}[(\Lambda_{\omega_\infty}F_{h(t)})^2]^{\frac{1}{2}})\int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)^2]^{\frac{1}{2}}\,\omega_\infty \\ &\leqslant -\int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)^2]\,\omega_\infty \\ &\quad +C\sup_X(e^{u(t)}-1)\left(\int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)^2]\,\omega_\infty\right)^{\frac{1}{2}} \\ &= -A^2 + BA \\ &= -A(A-B)\end{aligned}$$

with $A = (\int_X \operatorname{tr}[(\Lambda_{\omega_\infty} iF_h - \lambda\,\mathrm{id}_E)^2]\,\omega_\infty)^{\frac{1}{2}}$ and $B = C\sup_X(e^{u(t)}-1)$. By the assumptions on the convergence properties of $u(t)$, one deduces $B(t) = C\sup_X(e^{u(t)}-1) \to 0$. We consider two cases:

- $\forall t > 0 \exists t' > t$: $A(t') - B(t') \leqslant 0$. In this case there exists a sequence of times $t_i \to \infty$ such that $0 \leqslant A(t_i) \leqslant B(t_i)$, so $A(t_i)$ converges to 0.
- $\exists t' > 0 \forall t > t'$: $A(t) - B(t) > 0$. In this case $\partial_t M(h_0, h(t))$ is negative for $t > t'$. Since $M(h_0, h(t))$ is bounded from below, we then know that there exists a sequence of times $t_i \to \infty$ such that $\partial_t M(h_0, h(t_i))$ converges to 0, but reexamining (3.10), we see this is only possible if $A(t_i)$ goes to 0 (the other summand already tends to 0).

Either way, we find a sequence $t_i \to \infty$, such that $A(t_i) = \|\Lambda_{\omega_\infty} iF_{h(t_i)} - \lambda \operatorname{id}_E \|_{L^2(X,h,\omega_\infty)}$ converges to 0. The pointwise inner product on $\operatorname{End}(E)$ depends of course on h, but since $h(t)$ remained uniformly bounded for all times, we also have $\|\Lambda_{\omega_\infty} iF_{h(t_i)} - \lambda \operatorname{id}_E \|_{L^2(X,h_0,\omega_\infty)} \to 0$.

We can now rely on the usual arguments to obtain further information about $h(t)$ as $t \to \infty$. Since h and $\Lambda_\omega F_h$ are bounded in C^0, the blowup argument in Appendix C.3 gives boundedness of h in C^1. Recall from (3.8) that

$$\Delta_{\partial,h_0,\omega}\eta = \eta\Lambda_\omega iF_h + i\Lambda_\omega(\overline{\partial}\eta)\eta^{-1}(\partial_{h_0}\eta) - \eta\Lambda_\omega iF_{h_0}\,.$$

The right hand side is bounded in C^0, so by elliptic regularity it follows that $h(t)$ is bounded in L^p_2 (for any p). We turn back to the sequence $t_i \to \infty$ for which we had that $\Lambda_{\omega_\infty} F_{h(t_i)} \to \lambda \operatorname{id}_E$ in L^2. Since $L^p_2 \hookrightarrow C^1$ compactly (for sufficiently high p), we can assume h_i to be convergent in C^1 by passing to a subsequence. The same formula then implies that $\Delta_{h_0,\omega}\eta(t_i)$ converges in L^2, so by elliptic regularity $h(t_i)$ converges in L^2_2. Since $h(t_i)$ is a sequence satisfying $\Lambda_\infty F_{h(t_i)} \to \lambda \operatorname{id}_E$ in L^2, the limit has to be a Hermite–Einstein metric h_∞. Looking again at the convergence properties of $h(t)$, we observe that for the particular sequence $h(t_i)$ constructed above, $\sigma(h(t_i), k(t_i))$ converges to 0 (after a rescaling of $k(t)$, so that its limit is h_∞). But $\sigma(h(t), k(t))$ is decreasing, so $h(t)$ converges to h_∞ in C^0. Now since $h(t)$ is bounded in L^p_2 which embeds compactly into C^1, the convergence $h(t) \to h_\infty$ occurs in C^1 (by Lemma A.2.2 in Appendix A.2).

3.5.3. Convergence of $h(t)$ in C^∞

In order to establish the convergence of $h(t)$ in C^∞ we show boundedness of κ in C^k for any $k \in \mathbb{N}_0$ by slightly modifying the argument used to show uniform boundedness of κ in $t \in [0, T[$. Recall that the maximum principle argument in 3.4.2 showed boundedness of $\|\kappa\|_h$ uniformly for $t \in [0, \infty[$. Since $h(t)$ converges in C^1, this implies uniform boundedness of $|\kappa|_{h_0}$. By equation (3.8) $\Delta_{h_0}\eta$ is thus bounded uniformly in C^0 for all times. Now cover $\{t \in \mathbb{R} \,|\, t \geqslant 0\}$ by intervals $I_j = [j, j+2[$, $j \in \mathbb{N}_0$. On each of the intervals we invoke the $L^p_{1,2}$ regularity estimates of Proposition A.4.3 to obtain bounds $\|\kappa\|_{L^p_{1,2}(X\times[j+\varepsilon,j+2[,\omega_\infty,h_0)} < C$ with a constant C independent of j. Since the intervals overlap and κ is bounded for finite times in any C^k, we obtain that $\|\kappa\|_{L^p_{1,2}(X\times[j,j+2[,\omega_\infty,h_0)} < C$. Using the parabolic Sobolev embedding A.4.5 then gives $\|\kappa\|_{C^{0,0,\alpha}(X\times[j,j+2[,\omega_\infty,h_0)} < C$. Now repeating the argument with the

Schauder estimates of Proposition A.4.3, one has $\|\kappa\|_{C^{k,2k,\alpha}(X\times[j,j+2[,\omega_\infty,h_0)} < C$ and indeed $\|\kappa\|_{C^{k,2k,\alpha}(X\times[0,\infty[,\omega_\infty,h_0)} < C$ for any $k \in \mathbb{N}_0$. In particular, $h(t)$ is bounded in C^k. Lemma A.2.2 in Appendix A.2 then implies that the convergence of $h(t)$ as $t \to \infty$ occurs in C^∞.

3.6. Exponentiality of Convergence

The main result of this section is the following exponentiality result:

Proposition 3.6.1. *Under the assumptions in Theorem 3.1.1, the renormalised contracted curvature defined by a solution $h(t)$ to time-dependent Hermitian Yang–Mills flow, $\tilde\kappa := \Lambda_\omega iF_h - \lambda\,\mathrm{id}_E = \kappa - \lambda\,\mathrm{id}_E$, tends to zero at an exponential rate in each C^l. More precisely, for each $l \in \mathbb{N}$ there exist positive constants C_l, α_l, such that $\|\Lambda_\omega iF_h - \lambda\,\mathrm{id}_E\|_{C^l(X)} \leqslant C_l e^{-\alpha_l t}$.*

The idea of the proof is to show that the positive L^2-products $\langle\tilde\kappa, \Delta^k\tilde\kappa\rangle_{h,\omega}$ decay exponentially and deduce that all L^2_k-norms of $\tilde\kappa$ tends to 0 exponentially fast. Sobolev embeddings into C^l then give the behaviour as claimed.

Once the exponential decay of $\tilde\kappa$ is established, one then deduces exponential convergence of $h(t)$ as follows. Fix a $k \in \mathbb{N}_0$. Since $h(t)$ is bounded in C^k and $\partial_t h = -h\tilde\kappa$, one obtains that $\partial_t h$ and its k^{th} spatial derivatives tend to zero at an exponential rate. Estimating $h_\infty - h(t) = \int_t^\infty (\partial_t h)(s)\,ds$ then implies that $h(t) - h_\infty$ and its k^{th} spatial derivatives also tend to zero exponentially.

Some technical preparation is required to establish the claimed decay properties of $\tilde\kappa$.

3.6.1. Poincaré-Inequality on $\Omega^0(X, \mathrm{End}(E))$

Throughout this section let $\Delta = \Delta_{d,h,\omega}$ denote the full d-Laplacian on $\Omega^0(X, \mathrm{End}(E))$ or occasionally on $\Omega^1(X, \mathrm{End}(E))$ defined by ω and h. We claim that $\Delta^k\tilde\kappa(t)$, $k \geqslant 0$ is L^2-orthogonal (with respect to the metrics $\omega(t), h(t)$) to the kernel of d_h for any $t \in [0,\infty[$. For $k > 0$ this follows from the self-adjointness of Δ and for $k = 0$ from the fact that the only holomorphic endomorphisms of $[\omega]$-stable bundles over compact Kähler manifolds (X, ω) are multiples of id_E. As the kernel of d_h is contained in $H^0(X, \mathrm{End}(E))$, we find $\ker d_h \colon \Omega^0(X, \mathrm{End}(E)) \to \Omega^1(X, \mathrm{End}(E)) = \{c\,\mathrm{id}_E \,|\, c \in \mathbb{C}\}$ and compute

$$\langle\tilde\kappa, \mathrm{id}_E\rangle_{h,\omega} = \int_X \mathrm{tr}(\Lambda_\omega iF_h - \lambda\,\mathrm{id}_E)\omega = 0$$

to prove our claim. The importance in this observation lies in the fact that we have Poincaré-inequalities

$$\|\Delta^k\tilde\kappa\|_{L^2,h,\omega} \leqslant C\|d_h\Delta^k\tilde\kappa\|_{L^2,h,\omega}\,, \quad \|d_h\Delta^k\tilde\kappa\|_{L^2,h,\omega} \leqslant C\|\Delta^{k+1}\tilde\kappa\|_{L^2,h,\omega}$$

where the constant C can be chosen independent of t since convergence of the metrics $h(t)$ and $\omega(t)$ in C^∞ implies convergence of the spectrum of their associated Laplacians. Furthermore one has elliptic estimates of the form

$$\|\tilde\kappa\|_{L^2_{2k},h,\omega} \leqslant K\|\Delta^k\tilde\kappa\|_{L^2,h,\omega}$$

where K depends on k but can again be chosen independently of t.

3.6.2. Time dependence of the Laplacian

When taking the pointwise inner product of endomorphisms $\alpha, \beta \in \Omega^0(X, \operatorname{End}(E))$, the real part of $(\alpha, \beta)_{h,\omega}$ is given by the product of the self-adjoint parts of α and β while the imaginary part is the product of their skew-adjoint parts. The mixed products vanish. The same is true for the L^2-product. In our case we will only care about the real part and it is useful to split time-derivatives of $\Delta^k \tilde{\kappa}$ into self-adjoint and skew parts.

Let $\alpha(t)$ be a smooth one-parameter family in $\Omega^0(X, \operatorname{End}(E))$. If h is a smooth path of Hermitian metrics on E with $h^{-1}\partial_t h = \rho$, then

$$(\partial_t \alpha)^h = \partial_t(\alpha^h) + [\rho, \alpha^h]\,.$$

If in addition $\alpha(t)$ is $h(t)$-self-adjoint for each t, then

$$\partial_t \alpha = \left(\partial_t \alpha + \frac{1}{2}[\rho, \alpha]\right) - \frac{1}{2}[\rho, \alpha]$$

where the first summand is the h-self-adjoint and the second the h-skew-adjoint. In our case we have $\rho = -\tilde{\kappa}$ and a computation gives

$$\partial_t \Delta\alpha = \left[-\dot{u}\Delta\alpha + \Delta\left(\dot{\alpha} - \frac{1}{2}[\kappa, \alpha]\right) + i\Lambda_\omega[d_h\kappa, d_h\alpha]\right] + \frac{1}{2}[\kappa, \Delta\alpha]\,,$$

where the two summands represent again the self- and skew-adjoint parts. Now induction on k shows that

$$\partial_t \Delta^k \kappa = \left[-\frac{1}{2}\Delta^{k+1}\kappa + \sum_{j=0}^{k-1} \Delta^j i\Lambda_\omega[d_h\kappa, d_h\Delta^{k-1-j}\kappa] - \sum_{j=0}^{k} \Delta^j(\dot{u}\Delta^{k-j}\kappa)\right] + \frac{1}{2}[\kappa, \Delta^k\kappa]\,. \tag{3.11}$$

Note that one can replace κ by $\tilde{\kappa}$ in this formula and that the first summand is again self-adjoint while the second is skew. Using the Kähler identities we can simplify the expression $i\Lambda_\omega[d_h\kappa, d_h\Delta^{k-1-j}\kappa]$ a bit. One finds that for $\alpha, \beta \in \Omega^0(X, \operatorname{End}(E))$ it is

$$i\Lambda_\omega[d_h\alpha, d_h\beta] = d_h^*[(\overline{\partial} - \partial_h)\alpha, \beta] + [\operatorname{ad}_\kappa \alpha, \beta]\,,$$

which in the case at hand gives

$$i\Lambda_\omega[d_h\kappa, d_h\Delta^{k-1-j}\kappa] = d_h^*[(\overline{\partial} - \partial_h)\kappa, \Delta^{k-1-j}\kappa]\,. \tag{3.12}$$

3.6.3. Time-derivatives of $\langle\tilde{\kappa}, \Delta^k\tilde{\kappa}\rangle_{h,\omega}$

We use (3.11) to compute the time-derivative of the positive expression $f_k := \langle\tilde{\kappa}, \Delta^k\tilde{\kappa}\rangle_{h,\omega}$. The positivity stems from the self-adjointness of the Laplacians which permits to write the L^2-product as $\|\Delta^{k/2}\tilde\kappa\|^2_{h,\omega}$ if k is even or $\|d_h\Delta^{(k-1)/2}\tilde\kappa\|^2_{h,\omega}$ if k is odd. One obtains (omitting the subscripts of the pointwise inner product $(\cdot,\cdot)$) that

$$\begin{aligned}\partial_t\langle\tilde\kappa,\Delta^k\tilde\kappa\rangle_{h,\omega} &= \partial_t\int_X(\tilde\kappa,\Delta^k\tilde\kappa)\,\omega\\ &= \int(\partial_t\tilde\kappa,\Delta^k\tilde\kappa)\,\omega+\int_X(\tilde\kappa,\partial_t\Delta^k\tilde\kappa)\,\omega+\int_X(\tilde\kappa,\Delta^k\tilde\kappa)\dot u\,\omega\\ &= -\int_X(\tilde\kappa,\Delta^{k+1}\tilde\kappa)\,\omega-\sum_{j=0}^{k}\int_X(\tilde\kappa,\Delta^j(\dot u\Delta^{k-j}\tilde\kappa))\,\omega\\ &\quad+\sum_{j=0}^{k-1}\int_X(\tilde\kappa,\Delta^j i\Lambda_\omega[d_h\tilde\kappa,d_h\Delta^{k-1-j}\tilde\kappa])\,\omega\,.\end{aligned}\tag{3.13}$$

Examining the terms a bit closer one can find the symmetries

$$\int_X(\tilde\kappa,\Delta^j(\dot u\Delta^{k-j}\tilde\kappa))\,\omega=\int_X(\tilde\kappa,\Delta^{k-j}(\dot u\Delta^j\tilde\kappa))\,\omega\tag{3.14}$$

via integration by parts. Less obvious are the symmetries

$$\int_X(\tilde\kappa,\Delta^j i\Lambda_\omega[d_h\tilde\kappa,d_h\Delta^{k-1-j}\tilde\kappa])\,\omega=\int_X(\tilde\kappa,\Delta^{k-1-j}i\Lambda_\omega[d_h\tilde\kappa,d_h\Delta^j\tilde\kappa])\,\omega\,.\tag{3.15}$$

One way to see them is to compute

$$\begin{aligned}\partial_t\langle\Delta^l\tilde\kappa,\Delta^{k-l}\tilde\kappa\rangle_{h,\omega} &= -\int_X(\tilde\kappa,\Delta^{k+1}\tilde\kappa)\,\omega-\sum_{m=0}^{k}\int_X(\tilde\kappa,\Delta^m(\dot u\Delta^{k-m}\tilde\kappa))\,\omega\\ &\quad+\sum_{m=0}^{l-1}\int_X(\Delta^{k-l+m}\tilde\kappa,i\Lambda_\omega[d_h\tilde\kappa,d_h\Delta^{l-1-m}\tilde\kappa])\,\omega\\ &\quad+\sum_{m=0}^{k-l-1}\int_X(\Delta^{l+m}\tilde\kappa,i\Lambda_\omega[d_h\tilde\kappa,d_h\Delta^{k-l-1-m}\tilde\kappa])\,\omega\,.\end{aligned}$$

As the left hand side and the first two terms on the right hand side agree for all $l=0,\cdots,k$, so do the remaining terms. Taking their difference for $l=j$ and $l=j+1$ yields the desired relations.

We now estimate the terms of (3.13) separately.

- For the leading order term $-\int_X(\tilde\kappa,\Delta^{k+1}\tilde\kappa)\,\omega=-f_{k+1}$ we simply use the Poincaré-inequality $f_k\leqslant Cf_{k+1}$, so

$$-\int_X(\tilde\kappa,\Delta^{k+1}\tilde\kappa)\,\omega\leqslant -C^{-1}f_k\,.$$

- To estimate the absolute value of the second term, we can estimate each summand $\int_X(\tilde\kappa, \Delta^j(\dot u \Delta^{k-j}\tilde\kappa))\,\omega$ individually. Using the symmetry (3.14) we may assume $j \geqslant k/2$. If $k = 2q$ is even, then

$$\begin{aligned}
\int_X(\tilde\kappa, \Delta^j(\dot u\Delta^{k-j}\tilde\kappa))\,\omega &= \int_X(\Delta^q\tilde\kappa, \Delta^{j-q}(\dot u\Delta^{2q-j}\tilde\kappa))\,\omega \\
&\leqslant \|\Delta^q\tilde\kappa\,\|_{L^2}\|\Delta^{j-q}(\dot u\Delta^{2q-j}\tilde\kappa)\|_{L^2} \\
&\leqslant f_k^{1/2}\|\dot u\Delta^{2q-j}\tilde\kappa\|_{L^2_{2(j-q)}} \\
&\leqslant f_k^{1/2}c\|\dot u\|_{C^{2(j-q)}}\|\Delta^{2q-j}\tilde\kappa\|_{L^2_{2(j-q)}} \\
&\leqslant f_k^{1/2}c\|\dot u\|_{C^{2(j-q)}}\|\tilde\kappa\|_{L^2_k} \\
&\leqslant f_k^{1/2}cK\|\dot u\|_{C^{2(j-q)}}\|\Delta^q\tilde\kappa\|_{L^2} \\
&= cK\|\dot u\|_{C^{2(j-q)}}\cdot f_k\,.
\end{aligned}$$

  Similarly, if $k = 2q+1$ is odd, then

$$\begin{aligned}
\int_X(\tilde\kappa, \Delta^j(\dot u\Delta^{k-j}\tilde\kappa))\,\omega &= \int_X(d_h\Delta^q\tilde\kappa, d_h\Delta^{j-q}(\dot u\Delta^{2q-j}\tilde\kappa))\,\omega \\
&\leqslant \|d_h\Delta^q\tilde\kappa\,\|_{L^2}\|d_h\Delta^{j-q}(\dot u\Delta^{2q-j}\tilde\kappa)\|_{L^2} \\
&\leqslant f_k^{1/2}\|\dot u\Delta^{2q-j}\tilde\kappa\|_{L^2_{2(j-q)+1}} \\
&\leqslant f_k^{1/2}c\|\dot u\|_{C^{2(j-q)+1}}\|\Delta^{2q-j}\tilde\kappa\|_{L^2_{2(j-q)+1}} \\
&\leqslant f_k^{1/2}c\|\dot u\|_{C^{2(j-q)+1}}\|\tilde\kappa\|_{L^2_{k+1}} \\
&\leqslant f_k^{1/2}cK\|\dot u\|_{C^{2(j-q)+1}}\|d_h\Delta^q\tilde\kappa\|_{L^2} \\
&= cK\|\dot u\|_{C^{2(j-q)+1}}\cdot f_k\,.
\end{aligned}$$

  We keep in mind that $\|\dot u\|_{C^l} \to 0$ as $t \to \infty$ for any l.

- The remaining terms can be estimated quite crudely by recalling that we already know that $|\Delta^l\tilde\kappa| \to 0$ as $t \to \infty$ for any l. We use the symmetry (3.15) to assume that $j \leqslant (k-1)/2$. If $k = 2q+1$ is odd, then using (3.12) we compute

$$\begin{aligned}
\int_X(\tilde\kappa, \Delta^j i\Lambda_\omega[d_h\tilde\kappa, d_h\tilde\kappa\Delta^{2q-j}\tilde\kappa])\,\omega &= \int_X(d_h\Delta^j\tilde\kappa, [(\overline\partial - \partial_h)\tilde\kappa, \Delta^{2q-j}\tilde\kappa])\,\omega \\
&\leqslant \|d_h\Delta^j\tilde\kappa\|_{L^2}\left(\int_X|\operatorname{ad}_{\Delta^{2q-j}\tilde\kappa}(\overline\partial - \partial_h)\tilde\kappa|^2\,\omega\right)^{\frac12} \\
&\leqslant \|d_h\Delta^j\tilde\kappa\|_{L^2}\left(\int_X|\operatorname{ad}_{\Delta^{2q-j}\tilde\kappa}|^2_{op}\cdot|d_h\tilde\kappa|^2\,\omega\right)^{\frac12} \\
&\leqslant \|d_h\Delta^j\tilde\kappa\|_{L^2}\|d_h\tilde\kappa\|_{L^2}\sup_X|\operatorname{ad}_{\Delta^{2q-j}\tilde\kappa}|_{op} \\
&\leqslant C\|d_h\Delta^q\tilde\kappa\|_{L^2}\|d_h\Delta^q\tilde\kappa\|_{L^2}\sup_X|\operatorname{ad}_{\Delta^{2q-j}\tilde\kappa}|_{op} \\
&\leqslant C\sup_X|\operatorname{ad}_{\Delta^{2q-j}\tilde\kappa}|_{op}\cdot f_k\,.
\end{aligned}$$

If $k = 2q$ is even and $q \geqslant 1$, it suffices to simply replace all instances of $2q - j$ by $2q-1-j$ in the previous computation. Note that the prefactor $C \sup_X |\operatorname{ad}_{\Delta^{2q-j}\tilde{\kappa}}|_{op}$ of f_k is also a function tending to 0 as $t \to \infty$.

Combing the estimates for each term one finally obtains

$$\partial_t f_k \leqslant -C^{-1} f_k + r \cdot f_k \,,$$

where r is a smooth function of t converging to 0 as $t \to \infty$. Pick t' sufficiently big such that for all $t \geqslant t'$ one has $-C^{-1} + r(t) \leqslant -C^{-1}/2$. Then for $t \geqslant t'$ one has

$$\partial_t f_k \leqslant -\frac{1}{2C} f_k$$

implying $f_k \leqslant Ae^{-\alpha t}$ with $\alpha = 1/2C$ and A sufficiently big such that the inequality also holds for $0 \leqslant t < t'$. This means that $\|\tilde{\kappa}\|_{L^2_k,h,\omega}$ tends exponentially to 0. Again, $\omega(t)$ and $h(t)$ converge in C^∞, so $\|\tilde{\kappa}\|_{L^2_k,h_0,\omega_\infty}$ exhibit the same exponential decay. Choosing k sufficiently large so that $L^2_k \hookrightarrow C^l$ proves exponential convergence in C^k for any $k \in \mathbb{N}_0$ concluding the proof of Proposition 3.6.1.

4. Geometric Motivation and Adiabatic Limits

4.1. Introduction

The objective of this chapter is to motivate the development of twisted Calabi flow and time-dependent Hermitian Yang–Mills flow carried out in chapters 2 and 3 by showing that they arise naturally in the construction of adiabatic approximations to Calabi flow on certain fibrations.

Given a holomorphic submersion $\pi\colon Z \to X$ between two compact complex manifolds, the fibres $Z_x := \pi^{-1}(\{x\})$ are complex submanifolds of Z. Fibres over different points are always diffeomorphic, but need not be biholomorphic — their complex structures can change along the base. Assuming that X is Kähler and that there exists a holomorphic line bundle $L \to Z$ admitting a Hermitian metric with fibrewise positive curvature, one can construct Kähler structures on Z as follows. Let ω_X be a Kähler form on the base and h be a Hermitian metric on L with curvature F_h such that $\omega_0 := iF_h/2\pi$ is positive when restricted to the fibres. For $r \in \mathbb{R}$ sufficiently big, $\omega_r := \omega_0 + \pi^* r\omega_X$ is a closed and positive $(1,1)$-form on Z and hence Kähler. Moreover, with growing r, the base becomes approximately flat and one expects the curvature of (Z, ω_r) to be dominated by that of the fibres. Indeed, formally expanding the scalar curvature $S(\omega_r)$ into powers of r^{-1}, one finds that the leading order r^0-coefficient is given by the scalar curvature of the fibre whereas the scalar curvature of the base appears in the coefficient of r^{-1}.

If the fibres Z_x admit cscK metrics in $c_1(L|_{Z_x})$, one might attempt to find a cscK metric on the total space Z by having ω_0 restrict to the canonical metrics on each fibre and making r very large, hoping to be able to perturb ω_r to a cscK metric using the inverse function theorem. However, this procedure turns out to be too naïve. The reason for its failure lies in the interplay of the adiabatic geometry of (Z, ω_r) and the analytic details of the inverse function theorem. Roughly speaking, if $\Phi_r\colon A \to B$ are the maps from a suitable Banach space A of perturbations of ω_r to a Banach space B such that the norm of $\Phi_r(\psi)$ measures the deficiency of the metric ω_r perturbed by ψ having constant scalar curvature, then a cscK metric on Z corresponds to a zero of Φ_r. The curvature analysis can be translated to $\|\Phi_r(0)\|_B = \mathcal{O}(r^{-1})$ and provided that $(d\Phi_r)_0$ is surjective, the inverse function theorem guarantees the existence of a ball around $\Phi_r(0)$ onto which Φ_r maps surjectively. It turns out that the radius of that ball decreases like r^{-s} for some $s > 0$, so one cannot guarantee that it contains $0 \in B$, no matter the size of r. In order to make the inverse function argument work, one needs to construct better approximations $\omega_{r,k}$, such that the corresponding maps satisfy $\|\Phi_{r,k}(0)\|_B = \mathcal{O}(r^{-(k+1)})$ for $k > s - 1$.

A more precise analysis of the r^{-1} coefficient of $S(\omega_r)$ reveals that it is the sum of the scalar curvature of (X, ω_X) and a term reflecting how the moduli and fibrewise metrics change along the base. To construct the next best approximation $\omega_{r,1}$, one thus needs to modify the base metric to account for this basepoint dependence of the fibrewise structures. This amounts to solving a nonlinear elliptic equation that resembles that characterising a canonical metric on the base, modified by a twist reflecting the structure of the fibres. Subsequent approximations $\omega_{r,k}$, $k > 1$ can then be obtained by solving linear elliptic PDEs.

These ideas have been exploited by Fine [13, 14] and Hong [22]. Using the above adiabatic scheme, Fine constructed cscK metrics on the total space of Kodaira surfaces, i.e. X and the fibres of $Z \to X$ are Riemann surfaces of genus $\geqslant 2$, while Hong does the same on projectivised stable holomorphic vector bundles over compact Kähler manifolds. We consider a parabolic version of the adiabatic limits in these cases, where the elliptic problems of finding metrics $\omega_{r,k}$ approximating a cscK metric up to order $\mathcal{O}(r^{-k})$ are replaced by the parabolic problems of finding paths of metrics $\omega_{r,k}(t)$ representing $\mathcal{O}(r^{-k})$ approximations to Calabi flow on Z. The $\omega_{r,k}(t)$ are given as exact solutions to approximative equations which are defined by demanding that the defect $\partial_t \omega_{r,k}(t) + i\bar{\partial}\partial S(\omega_{r,t}(t))$ of $\omega_{r,k}(t)$ solving Calabi flow be $\mathcal{O}(r^{-(k+1)})$. The precise forms of the approximative equations are derived below. We show that the 1$^{\text{st}}$ order approximations $\omega_{r,1}(t)$ naturally lead to twisted Calabi flow in the case of Kodaira surfaces[1] and time-dependent Hermitian Yang–Mills flow in the case of ruled manifolds (projectivised bundles).

4.2. General Adiabatic Setup

We want to establish a precise notion of adiabatic limits in a general context and compute a pointwise expansion of scalar curvature of the adiabatic metrics ω_r in powers of r^{-1}. Some preparation is required.

4.2.1. Notation and Basic Properties

Denote by Z, X compact connected complex manifolds and by $\pi\colon Z \to X$ a holomorphic submersion. The fibres $Z_x = \pi^{-1}(\{x\})$ are complex submanifolds of Z whose holomorphic inclusions $Z_x \hookrightarrow Z$ are denoted by ι_x. The *vertical bundle* $V := \ker d\pi$ is a holomorphic subbundle of TZ and its fibrewise restrictions $\iota_x^* V$ are isomorphic to the tangent bundle TZ_x of the fibres. With the help of a real $(1,1)$-form ω_0 on Z whose fibrewise restrictions are nondegenerate one can define a complement to V in TZ:

$$H := \{w \in TZ \mid \forall\, v \in V : \omega_0(w, v) = 0\}\,.$$

Lemma 4.2.1. *A fibrewise nondegenerate two-form ω_0 induces a splitting $TZ = V \oplus H$ of complex vector bundles.*

[1] Unfortunately, the results of Chapter 2 are not strong enough. It is not evident that the twist appearing in the adiabatic analysis should be negative semidefinite. In addition, its cohomology class may vary in time.

Proof. The pointwise splittings $TZ_x = V_x \oplus H_x$ are linear algebra, but it remains to see that the splitting is one vector bundles. As the kernel of $d\pi$, V is manifestly a holomorphic subbundle of TZ. Over suitable open subsets $U \subset Z$ one can find smooth frames of V given by local sections $\{x_j\}$ and $\{y_j\}$ of TZ such that $\omega_0(x_i, y_j) = \delta_{ij}$ (one way to do this is to write Z as a local product of an open set of X with the underlying real manifold N of the fibres Z_x. In product charts, one can then simply Gram–Schmidt the coordinate vector fields of the N-coordinates with respect to the pseudo Riemannian metric $\omega_0(\cdot, J\cdot)$ with the X-coordinates as a parameter). For a tangent vector $\zeta \in TZ$ set

$$\begin{aligned} \mathrm{pr}_V^U(\zeta) &= \sum_i \left(\omega_0(x_i,\zeta)y_i - \omega_0(y_i,\zeta)x_i\right) , \\ \mathrm{pr}_H^U &= \mathrm{id} - \mathrm{pr}_V^U . \end{aligned}$$

This defines local projection maps of TZ to V and H respectively which in fact do not depend on the chosen coordinates so long as $\omega_0(x_i, y_j) = \delta_{ij}$ and hence extend to smooth sections pr_V and pr_H of $\mathrm{End}(TZ)$. Checking that $H = \ker \mathrm{pr}_V$ then shows that H is a real subvectorbundle of TZ. Since ω_0 is J-invariant, so is H, making it a complex subvectorbundle of TZ. □

We call H the *horizontal bundle* and remark that splitting $TZ = V \oplus H$ defines a decomposition of differential forms induced by $\Lambda^k(V^*\oplus H^*) = \bigoplus_{i+j=k} \Lambda^i V^* \otimes \Lambda^j H^*$. For a two-form α we write $\alpha = \alpha_{VV} + \alpha_{HH} + \alpha_{VH}$ for its decomposition into vertical-vertical, horizontal-horizontal and vertical-horizontal components.

For later use we also observe that the splitting of TZ into horizontal and vertical subspaces defines a connection on the fibration $\pi\colon Z \to X$. In particular, one can define its curvature to be the two-form F on X with values in vertical vector fields (sections of V) constructed as follows. Given $\xi_1, \xi_2 \in T_{\pi(z)}X$, extended locally to vector fields, set $F(\xi_1,\xi_2) := [\widetilde{\xi}_1, \widetilde{\xi}_2]_{vert} = [\widetilde{\xi}_1, \widetilde{\xi}_2] - \widetilde{[\xi_1,\xi_2]}$, where $\widetilde{\xi}$ denotes the unique horizontal lift of $\xi \in TX$ to H. This corresponds to the notion of curvature as infinitesimal holonomy around loops on the base.

For the adiabatic setup, a suitable ω_0 should in addition be closed, of type $(1,1)$ and positive on each fibre in addition to being fibrewise nondegenerate. In the two cases we consider, such forms arise as the curvature of a holomorphic line bundle over Z:

Definition 4.2.2. A holomorphic line bundle $L \to Z$ is called *relatively positive* if there exists a Hermitian structure h on L whose curvature $\omega_0 = i/(2\pi)F_h$ is positive on each fibre Z_x of Z.

Forms ω_0 with these properties can be used to construct Kähler metrics on Z provided that the base X is Kähler.

Lemma 4.2.3. *If ω_0 is a closed $(1,1)$-form on Z which is fibrewise positive and ω_X is a Kähler form on X, then for all sufficiently large r, $\omega_r := \omega_0 + r\pi^*\omega_X$ defines a Kähler form on Z.*

Proof. The forms ω_0 and ω_X are closed and J-invariant. Since π is holomorphic and the exterior derivative commutes with pullbacks, the same is true for $\pi^*\omega_X$ and hence for ω_r. The vertical-horizontal decomposition of ω_r is given by $\omega_r = \omega_{0VV} + (\omega_{0HH} + r\pi^*\omega_X)$. Note that by definition of H, ω_r has no vertical-horizontal component. As the decomposition $TZ = V \oplus H$ is J-invariant, it suffices to check that both the vertical-vertical piece ω_{0VV} and the horizontal-horizontal piece $\omega_{0HH} + r\pi^*\omega_X$ are positive for sufficiently large r. For ω_{0VV} this is true by assumption. For the horizontal-horizontal component consider the compact unit tangent bundle $UTX \subset TX$ with respect to the metric $g_X = \omega_X(\cdot, J, \cdot)$ and pull it back to $\widetilde{UTX} \subset H$ via horizontal lifts. Since the fibres of Z are compact, this is a compact subset of H and it suffices to check positivity of $\omega_{0HH} + r\pi^*\omega_X$ on $\widetilde{UTX}$. Denote by a the minimum of the function $w \to \omega_{0HH}(w, Jw)$ on $\widetilde{UTX}$ and choose $r > -a$. Then $(\omega_{0HH} + r\pi^*\omega_X)(w, Jw) > 0$ on $\widetilde{UTX}$ and hence on all of E. $\square$

The cohomology classes $\kappa_r := [\omega_0] + r[\pi^*\omega_X]$ are called *adiabatic classes*. Assuming ω_0 is the curvature of a relatively positive line bundle $L \to Z$ and the base X is polarised by a line bundle $Q \to X$ such that $\omega_X \in c_1(Q)$, then Z is polarised by the line bundle $L \otimes \pi^*Q^r$ and $\kappa_r = c_1(L \otimes \pi^*Q^r)$ for r large enough to imply that κ_r is a Kähler class. We shall henceforth make these assumptions.

4.2.2. Geometry of the Vertical-Horizontal Splitting

Via the splitting $TZ = V \oplus H$ one can define several geometric operations that are useful in computing the scalar curvature of the metrics ω_r in terms of data on the base and on the fibres. Let n be the (complex) dimension of X and $n + k$ that of Z. We denote by $\sigma_x = \iota_x^*\omega_0$ the fibrewise restriction of ω_0 to Z_x and permit the occasional omission of the basepoint-specifying subscript as well as the identification $\omega_{0VV} = \sigma$. For $\alpha \in \Lambda^2 V^*$ and $\beta \in \Lambda^2 H^*$ define the vertical and horizontal contractions as

$$\begin{aligned} \Lambda_V \alpha &:= k\frac{\alpha \wedge \sigma^{k-1}}{\sigma^k}, \\ \Lambda_H \beta &:= n\frac{\beta \wedge \pi^*\omega_X^{n-1}}{\pi^*\omega_X^n} \end{aligned}$$

and the corresponding vertical and horizontal Laplacians by

$$\begin{aligned} \Delta_V f &:= \Lambda_V(i\bar{\partial}\partial f)_{VV}, \\ \Delta_H f &:= \Lambda_H(i\bar{\partial}\partial f)_{HH} \end{aligned}$$

for a function f on Z. One has the compatibility rules $\Delta_H \pi^* g = \pi^*\Delta_{\omega_X} g$ for $g \in C^\infty(X)$ and $\Delta_{\sigma_x}(\iota_x^* f) = \iota_x^*(\Delta_V f)$ for $f \in C^\infty(Z)$, where Δ_{σ_x} is the Laplacian on the fibre Z_x defined by the metric σ_x. This is to say that Δ_V are the fibrewise Laplacians with respect to the metrics σ on the fibres and Δ_H is the pullback of the Laplacian on X. Next we examine how $\Lambda_V, \Lambda_H, \Delta_V$ and Δ_H behave under changes of the adiabatic parameter r.

Lemma 4.2.4. *For $\alpha \in \Lambda^2 T^*Z$ one has*

$$\Lambda_{\omega_r}\alpha = \Lambda_V \alpha_{VV} + r^{-1}\Lambda_H \alpha_{HH} + O(r^{-2}) .$$

Proof. In light of the splitting $\omega_r = \omega_{rVV} + \omega_{rHH} = \sigma + (r\pi^*\omega_X + \omega_{0HH})$ it is

$$\begin{aligned}
\omega_r^{n+k} &= \binom{n+k}{k}\omega_{rVV}^{\,k} \wedge \omega_{rHH}^{\,n} = \binom{n+k}{k}\sigma^k \wedge (r\pi^*\omega_X + \omega_{0HH})^n \\
\omega_r^{n+k-1} &= \binom{n+k-1}{k}\sigma^k \wedge (r\pi^*\omega_X + \omega_{0HH})^{n-1} \\
&+ \binom{n+k-1}{k-1}\sigma^{k-1} \wedge (r\pi^*\omega_X + \omega_{0HH})^n ,
\end{aligned}$$

with which one obtains

$$\begin{aligned}
\Lambda_{\omega_r}\alpha &= (n+k)\frac{\alpha \wedge \omega_r^{n+k-1}}{\omega_r^{n+k}} \\
&= k\frac{\alpha_{VV} \wedge \sigma^{k-1} \wedge (r\pi^*\omega_X + \omega_{0HH})^n}{\sigma^k \wedge (r\pi^*\omega_X + \omega_{0HH})^n} \\
&+ n\frac{\alpha_{HH} \wedge \sigma^{k} \wedge (r\pi^*\omega_X + \omega_{0HH})^{n-1}}{\sigma^k \wedge (r\pi^*\omega_X + \omega_{0HH})^n} \\
&= \Lambda_V \alpha_{VV} + \Lambda_{r\pi^*\omega_X + \omega_{0HH}}\alpha_{HH} .
\end{aligned}$$

Setting $s = r^{-1}$ the second term can be written as

$$n\frac{\alpha_{HH} \wedge \sigma^{k} \wedge (r\pi^*\omega_X + \omega_{0HH})^{n-1}}{\sigma^k \wedge (r\pi^*\omega_X + \omega_{0HH})^n} = ns\frac{\alpha_{HH} \wedge (\pi^*\omega_X + s\omega_{0HH})^{n-1}}{(\pi^*\omega_X + s\omega_{0HH})^n} ,$$

from where differentiation at $s = 0$ yields

$$\Lambda_{r\pi^*\omega_X + \omega_{0HH}}\alpha_{HH} = r^{-1}\Lambda_H \alpha_{HH} + O(r^{-2})$$

as claimed. □

Lemma 4.2.5. *The ω_r-Laplacian on functions satisfies the following adiabatic expansion property:*

$$\Delta_{\omega_r} = \Delta_V + r^{-1}\Delta_H + O(r^{-2}) .$$

Proof. It is $\Delta_{\omega_r} f = \Lambda_{\omega_r} i\bar\partial\partial f$. The claim follows from the adiabatic expansion properties of Λ_{ω_r}. □

4.2.3. Scalar Curvature Expansion of Adiabatic Metrics

We now derive the scalar curvature expansion of the adiabatic metrics ω_r into inverse powers of the adiabatic parameter r. Recall that on a Kähler manifold (Y, ω) of dimension n the Ricci curvature is J-invariant and defines a closed $(1,1)$-form, the Kähler–Ricci form $\rho(\omega)$, which is related to the Ricci tensor in the same way the Kähler form ω is related to the metric g. Moreover, $\rho(\omega)$ can be computed as i times the curvature of the anticanonical bundle $K^* = \Lambda^n(TY, J)$ endowed with the Hermitian metric $h = (n!)^{-1}\omega^n$ given by the volume form seen as a section of $K \otimes \overline{K}$ (here $\overline{K}$ means reversing the sign of the complex structure).

Lemma 4.2.6. *The scalar curvature of ω_r satisfies*

$$S(\omega_r) = S(\sigma) + r^{-1}[\pi^* S(\omega_X) + \Delta_V(\Lambda_H \omega_{0HH}) + i\Lambda_H F_{\Lambda^k V,\sigma\, HH}] + \mathcal{O}(r^{-2})\,, \tag{4.1}$$

where $F_{\Lambda^k V,\sigma}$ is the curvature of the vertical anticanonical bundle $\Lambda^k V$ with Hermitian metric induced by σ.

Proof. We define Hermitian metrics on $\Lambda^k V$ and $\Lambda^n H$ by

$$\begin{aligned} h_V &:= (k!)^{-1}(\omega_r)^k_{VV} = (k!)^{-1}\sigma^k\,,\\ h_H &:= (n!)^{-1}(\omega_r)^n_{HH} = (n!)^{-1}(r\pi^*\omega_X + \omega_{0HH})^n\,,\\ h_X &:= (n!)^{-1}\pi^*\omega_X^n\,. \end{aligned}$$

The Kähler–Ricci form of Z is i times the curvature of the anticanonical bundle $K^* = \Lambda^k V \otimes \Lambda^n H$ with Hermitian metric $h_V \otimes h_H$ which is locally given by

$$\begin{aligned} \rho(\omega_r) &= i\bar\partial\partial \log h_V \otimes h_H\\ &= i\bar\partial\partial \log h_V + i\bar\partial\partial \log(\pi^*\omega_X + r^{-1}\omega_{0HH})^n\\ &= i\bar\partial\partial \log h_V + i\bar\partial\partial \log((\pi^*\omega_X)^n + nr^{-1}\omega_{0HH}\wedge\omega_X^{n-1} + \mathcal{O}(r^{-2}))\\ &= i\bar\partial\partial \log h_V + i\bar\partial\partial \log([1 + r^{-1}\Lambda_H\omega_{0HH} + \mathcal{O}(r^{-2})]h_X)\\ &= i\bar\partial\partial \log h_V + i\bar\partial\partial \log h_X + r^{-1} i\bar\partial\partial(\Lambda_H\omega_{0HH}) + \mathcal{O}(r^{-2})\,. \end{aligned}$$

Observing that $i\bar\partial\partial \log h_X = \pi^*\rho(\omega_X)$ is purely horizontal-horizontal and in light of Lemma 4.2.4 one then computes

$$\begin{aligned} S(\omega_r) &= \Lambda_{\omega_r}\rho(\omega_r)\\ &= \Lambda_V\rho(\omega_r)_{VV} + r^{-1}\Lambda_H\rho(\omega_r)_{HH} + \mathcal{O}(r^{-2})\\ &= \Lambda_V(i\bar\partial\partial\log h_V)_{VV}\\ &+ r^{-1}[\Lambda_V(i\bar\partial\partial(\Lambda_H\omega_{0HH}))_{VV} + \Lambda_H(i\bar\partial\partial\log h_V)_{HH} + \Lambda_H i\bar\partial\partial\log h_X] + \mathcal{O}(r^{-2})\\ &= S(\sigma) + r^{-1}[\Delta_V(\Lambda_H\omega_{0HH}) + i\Lambda_H F_{\Lambda^k V,\sigma\,HH} + \pi^* S(\omega_X)] + \mathcal{O}(r^{-2}) \end{aligned}$$

which is as claimed. □

Remark. It will later occur that ω_0 and ω_X individually depend on r. In this case, one can first use the above expansion of $S(\omega_r)$ ignoring the individual r dependencies of ω_0 and ω_X and subsequently expand each term still depending on r into powers of r^{-1}. Owing to the analyticity of all involved operations, the resulting terms can then be grouped according to their powers of r^{-1} to yield the full expansion of $S(\omega_r)$. In essence, this is just the chain rule for composite functions.

We want to use this expansion to construct approximations to Calabi flow on Z in the adiabatic classes $\kappa_r = c_1(L \otimes \pi^* Q^r)$.

Definition 4.2.7. Let $k \in \mathbb{N}_0$. A family of smooth paths of Kähler metrics $\omega_{r,k}(t)$ in κ_r is said to be a (pointwise) order k or $\mathcal{O}(r^{-k})$ approximation to Calabi flow with initial condition ω_r if

$$\partial_t \omega_{r,k}(t) + i\bar{\partial}\partial S(\omega_{r,k}(t)) = \mathcal{O}(r^{-(k+1)})$$

and $\omega_{r,k}(0) = \omega_r$.

Remark. The term "initial condition" might be slightly confusing in this context, as the initial metric depends on the adiabatic parameter r. However, this dependence is given by $\omega_r = \omega_0 + r\pi^*\omega_X$ and fixing both ω_0 and ω_X prescribes initial data ω_r in each class κ_r. From here on, if no explicit mention is made to the contrary, the initial data are understood to be of this form.

The following sections exploit specific features of two types of complex manifolds Z fitting into the adiabatic framework and outline the construction of approximations to Calabi flow of arbitrary order in r^{-1} in these cases. The first type consists of Kodaira surfaces and the second of ruled manifolds. In light of the wealth of geometric data in both cases, it is useful to think of the construction of approximations $\omega_{r,k}(t)$ in terms of a scheme indexed by r:

- A fibrewise positive two-form ω_0 and a Kähler form ω_X on the base define the following data: the metric $\omega_r = \omega_0 + r\pi^*\omega_X$, the splitting $TZ = V \oplus H$ and the corresponding contractions and operators, a family of fibrewise metrics σ and the scalar curvatures of the base, the fibres and the total space. The forms ω_0 and ω_X may also individually depend on r.

- From this data elements in the tangent spaces to ω_0 and ω_X are constructed prescribing their infinitesimal evolution. The construction of infinitesimal changes may explicitly depend on r.

- After the infinitesimal evolution, the forms ω_0 and ω_X have changed and along with them all dependent quantities outlined in the first point, in particular ω_r.

4.3. Adiabatic Scheme for Kodaira Surfaces

In [13] J. Fine used elliptic adiabatic techniques to construct cscK metrics on Kodaira surfaces, i.e. compact complex surfaces Z admitting a holomorphic submersion $\pi\colon Z \to X$ to a Riemann surface. The additional requirement that the fibres have genus at least 2 ensures that the fibres admit no nonzero holomorphic vector fields and that the cscK metrics thereon are unique in each Kähler class. In an analysis similar to that of [13], we derive for each $k \in \mathbb{N}$ parabolic equations for families of metrics $\omega_{r,k}(t)$ that agree (pointwise) with Calabi flow up to order k in r^{-k}.

For the remainder of this section $\pi\colon Z \to X$ denotes such a Kodaira surface with high genus fibres and base. We begin by showing that Z admits a relatively positive line bundle and the previously presented adiabatic framework can be applied.

Proposition 4.3.1. *Denote by $V := \ker d\pi$ the vertical bundle of the fibration $\pi\colon Z \to X$. The relative canonical bundle $V^* \to Z$ is relatively positive.*

Proof. The restriction of V^* to a fibre Z_x is the canonical bundle of that fibre which is positive owing to the genus assumption $g(Z_x) \geqslant 2$. Indeed, it is $\int_{Z_x} c_1(K_{Z_x}) = -\int_{Z_x} c_1(Z_x) = -2\chi(Z_x) = 4(g-1) > 0$ and hence $c_1(K_{Z_x}) \in H^{1,1}(X,\mathbb{Z})$ is positive. By the uniformisation theorem, there exists a unique Kähler metric with constant negative scalar curvature in $c_1(K_{Z_x})$ which can be interpreted as a negatively curved Hermitian structure on TZ_x. The induced Hermitian structure on $K_{Z_x} = T^*Z_x$ thus is positively curved. It is standard that the constant scalar curvature metrics with given volume on a Riemann surface depend smoothly on the complex structure, so the induced Hermitian structures on K_{Z_x} smoothly piece together to a Hermitian structure on V^* (cf. e.g. [14]).

Alternatively, one can pick any J-invariant Riemannian metric g on Z such that its fundamental two-form ω has fibrewise restrictions $\sigma_x := \omega|_{Z_x} \in c_1(K_{Z_x})$. Since the restriction of g to a fibre is necessarily Kähler, one has that $\sigma_x > 0$. The two-form ω itself need not be closed, but making a smooth choice of Hermitian metrics h_x on $V^*|_{Z_x}$ such that $i/(2\pi)F_{h_x} = \sigma_x$ and then gluing the h_x to a Hermitian metric h on V^* yields a closed two-form $\omega_0 \in c_1(V^*)$ with fibrewise restrictions σ_x. □

We fix an initial Hermitian metric h on V^* with curvature $\omega_0 := i/(2\pi)F_h$ having fibrewise positive restrictions. It is also natural to assume that the initial base metric lies in $c_1(K_X) \in H^{1,1}(X,\mathbb{Z})$ (this can be achieved by rescaling). The Kähler manifold Z is then polarised by $V^* \otimes \pi^*K_X^r$ for large r. Lemma 4.2.6 is applicable to the metrics $\omega_r := \omega_0 + r\pi^*\omega_X \in c_1(V^* \otimes \pi^*K_X^r)$ and the scalar curvature expansion can be used to define approximations to Calabi flow on Z.

4.3.1. 0th Order Approximation to Calabi Flow

The order $\mathcal{O}(r^0)$ approximation to Calabi flow on Z is given by fibrewise Calabi flow. For $x \in X$ denote by h_x the Hermitian product on the line bundle $K_{X_x} \to Z_x$ obtained by restricting h and V^* to Z_x. By assumption $\sigma_x = i/(2\pi)F_{h_x}$ is a Kähler form on Z_x and Calabi flow on that fibre is given by $\partial_t\sigma_x = -i\overline{\partial}\partial S(\sigma_x)$. Locally expressing the curvature of h_x as $F_{h_x} = \overline{\partial}\partial \log h_x$, fibrewise Calabi flow can be written as $\partial_t h_x = 2\pi S(\sigma_x)h_x$. By [8] and [5], that flow exists for all times and its solution $h_x(t)$ depends smoothly on the initial conditions and the complex structure on the fibre. This implies that the $h_x(t)$ piece together to a smooth Hermitian metric $h(t)$ on V^*. Denote by $\omega_0(t) = i/(2\pi)F_{h(t)}$ the rescaled curvature of $h(t)$ and set

$$\omega_{r,0}(t) := \omega_0(t) + r\pi^*\omega_X\,.$$

It remains to check that $\omega_{r,0}(t)$ is the desired $\mathcal{O}(r^0)$ approximation to Calabi flow. Indeed,

$$\begin{aligned}\partial_t\omega_{r,0}(t) + i\bar\partial\partial S(\omega_{r,0}(t)) &= \frac{i}{2\pi}\partial_t F_h + i\bar\partial\partial S(\omega_{r,0}(t)) \\ &= \frac{i}{2\pi}\bar\partial\partial\frac{\partial_t h}{h} + i\bar\partial\partial S(\omega_{r,0}(t)) \\ &= i\bar\partial\partial(-S(\sigma) + S(\omega_{r,0}(t))) \\ &= \mathcal{O}(r^{-1}),\end{aligned}$$

where in the last step we used the scalar curvature expansion of Lemma 4.2.6. As the fibrewise restrictions $h_x(t)$ converge for $t \to \infty$, so does $h(t)$.

Remark. As the horizontal-horizontal parts of $\omega_0(t)$ can change, it is not a priori clear that $\omega_{r,0}(t)$ remains positive for all t. However, since $\omega_0(t)$ converges, its horizontal-horizontal part varies in a compact family. The arguments in Lemma 4.2.3 can thus be adapted to hold uniformly in t, albeit for a possibly larger threshold for r. Similar situations arise throughout the analysis and we tacitly treat them the same way without explicit mention.

4.3.2. 1st Order Approximation to Calabi Flow

Denote by ψ_1 the r^{-1} term in the expansion of $S(\omega_{r,0})$ which is given by

$$\psi_1 = \pi^* S(\omega_X) + \Delta_V(\Lambda_H \omega_{0\,HH}) + i\Lambda_H F_{V,\sigma\,HH}\,.$$

The fibrewise metrics σ define an r-independent, $L^2(Z,\omega_r)$-orthogonal decomposition $C^\infty(Z) = \pi^* C^\infty(X) \oplus C^\infty_\perp(Z)$ of functions on Z into functions pulling back from the base and those having fibrewise zero integral. Let $\psi_1 = \psi_{\perp,1} + \psi_{X,1}$ be the corresponding decomposition of ψ. The part pulling back from the base is given by

$$\psi_{X,1}(x) = \mathrm{Vol}(Z_x)^{-1}\int_{Z_x}\psi_1\,\sigma_x = \pi^* S(\omega_X) + \mathrm{Vol}(Z_x)^{-1}\int_{Z_x}(\Lambda_H i F_{V,\sigma\,HH})\,\sigma_x\,.$$

The rightmost term can be expressed as $A_x^{-1}\Lambda_{\omega_X}\pi_*(iF_{V,\sigma}\wedge\sigma)$, where $A_x = \mathrm{Vol}(Z_x)$ and π_* denotes the pushdown of forms on Z to forms on X which is characterised by $\int_X \theta\wedge\pi_*\eta = \int_Z \pi^*\theta\wedge\eta$ for forms $\theta\in\Omega^l(X)$ and $\eta\in\Omega^{4-l}(Z)$. We set

$$\alpha := A_x^{-1}\pi_*(iF_{V,\sigma}\wedge\sigma)\in\Omega^2(X)$$

and observe that α depends only on the fibrewise restrictions σ of ω_0. Suppose ω_X solves the twisted Calabi flow equation

$$(\partial_t\omega_X)(t) = -r^{-2} i\bar\partial\partial(S(\omega_X(t)) + \Lambda_{\omega_X(t)}\alpha(t))\,,$$

then the path of Kähler metrics

$$\omega'_{r,0} := \omega_0(t) + r\pi^*\omega_X(t)$$

satisfies

$$\partial_t\omega'_{r,0} + i\bar\partial\partial S(\omega'_{r,0}) = r^{-1} i\bar\partial\partial\psi_{\perp,1} + \mathcal{O}(r^{-2})$$

and thus gives an $\mathcal{O}(r^{-1})$ approximation to Calabi flow up to the remaining term $\psi_{\perp,1}$.

Remark. Unfortunately, the results of Chapter 2 are not directly applicable, since we do not know whether the twist α is pointwise nonpositive or stays within a given cohomology class.

To deal with this remaining error at r^{-1}, we add a time-dependent Kähler potential $\phi_{\perp,1}$ with $\int_{Z_x} \phi_{\perp,1}\, \sigma_x = 0$ to $\omega'_{r,0}$ and set

$$\omega_{r,1} := \omega'_{r,0} + r^{-1} i\bar{\partial}\partial\phi_{\perp,1}\,.$$

The effect of adding the Kähler potential $r^{-1}\phi_{\perp,1}$ on the scalar curvature can be determined by expanding

$$S(\omega_{r,1}) = S(\omega'_{r,0}) + r^{-1} d\,\mathrm{Sc}_{\omega'_{r,0}}\,\phi_{\perp,1} + \mathcal{O}(r^{-2})\,.$$

Using the general equation for the linearisation of the scalar curvature map found in Appendix A.3 and expanding the r-dependent quantities into powers of r using Lemma 4.2.5 and the formulae in Lemma 4.2.6 one finds that

$$\begin{aligned} d\,\mathrm{Sc}_{\omega'_{r,0}}\,\phi_{\perp,1} &= \Delta^2_{\omega'_{r,0}}\phi_{\perp,1} - S(\omega'_{r,0})\Delta_{\omega'_{r,0}}\phi_{\perp,1} + 2\frac{i\bar{\partial}\partial\phi_{\perp,1}\wedge\rho(\omega'_{r,0})}{\omega'^2_{r,0}} \\ &= \Delta_V^2\phi_{\perp,1} - S(\sigma)\Delta_V\phi_{\perp,1} + \mathcal{O}(r^{-1})\,, \end{aligned} \tag{4.2}$$

so in leading order, the linearisation of the scalar curvature map is the linearised fibrewise scalar curvature. Setting $D_0 := \Delta_V^2 - S(\sigma)\Delta_V$, this implies $S(\omega_{r,1}) = S(\omega'_{r,0}) + r^{-1}D_0\phi_{\perp,1} + \mathcal{O}(r^{-2})$, so if $\phi_{\perp,1}$ solves the linear parabolic equation

$$(\partial_t + D_0)\phi_{\perp,1} = -\psi_{\perp,1} \tag{4.3}$$

with initial condition $\phi_{\perp,1}(0) = 0$, then one has

$$\partial_t\omega_{r,1} + i\bar{\partial}\partial S(\omega_{r,1}) = r^{-1} i\bar{\partial}\partial(\partial_t\phi_{\perp,1} + D_0\phi_{\perp,1} + \psi_{\perp,1}) + \mathcal{O}(r^{-2}) = \mathcal{O}(r^{-2})$$

and $\omega_{r,1}$ is the desired $\mathcal{O}(r^{-1})$ approximation to Calabi flow.

Remark. The fibres have no holomorphic vector fields, so generically $\ker D_0$ consists of fibrewise constant functions and the inhomogeneity $-\psi_{\perp,1}$ is orthogonal to $\ker_{D_0}$. Moreover, as the fibrewise metrics $\sigma(t)$ converge as $t \to \infty$, so does the generator D_0 of (4.3). The solution $\phi_{\perp,1}$ to (4.3) can thus be expected to converge.

4.3.3. 2nd Order Approximation to Calabi Flow

Denote by ψ_2 the r^{-2} coefficient function of the expansion of $S(\omega_{r,1})$ into powers of r^{-1} and by $\psi_2 = \psi_{X,2} + \psi_{\perp,2}$ its decomposition according to $C^\infty(Z) = \pi^*C^\infty(X) \oplus C^\infty_\perp(Z)$ defined by $\sigma(t)$. Each of the two terms will be dealt with by adding appropriate Kähler potentials to $\omega_{r,1}$. This requires more detailed knowledge of how Kähler potentials in $\pi^*C^\infty(X)$ and $C^\infty_\perp(Z)$ affect the scalar curvature.

Since the leading order term of the scalar curvature expansion (4.1) is the fibrewise curvature $S(\sigma)$, adding a potential $r^{-j}i\bar{\partial}\partial\phi_\perp$ to $\omega'_{r,0}$ with $\phi_\perp \in C^\infty_\perp(Z)$ entails a change in the total scalar curvature whose lowest order contribution in r^{-j} is given by the linearisation of the fibrewise scalar curvature map, i.e. $r^{-j}D_0\phi_\perp = r^{-j}(\Delta_V^2 - S(\sigma)\Delta_V)\phi_\perp$. Higher orders in r^{-1} of course also undergo changes. This is merely a slight variation of the required modification for passing from $\omega'_{r,0}$ to $\omega_{r,1}$.

The situation changes if the Kähler potential is pulled back from the base. Adding a potential $i\bar{\partial}\partial\phi_X$ to $\omega'_{r,0} = \omega_0 + r\pi^*\omega_X$ is the same as replacing the base metric ω_X by $\omega_X + r^{-1}i\bar{\partial}\partial f$, where f satisfies $\pi^* f = \phi_X$. The lowest order in which the scalar curvature of $\omega'_{r,0}$ is sensitive to the base metric is r^{-1}, so the addition of $r^{-1}i\bar{\partial}\partial f$ to ω_X affects $S(\omega_{r,0})$ only from order r^{-2} onwards with the change being the r^{-2} part of

$$r^{-1}\left[\pi^* S(\omega_X + r^{-1}i\bar{\partial}\partial f) + \Delta_V(\Lambda'_H\omega_{0HH}) + i\Lambda'_H F_{V,\sigma HH}\right],$$

where Λ'_H is horizontal contraction defined by $\omega_X + r^{-1}i\bar{\partial}\partial f$. Linearising the above expression shows that this r^{-2} part is given by

$$\pi^*(\Delta^2_{\omega_X} f - S(\omega_X)\Delta_{\omega_X} f) + (\Delta_V(\Lambda_H\omega_{0HH}) + i\Lambda_H F_{V,\sigma HH})\pi^*\Delta_{\omega_X} f$$

and we set $D_X\phi_X$ to be the $\pi^*C^\infty(X)$ part of this, i.e.

$$D_X\phi_X = \Delta^2_H\phi_X - \pi^*S(\omega_X)\Delta_H\phi_X + \pi^*\pi_*(\Delta_V(\Lambda_H\omega_{0HH}) + i\Lambda_H F_{V,\sigma HH})\Delta_H\phi_X\,.$$

The upshot of this analysis is that the modification of $\omega'_{r,0}$ by $i\bar{\partial}\partial\phi_X$ entails an $\mathcal{O}(r^{-2})$ change in the scalar curvature, the $\pi^*C^\infty(X)$-part of which is $D_X\phi_X$ in order r^{-2}. Adding the Kähler potential $r^{-j}i\bar{\partial}\partial\phi_X$ instead, all changes to the scalar curvature are $\mathcal{O}(r^{-(j+2)})$ with the order $r^{-(j+2)}$ change in $\pi^*C^\infty(X)$ being $D_X\phi_X$.

With these preparations we can define second and higher order order approximations to Calabi flow. Set

$$\omega'_{r,1} := \omega_{r,1} + i\bar{\partial}\partial\phi_{X,2}$$

with $\phi_{X,2} \in \pi^*C^\infty(X)$ solving the fourth order linear parabolic equation

$$(\partial_t + D_X)\phi_{X,2} = -\psi_{X,2} \tag{4.4}$$

with initial condition $\phi_{X,2}(0) = 0$. By construction one has $\partial_t\omega'_{r,1} + i\bar{\partial}\partial S(\omega'_{r,1}) = r^{-2}i\bar{\partial}\partial\psi'_{\perp,2} + \mathcal{O}(r^{-3})$, where $\psi'_{\perp,2}$ is the sum of $\psi_{\perp,2}$ and the additional error term in $C^\infty_\perp(Z)$ at r^{-2} due to the Kähler potential $i\bar{\partial}\partial\phi_{X,2}$. Now have $\phi_{\perp,2}$ solve the linear parabolic equation

$$(\partial_t + D_0)\phi_{\perp,2} = -\psi'_{\perp,2}\,,$$

again with initial condition $\phi_{\perp,2}(0) = 0$. The path of Kähler metrics

$$\omega_{r,2} := \omega'_{r,1} + r^{-2}i\bar{\partial}\partial\phi_{\perp,2}$$

then is an $\mathcal{O}(r^{-2})$ approximation to Calabi flow.

4.3.4. Higher Order Approximation to Calabi Flow

The method for obtaining the 2[nd] order approximation can be used inductively to construct higher order approximations. Suppose that

$$\omega_{r,k} = \omega'_{r,0} + \sum_{l=1}^{k} r^{-l} i\bar{\partial}\partial\phi_{\perp,l} + \sum_{l=2}^{k} r^{-(l-2)} i\bar{\partial}\partial\phi_{X,l}$$

is an $\mathcal{O}(r^{-k})$ approximation to Calabi flow, where the $\psi_{\perp,l}$ have fibrewise zero integral and $\phi_{X,l}$ are pulled back from the base. Denote by $\psi_{k+1} = \psi_{X,k+1} + \psi_{\perp,k+1}$ the decomposition of the $r^{-(k+1)}$ term in $S(\omega_{r,k})$ into summands in $\pi^* C^\infty(X)$ and $C^\infty_\perp(Z)$. Have $\phi_{X,k+1}$ solve $(\partial_t + D_X)\phi_{X,k+1} = -\psi_{X,k+1}$ with zero initial condition and set $\omega'_{r,k} := \omega_{r,k} + r^{-(k-1)} i\bar{\partial}\partial\phi_{X,k+1}$. As for the corresponding correction term in the 2$^{\text{nd}}$ order approximation, the addition of the Kähler potential $\phi_{X,k+1}$ at order $r^{-(k-1)}$ only affects the scalar curvature at order $(k+1)$ and higher in r^{-1}. Denote $\psi'_{\perp,k+1}$ to be the order $r^{-(k+1)}$ term in the scalar curvature of $\omega'_{r,k}$ and let $\phi_{\perp,k+1}$ be the solution to $(\partial_t + D_0)\phi_{\perp,k+1} = -\psi'_{\perp,k+1}$. Thenx

$$\omega_{r,k+1} := \omega'_{r,k} + r^{-(k+1)} i\bar{\partial}\partial\phi_{\perp,k+1}$$

is an $\mathcal{O}(r^{-(k+1)})$ approximation to Calabi flow.

4.4. Adiabatic Scheme for Ruled Manifolds

Another case in which adiabatic techniques have successfully been used to construct cscK metrics is that of ruled manifolds. A ruled manifold is the projectivisation of a holomorphic vector bundle $E \to X$, i.e. a holomorphic fibration $\pi\colon \mathbb{P}E \to X$ where the fibres are copies of complex projective space. Suppose that the base X is compact with no holomorphic vector fields and that κ is a Kähler class on X admitting a cscK metric. Further suppose that the bundle $E \to X$ is slope stable with respect to κ. Using elliptic adiabatic techniques, Hong proved the existence of cscK metrics on $\mathbb{P}E$ in [22]. In [21], the construction was extended to certain ruled manifolds coming from polystable bundles. Later, in [3] Brönnle considered the case of E splitting into a direct sum of stable bundles with pairwise different slopes and proved the existence of non-cscK extremal Kähler metrics on $\mathbb{P}E$.

With the objective of outlining a parabolic version of the adiabatic analysis, we adopt the setting in [22]. To this end, we consider a compact connected Kähler base manifold X with no nonzero holomorphic vector fields and Kähler class κ containing a cscK metric on which Calabi flow is assumed to admit a converging long-time solution. The holomorphic vector bundle E is assumed to be a κ-slope stable $\mathrm{SU}(k+1)$-bundle. In order to produce a suitable adiabatic class of Kähler metrics, we specify a relatively positive line bundle on $\mathbb{P}E$.

Proposition 4.4.1. *The relative hyperplane bundle $\widetilde{\mathcal{O}(1)} \to \mathbb{P}E$ obtained by gluing together the hyperplane bundles $\mathcal{O}(1)$ over each fibre is a relatively positive holomorphic line bundle on $\mathbb{P}E$.*

Proof. The holomorphic local trivialisations $E|_U \cong \mathbb{C}^{k+1} \times U$ for $U \subset X$ open induce holomorphic local trivialisations $\mathbb{P}E|_U \cong \mathbb{CP}^k \times U$. Setting $\widetilde{\mathcal{O}(1)}|_U := \mathcal{O}(1) \times U$ gives a holomorphic line bundle over $\mathbb{P}E|_U$. The holomorphic transition functions between local trivialisations $E|_U$ and $E|_V$ induce holomorphic transition functions between $\widetilde{\mathcal{O}(1)}|_U$ and $\widetilde{\mathcal{O}(1)}|_V$ which establishes that $\widetilde{\mathcal{O}(1)}$ is a holomorphic line bundle. To show that $\widetilde{\mathcal{O}(1)}$ is relatively positive, observe that a choice of Hermitian metric h in E defines a Hermitian structure $\widetilde{h}^{-1}$ on the relative hyperplane bundle $\widetilde{\mathcal{O}(1)} \to \mathbb{P}E$ whose curvature $\omega_0(h) := i/(2\pi)F_{\widetilde{h}^{-1}}$ restricts to the Fubini–Study metric on each fibre $\mathbb{P}E_x$ defined by h at $x \in X$. □

Remark. The holomorphic submersion $\pi\colon \mathbb{P}E \to X$ inducing local holomorphic trivialisations is not a generic feature. As seen in the case of Kodaira surfaces, a model holomorphic fibre need not exist. In contrast to high genus Riemann surfaces, complex projective spaces don't have moduli and $\mathbb{CP}^n$ serves as a model fibre for $\pi\colon \mathbb{P}E \to X$.

Remark. We shall lighten the notation by also denoting the relative hyperplane bundle and its powers by $\mathcal{O}(l)$, $l \in \mathbb{Z}$.

We now fix an initial Kähler metric $\omega_X \in \kappa$ and a Hermitian metric h on E and consider the adiabatic metrics

$$\omega_r := \omega_0(h) + r\pi^*\omega_X$$

as initial data for the to be constructed approximations to Calabi flow. As the fibrewise metrics have constant scalar curvature, the constant path of Kähler metrics defined by ω_r is already an $\mathcal{O}(r^0)$ approximation to Calabi flow. Constructing higher order approximations to Calabi flow, however, is slightly more involved than in the case of Kodaira surfaces, owing to the presence of nontrivial holomorphic vector fields on the fibres. Their existence entails the nonuniqueness of fibrewise cscK metrics in their cohomology class and the kernel of the linearised fibrewise scalar curvature map contains strictly more than fibrewise constant functions. The corresponding terms in the scalar curvature expansion into powers of r^{-1} need to be treated separately when constructing approximative flows. A detailed discussion requires some background on the parametric geometry of $\mathbb{P}E$.

4.4.1. Notation and Parametric Geometry of $\mathbb{P}E$

We first recall some standard theory of complex projective space and its symplectic geometry. Let $\mathbb{C}^{k+1}$ be equipped with the standard Hermitian product (inducing the standard inner product on $\mathbb{R}^{2k+2}$ as its real part) and $\mathrm{U}(k+1)$ the unitary group with respect to that Hermitian product. We consider $\mathbb{CP}^k = (\mathbb{C}^{k+1}\backslash\{0\})/\mathbb{C}^\times$. The $\mathrm{U}(k+1)$-action on $\mathbb{C}^{k+1}$ commutes with the $\mathbb{C}^\times$-action, so the action of $U(k+1)$ descends to $\mathbb{CP}^k$. The quotient $p\colon \mathbb{C}^{k+1}\backslash\{0\} \to \mathbb{CP}^k$ factors through the unit sphere $S^{2k+1} \subset \mathbb{R}^{2k+2} \cong \mathbb{C}^{k+1}$ via

$$\mathbb{C}^{k+1}\backslash\{0\} \xrightarrow{p_1} S^{2k+1} \xrightarrow{p_2} \mathbb{CP}^k\,,$$

where the map p_1 is given by $z \mapsto z/|z|$ and p_2 is the obvious quotient map. The fibre $p_2^{-1}(\{l\})$ in S^{2k+1} over a point $l \in \mathbb{CP}^k$ consists of those $z \in l$ with $|z| = 1$ and hence is a copy of S^1. This allows for an elegant description of a Fubini–Study form in the symplectic picture. Denote by $\Omega = \frac{i}{2}\sum_{j=0}^k dz_j \wedge d\bar{z}_j = \sum_{j=0}^k dx_j \wedge dy_j$ the standard symplectic form on $\mathbb{C}^{k+1} \cong \mathbb{R}^{2k+2}$. The (symplectic geometric) Fubini–Study form on $\mathbb{CP}^k$ with standard Hermitian metric is the unique $\mathrm{U}(k+1)$-invariant closed two-form such that $p_2^*\omega' = \iota^*\Omega$, where $\iota\colon S^{2k+1} \to \mathbb{C}^{2k+2}$ is the inclusion map.

To define ω' at $l \in \mathbb{CP}^k$ we set $\omega'_l(u,v) := \Omega_z(\tilde{u},\tilde{v})$ for a choice of $z \in p_2^{-1}(\{l\})$ and lifts $\tilde{u},\tilde{v}$ of $u,v \in T_l\mathbb{CP}^k$ to T_zS^{2k+1}. We need to show that this is well defined, i.e. independent of the choice of $z \in p_2^{-1}(\{l\}) = l \cap S^{2k+1}$ and independent of the choice of lift $\tilde{u},\tilde{v}$. First fix a $z \in l \cap S^{2k+1}$. The orthogonal (Euclidian) complement to T_zS^{2k+1} inside $T_z\mathbb{C}^{k+1}$ is just the real line $\mathbb{R}z$, so the symplectic complement with respect to Ω is $i\mathbb{R}z$. Since the S^1 orbit of z is $\{e^{i\theta}z\}$, its tangent space at z is also given by $i\mathbb{R}z$. This implies that Ω_z as a map

$$\Omega_z\colon T_zS^{2k+1} \times T_zS^{2k+1} \to \mathbb{R}$$

descends to a map $T_zS^{2k+1}/T_zS^1 \times T_zS^{2k+1}/T_zS^1 \to \mathbb{R}$. Hence $\Omega_z(\tilde{u},\tilde{v})$ is independent of the choice of lifts. Furthermore, since Ω is $\mathrm{U}(k+1)$-invariant, it is in particular S^1-invariant, so $\Omega_{e^{i\theta}z}(e^{i\theta}\tilde{u}, e^{i\theta}\tilde{v}) = \Omega_z(\tilde{u},\tilde{v})$. The relation $p_2^*\omega' = \iota^*\Omega$ holds by construction and ω' is $\mathrm{U}(k+1)$-invariant since Ω is. In addition ω' is closed since $p_2^*d\omega' = \iota^*d\Omega = 0$ and p^* is injective. The latter injectivity also gives uniqueness of ω' with the property $p_2^*\omega' = \iota^*\Omega$.

Remark. It turns out that the symplectic geometric Fubini–Study form ω' obtained this way is not the most natural from a Kähler geometric viewpoint. The description of $\mathbb{CP}^k$ as a Riemannian quotient $\pi_2\colon S^{2k+1} \to \mathbb{CP}^k$ with fibre $S^1 \subset \mathbb{R}^{2k+2}$ implies $\mathrm{Vol}(S^{2k+1}) = \mathrm{Vol}(\mathbb{CP}^k,\omega') \cdot \mathrm{Vol}(S^1)$. Since $\mathrm{Vol}(S^{2k+1}) = 2\pi \cdot \pi^k/k!$, the volume of $\mathbb{CP}^k$ with respect to the volume form $\omega'^k/k!$ induced by the Fubini–Study metric defined as above is $\mathrm{Vol}(\mathbb{CP}^k,\omega') = \pi^k/k!$, so ω' does not lie in an integral class. Rescaling ω' by setting $\omega := \omega'/\pi$ yields a Fubini–Study form satisfying $\omega \in c_1(\mathcal{O}(1))$ which can alternatively be described as $\omega = i/(2\pi)F$, where F is the curvature of the Hermitian metric on $\mathcal{O}(1)$ induced by the standard Hermitian structure of $\mathbb{C}^{k+1}$. From now on we shall work with the integral Fubini–Study form ω and we remark that ω satisfies $p^*\omega = 1/\pi \cdot \iota^*\Omega$.

Next, we want to show that the $\mathrm{U}(k+1)$-action on $\mathbb{CP}^k$ is Hamiltonian and describe the moment map. Let $A \in \mathfrak{u}(k+1)$ be a skew-Hermitian endomorphism. The infinitesimal action of A on $\mathbb{C}^{k+1}$ is given by the vector field $X_z^A = \partial_t|_{t=0}(e^{tA}z) = Az \in \mathbb{C}^{k+1} \cong T_z\mathbb{C}^{k+1}$. The map

$$\langle \mu_{\mathbb{C}^{k+1}}, A\rangle\colon \mathbb{C}^{k+1} \to \mathbb{R}\,, \quad z \mapsto \frac{1}{2}\Omega_z(Az,z)$$

defines a moment map for the $\mathrm{U}(k+1)$-action on $\mathbb{C}^{k+1}$. One computes

$$\begin{aligned}(d\langle \mu_{\mathbb{C}^{k+1}}, A\rangle)_z(w) &= \frac{1}{2}\left(\Omega_z(Aw, z) + \Omega_z(Az, w)\right) \\ &= \frac{1}{2}\left(\Omega_z(w, A^*z) + \Omega_z(Az, w)\right) \\ &= \frac{1}{2}\left(-\Omega_z(w, Az) + \Omega_z(Az, w)\right) \\ &= \frac{1}{2}\left(\Omega_z(Az, w) + \Omega_z(Az, w)\right) \\ &= \Omega_z(Az, w) \\ &= (\iota_{X^A}\Omega)_z(w)\,,\end{aligned}$$

where ι_{X^A} denotes contraction with the vector field X^A (the double use if ι as a contraction of tensors and an inclusion of $S^{2k+1} \hookrightarrow \mathbb{C}^{k+1}$ should not cause any ambiguity). To see that $\mu_{\mathbb{C}^{k+1}}$ is equivariant, let $U \in \mathrm{U}(k+1)$ and compute

$$\langle \mu_{C^{k+1}}(Uz), A\rangle = \frac{1}{2}\Omega(AUz, Uz) = \frac{1}{2}\Omega(U^{-1}AUz, z) = \langle \mu_{C^{k+1}}(z), \mathrm{Ad}_{U^{-1}} A\rangle\,.$$

The moment map for the $\mathrm{U}(k+1)$-action on $\mathbb{C}^{k+1}$ can be used to construct one for the action on $\mathbb{CP}^k$. We set

$$\langle \mu_{\mathbb{CP}^k}([z]), A\rangle := \frac{1}{2\pi}\frac{\Omega_z(Az, z)}{|z|^2} = \frac{1}{2\pi}\Omega_{p_1(z)}(Ap_1(z), p_1(z))\,.$$

It is apparent that this is well defined and one has $\mu_{\mathbb{CP}^k}([z]) = 1/\pi \cdot \mu_{\mathbb{C}^{k+1}}(p_1(z))$. In essence, the moment map on $\mathbb{CP}^k$ at $[z]$ is given by the restriction of $\mu_{\mathbb{C}^{k+1}}$ to S^{2n+1} evaluated at any unit length representative of $[z]$. Observe that if $w \in S^{2k+1}$, then $X^A_w = Aw \in T_w S^{2k+1}$. Keeping this in mind we compute

$$\begin{aligned}d(\langle \mu_{\mathbb{CP}^k} \circ p, A\rangle)_z &= 1/\pi \cdot d(\langle \mu_{\mathbb{C}^{k+1}}, A\rangle)_{p_1(z)} \cdot (dp)_z \\ &= 1/\pi \cdot (\iota_{X^A}\Omega)_{p_1(z)} \cdot (dp_1)_z \\ &= 1/\pi \cdot (\iota_{X^A}\iota^*\Omega)_{p_1(z)} \cdot (dp_1)_z \\ &= (\iota_{X^A} p^*\omega)_{p_1(z)} \cdot (dp_1)_z \\ &= (\iota_{dpX^A}\omega)_{[z]} \cdot (dp)_z\,.\end{aligned}$$

Since dpX^A is the vector field on $\mathbb{CP}^k$ representing the infinitesimal action of $A \in \mathfrak{u}(k+1)$ on $\mathbb{CP}^k$, this affirms that $\mu_{\mathbb{CP}^k}$ is indeed a moment map. Equivariance of $\mu_{\mathbb{CP}^k}$ follows from the equivariance of $\mu_{\mathbb{C}^{k+1}}$.

A computation shows that for $A \in \mathfrak{u}(k+1)$ one has

$$\int_{\mathbb{CP}^k} \langle \mu_{\mathbb{CP}^k}, A\rangle \frac{\omega^k}{k!} = \frac{i}{2\pi}\frac{\mathrm{tr}\, A}{(k+1)!}\,,$$

so the Hamiltonians of the $\mathrm{SU}(k+1)$-action on $\mathbb{CP}^k$ have zero integral.

Remark. The fibrewise Hamiltonians automatically having zero integral is the reason we chose to work with an $\mathrm{SU}(k+1)$-bundle E. One could modify the moment map $\mu_{\mathbb{CP}^k}$ by subtracting the trace term to obtain a moment map for $\mathrm{U}(k+1)$-bundles with the same property, but as we will see this turns out to be unnatural when considering the parametric geometry of $\mathbb{P}E$.

We finish this summary by $\mathbb{C}$-linearly extending $\mu_{\mathbb{CP}^k}$ to a map $\mathbb{CP}^k \to \mathfrak{gl}_{\mathbb{C}}(k+1)^*$ (using the decomposition $\mathfrak{gl}_{\mathbb{C}}(k+1) = \mathfrak{u}(k+1) \oplus i\mathfrak{u}(k+1)$). Concretely, this extension is given by

$$\langle \mu_{\mathbb{CP}^k}([z]), A\rangle = \frac{i}{2\pi}\frac{h_{std}(Az,z)}{h_{std}(z,z)}\,,$$

where h_{sdt} is the standard Hermitian product on $\mathbb{C}^{k+1}$ (with $\mathbb{C}$-linearity in the first argument) of which Ω is the negative imaginary part.

We now introduce a parametric moment map for the fibration $\mathbb{P}E \to X$, where each fibre carries the Fubini–Study metric induced by the Hermitian metric h in E on that fibre. Given an element $u \in \Gamma(X, \mathfrak{u}(E,h))$, i.e. a smooth section of the bundle $\mathfrak{u}(E,h) \to X$ of h-skew-adjoint endomorphisms of E, we get fibrewise Hamiltonians $\langle \mu_{\mathbb{P}E_x}, u_x\rangle$ with zero integral over the fibre $\mathbb{P}E_x$ (with respect to the Fubini–Study metric induced by h_x) for each $x \in X$.

Proposition 4.4.2. *The fibrewise Hamiltonions $\langle \mu_{\mathbb{P}E_x}, u_x\rangle$ glue to a smooth function $M(h,u) \in C^\infty(\mathbb{P}E)$ on the total space.*

Proof. Let $U \subset X$ be an open set over which E is trivialised by a $\Phi\colon E|_U \to U \times \mathbb{C}^{k+1}$ such that for each $x \in U$ the associated fibre map $\Phi_x\colon (E_x, h_x) \to (\mathbb{C}^{k+1}, h_{std})$ is a unitary isomorphism. The induced trivialisation $\phi\colon \mathbb{P}E|_U \to U \times \mathbb{CP}^k$ then has fibre maps $\phi_x\colon \mathbb{P}E_x \to \mathbb{CP}^k$ that are holomorphic isometries with respect to the respective Fubini–Study metrics. We use Φ to locally interpret u as a smooth map $U \to \mathrm{U}(k+1)$, more precisely set

$$\hat{u}\colon U \to \mathrm{U}(k+1)\,, \quad x \mapsto \Phi_x u_x \Phi_x^{-1}\,.$$

For each $x \in U$ the action of $\hat{u}_x$ on $\mathbb{CP}^k$ has the zero-integral Hamiltonian

$$\widehat{M}(h,u)_x := \langle \mu_{\mathbb{P}E_x}, u_x\rangle \circ \phi_x^{-1}\,.$$

This should be clear since Φ_x and ϕ_x preserve all relevant structure. For a more detailed proof denote by σ_x, ω the Fubini–Study forms on $\mathbb{P}E_x$ and $\mathbb{CP}^k$ respectively and compute

$$\begin{aligned} d\widehat{M}(h,u)_x(\cdot) &= d(\langle \mu_{\mathbb{P}E_x}, u_x\rangle)\cdot d\phi_x^{-1}(\cdot) \\ &= \sigma_x(X^{u_x}, d\phi_x^{-1}\cdot) \\ &= (\phi_x^{-1*}\sigma_x)(d\phi_x X^{u_x}, \cdot) \\ &= \omega(X^{\hat{u}_x}, \cdot) \end{aligned}$$

The zero integral property of $\widehat{M}(h,u)_x$ follows from that of $M(h,u)_x$ by the diffeomorphism invariance of the integral of forms. We observe that the right hand side of the moment map relation for $\widehat{M}$ is a smooth family of one-forms on $\mathbb{CP}^k$ indexed by U. Applying d^* (the formal L^2-adjoint of d with respect to the Fubini–Study metric on $\mathbb{CP}^k$) to both sides of the equality gives

$$\Delta\widehat{M}(h,u)_x = d^*\omega(X^{\hat{u}_x},\cdot)$$

and since $\widehat{M}(h,u)_x$ has zero integral, we can invert the Laplacian and obtain

$$\widehat{M}(h,u)_x = \Delta^{-1}d^*\omega(X^{\hat{u}_x},\cdot)\,.$$

The right hand side defines a smooth function on $U \times \mathbb{CP}^k$, so the same is true for $\widehat{M}(h,u)_x$. We immediately infer that the collection of fibrewise moment maps $M(h,u)_x := \langle \mu_{\mathbb{P}E_x}, u_x\rangle = \widehat{M}(h,u)_x \circ \phi_x$ is smooth in x. This is independent of the choice of U and ϕ, so $M(h,u) \in C^\infty(\mathbb{P}E)$ as claimed. □

As in the nonparametric model case we can $\mathbb{C}$-linearly extend $M(h,\cdot)$ to a map

$$M(h,\cdot)\colon \Gamma(X,\mathrm{End}(E)) \to C^\infty(\mathbb{P}E,\mathbb{C})$$

and even more generally to a map

$$M(h,\cdot)\colon \Omega^p(X,\mathrm{End}(E)) \to \Omega^p(\mathbb{P}E)$$

by applying $M(h,\cdot)$ to the endomorphism factor and pulling back the form part to $\mathbb{P}E$.

With these preparations we can now describe $\omega_0(h) = i/(2\pi)F_{\mathcal{O}(1),\tilde{h}^{-1}}$ in terms of the horizontal-vertical decomposition of $T\mathbb{P}E = H \oplus V$.

Proposition 4.4.3. *In the decomposition $\omega_0(h) = \omega_0(h)_{VV} + \omega_0(h)_{HH} + \omega_0(h)_{HV}$, the vertical-vertical component $\omega_0(h)_{VV}$ is given by the fibrewise Fubini–Study metrics σ defined by the Hermitian metric h. The horizontal-horizontal component is $\omega_0(h)_{HH} = -M(h,F_{E,h})$, where $F_{E,h} \in \Omega^2(X,\mathfrak{u}(E,h))$ is the curvature of the Chern connection on (E,h). The mixed component vanishes.*

Proof. The restriction of $\omega_0(h)$ to a fibre $\mathbb{P}E_x$ is the Fubini-Study metric on that fibre defined by h_x, i.e. $\omega_0(h)_{VV} = \sigma$ and the mixed part vanishes by definition of the connection $T\mathbb{P}E = H \oplus V$. It remains to show that $\omega_0(h)_{HH} = -M(h,F_{E,h})$.

The calculation in Appendix D.1 computes the decomposition of $\omega_0(h)$ in terms of the connection $T\mathbb{P}E = H' \oplus V'$ on $\mathbb{P}E$ that is induced by the Chern connection on (E,h) (one has $V = V'$, the prime merely indicates that the projections $T\mathbb{P}E \to V$ might a priori differ). It is shown that with respect to that decomposition $\omega_0(h)_{V'V'} = \sigma$, $\omega_0(h)_{H'V'} = 0$ and $\omega_{0\,H'H'} = -M(h,F_{E,h})$. In particular, the two connections on $\mathbb{P}E$ coincide and $\omega_{0\,HH} = -M(h,F_{E,h})$. □

We can now derive an explicit form of the scalar curvature expansion (4.2.6) for $\omega_r = \omega_0(h) + r\pi^*\omega_X$ on $\mathbb{P}E$.

Proposition 4.4.4. *The expansion of ω_r in powers of r^{-1} is given by*

$$S(\omega_r) = 2\pi k(k+1) + r^{-1}[-4\pi(k+1)M(h,\Lambda_{\omega_X}F_{E,h}) + \pi^* S(\omega_X)] + \mathcal{O}(r^{-2}) \tag{4.5}$$

Proof. The order r^0 term is just the fibrewise scalar curvature $S(\sigma)$, which is constant for the Fubini–Study metric. The integral Fubini–Study metric is Kähler–Einstein with $\rho = 2\pi(k+1)\omega$, so $S(\sigma) = 2\pi k(k+1)$. Apart from the pulled back scalar curvature of the base, the r^{-1} term consists of two parts, the vertical Laplacian applied to the contracted horizontal-horizontal part of the curvature of $\mathcal{O}(1)$, $\Delta_V \Lambda_H \omega_{0HH}$, and horizontal contraction of the curvature of the vertical anticanonical bundle $i\Lambda_H F_{\Lambda^k V,\sigma HH}$. Proposition 4.4.3 shows that $\Lambda_H\omega_0(h)_{HH} = -M(h,\Lambda_{\omega_X}F_{E,h})$ and since the fibrewise SU$(k+1)$-Hamiltonians constitute the first eigenspace of the Fubini–Study Laplacian with eigenvalue $2\pi(k+1)$ (cf. Proposition D.2.2 and the following remark in Appendix D.2), one has $\Delta_V\Lambda_H\omega_{0HH} = -2\pi(k+1)M(h,\Lambda_{\omega_X}F_{E,h})$. For the second term, observe that $\Lambda^k V = \mathcal{O}(k+1)$ (the canonical bundle of $\mathbb{CP}^k$ is $\mathcal{O}(-k-1)$) and that the Hermitian metric on $\Lambda^k V$ induced by σ corresponds to $\widetilde{h}^{-\otimes(k+1)}$. Since by definition $i/(2\pi)F_{\mathcal{O}(1),\widetilde{h}^{-1}} = \omega_0(h)$ this implies $i\Lambda_H F_{\Lambda^k V,\sigma HH} = 2\pi(k+1)\Lambda_H\omega_0(h)_{HH} = -2\pi(k+1)M(h,\Lambda_{\omega_X}F_{E,h})$. The total $\mathcal{O}(r^{-1})$ term is given by $-4\pi(k+1)M(h,\Lambda_{\omega_X}F_{E,h}) + \pi^*S(\omega_X)$ as claimed. □

4.4.2. 1st Order Approximation to Calabi Flow

To define an $\mathcal{O}(r^{-1})$ approximation to Calabi flow we first need to understand how $\omega_0(h)$ evolves if $h(t)$ is a smooth path of Hermitian metrics on E.

Lemma 4.4.5. *Let $h(t)$ be a smooth family of Hermitian metrics on E and $u(t) := ih^{-1}(t)(\partial_t h)(t)$ (this is an element of $\Gamma(X,\mathfrak{u}(E,h(t)))$ for each t). Then $\omega_0(h(t))$ satisfies*

$$\partial_t\omega_0(h(t)) = i\overline{\partial}\partial M(h(t),u(t)) .$$

Proof. First consider the nonparametric case of E being a vector space with a family of Hermitian structures $h(t)$ (we omit t from here on) with reference structure h_0. Each h defines a real scalar product g_h and a symplectic from Ω_h on E via $g_h = \operatorname{Re} h$ and $\Omega_h = -\operatorname{Im} h$. (in this convention one has $g_h(u,v) = \Omega_h(u,iv)$ and $\Omega_h = \sum_i dx_i \wedge dy_i$ in a g_h-orthonormal $\mathbb{R}$-basis $\{x_i, y_i\}$ with $y_i = ix_i$). The moment map for the U(E,h)-action on $\mathbb{P}E$ with integral Fubini–Study metric defined by h is given by

$$\langle \mu_{\mathbb{P}E}, A\rangle([z]) = \frac{1}{2\pi}\frac{\Omega_h(Az,z)}{h(z,z)} .$$

If $A = ih^{-1}(\partial_t h)$, then

$$\begin{aligned} \langle \mu_{\mathbb{P}E}, ih^{-1}(\partial_t h)\rangle([z]) &= \frac{1}{2\pi}\frac{\Omega_h(ih^{-1}(\partial_t h)z, z)}{h(z,z)} \\ &= -\frac{1}{2\pi}\frac{\operatorname{Im} h(ih^{-1}(\partial_t h)z, z)}{h(z,z)} \\ &= -\frac{1}{2\pi}\frac{\operatorname{Im} i(\partial_t h)(z,z)}{h(z,z)} \\ &= -\frac{1}{2\pi}\frac{(\partial_t h)(z,z)}{h(z,z)}\,. \end{aligned}$$

Keeping this in mind, we define a family of positive functions $e^{f(t)}$ on $\mathbb{P}E$ by setting

$$e^{f(t,[z])} = \frac{h(z,z)}{h_0(z,z)}\,.$$

The derivative of $f(t)$ with respect to the parameter t is then given by

$$(\partial_t f)(t,[z]) = \frac{(\partial_t h)(z,z)}{h(z,z)}\,.$$

If $\widetilde{h}^{-1}$ and $\widetilde{h}_0^{-1}$ denote the induced metrics on $\mathcal{O}(1) \to \mathbb{P}E$, then one finds that they are related by $\widetilde{h}^{-1} = e^{-f}\widetilde{h}_0^{-1}$, so their curvatures satisfy $F_{\widetilde{h}^{-1}} = F_{\widetilde{h}_0^{-1}} - \overline{\partial}\partial f$. Putting everything together one finds that

$$\partial_t i F_{\widetilde{h}^{-1}} = -i\overline{\partial}\partial(\partial_t f) = 2\pi i\overline{\partial}\partial\langle \mu_{\mathbb{P}E}, ih^{-1}(\partial_t h)\rangle\,.$$

For the parametric case ($E \to X$ now being a vector bundle), observe that the change of moment map and the change of the Hermitian metric $\widetilde{h}^{-1}$ on $\mathcal{O}(1)$ occur fibrewise, so the nonparametric computation also applies in this case up until $\widetilde{h}^{-1} = e^{-f}\widetilde{h}_0^{-1}$. The curvature relations also hold for $\mathcal{O}(1) \to \mathbb{P}E$, so $\omega_0(h) = i/(2\pi)F_{\widetilde{h}^{-1}} = \omega_0(h_0) - i/(2\pi)\overline{\partial}\partial f$ and

$$\partial_t \omega_0(h) = i\overline{\partial}\partial M(h, u(t))$$

as claimed. □

We can now derive equations for paths in metrics on the base $\omega_X(t)$ and Hermitian metrics $h(t)$ on E whose solutions will make $\omega_r(t) = \omega_0(h(t)) + r\pi^*\omega_X(t)$ an $\mathcal{O}(r^{-1})$ approximation to Calabi flow. Comparing

$$\partial_t \omega_r(t) = i\overline{\partial}\partial M(h, ih^{-1}\partial_t h) + r\pi^*\partial_t\omega_X$$

to the scalar curvature expansion from Proposition 4.5

$$i\overline{\partial}\partial S(\omega_r) = r^{-1} i\overline{\partial}\partial[-4\pi(k+1)M(h, \Lambda_{\omega_X}F_{E,h}) + \pi^* S(\omega_X)] + \mathcal{O}(r^{-2})$$

shows that if $\omega_X(t)$ solves Calabi flow on the base, i.e. $\partial_t\omega_X = -i\bar\partial\partial S(\omega_X)$ and if $h(t)$ solves the time-dependent Hermitian Yang–Mills flow

$$ih^{-1}\partial_t h = 4\pi(k+1)\Lambda_{\omega_X(t/r)}F_{E,h}\,, \tag{4.6}$$

then

$$\omega_{r,1} := \omega_0(h(t/r)) + r\pi^*\omega_X(t/r^2)$$

defines the desired approximation.

Remark. The $\mathcal{O}(r^{-1})$ approximation to Calabi flow being given by Hermitian Yang–Mills flow with respect to an evolving base metric is the reason why we have assumed the bundle $E \to X$ to be slope stable. This assumption guarantees the existence and convergence of the Hermitian Yang–Mills flow, at least when the base is a Riemann surface. In the general case, slope stability is a necessary and sufficient condition for the existence of a suitable limit object (a Hermite–Einstein metric) by the Hitchin–Kobayashi correspondence.

4.4.3. 2nd Order Approximation to Calabi Flow

In the adiabatic scheme for Kodaira surfaces, higher order approximations to Calabi flow in r^{-1} could be obtained by solving linear parabolic equations for two types of perturbations of ω_r, one for each summand in the decomposition $C^\infty(Z) = \pi^*C^\infty(X) \oplus C^\infty_\perp(Z)$ of functions on Z into functions that are fibrewise constant and those that have zero integral on each fibre. The addition of a path of Kähler potentials on the base could compensate for the failure of the $\mathcal{O}(r^{-k})$ approximation $\omega_{r,k}$ to be $\mathcal{O}(r^{-(k+1)})$ in $\pi^*C^\infty(X)$ and the subsequent addition of a Kähler potential with fibrewise zero integral correcting to remaining error term in $C^\infty_\perp(X)$ would then give an $\mathcal{O}(r^{-(k+1)})$ approximation $\omega_{r,k+1}$ to Calabi flow.

In the case of ruled manifolds, similar techniques can be used. However, one cannot hope to correct the full failure of $\omega_{r,1}$ being an $\mathcal{O}(r^{-2})$ approximation to Calabi flow in the L^2-complement to $\pi^*C^\infty(X)$ in $C^\infty(\mathbb{P}E)$ via the addition of fibrewise mean value zero Kähler potentials. The reason for this is that the cscK metrics on the fibres are not unique — the gradients of Hamiltonians for the $\mathrm{U}(k+1)$-action define holomorphic vector fields, so the pieced together fibrewise Hamiltonians for the action of sections in $\Gamma(X, \mathfrak{u}(E,h))$ lie in the kernel of the linearised fibrewise scalar curvature map. The corresponding error terms need to be treated separately.

We split the $\mathcal{O}(r^{-2})$ term ψ_2 of $S(\omega_{r,1}(t))$ into three parts, $\psi_2 = \psi_{X,2} + \psi_{\mathfrak{u},2} + \psi_{\perp,2}$ according to the splitting

$$C^\infty(\mathbb{P}E) = \pi^*C^\infty(X) \oplus M(h, \Gamma(X, \mathfrak{u}(E,h))) \oplus C^\infty_{\perp,h}\,,$$

where the second summand consists of mean value zero Hamiltonians for infinitesimal isometries of the fibre (with respect to the Fubini–Study metric depending on h) and the last summand is the L^2 orthogonal complement to the other two. The decomposition only depends on the Hermitian metric $h(t)$. Each of the three $\mathcal{O}(r^{-2})$ error terms necessitates a specific correction of $\omega_{r,1}$.

- $\psi_{X,2} \in \pi^* C^\infty(X)$ is dealt with by perturbing Calabi flow on X. This correction introduces a new error term in $M(h, \Gamma(X, \mathfrak{u}(E,h)))$ at r^{-2} which we subsume along with the original r^{-2} term $\psi_{\mathfrak{u},2}$ to $\psi'_{\mathfrak{u},2} \in M(h, \Gamma(X, \mathfrak{u}(E,h)))$.

- $\psi'_{\mathfrak{u},2} \in M(h, \Gamma(X, \mathfrak{u}(E,h)))$ requires an adjustment of $h(t)$,

- $\psi_{\perp,2} \in \pi^* C^\infty_{\perp,h}$ is compensated for by a Kähler potential in $C^\infty_\perp(\mathbb{P}E)$.

These changes accumulate to give the desired $\mathcal{O}(r^{-2})$ approximation $\omega_{r,2}$ of Calabi flow. A key features of the adjustments is that they leave the scalar curvature in lower order unchanged, i.e. the scalar curvatures of $\omega_{r,1}$ and $\omega_{r,2}$ agree up to (including) order r^{-1}, only their higher order parts differ.

The $\pi^* C^\infty(X)$-Correction

Set $\omega_{X,1} := \omega_X + r^{-1} i\bar\partial\partial f$ and $\omega'_{r,1} := \omega_{r,1} + \pi^* i\bar\partial\partial f = \omega_0(h) + r\pi^*\omega_{X,1}$. Passing from $S(\omega_{r,1})$ to $S(\omega'_{r,1})$ amounts to replacing ω_X by $\omega_{X,1}$ in the scalar curvature expansion (4.5), so the change in scalar curvature caused by the addition of the Kähler potential $\pi^* f$ is $\mathcal{O}(r^{-2})$. The precise change at r^{-2} in the scalar curvature is obtained by linearising the quantities at r^{-1} involving $\omega_{X,1}$. We expand

$$\begin{aligned} S(\omega_{X,1}) &= S(\omega_X) + r^{-1}(d\,\mathrm{Sc})_{\omega_X} f + \mathcal{O}(r^{-2})\,, \\ \Lambda_{\omega_{X,1}} F_{E,h} &= \Lambda_{\omega_X} F_{E,h} + r^{-1}\left[\frac{1}{2}\Lambda^2_{\omega_X} F_{E,h} \wedge i\bar\partial\partial f - \Lambda_{\omega_X} F_{E,h} \cdot \Delta_{\omega_X} f\right] + \mathcal{O}(r^{-2})\,, \end{aligned}$$

which gives

$$\begin{aligned} S(\omega'_{r,1}) \;=\; & S(\omega_{r,1}) + r^{-2}\left[\pi^*(d\,\mathrm{Sc})_{\omega_X} f\right] \\ & - r^{-2} 2\pi(k+1)\left[M(h, \Lambda^2_{\omega_X} F_{E,h} \wedge i\bar\partial\partial f) - 2M(h, \Lambda_{\omega_X} F_{E,h}) \cdot \pi^*\Delta_{\omega_X} f\right] + \mathcal{O}(r^{-3})\,. \end{aligned}$$

Observe that the third term lies in $M(h, \Gamma(X, \mathfrak{u}(E,h)))$. Now define $g \in C^\infty(X)$ by $\pi^* g = \psi_{X,2}$ and have f solve the linear parabolic PDE

$$(\partial_t + d\,\mathrm{Sc}_{\omega_X(t)}) f = -g(r^2 t)$$

with zero initial condition. Then $\omega'_{r,1}$ with h evaluated at t/r and $\omega_{X,1}$ evaluated at t/r^2 satisfies

$$\partial_t \omega'_{r,1} + i\bar\partial\partial S(\omega'_{r,1}) = r^{-2} i\bar\partial\partial[\psi'_{\mathfrak{u},2} + \psi_{\perp,2}] + \mathcal{O}(r^{-3})\,,$$

where $\psi'_{\perp,2} = \psi_{\perp,2} - 2\pi(k+1)\left[M(h, \Lambda^2_{\omega_X} F_{E,h} \wedge i\bar\partial\partial f) - 2M(h, \Lambda_{\omega_X} F_{E,h}) \cdot \pi^*\Delta_{\omega_X} f\right]$.

The $M(h, \Gamma(X, \mathfrak{u}(E,h)))$-Correction

Adjust $h(t)$ by setting

$$h_1(t) := h(t)(\mathrm{id}_E + r^{-1}\vartheta(t))$$

for a family of $h(t)$-self-adjoint endomorphisms $\vartheta(t) \in i\mathfrak{u}(E, h(t))$ of E. We derive an evolution equation for ϑ such that the new family

$$\omega''_{r,1}(t) := \omega_0(h_1) + r\pi^*\omega_{X,1}\,,$$

again with h_1 evaluated at t/r and $\omega_{X,1}$ evaluated at t/r^2, is an $\mathcal{O}(r^{-2})$ approximation to Calabi flow up to a part in $C^\infty_{\perp,2}$. Using Lemma 4.4.5 to compare the time-derivatives of $\omega'_{r,1}$ and $\omega''_{r,1}$ yields

$$\partial_t\omega''_{r,1} - \partial_t\omega'_{r,1} = r^{-1}i\bar\partial\partial[M(h_1, i{h_1}^{-1}\partial_t h_1) - M(h, ih^{-1}\partial_t h)]\,. \tag{4.7}$$

On the other hand,

$$S(\omega''_{r,1}) - S(\omega'_{r,1}) = -r^{-1}4\pi(k+1)[M(h_1, \Lambda_{\omega_X}F_{E,h_1}) - M(h, \Lambda_{\omega_X}F_{E,h})] + \mathcal{O}(r^{-3})\,, \tag{4.8}$$

where the remainder term is $\mathcal{O}(r^{-3})$ because the entirety of the r^{-2} term stems from the linearisation of the r^{-1} term. Expressing the linearisations of $M(h_1, i{h_1}^{-1}\partial_t h_1)$ and $M(h', \Lambda_{\omega_X}F_{E,h_1})$ in r^{-1} as

$$\begin{aligned} M(h_1, i{h_1}^{-1}\partial_t h_1) &= M(h, ih^{-1}\partial_t h) + r^{-1}Q(h)(\vartheta, \partial_t\vartheta) + \mathcal{O}(r^{-2}) \\ M(h_1, \Lambda_{\omega_X}F_{E,h_1}) &= M(h, F_{E,h}) + r^{-1}P(h)(\vartheta) + \mathcal{O}(r^{-2}) \end{aligned}$$

we can combine (4.7) and (4.8) to

$$\partial_t\omega''_{r,1} + i\bar\partial\partial S(\omega''_{r,1}) = r^{-2}i\bar\partial\partial[Q(h)(\vartheta, \partial_t\vartheta) - 4\pi(k+1)P(h)(\vartheta) + \psi'_{\mathfrak{u},2} + \psi_{\perp,2}] + \mathcal{O}(r^{-3})\,.$$

It remains to compute P and Q and to check that the linear equation

$$Q(h)(\vartheta, \partial_t\vartheta) - 4\pi(k+1)P(h)(\vartheta) + \psi'_{\mathfrak{u},2} = 0 \tag{4.9}$$

is parabolic and can be solved. In order to do this we need to linearise M as a map

$$M\colon \mathcal{H} \times \mathfrak{gl}_{\mathbb{C}}(E) \to C^\infty(\mathbb{P}E, \mathbb{C})\,,$$

where $\mathcal{H} = \mathrm{Gl}_{\mathbb{C}}(E)/\,\mathrm{U}(E, h_0)$ is the space of Hermitian inner products on E parametrised by the transitive $\mathrm{Gl}_{\mathbb{C}}(E)$-action on a reference product h_0. As M only depends on fibrewise restrictions of the relevant quantities, it suffices to think of E as a vector space.

Remark. The reason to look at the $\mathbb{C}$-linear extension of $M(h, \cdot)$ is that the space of arguments $\mathfrak{gl}_{\mathbb{C}}(E) = \mathfrak{u}(E, h) \oplus i\mathfrak{u}(E, h)$ does not itself depend on h making calculations conceptionally easier.

With respect to the group action of $\mathrm{Gl}_{\mathbb{C}}(E)$ on $\mathcal{H}$ via $g \cdot h := h(g^{-1}\cdot, g^{-1}\cdot)$, the adjoint action on $\mathfrak{gl}_{\mathbb{C}}(E)$ and precomposition by $\phi_{g^{-1}}$ (the action of $\mathrm{Gl}_{\mathbb{C}}(E)$ on $\mathbb{P}E$) on

$C^\infty(\mathbb{P}E)$, the map M is equivariant. Recalling $M(h,A)([z]) = i/(2\pi)\cdot h(Az,z)/h(z,z)$ one computes

$$\begin{aligned} M(g\cdot h, \operatorname{Ad}_g A)([z]) &= \frac{i}{2\pi}\left[\frac{(g\cdot h)(\operatorname{Ad}_g(A)z,z)}{(g\cdot h)(z,z)}\right] \\ &= \frac{i}{2\pi}\left[\frac{h(Ag^{-1}z, g^{-1}z)}{h(g^{-1}z,g^{-1}z)}\right] \\ &= M(h,A)(g^{-1}\cdot[z])\,. \end{aligned}$$

We now compute the derivatives D_1M and D_2M of M with respect to the $\mathcal{H}$ and the $\mathfrak{gl}_{\mathbb{C}}(E)$-argument. Since M is linear in the second argument, we have

$$(D_2M)(h_0,A)(B) = M(h_0,B)\,.$$

Computing the derivative with respect to the first argument is done by differentiating the equivariance property. Using the transitive action $\mathrm{Gl}_{\mathbb{C}}(E)$ with stabiliser $\mathrm{U}(E,h_0)$ at h_0 on $\mathcal{H}$, we identify the tangent space $T_{h_0}\mathcal{H}$ with $i\mathfrak{u}(E,h_0)$ (Note: Instead of the transitive $\mathrm{Gl}_{\mathbb{C}}(E)$-action one can parametrise $\mathcal{H}$ via $h_0\zeta$ for a positive h_0-self-adjoint ζ. The resulting identification of $T_{h_0}\mathcal{H}$ with $i\mathfrak{u}(E,h)$ differs from the previous one by a factor of -2). Let g_t be a path in $\mathrm{U}(k+1)$ with $g_0 = \mathrm{id}_E$ and $(\partial_t g_t)|_{t=0} = \eta = \eta_{\mathfrak{u}} + \eta_{i\mathfrak{u}} \in \mathfrak{u}(E,h_0)\oplus i\mathfrak{u}(E,h_0)$. Differentiating the left hand side of the equivariance property for g_t at $t=0$ yields

$$\begin{aligned} \partial_t|_{t=0}M(g_t\cdot h_0, \operatorname{Ad}_{g_t}A) &= (D_1M)(h_0,A)(\partial_t|_{t=0}g_t\cdot h_0) + (D_2M)(h_0,A)(\partial_t|_{t=0}\operatorname{Ad}_{g_t}A) \\ &= (D_1M)(h_0,A)(\eta_{i\mathfrak{u}}) + M(h_0,[\eta,A])\,. \end{aligned}$$

On the other hand

$$\begin{aligned} \partial_t|_{t=0}M(g_t\cdot h_0, \operatorname{Ad}_{g_t}A) &= \partial_t|_{t=0}(\phi_{g^{-1}})^*M(h,A) \\ &= \partial_t|_{t=0}(\phi_g^{-1})^*M(h_0,A) \\ &= -\mathscr{L}_{X^\eta}M(h_0,A)\,, \end{aligned}$$

where X^η is the holomorphic vector field generated by η (which is Killing and symplectic if $\eta_{i\mathfrak{u}} = 0$). Combining the two gives

$$(D_1M)(h_0,A)(\eta_{i\mathfrak{u}}) = -\mathscr{L}_{X^\eta}M(h_0,A) - M(h_0,[\eta,A])\,. \tag{4.10}$$

It seems as if though the right hand side seems to "see more" of η, but this is resolved by observing that for a path g_t in $\mathrm{U}(E,h_0)$, we have $g_t\cdot h_0 = h_0$ and $\eta = \eta_{\mathfrak{u}}$. The invariance property then gives just the usual $\mathrm{U}(E,h_0)$-equivariance property of the moment map $M(h,\cdot)$ which reads

$$\mathscr{L}_{X^{\eta_{\mathfrak{u}}}}M(h_0,A) + M(h_0,[\eta_{\mathfrak{u}},A]) = 0\,,$$

so (4.10) reduces to the more natural looking form

$$(D_1M)(h_0,A)(\eta_{i\mathfrak{u}}) = -\mathscr{L}_{X^{\eta_{i\mathfrak{u}}}}M(h_0,A) - M(h_0,[\eta_{i\mathfrak{u}},A])\,. \tag{4.11}$$

Now let h_s be a family of paths in Hermitian metrics of the form $h_s = h_0(\mathrm{id}_E + s\vartheta)$ for $\vartheta(t) \in i\mathfrak{u}(E, h_0(t))$. We want to compute $\partial_s|_{s=0} M(h_s, ih_s^{-1}(\partial_t h_s))$. Since the derivative only depends on h_s up to first order in s we can instead look at $h_s = h_0 e^{s\vartheta}$ which has the advantage that we can write $h_s = g_s \cdot h_0$ for $g_s := e^{-\frac{1}{2}s\vartheta}$. One computes

$$\begin{aligned}
\partial_s|_{s=0} M(h_s, ih_s^{-1}(\partial_t h_s)) &= (D_1 M)(h_0, ih_0^{-1}(\partial_t h_0))(\partial_s|_{s=0} g_s \cdot h_0) \\
&\quad + (D_2 M)(h_0, ih_0^{-1}(\partial_t h_0))(\partial_s|_{s=0} ih_s^{-1}(\partial_t h_s)) \\
&= \frac{1}{2}\mathscr{L}_{X^\vartheta} M(h_0, ih_0^{-1}(\partial_t h_0)) + \frac{1}{2} M(h_0, [\vartheta, ih_0^{-1}(\partial_t h_0)]) \\
&\quad + M(h_0, [ih_0^{-1}(\partial_t h_0), \vartheta]) + M(h_0, i\partial_t \vartheta) \\
&= \frac{1}{2}\mathscr{L}_{X^\vartheta} M(h_0, ih_0^{-1}(\partial_t h_0)) + M(h_0, i\partial_t\vartheta) \\
&\quad - \frac{1}{2} M(h_0, [\vartheta, ih_0^{-1}(\partial_t h_0)])\,.
\end{aligned}$$

The fact that the last summand doesn't directly cancel in the computation is not a mistake, but it vanishes nonetheless. The easiest way to see this is to observe that the left hand side and all other summands on the right hand side are real, whereas the last summand is purely imaginary since $[\theta, ih_0^{-1}(\partial_t h_0)]$ is h_0-skew-adjoint. Another way to see this is to explicitly compute

$$\begin{aligned}
\partial_s|_{s=0} M(h_s, ih_s^{-1}(\partial_t h_s))([z]) &= \partial_s|_{s=0} \frac{i}{2\pi}\left[\frac{h_s(ih_s^{-1}(\partial_t h_s)z, z)}{h_s(z,z)}\right] \\
&= \frac{i}{2\pi}\left[\frac{\partial_t h_0(i\vartheta z, z)}{h_0(z,z)} - \frac{\partial_t h_0(z,z)}{h_0(z,z)}\frac{h_0(i\vartheta z, z)}{h_0(z,z)} + \frac{h_0(i\partial_t\vartheta z, z)}{h_0(z,z)}\right] \\
&= M(h_0, i\partial_t\vartheta) + \frac{i}{2\pi}\left[\frac{1}{2}\frac{d}{ds}\bigg|_{s=0} \frac{i(\partial_t h_0)(e^{s\vartheta}z, e^{s\vartheta}z)}{h_0(e^{s\vartheta}z, e^{s\vartheta}z)}\right] \\
&= M(h_0, i\partial_t\vartheta) + 1/2 \cdot \mathscr{L}_{X^\vartheta} M(h_0, ih_0^{-1}(\partial_t h_0))\,.
\end{aligned}$$

In this computation the term $M(h_0, [\vartheta, ih_0^{-1}(\partial_t h_0)])$ does not even appear. Applied to our case this computation yields

$$Q(h)(\vartheta, \partial_t\vartheta) = M(h, i\partial_t\vartheta) + 1/2 \cdot \mathscr{L}_{X^\vartheta} M(h, ih^{-1}\partial_t h)\,. \tag{4.12}$$

In order to determine P, consider

$$\begin{aligned}
F_{E,h_s} &= \overline{\partial}(h_s^{-1}\partial h_s) \\
&= F_{E,h_0} + s[-\overline{\partial}(\vartheta h_0^{-1}\partial h_0) + \overline{\partial}(h_0^{-1}\partial(h_0\vartheta))] + \mathcal{O}(s^2) \\
&= F_{E,h_0} + s\overline{\partial}[\partial\vartheta + [h_0^{-1}\partial h_0, \vartheta]] + \mathcal{O}(s^2) \\
&= F_{E,h_0} + s\overline{\partial}\partial_{h_0}\vartheta + \mathcal{O}(s^2)\,,
\end{aligned}$$

which implies

$$\Lambda_{\omega_X} F_{E,h_s} = \Lambda_{\omega_X} F_{E,h_0} - si\Delta_{\overline{\partial},h_0,\omega_X}\vartheta + \mathcal{O}(s^2)\,,$$

where $\Delta_{\overline{\partial},h_0,\omega_X}$ is the $\overline{\partial}$-bundle Laplacian on $\mathrm{End}(E)$ defined by ω_X and h_0. Again using $h_s = g_s \cdot h_0$ with $g_s = e^{-\frac{1}{2}s\vartheta}$ instead of $h_s = (\mathrm{id}_E + s\vartheta)$, the desired first order expansion of $M(h_s, \Lambda_{\omega_X} F_{E,h_s})$ can be computed via

$$\begin{aligned}
\partial_s|_{s=0} M(h_s, \Lambda_{\omega_X} F_{E,h_s}) &= (D_1 M)(h_0, \Lambda_{\omega_X} F_{E,h_0})(\partial_s|_{s=0} g_s \cdot h_0) \\
&+ (D_2 M)(h_0, \Lambda_{\omega_X} F_{E,h_0})(\partial_s|_{s=0} \Lambda_{\omega_X} F_{E,h_s}) \\
&= \frac{1}{2}\mathscr{L}_{X^\vartheta} M(h_0, \Lambda_{\omega_X} F_{E,h_0}) + \frac{1}{2} M(h_0, [\vartheta, \Lambda_{\omega_X} F_{E,h_0}]) \\
&- M(h_0, \Delta_{\overline{\partial},h_0,\omega_X} i\vartheta) \\
&= \frac{1}{2}\mathscr{L}_{X^\vartheta} M(h_0, \Lambda_{\omega_X} F_{E,h_0}) - M(h_0, \Delta_{\overline{\partial},h_0,\omega_X} i\vartheta)
\end{aligned}$$

which yields

$$P(h)(\vartheta) = \frac{1}{2}\mathscr{L}_{X^\vartheta} M(h, \Lambda_{\omega_X} F_{E,h}) - M(h, \Delta_{\overline{\partial},h,\omega_X} i\vartheta)\,. \tag{4.13}$$

With the expression (4.13) and (4.12) for P and Q, the equation (4.9) for the correction ϑ becomes

$$\begin{aligned}
M(h, i\partial_t\vartheta) + 4\pi(k+1) M(h, \Delta_{\overline{\partial},h,\omega_X} i\vartheta) + \psi'_{\mathfrak{u},2} & \\
+\frac{1}{2}\mathscr{L}_{X^\vartheta}[M(h, ih^{-1}\partial_t h) - 4\pi(k+1) M(h, \Lambda_{\omega_X} F_{E,h})] &= 0
\end{aligned}$$

and in light of (4.6) the Lie-derivative term vanishes automatically. Writing $\psi'_{\mathfrak{u},2} = M(h, iA(t))$, this equation is satisfied if and only if

$$(\partial_t + 4\pi(k+1)\Delta_{\overline{\partial},h,\omega_X})\vartheta = -A(t)\,, \tag{4.14}$$

which is a bundle valued heat equation and indeed parabolic. If we have a solution ϑ with initial value $\vartheta(0) = 0$, then the corresponding $\omega''_{r,1}$ solves

$$\partial_t \omega''_{r,1} + i\overline{\partial}\partial S(\omega''_{r,1}) = r^{-2} i\overline{\partial}\partial\psi_{\perp,2} + \mathcal{O}(r^{-3})\,.$$

The $C^\infty_\perp(\mathbb{P}E)$-Correction

To correct the remaining deficiency of $\omega''_{r,1}$ not solving Calabi flow up to (and including) order r^{-2}, we use the same technique as in the case of Kodaira surfaces and modify $\omega''_{r,1}$ by a Kähler potential $\phi \in C^\infty_\perp(\mathbb{P}E)$. Set

$$\omega_{r,2} := \omega''_{r,1} + r^{-2} i\overline{\partial}\partial\phi$$

and observe from equation (4.1) (cp. also (4.2)) that the addition of the Kähler potential $r^{-2} i\overline{\partial}\partial\phi$ changes the scalar curvature at order r^{-2} by the linearised fibrewise curvature $d\,\mathrm{Sc}_F$, i.e.

$$S(\omega_{r,2}) = S(\omega''_{r,1}) + r^{-2}(d\,\mathrm{Sc}_F)_\sigma\phi + \mathcal{O}(r^{-3})\,.$$

Observe that $(d\,\mathrm{Sc}_F)_\sigma$ restricted to $C^\infty_\perp(\mathbb{P}E)$ has trivial kernel. This is due to $C^\infty_\perp(\mathbb{P}E)$ being orthogonal to $\pi^* C^\infty(X) \oplus M(h, \Gamma(\mathfrak{u}(E,h)))$ and the fibrewise linearisations of the scalar curvature taking place at cscK metrics, so $\ker(d\,\mathrm{Sc}_F)_\sigma$ is precisely $C^\infty(X) \oplus M(h, \Gamma(\mathfrak{u}(E,h)))$. The linear parabolic equation

$$(\partial_t \phi + (d\,\mathrm{Sc}_F)_\sigma)\phi = -\psi_{\perp,2}$$

should thus be solvable and taking ϕ to be its solution with zero initial condition makes $\omega_{r,2}$ the desired $\mathcal{O}(r^{-3})$ approximation to Calabi flow.

4.4.4. Higher Order Approximation to Calabi Flow

The procedure described in 4.4.3 can adapted inductively to arbitrarily high orders. Set

$$\begin{aligned} \omega_{X,p} &= \omega_X + i\bar{\partial}\partial \sum_{l=1}^{p} r^{-l} f_l\,, \\ h_p &= h\left(\mathrm{id}_E + \sum_{l=1}^{p} r^{-l}\vartheta_l\right)\,, \\ \Phi_p &= r^{-1}\sum_{l=1}^{p} r^{-l}\phi_l \end{aligned}$$

and suppose that

$$\omega_{r,p}(t) = \omega_0(h_{p-1}(t/r)) + r\pi^*\omega_{X,p-1}(t/r^2) + i\bar{\partial}\partial\Phi_{p-1}(t)$$

solves Calabi flow up to order r^{-p}, i.e.

$$\partial_t\omega_{r,p} + i\bar{\partial}\partial S(\omega_{r,p}) = r^{-(p+1)} i\bar{\partial}\partial(\psi_{X,p+1} + \psi_{\mathfrak{u},p+1} + \psi_{\perp,p+1}) + \mathcal{O}(r^{-(p+2)})\,,$$

where the $r^{-(p+1)}$ error has been decomposed according to $C^\infty\mathbb{P}E = \pi^* C^\infty(X) \oplus M(h, \Gamma(X, \mathfrak{u}(E,h))) \oplus C^\infty_\perp(\mathbb{P}E)$. The corrections ϑ_p, f_p and ϕ_p such that the corresponding $\omega_{r,p+1}$ is an order $r^{-(p+1)}$ approximation to Calabi flow are found exactly as before. For completeness, we state the equations that need to be solved (with zero initial conditions) and refer to the discussion of the second order approximation for details.

- Set $\omega_{X,p} := \omega_{X,p-1} + i\bar{\partial}\partial r^{-p} f_p$ and $\omega'_{r,p}(t) := \omega_0(h_{p-1}(t/r)) + r\pi^*\omega_{X,p}(t/r^2) + i\bar{\partial}\partial\Phi_{p-1}(t)$, write $\psi_{X,p+1} = \pi^* g$ and have f_p solve

  $$(\partial_t + (d\,\mathrm{Sc})_{\omega_X(t)}) f_p = -g(r^2 t)\,.$$

  Then

  $$\partial_t\omega'_{r,p} + i\bar{\partial}\partial S(\omega'_{r,p}) = r^{-(p+1)} i\bar{\partial}\partial(\psi'_{\mathfrak{u},p+1} + \psi_{\perp,p+1}) + \mathcal{O}(r^{-(p+2)})\,,$$

  where $\psi'_{\perp,p+1} = \psi_{\perp,p} - 2\pi(k+1)\left[M(h, \Lambda^2_{\omega_X} F_{E,h} \wedge i\bar{\partial}\partial f_p - 2\Lambda_{\omega_X} F_{E,h}\Delta_{\omega_X} f_p)\right]$.

- Set $h_p := h_{p-1} + r^{-p}h\vartheta_p$ and $\omega''_{r,p}(t) := \omega_0(h_p(t/r)) + r\pi^*\omega_{X,p}(t/r^2) + i\bar{\partial}\partial\Phi_{p-1}(t)$, write $\psi'_{\mathrm{u},p+1} = M(h, iA)$ and have ϑ_p solve

$$(\partial_t + 4\pi(k+1)\Delta_{\bar{\partial},h,\omega_X})\vartheta = -A\,.$$

  Then

$$\partial_t\omega''_{r,p} + i\bar{\partial}\partial S(\omega''_{r,p}) = r^{-(p+1)}i\bar{\partial}\partial\psi_{\perp,p+1} + \mathcal{O}(r^{-(p+2)})\,.$$

- Set $\Phi_p := \Phi_{p-1} + r^{-(p+1)}\varphi_p$ and $\omega_{r,p+1}(t) := \omega_0(h_p(t/r)) + r\pi^*\omega_{X,p}(t/r^2) + i\bar{\partial}\partial\Phi_p(t)$. Having ϕ_p solve

$$(\partial_t + (d\,\mathrm{Sc}_F)_\sigma)\phi_p = -\psi_{\perp,p+1}$$

  gives the desired $\mathcal{O}(r^{-(p+1)})$ approximation to Calabi flow:

$$\partial_t\omega_{r,p+1} + i\bar{\partial}\partial S(\omega_{r,p+1}) = \mathcal{O}(r^{-(p+2)})\,.$$

4.5. Outlook

The adiabatic analysis presented here does not constitute a complete proof of the existence and convergence of Calabi flow on Kodaira surfaces and projectivised stable bundles with initial condition given by ω_r for sufficiently large r; it has been included in this thesis to motivate the development of twisted Calabi flow and time-dependent Hermitian Yang–Mills flow in the previous chapters. However, it should be possible to extend the analysis to give a full proof. We present a brief account of the work required to fill in the gaps.

4.5.1. The Inverse Function Theorem

The Banach space version of the inverse function theorem states that if $\Phi\colon A \to B$ is a continuously differentiable map between Banach spaces A, B and $d\Phi_0$ is an isomorphism, then there exists an open ball $B_\delta(\Phi(0)) \subset B$ and an open neighbourhood $U \subset A$ of 0, such that $\Phi|_U\colon U \to B_\delta(\Phi(0))$ is bijective. Moreover, the size of δ is controlled by $\|(d\Phi)_0^{-1}\|_{op}$ — the smaller $\|(d\Phi)_0^{-1}\|_{op}$, the larger δ.

As outlined in the introduction, the intended application of the inverse function theorem is to perturb a sufficiently good approximation $\omega_{r,k}$ to Calabi flow to a genuine solution. Writing the approximative metrics as $\omega_{r,n} = \omega_r(0) + i\bar{\partial}\partial\varphi_{r,n}(t)$, define the map $\Phi_{r,n}$ by

$$\Phi_{r,n}(\psi) = \partial_t(\varphi_{r,n} + \psi) + \mathrm{Sc}(\varphi_{r,n} + \psi) - \underline{S}_r\,,$$

where Sc denotes the scalar curvature map of the background metric $\omega_r(0)$ and $\underline{S}_r$ is the average scalar curvature of the adiabatic class κ_r. By construction $\Phi_{r,n}(\psi)$ measures the failure of $\omega_{r,n} + i\bar{\partial}\partial\psi$ to satisfy Calabi flow. To invoke the inverse function theorem, the $\Phi_{r,n}$ need to be set up as maps between suitable Banach spaces $A_{r,n}$ and $B_{r,n}$ such that the following criteria are satisfied:

1. The maps $\Phi_{r,n}\colon A_{r,n} \to B_{r,n}$ are differentiable with invertible derivative at 0.

2. The sequences $\Phi_{r,n}(0)$ measuring how close $\omega_{r,n}$ is to being a solution to Calabi flow in $B_{r,n}$ satisfy $\|\Phi_{r,n}(0)\|_{B_r} \leqslant Cr^{-(n-a)}$ for constants C, a independent of r and n.

3. The operator norms of the inverse of $(d\Phi_{r,n})_0$ (controlling the size $\delta_{r,n}$ of the balls onto which $\Phi_{r,n}$ maps surjectively) grow at most at a rate such that $\delta_{r,n} \geqslant Cr^{-b}$ with constants C, b independent of r and n.

4. The topologies of $A_{r,n}$ and $B_{r,n}$ are strong enough to ensure that $\Phi_{r,n}(\psi) = 0$ implies that $\omega_{r,n} + i\bar{\partial}\partial\psi$ is a classical solution to Calabi flow.

The first three conditions ensure that by choosing $n > a + b$ one has that δ-balls around $\Phi_{r,n}(0)$ onto which a suitable restriction of $\Phi_{r,n}$ maps bijectively contain 0 for all sufficiently large r. The last condition then implies that the metric given by the preimage of 0 defines a classical solution to Calabi flow.

Good candidates for $A_{r,n}$ and $B_{r,n}$ are parabolic Sobolev spaces P^0_{k+1} (the upper 0 means zero initial value) and P_k whose norms are comprised of L^2-norms of mixed derivatives up to order $4k$, where a time-derivative counts for four spatial derivatives and the spatial L^2-norms are taken with respect to $\omega_{r,n}$ (an exponential damping term in t is possibly necessary to account for the noncompactness of $[0, \infty[$). For sufficiently high k, parabolic Sobolev embeddings guarantee that condition 4 is satisfied. Condition 1 can be verified by setting up the theory of linear parabolic PDEs in a suitable matter. Condition 2 requires the translation of the pointwise estimates for the $\mathcal{O}(r^{-n})$ approximations $\omega_{r,n}$ to genuine estimates in P_k. In the elliptic case, this can be achieved by estimates in suitably constructed local models (see [13] and also [3]) and a similar construction should work in the parabolic case. One of the difficulties lies in the fact that most corrections used to define the approximative flows depend themselves on r — a nuisance we clandestinely ignored in our analysis (for instance $h(t)$ defined via (4.6) depends on r via the rescaling of time in ω_X). The construction of the local model requires that this dependence can be controlled uniformly in r. In many cases this can be reduced to analysing the behaviour of solutions to linear parabolic PDEs under rescaling of the time parameter in the inhomogeneity and the elliptic generator at different rates. Lastly, condition 3 amounts to controlling the operator norm of $(d\Phi_{r,n})_0^{-1}$. For this, one has to solve the linear parabolic PDEs $\partial_t\psi + (d\,\mathrm{Sc})_{\varphi_{r,n}} \cdot \psi = f$ and establish estimates of the form $\|\psi\|_{A_{r,n}} \leqslant C' r^{b'} \|f\|_{B_{r,n}}$, which requires precise control over the coefficients of the generator $(d\,\mathrm{Sc})_{\varphi_{r,n}}$ and the norms on both sides. The resulting estimate $\|(d\Phi_{r,n})_0^{-1}\|_{op} \leqslant C' r^{b'}$ then needs to be translated into the desired control over $\delta_{r,n}$ with C, b depending on C', b'.

5. Symplectic Curvature Flow

5.1. Introduction

In this chapter we present explicit non-Kähler solutions to symplectic curvature flow (or SCF) recently introduced by J. Streets and G. Tian in [37]. Symplectic curvature flow on an almost Kähler manifold (M, ω_0, J_0) of real dimension $2n$ is given by a system of coupled evolution equations for the symplectic structure ω and the almost complex structure J with initial conditions $\omega(0) = \omega_0$, $J(0) = J_0$. Explicitly,

$$\begin{aligned} \partial_t \omega &= -2P\,, \\ \partial_t J &= -2g^{-1}\left[P^{(2,0)+(0,2)}\right] + \mathcal{R}\,. \end{aligned}$$

Here, P denotes the Chern–Ricci form given by $2i$ times the curvature of the Chern connection on the almost anticanonical bundle $\Lambda^{n,0}(TM)$ and $P^{(2,0)+(0,2)}$ is the sum of the $(2,0)$ and $(0,2)$-part of P. The musical isomorphism g^{-1} raises the second index, i.e. $g(g^{-1}P^{(2,0)+(0,2)}\xi, \eta) = P^{(2,0)+(0,2)}(\xi, \eta)$. Finally, $\mathcal{R} := [Rc, J]$ is the J-antilinear part of Rc, where Rc denotes the Riemann–Ricci curvature tensor Ric viewed as an endomorphism of the tangent bundle via g.

Key properties of this flow proved in [37] include parabolicity, short-time existence and preservation of the almost Kähler property of ω and J. Furthermore, if the initial J_0 is integrable, i.e. (M, ω_0, J_0) is Kähler, then P is the Kähler–Ricci form and $\partial_t J = 0$, so in this case SCF reduces to Kähler–Ricci flow.

Seeing symplectic curvature flow as a generalisation of Kähler–Ricci flow to almost Kähler geometry, one might hope to study canonical structures on almost Kähler manifolds in terms of limiting objects of the flow. In section 5.2 we show that SCF on certain twistor fibrations over hyperbolic space leads to compact non-Kähler static solutions, thus providing first examples to limiting structures of symplectic curvature flow that are genuinely outside the realm of Kähler geometry and Kähler–Ricci flow. Section 5.3 concerns certain invariant structures on nilmanifolds for which the flow equations reduce to an ODE. For these structures we solve the flow explicitly and compute the asymptotic behaviour of the Riemann and the Nijenhuis tensors.

5.2. Compact non-Kähler Static Solutions to SCF

The SCF equations can be readily solved if $\partial_t \omega = \lambda \omega_0$, $\partial_t J = 0, \lambda \in \mathbb{R}$, in which case the flow acts by rescaling the metric: $\omega(t) = (1 + \lambda t)\omega_0$, $J(t) = J_0$. Such solutions are called *static* and in the Kähler setting this behaviour is exhibited by Kähler–Einstein metrics.

We present examples of compact static solutions to SCF in dimensions $n(n+1)$ which cannot be Kähler if $n > 1$. They are constructed from the twistor fibrations

$$\pi : Z_{2n} \to H^{2n},$$

where the fibre over each point in $2n$-dimensional hyperbolic space H^{2n} consists of all almost complex structures compatible with the standard hyperbolic metric on H^{2n} and inducing a fixed orientation. These spaces are examples of symplectic twistor spaces described by A. Reznikov in [32]. J. Fine and D. Panov showed in [15] that Z_{2n} can be realised as a coadjoint orbit. We follow their approach to define a symplectic structure ω and a compatible almost complex structure I on Z_{2n} and show that (Z_{2n}, ω, I) is a static solution to SCF. Furthermore, these static solutions descend to compact quotients of Z_{2n} with hyperbolic fundamental group which cannot support any Kähler structures if $n > 1$. We find that the flow shrinks the metric if $n > 2$, expands it if $n = 1$ and leaves it invariant in the case $n = 2$.

5.2.1. Coadjoint Orbit Description

Consider $\mathrm{SO}(2n, 1)$, the identity component of the group of isometries of $\mathbb{R}^{2n+1}$ with Lorentzian metric. Its Lie algebra is given by

$$\mathfrak{so}(2n,1) = \left\{ (\mathbf{u}, A) := \begin{pmatrix} 0 & \mathbf{u}^t \\ \mathbf{u} & A \end{pmatrix} \middle| \, \mathbf{u} \in \mathbb{R}^{2n},\, A \in \mathfrak{so}(2n) \right\} .$$

In this description, $\mathrm{SO}(2n)$ can be seen as a subgroup of $\mathrm{SO}(2n, 1)$ defined as the stabiliser of $(1, 0) \in \mathbb{R} \times \mathbb{R}^{2n} = \mathbb{R}^{2n+1}$. A choice of almost complex structure $J_0 \in \mathfrak{so}(2n)$ on $\mathbb{R}^{2n}$ defines an element

$$\xi_0 := \begin{pmatrix} 0 & 0 \\ 0 & J_0 \end{pmatrix} \in \mathfrak{so}(2n,1)$$

and singles out a copy of $\mathrm{U}(n)$ inside $\mathrm{SO}(2n) \subset \mathrm{SO}(2n, 1)$ as the stabiliser of ξ_0 under the adjoint action (the matrices in $\mathrm{SO}(2n, 1)$ commuting with ξ_0 are precisely those $A \in \mathrm{SO}(2n) \subset \mathrm{SO}(2n, 1)$ with $AJ_0 = J_0A$). Denote by

$$\mathcal{O}(\xi_0) \cong \mathrm{SO}(2n,1)/\mathrm{U}(n)$$

the adjoint orbit of ξ_0. The Killing form on $\mathfrak{so}(2n, 1)$ is nondegenerate and defines an isomorphism $\mathfrak{so}(2n, 1) \cong \mathfrak{so}(2n, 1)^*$ intertwining the adjoint and coadjoint action of $\mathrm{SO}(2n, 1)$, so $\mathcal{O}(\xi_0)$ can be seen as a coadjoint orbit. Standard theory then endows $\mathcal{O}(\xi_0)$ with a $\mathrm{SO}(2n, 1)$-invariant symplectic structure ω.

An explicit description of the tangent space of $\mathcal{O}(\xi_0)$ can be given with the help of the following lemma.

Lemma 5.2.1. *As a* $\mathrm{U}(n)$ *representation space,* $\mathfrak{so}(2n, 1)$ *admits the equivariant decomposition*

$$\mathfrak{so}(2n,1) \cong \mathfrak{u}(n) \oplus \Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n ,$$

where $\mathbb{C}^n = (\mathbb{R}^{2n}, J_0)$*.*

Proof. We only sketch the proof here and refer to [15] for details. Let $U \in \mathrm{U}(n) \subset \mathrm{SO}(2n) \subset \mathrm{SO}(2n,1)$. The adjoint action of U on $\mathfrak{so}(2n,1)$ is given by

$$\mathrm{Ad}_U \begin{pmatrix} 0 & \mathbf{u}^t \\ \mathbf{u} & A \end{pmatrix} = \begin{pmatrix} 0 & (U\mathbf{u})^t \\ U\mathbf{u} & UAU^{-1} \end{pmatrix} ,$$

i.e. $\mathrm{Ad}_U(\mathbf{u}, A) = (U\mathbf{u}, \mathrm{Ad}_U A)$, from which the equivariant splitting $\mathfrak{so}(2n,1) \cong \mathfrak{so}(2n) \oplus \mathbb{C}^n$ can be inferred. Those elements in $\mathfrak{so}(2n)$ commuting with J_0 constitute $\mathfrak{u}(n)$ as a subset of $\mathfrak{so}(2n)$. The $\mathrm{U}(n)$-invariant complement of $\mathfrak{u}(n)$ in $\mathfrak{so}(2n)$ can be naturally identified with $\Lambda^2(\mathbb{C}^n)^*$ giving the desired decomposition. □

Viewing $\mathcal{O}(\xi_0)$ as $\mathrm{SO}(2n,1)/\mathrm{U}(n)$, the lemma implies that

$$T_{\xi_0}\mathcal{O}(\xi_0) \cong T_{E\cdot \mathrm{U}(n)}(\mathrm{SO}(2n,1)/\mathrm{U}(n)) \cong T_E\mathrm{SO}(2n,1)/T_E\mathrm{U}(n) \cong \Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n ,$$

where E denotes the identity in $\mathrm{SO}(2n,1)$. It is apparent that the (real) dimension of $\mathcal{O}(\xi_0)$ is $n(n+1)$. Since $\mathcal{O}(\xi_0)$ is a homogeneous space, the same description is valid for the tangent spaces at other points as well. However, the almost complex structure J determining the identification $\mathbb{C}^n \cong (\mathbb{R}^{2n}, J)$ will depend on the chosen point. The following consideration makes this clearer.

Observe that the different points in the adjoint orbit of ξ_0 under $\mathrm{SO}(2n) \subset \mathrm{SO}(2n,1)$ are of the form

$$\xi = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} ,$$

where $J = AJ_0A^{-1}$ with $A \in \mathrm{SO}(2n)$. The stabiliser of ξ under the $\mathrm{SO}(2n)$-action is again $\mathrm{U}(n)$, so the orbit is given by $\mathrm{SO}(2n)/\mathrm{U}(n)$, which amounts to all possible choices of orientation preserving almost complex structures compatible with the given inner product on $\mathbb{R}^{2n}$.

From a slightly different point of view this can be formulated as follows: The inclusion $\mathrm{U}(n) \to \mathrm{SO}(2n)$ induces a fibre map $\pi : \mathcal{O}(\xi) \cong \mathrm{SO}(2n,1)/\mathrm{U}(n) \to \mathrm{SO}(2n,1)/\mathrm{SO}(2n) \cong H^{2n}$ with fibre isomorphic to $\mathrm{SO}(2n)/\mathrm{U}(n)$; the adjoint orbit $\mathcal{O}(\xi_0)$ fibres over hyperbolic space with the fibre over a point $x \in H^{2n}$ consisting of all almost complex structures compatible with the hyperbolic metric on H^{2n} at x. This gives the identification $Z_{2n} \cong \mathcal{O}(\xi_0)$. From here on, Z_{2n} will be used to denote the adjoint orbit $\mathcal{O}(\xi_0)$, the corresponding coadjoint orbit, the homogeneous space $\mathrm{SO}(2n,1)/\mathrm{U}(n)$ and the total space of the twistor fibration $\pi : Z_{2n} \to H^{2n}$.

If $(x, J) \in Z_{2n}$ with $x \in H^{2n}$ and J in the fibre over x, the tangent space at (x, J) is

$$T_{(x,J)}Z_{2n} \cong \Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n , \quad \mathbb{C}^n = (\mathbb{R}^{2n}, J) .$$

We endow Z_{2n} with an almost complex structure I by demanding this identification to be $\mathbb{C}$-linear with respect to the usual linear complex structure on $\mathbb{C}^n$ and the sign-reversed linear complex structure on $\Lambda^2(\mathbb{C}^n)^*$. The resulting almost complex structure is the "Eells–Salamon" structure of the twistor space $Z_{2n} \to H^{2n}$. (cf. [11]).

As a coadjoint orbit, Z_{2n} has already been endowed with $\mathrm{SO}(2n,1)$-invariant symplectic form ω. It follows from homogeneous space description that $\mathrm{SO}(2n,1)$-invariant forms on Z_{2n} are in one-to-one correspondence with $\mathrm{U}(n)$-invariant forms on $T_{\xi_0}Z_{2n} \cong \Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n$. In the following, we will show that the space of closed $\mathrm{U}(n)$-invariant real two-forms on $\Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n$ is one-dimensional.

Lemma 5.2.2. *For $n > 1$ the space of $\mathrm{U}(n)$-invariant real two-forms on $\mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n)^*$ is two-dimensional. Invariant forms are linear combinations of the standard Hermitian forms on $P := \mathbb{C}^n$ and on $Q := \Lambda^2(\mathbb{C}^n)^*$. If $n = 1$, $\mathrm{U}(n)$-invariant real two-forms are multiples of the standard Hermitian form on P.*

Proof. Let Ω be a real invariant two-form on $P \oplus Q$. In the obvious notation Ω can be written as

$$\Omega = \left(\begin{array}{c|c} \Omega_{P\times P} & \Omega_{Q\times P} \\ \hline \Omega_{P\times Q} & \Omega_{Q\times Q} \end{array} \right) .$$

As $\mathrm{U}(n)$ is compact, P and Q are equivalent to their dual representations. Furthermore, if $n > 1$, P and Q are irreducible and inequivalent, so by Schur's lemma we have $\Omega_{Q\times P} = 0$, $\Omega_{P\times Q} = 0$ and $\Omega_{P\times P} = \lambda_1\Omega_1$ and $\Omega_{Q\times Q} = \lambda_2\Omega_2$, where Ω_1 and Ω_2 are the standard Hermitian forms on P and Q respectively. Since Ω is real, so are λ_1, λ_2.

If $n = 1$, then $Q = 0$ and the above argument shows that $\Omega = \lambda_1\Omega_1$ for $\lambda_1 \in \mathbb{R}$. □

Lemma 5.2.3. *The "Eells-Salamon" almost complex structure I on Z_{2n} is compatible with the symplectic structure ω.*

Remark. If $n = 2$, this is a special case of Theorem 4.4 in [16].

Proof. At ξ_0, I is given by the $\mathrm{U}(n)$-invariant linear complex structure of $(\Lambda^2(\mathbb{C}^n)^*, -i) \oplus (\mathbb{C}^n, i)$, with respect to which the standard Hermitian forms on $\Lambda^2(\mathbb{C}^n)^*$ and $\mathbb{C}^n$ are invariant. Being a linear combination of the latter two, ω is I-invariant at ξ_0.

A direct computation shows that $\omega_{\xi_0}(I\cdot, \cdot)$ is positive definite. More precisely, for $(\mathbf{u}, A), (\mathbf{v}, B) \in \mathfrak{so}(1, 2n)$ one finds

$$\omega_{\xi_0}(I(\mathbf{u}, A), (\mathbf{v}, B)) = 2(2n-1)[\langle \mathbf{u}, \mathbf{v}\rangle + \operatorname{tr} AB^t] ,$$

so ω is compatible with I at ξ_0. By $\mathrm{SO}(2n, 1)$-invariance of ω and I, the compatibility is global. □

Proposition 5.2.4. *The space of closed $\mathrm{SO}(2n, 1)$-invariant real two-forms on Z_{2n} is one-dimensional consisting of real multiples of the standard symplectic form ω on the adjoint orbit Z_{2n}.*

Proof. Standard theory endows the adjoint orbit $Z_{2n} = \mathcal{O}(\xi_0)$ with a $\mathrm{SO}(2n, 1)$-invariant symplectic form ω and a moment map $\mu : Z_{2n} \to \mathfrak{so}(2n, 1)$ which is the inclusion of the adjoint orbit. Let ω' be another $\mathrm{SO}(2n, 1)$-invariant symplectic form on Z_{2n} and assume it is not a real multiple of ω. As $\mathrm{SO}(2n, 1)$ is semisimple, its symplectic action on the simply connected space (Z_{2n}, ω') admits a moment map $\mu' : Z_{2n} \to \mathfrak{so}(2n, 1)$ whose image is the adjoint orbit of $\xi_0' := \mu'(\xi_0)$. The elements ξ_0 and ξ_0' are linearly independent in $\mathfrak{so}(2n, 1)$ for $\xi_0' = \lambda\xi_0$ would imply $\mu' = \lambda\mu$ and hence $\omega' = \lambda\omega$. Their span is a two-dimensional subspace of $\mathfrak{so}(2n, 1)$ on which the isotropy group $\mathrm{U}(n)$ of ξ_0

acts trivially, but the space of all elements in $\mathfrak{so}(2n,1)$ on which $\mathrm{U}(n)$ acts trivially is one-dimensional, consisting of imaginary multiples of the identity matrix in $\mathfrak{u}(n) \subset \mathfrak{so}(2n,1)$ (cf. Lemma 5.2.1). This is a contradiction, so ω' is a real multiple of ω.[1]

In terms of $\mathrm{U}(n)$-invariant real two-forms $\Omega = \lambda_1\Omega_1 + \lambda_2\Omega_2$ on $\mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n)^*$ for $n > 1$, this means that closedness imposes a fixed ratio between λ_1 and λ_2. In particular, neither Ω_1 nor Ω_2 can be closed and the only closed invariant real two-forms are real multiples of ω. The case $n = 1$ is trivial. □

5.2.2. Symplectic Curvature Flow on (Z_{2n}, ω)

In order to run SCF on Z_{2n} with the $\mathrm{SO}(2n,1)$-invariant almost Kähler structure (ω, I) serving as initial data, the Chern–Ricci curvature and the I-anti-invariant part of the Riemann–Ricci tensor need to be determined.

The Riemann–Ricci tensor Ric is determined by a $\mathrm{U}(n)$-invariant metric, so it is itself invariant. This is enough to see that Ric has to be I-invariant: If multiplication by $(-i, i)$ on $\Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n$ were represented by an element in $\mathrm{U}(n)$ this would be immediate. This is not the case, but there is an easy work-around. Set $z^{ij} := z^i \wedge z^j$ and consider the basis (z^{ij}, z_k) of $\Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n$. Since Ric is symmetric bilinear, it is determined by its values on $((z^{ij},0),(z^{i'j'},0))$, $((0,z_k),(0,z_{k'}))$ and $((z^{ij},0),(0,z_k))$. For each of these pairs of arguments there exists an element $\mathrm{diag}(e^{i\lambda_1},\dots,e^{i\lambda_n}) \in \mathbb{T}^n \subset \mathrm{U}(n)$ acting by multiplication by $(-i, i)$, so Ric has to be I-invariant. Consequently, $\mathcal{R} = [Rc, I] = 0$.

The Chern–Ricci tensor P is a closed $\mathrm{SO}(2n,1)$-invariant two-form, so by Proposition 5.2.4, P is a multiple of ω. In [15] J. Fine and D. Panov determined the first Chern class of Z_{2n}: $c_1(Z_{2n}) = (n-2)[\omega]$. As $(1/4\pi)P$ represents the first Chern class (P is $2i$ times the curvature of the Chern connection on the anticanonical bundle, $i/2\pi$ times which represents the first Chern class), we have $P = 4\pi(n-2)\cdot\omega$. In particular, P has no $(2,0)$ and $(0,2)$-parts.

With this result, SCF for (Z_{2n}, ω) becomes

$$\partial_t\omega(t) = 8\pi(2-n)\cdot\omega(0)\,, \qquad \partial_t I = 0\,.$$

It is manifest that SCF collapses (Z_{2n}, ω) in finite time if $n > 2$, expands it if $n = 1$ and leaves the almost Kähler structure unchanged if $n = 2$.

5.2.3. Non-Kähler Quotients

The symplectic form ω and the almost complex structure I on Z_{2n} are $\mathrm{SO}(2n,1)$-invariant, so the almost Kähler structure will descent to quotients of Z_{2n} by subgroups $\Gamma \subset \mathrm{SO}(2n,1)$. In the adjoint orbit description of Z_{2n}, Γ acts by conjugation. Viewing $Z_{2n} \cong \mathrm{SO}(2n,1)/\mathrm{U}(n)$ ($\mathrm{U}(n)$ acting from the right), this corresponds to Γ acting by left multiplication, so the actions of $\mathrm{U}(n)$ and Γ on $\mathrm{SO}(2n,1)$ commute.

Choosing $\Gamma \subset \mathrm{SO}(2n,1)$ to be the fundamental group of a compact hyperbolic manifold M of dimension $2n$, one obtains two quotients: $\Gamma\backslash H^{2n} \cong \Gamma\backslash\mathrm{SO}(2n,1)/\mathrm{SO}(2n) =: M$

[1] We thank M. Cahen for this moment map trick.

and $\Gamma\backslash Z_{2n} \cong \Gamma\backslash\mathrm{SO}(2n,1)/\mathrm{U}(2n)$. The action of Γ on Z_{2n} and H^{2n} commutes with the projection $\pi : Z_{2n} \to H^{2n}$, so $\Gamma\backslash Z_{2n}$ fibres over M with fibre $\mathrm{SO}(2n)/\mathrm{U}(n)$. This shows that $\Gamma\backslash Z_{2n}$ is a fibre bundle with compact base and fibre, so it is itself compact. Furthermore, the fibre $\mathrm{SO}(2n)/\mathrm{U}(n)$ is connected and simply connected, so $\Gamma\backslash Z_{2n}$ and M have isomorphic fundamental groups $\pi_1(\Gamma\backslash Z_{2n}) \cong \pi_1(M) \cong \Gamma$, but compact Kähler manifolds cannot have fundamental group isomorphic to that of a compact hyperbolic manifold in dimension greater than 2 (see e.g. [39]). Hence, $\Gamma\backslash Z_{2n}$ cannot be Kähler if $n > 1$.

5.3. SCF on Left-invariant Structures on Select Nilmanifolds

In the case of left-invariant almost Kähler structures on a nilpotent Lie group, SCF reduces to an ODE on the corresponding nilpotent Lie algebra. Moreover, if the structure coefficients of a connected, simply connected Lie group's Lie algebra can be chosen rational, the Lie group admits cocompact lattices (Theorem 7 in [27]). As non-abelian nilpotent Lie algebras are never (cf. [20]), taking quotients by such lattices results in compact non-Kähler manifolds on which we can hope to explicitly solve SCF.

This section presents such explicit solutions for SCF on three different nilalgebras. For the computations involved, the expression for the Chern–Ricci form provided in the following lemma is useful.

Lemma 5.3.1. *Let (M, g, J, ω) be an almost Kähler manifold. Denote by A the connection one-form of the Levi–Civita connection in a local complex frame (a local frame in which J is constant). In that frame the Chern–Ricci form has the following expression:*

$$P = d\,\mathrm{tr}(AJ)\,.$$

Proof. The Chern connection on an almost Hermitian manifold (M, g, J, ω) is the unique connection ∇ with respect to which g and J are parallel and whose torsion has vanishing $(1,1)$-part. In the almost Kähler case it is given by $\nabla_X Y = D_X Y - \frac{1}{2}J(D_X J)Y$, where D denotes the Levi–Civita connection. If A and C are the connection one-forms of the Levi–Civita connection and the Chern connection in a local complex frame, then the formula for the Chern connection can be expressed as

$$C = A - \frac{1}{2}J(DJ) = A - \frac{1}{2}J[A, J] = \frac{1}{2}(A - JAJ)$$

Denote by F the full curvature tensor of ∇ given by the endomorphism-valued two-form $F = dC + C \wedge C$. The Chern–Ricci tensor is derived from F via $P_{kl} = \omega^{ij}F_{ijkl}$, where ij are the endomorphism indices (i lowered via the metric) and kl the form indices. Omitting the form indices, a brief calculation yields

$$P = \omega^{st}F_{st} = \omega_{it}F^{it} = g_{jt}J^j_i F^{it} = J^j_i F^i_j = \mathrm{tr}(JF)\,.$$

Application of the two-form $C \wedge C$ to a pair of tangent vectors u, v gives $(C \wedge C)(u, v) = [C_u, C_v]$. The fact that for two endomorphisms A, B one has $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ in conjunction with $[J, C] = 0$ implies

$$(\operatorname{tr} JC \wedge C)(u,v) = \operatorname{tr}(J[C_u, C_v]) = \operatorname{tr}(JC_uC_v) - \operatorname{tr}(JC_vC_u) = 0\,,$$

i.e. $\operatorname{tr}(JC \wedge C) = 0$. Since $dJ = 0$, the remaining contribution to the Chern–Ricci form is

$$P = \operatorname{tr}(JdC) = d\operatorname{tr}(JC) = \frac{1}{2}d\left[\operatorname{tr}(JA) + \operatorname{tr}(AJ)\right] = d\operatorname{tr}(AJ)$$

as claimed. □

We want to apply this result to left-invariant almost Kähler structures on Lie groups, in which case left-invariant frames are complex frames. With the help of the next lemma, the Chern–Ricci form P can be expressed directly in terms of the Lie algebra and the almost complex structure.

Lemma 5.3.2. *Let G be a Lie group and (g, J, ω) a left-invariant almost Kähler structure. If A is the connection one-form of the Levi–Civita connection in a left-invariant frame, then for any left-invariant vector field $Z \in \mathfrak{g}$ one has*

$$\operatorname{tr} A_Z J = \frac{1}{2}\operatorname{tr}(\operatorname{ad}_Z \circ J + J \circ \operatorname{ad}_Z) - \operatorname{tr}\operatorname{ad}_{JZ}\ .$$

Proof. Viewing the almost Kähler structure (g, J, ω) as algebraic data on the Lie algebra $\mathfrak{g}$ of G, the condition that the alternating bilinear form $\omega = g(J\cdot, \cdot)$ be closed means that $0 = d\omega(X, Y, Z) = -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X)$ for any $X, Y, Z \in \mathfrak{g}$.

Now let (e_i) be an orthonormal left-invariant frame of $\mathfrak{g}$. Using the Koszul formula

$$2g(Y, A_Z X) = g([Z, X], Y) - g([Z, Y], X) - g([X, Y], Z)\,,$$

the desired result follows from a straightforward computation:

$$\begin{aligned}
2\operatorname{tr} A_Z J &= 2\sum_i g(e_i, A_Z J e_i) \\
&= \sum_i g([Z, Je_i], e_i) - g([Z, e_i], Je_i) - g([Je_i, e_i], Z) \\
&= \sum_i g(\operatorname{ad}_Z \circ J(e_i), e_i) + g(J \circ \operatorname{ad}_Z(e_i), e_i) - g([Je_i, e_i], Z) \\
&= \operatorname{tr}(\operatorname{ad}_Z \circ J + J \circ \operatorname{ad}_Z) - \sum_i g([Je_i, e_i], Z)\,.
\end{aligned}$$

We use the closedness of ω to express the second term on the right as $-2\operatorname{tr}\operatorname{ad}_{JZ}$:

$$\begin{aligned}
\sum_i g([Je_i, e_i], Z) &= \sum_i \omega([Je_i, e_i], JZ) \\
&= \sum_i \omega([Je_i, JZ], e_i) - \omega([e_i, JZ], Je_i) \\
&= \sum_i g(J[Je_i, JZ], e_i) - g(J[e_i, JZ], Je_i) \\
&= \sum_i g(\operatorname{ad}_{JZ} Je_i, Je_i) + g(\operatorname{ad}_{JZ} e_i, e_i) \\
&= 2\operatorname{tr}\operatorname{ad}_{JZ}\ .
\end{aligned}$$

□

Since for any left-invariant one-form $\theta \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$ the relation $d\theta(X,Y) = -\theta([X,Y])$ holds, Lemmas 5.3.1 and 5.3.2 combine to

$$P(X,Y) = (d \operatorname{tr} AJ)(X,Y) = -\operatorname{tr} A_{[X,Y]}J = -\frac{1}{2}\operatorname{tr}(\operatorname{ad}_{[X,Y]} \circ J + J \circ \operatorname{ad}_{[X,Y]}) + \operatorname{tr} \operatorname{ad}_{J[X,Y]}\,.$$

This has a very useful consequence:

Proposition 5.3.3. (L. Vezzoni)[1] *All left-invariant almost Kähler structures on two-step nilpotent Lie groups are Chern–Ricci flat.*

Proof. Let G be a two-step nilpotent Lie group with fixed almost Kähler structure and $\mathfrak{g}$ the Lie algebra of G. The assumption that G is two-step then means that $[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] = 0$, i.e. for any $X, Y \in \mathfrak{g}$ we have $\operatorname{ad}_{[X,Y]} = 0$, so

$$P(X,Y) = -\frac{1}{2}\operatorname{tr}(\operatorname{ad}_{[X,Y]} \circ J + J \circ \operatorname{ad}_{[X,Y]}) + \operatorname{tr}\operatorname{ad}_{J[X,Y]} = +\operatorname{tr}\operatorname{ad}_{J[X,Y]}\ .$$

Now choose an orthonormal basis (e_i) of $\mathfrak{g}$ with the property that each e_j lies either in $[\mathfrak{g},\mathfrak{g}]$ or in $[\mathfrak{g},\mathfrak{g}]^{\perp}$. Then the summands of

$$\operatorname{tr}\operatorname{ad}_Z = \sum_i g([Z,e_i],e_i)$$

vanish since either $e_i \in [\mathfrak{g},\mathfrak{g}]$ and therefore $[\cdot, e_i] = 0$ (two-step property) or $e_i \in [\mathfrak{g},\mathfrak{g}]^{\perp}$ and $g([\cdot,e_i],e_i) = 0$. □

It should be noted that on manifolds with Chern–Ricci flat almost Kähler structure symplectic curvature flow reduces to anti-complexified Ricci flow introduced by H.V. Le and G. Wang in [25].

5.3.1. Kodaira–Thurston Manifold

The simplest example of a symplectic nilmanifold is the Kodaira–Thurston manifold which can be realised as a product of S^1 and the quotient of the three-dimensional Heisenberg group

$$H_3 := \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}$$

by the obvious integral lattice $\Gamma := H_3 \cap \mathrm{Gl}(3,\mathbb{Z}) \subset H_3$. Topologically, the Kodaira–Thurston manifold is a S^1-bundle over a three-torus where the fibers are given by the central direction in H_3 and the base by the two unpreferred directions in H_3 and the additional S^1-direction.

[1] This was brought to the our attention by Luigi Vezzoni in a private conversation. His proof will be published in *A note on canonical Ricci forms on 2-step nilmanifolds* in Proc. AMS.

The Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}$ of $H_3 \times \mathbb{R}$ is given by generators $e_1, ..., e_4$ with $[e_1, e_2] = e_3$ as the only nontrivial Lie bracket. Equivalently, of the dual basis vectors $e^1, ..., e^4$ of $(\mathfrak{h}_3 \oplus \mathbb{R})^*$ the only one whose corresponding left invariant one-form is not closed is e^3 with $de^3 = -e^1 \wedge e^2$. By Proposition 5.3.3, any left-invariant almost Kähler structure defined on the 2-step nilalgebra $\mathfrak{h}_3 \oplus \mathbb{R}$ is Chern–Ricci flat and SCF leaves the symplectic form of an initial left-invariant almost Kähler structure unchanged. The evolution equation then is just $\partial_t J = \mathcal{R}$ or equivalently $\partial_t g = -\operatorname{Ric} + \operatorname{Ric}(J\cdot, J\cdot)$ (the equivalence can be seen by observing $0 = \partial_t \omega = (\partial_t g)(J\cdot, \cdot) + g(\partial_t J\cdot, \cdot)$).

Consider the following two-parameter family of almost Kähler structures (matrices interpreted in the e_i/e^j basis) with positive parameters α, β:

$$\omega = e^1 \wedge e^3 - e^2 \wedge e^4 ,$$

$$J = (J^i_j) = \begin{pmatrix} 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & \beta \\ \alpha^{-1} & 0 & 0 & 0 \\ 0 & -\beta^{-1} & 0 & 0 \end{pmatrix} \quad g = (g_{ij}) = \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}.$$

Computing the connection one-form A of the Levi–Civita connection D via the Koszul formula gives

$$A = \frac{1}{2} \begin{pmatrix} 0 & \alpha^2 e^3 & \alpha^2 e^2 & 0 \\ -\alpha\beta e^3 & 0 & -\alpha\beta e^1 & 0 \\ -e^2 & e^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Ricci Tensor is then given by $\operatorname{Ric}_{jk} = R^k_{\ j\,kl}$, where $R = dA + A \wedge A$ is the full Riemann curvature tensor:

$$\operatorname{Ric} = \frac{1}{2} \begin{pmatrix} -\alpha\beta & 0 & 0 & 0 \\ 0 & -\alpha^2 & 0 & 0 \\ 0 & 0 & \alpha^3\beta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, SCF is determined by $\partial_t g = -\operatorname{Ric} + \operatorname{Ric}(J\cdot, J\cdot)$, so

$$\partial_t \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2\alpha\beta & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & -2\alpha^3\beta & 0 \\ 0 & 0 & 0 & -\alpha^2\beta^2 \end{pmatrix}.$$

The resulting equations $\partial_t \alpha = -\alpha^3\beta$, $\partial_t \beta = -\frac{1}{2}\alpha^2\beta^2$ can easily be integrated observing that $\partial_t(\alpha^{-\frac{2}{3}}\beta^{\frac{4}{3}}) = 0$. The general solution for initial values $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$ is given by

$$\alpha(t) = \alpha_0 \left(1 + \frac{5}{2}\alpha_0^2\beta_0 \cdot t\right)^{-\frac{2}{5}}, \quad \beta(t) = \beta_0 \left(1 + \frac{5}{2}\alpha_0^2\beta_0 \cdot t\right)^{-\frac{1}{5}}.$$

Geometrically, this means that symplectic curvature flow shrinks the central directions of $H_3 \times \mathbb{R}$ while expanding the unpreferred directions at inverse rates. The shrinking of the central direction in H_3 and that of $\mathbb{R}$ occur at different rates, the former collapsing faster than the latter. The corresponding unequal expansion of the unpreferred directions e_1, e_2 is due to the choice of symplectic form which couples e_1, e_3 and e_2, e_4.

Two quantities whose behaviour under SCF might be of interest are the (pointwise) norms of the Nijenhuis tensor and the Riemann curvature tensor. One finds

$$\|N\|^2 = 8\alpha^2\beta = \frac{8\alpha_0^2\beta_0}{1+\frac{5}{2}\alpha_0^2\beta_0\cdot t}\,, \quad \|R\|^2 = \frac{11}{4}\alpha^4\beta^2 = \frac{11}{4}\frac{\alpha_0^4\beta_0^2}{(1+\frac{5}{2}\alpha_0^2\beta_0\cdot t)^2}\,.$$

Symplectic curvature flow on the Kodaira–Thurston manifold has also been considered in [25] as an instance of anti-complexified Ricci flow, but it appears the example therein is faulty (e.g. the given solution does not satisfy the initial conditions) and we felt it would be worth including our own computation.

5.3.2. Sum of two Heisenberg algebras

The computation is similar for the product of two Heisenberg groups. The generators $e_1, ..., e_6$ of its Lie algebra can be chosen such that $[e_1, e_2] = e_5$ and $[e_3, e_4] = e_6$. The Lie algebra $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ is two-step, so Lemma 5.3.3 can be applied. In the following, all matrices are with respect to the e_i/e^j basis.

Consider the following three-parameter family of almost Kähler structures for positive parameters α, β, γ:

$$\begin{aligned}
\omega &= e^1\wedge e^5 + e^2\wedge e^4 + e^3\wedge e^6\,,\\
g &= \alpha^{-1}e^1\otimes e^1 + \beta^{-1}e^2\otimes e^2 + \gamma^{-1}e^3\otimes e^3 + \beta e^4\otimes e^4 + \alpha e^5\otimes e^5 + \gamma e^6\otimes e^6\,,\\
J &= \begin{pmatrix} & 0_3 & & 0 & -\alpha & 0\\ & & & -\beta & 0 & 0\\ & & & 0 & 0 & -\gamma\\ 0 & \beta^{-1} & 0 & & & \\ \alpha^{-1} & 0 & 0 & & 0_3 & \\ 0 & 0 & \gamma^{-1} & & & \end{pmatrix}.
\end{aligned}$$

As in the case of $\mathfrak{h}_3\oplus\mathbb{R}$, the flow equation of SCF can be written as $\partial_t g = -\operatorname{Ric} + \operatorname{Ric}(J\cdot, J\cdot)$, where the Ricci tensor is computed from the connection one-form of the Levi–Civita connection via the full Riemann curvature tensor. We obtain

$$\operatorname{Ric} = \frac{1}{2}\begin{pmatrix} -\alpha\beta & 0 & 0 & 0 & 0 & 0\\ 0 & -\alpha^2 & 0 & 0 & 0 & 0\\ 0 & 0 & -\gamma\beta^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & -\gamma^2 & 0 & 0\\ 0 & 0 & 0 & 0 & \alpha^3\beta & 0\\ 0 & 0 & 0 & 0 & 0 & \gamma^3\beta^{-1} \end{pmatrix}$$

and

$$\partial_t g = \frac{1}{2}\begin{pmatrix} 2\alpha\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma^2\beta^{-2}+\alpha^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\gamma\beta^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha^2\beta^2+\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\alpha^3\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\gamma^3\beta^{-1} \end{pmatrix}.$$

The resulting equations for α, β, γ are

$$\partial_t\alpha = -\alpha^3\beta, \qquad \partial_t\beta = -\frac{1}{2}\alpha^2\beta^2 + \frac{1}{2}\gamma^2, \qquad \partial_t\gamma = -\gamma^3\beta^{-1}$$

with initial conditions $\alpha(0) = \alpha_0, \beta(0) = \beta_0, \gamma(0) = \gamma_0$. The equation for $\partial_t\beta$ can be rewritten as $2\partial_t \log\beta = \partial_t \log\alpha/\gamma$, so $\beta/\beta_0 = (\alpha/\alpha_0)^{\frac{1}{2}}(\gamma/\gamma_0)^{-\frac{1}{2}}$. With this expression for β the other two equations read

$$\partial_t\alpha = -L\alpha^{\frac{7}{2}}\gamma^{-\frac{1}{2}}, \qquad \partial_t\gamma = -L^{-1}\gamma^{\frac{7}{2}}\alpha^{-\frac{1}{2}},$$

where $L = \beta_0(\gamma_0/\alpha_0)^{\frac{1}{2}}$.

In the case $\beta_0 = \gamma_0/\alpha_0$, these equations can be integrated without much difficulty and the solutions are

$$\alpha(t) = \alpha_0(1+2\alpha_0\gamma_0\cdot t)^{-\frac{1}{2}}, \qquad \beta(t) = \beta_0, \qquad \gamma(t) = \gamma_0(1+2\alpha_0\gamma_0\cdot t)^{-\frac{1}{2}}.$$

As on the Kodaira–Thurston manifold, symplectic curvature flow shrinks the central directions in each of the copies of H_3 and expands the base direction coupled to the central ones by the symplectic form at the inverse rate.

For general initial conditions, integration of the equations for α and γ becomes more difficult. One may substitute $\xi := L^{-1}\alpha^{-3}$, $\eta := L\gamma^{-3}$. Then $\partial_t\xi = \partial_t\eta$, so $\xi = \eta + c$, where $c = L^{-1}\alpha_0^{-3} - L\gamma_0^{-3}$. The case $c = 0$ corresponds exactly to the "easy" case considered previously. The equation for η reads

$$\partial_t\eta = 3\eta^{\frac{1}{6}}(\eta+c)^{\frac{1}{6}}.$$

Integration is possible in terms of hypergeometric series, but we have not pursued the analysis. Qualitatively, the behaviour is expected to be similar to the easy case with the central directions collapsing, the two base directions coupled to the central directions by the symplectic form expanding at inverse rates and the remaining two base directions coupled to each other tending to a finite scale.

The pointwise norms of the Nijenhuis and Riemann tensors are given by

$$\|N\|^2 = 8(\alpha^2\beta + \gamma^2\beta^{-1}), \quad \|R\|^2 = \frac{11}{4}(\alpha^4\beta^2 + \gamma^4\beta^{-2}).$$

In the case where $\beta_0 = \gamma_0/\alpha_0$, these reduce to

$$\|N\|^2 = 16\alpha\gamma = 16\frac{\alpha_0\gamma_0}{1+2\alpha_0\gamma_0\cdot t}, \quad \|R\|^2 = \frac{11}{2}\frac{\alpha_0^2\gamma_0^2}{(1+2\alpha_0\gamma_0\cdot t)^2}.$$

5.3.3. The Nilalgebra $\mathfrak{n}_4$

The situation changes for the nilalgebra $\mathfrak{n}_4$ with generators $e_1, ..., e_4$ and $[e_1, e_2] = e_3$ and $[e_2, e_3] = e_4$ as the only nonvanishing commutators. This nilalgebra is three-step, so Lemma 5.3.3 does not hold and SCF turns out to evolve both ω and J nontrivially.

The initial almost Kähler structure considered is $\omega_0 = e^1 \wedge e^3 + e^2 \wedge e^4$ and $J_0 = e_3 \otimes e^1 + e_4 \otimes e^2 - e_1 \otimes e^3 - e_2 \otimes e^4$, $e^i \in \mathfrak{n}_4^*$. The symplectic form ω_0 is closed since $de^1 = de^2 = 0$ and $de^3 = -e^1 \wedge e^2$ and $de^4 = -e^2 \wedge e^3$.

To run symplectic curvature flow, $\partial_t(\omega, J)$ needs to be known on a sufficiently large space of almost Kähler structures on $\mathfrak{n}_4$. For computational convenience the following familiy of almost Kähler structures was chosen:

$$\omega = e^1 \wedge e^3 + e^2 \wedge e^4 + \gamma e^1 \wedge e^2, \quad J = \begin{pmatrix} 0 & a' & b' & 0 \\ a & 0 & 0 & c' \\ b & 0 & 0 & d' \\ 0 & c & d & 0 \end{pmatrix}.$$

The matrix J is to be understood as an endomorphism of $\mathfrak{g}$ in the e_i/e^j basis. The fact that J is an almost complex structure imposes algebraic relations on $a, b, c, d, a', b', c', d'$:

$$\begin{aligned} aa' + bb' &= -1, & ac + bd &= 0 \\ aa' + cc' &= -1, & a'c' + b'd' &= 0 \\ bb' + dd' &= -1, & ab' + c'd &= 0 \\ cc' + dd' &= -1, & a'b + cd' &= 0\,. \end{aligned}$$

The equations on the right hand side are all equivalent in light of the ones on the left, of which only three are independent. Furthermore, the compatibility condition $\omega(J\cdot, J\cdot) = \omega$ fixes γ by $b'\gamma = a' + d$, so the above defines a four-dimensional space of almost Kähler structures on $\mathfrak{n}_4$.

The metric associated to ω, J in the e_i/e^j basis is given by

$$(g_{ij}) = \begin{pmatrix} g_{11} & 0 & 0 & g_{14} \\ 0 & g_{22} & g_{23} & 0 \\ 0 & g_{23} & g_{33} & 0 \\ g_{14} & 0 & 0 & g_{44} \end{pmatrix} = \begin{pmatrix} b + \gamma a & 0 & 0 & -a \\ 0 & c - \gamma a' & -a' & 0 \\ 0 & -a' & -b' & 0 \\ -a & 0 & 0 & -c' \end{pmatrix}.$$

Infinitesimal changes of these almost Kähler structures under SCF are determined by the Chern–Ricci form and the Ricci curvature (more precisely, the $(2,0) + (0,2)$-part of the Ricci curvature, since $2g^{-1}\operatorname{Ric}^{(2,0)+(0,2)} = J\mathcal{R}$).

To compute them, let D denote the Levi–Civita connection of the left-invariant metric g. Its connection one-form A in the e_i/e^j is the element of $\operatorname{End}(\mathfrak{n}_4) \otimes \mathfrak{n}_4^*$ given by

$$2g(e_k, D_{e_j} e_i) = 2g(e_k, A^l_{ij} e_l) = g([e_j, e_i], e_k) - g([e_j, e_k], e_i) - g([e_i, e_k], e_j)$$

or, more explicitly, by

$$\begin{aligned}
2d_{14}d_{23}A \; = \; & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -g_{23}(g_{33}-g_{14})d_{14} & -g_{33}(g_{33}-g_{14})d_{14} & 0 \\ 0 & g_{22}(g_{33}-g_{14})d_{14} & g_{23}(g_{33}-g_{14})d_{14} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^1 \\
& + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & g_{23}g_{44}d_{14} & g_{33}g_{44}d_{14} & 0 \\ 0 & -g_{22}g_{44}d_{14} & -g_{23}g_{44}d_{14} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^4 \\
& + \begin{pmatrix} 0 & 2g_{23}g_{44}d_{23} & g_{33}g_{44}d_{23} & 0 \\ -g_{23}(g_{33}-g_{14})d_{14} & 0 & 0 & g_{23}g_{44}d_{14} \\ -(d_{23}+g_{22}g_{14}-g_{23}^2)d_{14} & 0 & 0 & -g_{22}g_{44}d_{14} \\ 0 & -2g_{23}g_{14}d_{23} & (d_{14}-g_{14}g_{33})d_{23} & 0 \end{pmatrix} e^2 \\
& + \begin{pmatrix} 0 & g_{33}g_{44}d_{23} & 0 & 0 \\ -g_{33}(g_{33}-g_{14})d_{14} & 0 & 0 & g_{33}g_{44}d_{14} \\ g_{23}(g_{33}-g_{14})d_{14} & 0 & 0 & -g_{23}g_{44}d_{14} \\ 0 & -(d_{14}+g_{14}g_{33})d_{23} & 0 & 0 \end{pmatrix} e^3 .
\end{aligned}$$

Here $d_{14} = g_{11}g_{44} - g_{14}^2$ and $d_{23} = g_{22}g_{33} - g_{23}^2$. Observe $d_{14}d_{23} = \det g_{ij} = \det \omega_{ij} \cdot \det J$. For ω, J in the considered family, it is $\det \omega = 1$ and $\det J = \det J_0 = 1$, so $d_{14}d_{23} = 1$.

With A known, the Riemann curvature F_D is then given by the endomorphism valued two-form $A \wedge A + dA$. The Ricci curvature viewed as an endomorphism of $\mathfrak{n}_4$ by means of g turns to out to be

$$Rc = g^{-1}\,\mathrm{Ric} = \begin{pmatrix} -g_{33}^2 g_{44} & 0 & 0 & 0 \\ 0 & -g_{44}(g_{33}^2+d_{14}) & 0 & 0 \\ 0 & 2g_{23}g_{33}g_{44} & g_{44}(g_{33}^2-d_{14}) & 0 \\ g_{14}(g_{33}^2+d_{14}) & 0 & 0 & g_{44}d_{14} \end{pmatrix} .$$

Computing the commutator $[Rc, J]$ and expressing the g_{ij} in terms of entries of J yields for $2\mathcal{R}$:

$$\begin{pmatrix} 0 & a'c'(b'^2-d_{14}) & b'c'(2b'^2-d_{14}) & 0 \\ ac'(b'^2+2d_{14}) & 0 & 0 & c'^2(b'^2+2d_{14}) \\ -2aa'b'c'-bc'(2b'^2-d_{14})+ad'(b'^2+d_{14}) & 0 & 0 & -2ca'b'c'-d'c'(b'^2-2d_{14}) \\ 0 & -aa'(b'^2+d_{14})-cc'(b'^2+2d_{14})+2da'b'c' & -ab'(b'^2+d_{14})+dc'(b'^2-2d_{14}) & 0 \end{pmatrix} \tag{5.1}$$

The second quantity required to write out the SCF equations explicitly is the Chern–Ricci tensor P, for which a convenient expression was derived in Lemma 5.3.1:

$$P = \mathrm{tr}(J\,dA) .$$

With the A given above it is $P = c'e^1 \wedge e^2$. Furthermore,

$$-2g^{-1}P^{(2,0)+(0,2)} = \begin{pmatrix} 0 & -b'c' & 0 & 0 \\ c'^2 & 0 & 0 & 0 \\ d'c' + ac' & 0 & 0 & c'^2 \\ 0 & b'd' + ab' & -b'c' & 0 \end{pmatrix}. \tag{5.2}$$

Along with the expression for $\mathcal{R}$ found in equation (5.1) this constitutes the evolution equation $\partial_t J = -2g^{-1}P^{(2,0)+(0,2)} + \mathcal{R}$. Setting $y(t) = (1+5/2\cdot t)^{1/5}$, the explicit solution to this ODE with the initial condition $J(0) = J_0$ is given by

$$\begin{aligned} a &= y^{-1} - y^{-3}, & b &= 2y^{-1} - y^{-3}, \\ c &= 2y - y^{-1}, & d &= -y + y^{-1}, \\ a' &= -y + y^{-1}, & b' &= -y^{-1}, \\ c' &= -y^{-3}, & d' &= y^{-1} - y^{-3}. \end{aligned}$$

For the evolution of ω according to $\partial_t \omega = -2P$ with $\omega(0) = \omega_0$ one obtains

$$\omega(t) = e^1 \wedge e^3 + e^2 \wedge e^4 + 2(y^2 - 1)e^1 \wedge e^2$$

and the metric evolves as

$$(g_{ij}) = g(e_i, e_j) = \begin{pmatrix} 2y - 2y^{-1} + y^{-3} & 0 & 0 & -y^{-1} + y^{-3} \\ 0 & 2y^3 - 2y + y^{-1} & y - y^{-1} & 0 \\ 0 & y - y^{-1} & y^{-1} & 0 \\ -y^{-1} + y^{-3} & 0 & 0 & y^{-3} \end{pmatrix}.$$

The Nijenhuis tensor (N_{ij}) in the e_i/e^j basis is given by

$$\begin{pmatrix} 0 & (2y^{-4} - y^{-6})(e_2 + e_3) & -(y^{-4} - y^{-6})(e_2 + e_3) & y^{-4}(e_1 - e_4) \\ -(2y^{-4} - y^{-6})(e_2 + e_3) & 0 & -y^{-2}(e_1 - e_4) & -(y^{-4} - y^{-6})(e_2 + e_3) \\ (y^{-4} - y^{-6})(e_2 + e_3) & y^{-2}(e_1 - e_4) & 0 & -y^{-6}(e_2 + e_3) \\ -y^{-4}(e_1 - e_4) & (y^{-4} - y^{-6})(e_2 + e_3) & y^{-6}(e_2 + e_3) & 0 \end{pmatrix}$$

from which its norm can be computed with a bit of work. The leading order turns out to be y^{-5} or equivalently t^{-1} as in the Kodaira–Thurston case.

5.4. Outlook

It has been conjectured in [37] that SCF exists for as long as long as the cohomology class of $\omega(t)$ stays inside the symplectic cone $\mathcal{C} \subset H^2(X, \mathbb{R})$. In the case of left-invariant almost Kähler structures on Lie groups, the tangent bundle is trivial and the first Chern class, represented by a multiple of P, vanishes. This means that the symplectic class is stable under SCF and the conjecture then says that the flow should exist for all times. We have confirmed the long-time existence for the examples examined in the second part of this chapter and it would be interesting to see whether this is true in general for SCF on left-invariant almost Kähler structures on Lie groups. In any case, one might hope to express the limiting structure or the singularity formation in terms of the initial data, ideally of the symplectic class and the Lie algebra.

Is is known that, topologically, compact Nilmanifolds are iterated torus bundles (cf [33]). In the cases examined in this chapter, it seems that — in some imprecise sense — these fibres collapse under SCF. Studying the interaction between the iterated bundle structure and the flow might help answer the questions on the limiting structures and long-time existence of SCF.

A. Background Material

The material presented here is standard. It can be found in textbooks or is folklore.

A.1. Some Riemannian Geometry

Let (M, g) be a Riemannian manifold. Using summation convention, a (p, q)-tensor in local coordinates is given by

$$T = T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \partial_{x_{i_1}} \otimes \cdots \partial_{x_{i_p}} \otimes dx^{j_1} \cdots \otimes dx^{j_q} \, .$$

When dealing with tensors, abstract index notation is used, i.e. indices of $T^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ are interpreted as open arguments of T. When a given quantity does not define a tensor, indices are interpreted as concrete indices, i.e. coefficients of a coordinate representation. Let ∇ be the Levi–Civita connection on M. Locally, the connection one-form is given by the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

(this does not define a tensor). With this, the covariant derivative of a (p, q)-tensor T is

$$\nabla_m T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = \partial_m T^{i_1 \cdots i_p}_{j_1 \cdots j_q} + \sum_{k=1}^{p} T^{i_1 \cdots i_{k-1} l i_{k+1} \cdots i_p}_{j_1 \cdots j_q} \Gamma^{i_k}_{ml} - \sum_{k=1}^{q} T^{i_1 \cdots i_p}_{j_1 \cdots j_{k-1} l j_{k+1} \cdots j_q} \Gamma^l_{m j_k} \, .$$

Note that even though not every term on the right hand side defines a tensor, their sum does. The Riemannian curvature tensor is given by

$$R_{ijk}{}^l = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik}$$

from which the Ricci and scalar curvature are derived via

$$R_{jk} = R_{ljk}{}^l \, , \quad R = g^{jk} R_{jk} = R^l_l \, .$$

If M is a surface, the full Riemannian curvature is already determined by the scalar curvature. One has

$$R^l_{ijk} = \frac{1}{2}(\delta^l_i g_{jk} - \delta^l_j g_{ik}) \, , \quad R_{jk} = \frac{1}{2} R g_{jk} \, .$$

The following relations concern commutators of covariant derivatives

$$\nabla_i \nabla_j T^{i_1 \cdots i_p}_{j_1 \cdots j_q} - \nabla_j \nabla_i T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = \sum_{k=1}^{p} R_{ijl}{}^{i_k} T^{i_1 \cdots i_{k-1} l i_{k+1} \cdots i_p}_{j_1 \cdots j_q} - \sum_{k=1}^{q} R_{ij j_k}{}^l T^{i_1 \cdots i_p}_{j_1 \cdots j_{k-1} l j_{k+1} \cdots j_q} \, .$$

Observe that the left hand side is a priori second order in T, but the left hand side is in fact zeroth order. The tensor-Laplacian (or rough Laplacian) is defined by

$$\Delta T = -g^{ij}\nabla_i\nabla_j T^{i_1\cdots i_p}_{j_1\cdots j_q} = -\nabla^i\nabla_j T^{i_1\cdots i_p}_{j_1\cdots j_q}\,.$$

It is possible to commute (higher) covariant derivatives and the rough Laplacian acting on tensors and in general the difference

$$\nabla_{k_1}\cdots\nabla_{k_r}\Delta T^{i_1\cdots i_p}_{j_1\cdots j_q} - \Delta\nabla_{k_1}\cdots\nabla_{k_r} T^{i_1\cdots i_p}_{j_1\cdots j_q}$$

will be an expression depending on the curvature, its covariant derivatives and linearly on T in up to r covariant derivatives.

The pointwise inner product induced on (p,q)-tensors by the metric g is given by

$$(T,S)_g = g^{j_1 j_1'}\cdots g^{j_q j_q'} g_{i_1 i_1'}\cdots g_{i_p i_p'} T^{i_1\cdots i_p}_{j_1\cdots j_q} S^{i_1'\cdots i_p'}_{j_1'\cdots j_q'}\,.$$

Of special interest is the case where one of the tensors is a covariant derivative of a $(p,q-1)$-tensor. For notational convenience we omit the covariant indices and the corresponding contraction with the metric. If T is a $(p,q-1)$-tensor and S a (p,q)-tensor, then

$$g^{i_1 i_1'}\nabla_{i_1}(g^{i_2 i_2'}\cdots g^{i_q i_q'} T_{i_2\cdots i_q} S_{i_1'\cdots i_q'}) = (\nabla T, S)_g + (T,\delta S)_g\,,$$

where δS is the $(0,q-1)$ tensor given by $g^{i_1 i_1'}\nabla_{i_1} S_{i_1'\cdots i_q'}$. Defining the vector field $X^{i_1} = g^{i_1 i_1'}\cdots g^{i_q i_q'} T_{i_2\cdots i_q} S_{i_1'\cdots i_q'}$, the left hand side can be written as $\operatorname{div} X = \nabla_i X^i$ and a computation shows that $\operatorname{div} X\cdot d\mathrm{Vol}_g = d(\iota_X d\mathrm{vol}_g)$. Stoke's theorem then implies the relation

$$\int_M (\nabla T, S)_g\, d\mathrm{Vol}_g = -\int_M (T,\delta S)_g\, d\mathrm{Vol}_g\,,$$

which is valid for (p,q)-tensors S and $(p,q-1)$-tensors T. A consequence of this is that for (p,q)-tensors S and T one has

$$\int_M (\Delta T, S)_g\, d\mathrm{Vol}_g = \int_M (\nabla T,\nabla S)_g\, d\mathrm{Vol}_g$$

and in particular the rough Laplacian is self-adjoint. Suppressing superfluous indices, the formulae for commuting covariant derivatives and integrating by parts imply that the difference

$$\begin{aligned}&\int_M (\Delta T,\Delta S)_g - (\nabla^2 T,\nabla^2 S)_g \mathrm{Vol}_g \\ &= \int_M g^{ab}g^{a'b'}[(\nabla_a\nabla_b T,\nabla_{a'}\nabla_{b'}S)_g - (\nabla_a\nabla_{a'}T,\nabla_b\nabla_{b'}S)_g]\, d\mathrm{Vol}_g\end{aligned} \tag{A.1}$$

is an expression of two orders lower involving the curvature of the metric g.

A.2. Some Functional Analysis

Lemma A.2.1. *Let E be a Banach space, $x \in E$ and (x_i) a sequence in E such that every subsequence of (x_i) has a subsequence converging to x. Then (x_i) converges to x.*

Proof. Suppose (x_i) does not converge to x, i.e. there exists $\varepsilon > 0$ and a subsequence (x_{i_j}) such that $\|x_{i_j} - x\| \geqslant \varepsilon$ for all j. But (x_{i_j}) has a subsequence converging to x, a contradiction. □

Lemma A.2.2. *Let E, F, G be Banach spaces, $T\colon E \to F$ a compact operator and $S\colon F \to G$ a linear bounded injection. If (x_i) is a bounded sequence in E such that (STx_i) converges in G, then (Tx_i) converges in F.*

Proof. Denote $z \in G$ the limit of (STx_i). By compactness of T, any subsequence of (Tx_i) has a convergent subsequence in F whose limit point maps to z under S owing to the continuity of S. By injectivity of S, the limit points of (Tx_i) have to be $y := S^{-1}(z)$ and hence (Tx_i) converges to y by Lemma A.2.1. □

A.3. Linearisation of the Scalar Curvature Map

We compute the linearisation of the scalar curvature as a map

$$\mathrm{Sc}\colon \mathcal{H} \to C^\infty(X)\,, \quad \mathrm{Sc}(\varphi) := S(\omega_0 + i\bar\partial\partial\varphi)$$

from Kähler potentials $\mathcal{H}$ on a n-dimensional compact Kähler manifold X with reference metric ω_0 to $C^\infty(X)$. A metric $\omega_\varphi = \omega_0 + i\bar\partial\partial\varphi$ induces a volume form $\Omega_\varphi := 1/n! \cdot \omega_\varphi^n$ which can be seen as a Hermitian metric on the anticanonical bundle $K^* = \Lambda^n T^{(1,0)}X$. The Kähler–Ricci form ρ_φ of ω_φ is then given by the curvature of the Chern connection of Ω_φ, i.e. $\rho_\varphi = i\bar\partial\partial \log \Omega_\varphi$, and the scalar curvature by $\mathrm{Sc}(\varphi) = \Lambda_\varphi \rho_\varphi$. We fix a tangent direction ψ at $\varphi = 0 \in \mathcal{H}$ and note that for small enough s, the function $s\psi$ defines a Kähler potential. Using the subscript s instead of $s\psi$ we expand Ω_s and Λ_s into powers of s:

$$\Omega_s = 1/n! \cdot (\omega_0 + si\bar\partial\partial\psi)^n = \Omega_0 + s\Lambda_0 i\bar\partial\partial\psi \cdot \Omega_0 + \mathcal{O}(s^2) = \Omega_0 + s\Delta_0\psi \cdot \Omega_0 + \mathcal{O}(s^2)\,,$$

$$\begin{aligned}
\Lambda_s\alpha &= n\frac{\alpha \wedge \omega_s^{n-1}}{\omega_s^n} \\
&= \Lambda_0\alpha + s\left[n(n-1)\frac{\alpha \wedge i\bar\partial\partial\psi \wedge \omega_0^{n-2}}{\omega_0^n} - n^2\frac{\alpha \wedge \omega_0^{n-1}}{\omega_0^n}\frac{i\bar\partial\partial\psi \wedge \omega_0^{n-1}}{\omega_0^n}\right] + \mathcal{O}(s^2) \\
&= \Lambda_0\alpha + s\left[1/2 \cdot \Lambda_0^2\alpha \wedge i\bar\partial\partial\psi - \Lambda_0\alpha \cdot \Delta_0\psi\right] + \mathcal{O}(s^2)\,,
\end{aligned}$$

which is valid for any two-form α. We have also used that the k^{th} power of the adjoint of wedging with ω is given by

$$1/k! \cdot \Lambda_\omega^k \alpha_1 \wedge \cdots \wedge \alpha_k = \frac{n!}{(n-k)!}\frac{\alpha_1 \wedge \cdots \alpha_k \wedge \omega^{n-k}}{\omega^n}$$

for two-forms $\alpha_1, ..., \alpha_k$. This follows from the linear algebra in e.g. [23]. For the scalar curvature one can compute locally

$$\begin{aligned}
\mathrm{Sc}_s &= \Lambda_s i\bar{\partial}\partial \log \Omega_s \\
&= \Lambda_s \left[i\bar{\partial}\partial \log \Omega_0 + s i\bar{\partial}\partial \Delta_0 \psi + \mathcal{O}(s^2) \right] \\
&= \Lambda_0 i\bar{\partial}\partial \log \Omega_0 + s \left[\Delta_0^2 \psi + 1/2 \cdot \Lambda_0^2 i\bar{\partial}\partial \log \Omega_0 \wedge i\bar{\partial}\partial\psi - \Lambda_0 i\bar{\partial}\partial \log \Omega_0 \cdot \Delta_0 \psi \right] + \mathcal{O}(s^2) \\
&= \mathrm{Sc}_0 + s \left[\Delta_0^2 \psi - \mathrm{Sc}_0 \, \Delta_0 \psi + 1/2 \cdot \Lambda_0^2 \rho_0 \wedge i\bar{\partial}\partial\psi \right] + \mathcal{O}(s^2) \,,
\end{aligned}$$

so

$$(d\,\mathrm{Sc})_0 \cdot \psi = \Delta_0^2 \psi - \mathrm{Sc}_0 \, \Delta_0 \psi + 1/2 \cdot \Lambda_0^2 \rho_0 \wedge i\bar{\partial}\partial\psi \,.$$

A.4. Parabolic Hölder and Sobolev Norms on Manifolds

The role of this appendix is to outline a proof of how to transfer parabolic estimates in Sobolev and Schauder spaces (adapted to 2^{nd} order equations) from flat domains in $\mathbb{R}^n$ to compact manifolds. We denote by (X, g) a compact n-dimensional Riemannian manifold with Levi–Civita connection ∇^g and by (E, h) a rank m vector bundle over X with inner product h and a connection ∇^h with respect to which h is parallel. For $T > 0$ set $X_T := X \times [0, T[$ and define $\Gamma(X_T, E \otimes (T^*X)^{\otimes k})$ to be smoothly time-dependent C^∞-sections of $E \otimes (T^*X)^{\otimes k}$ over X. On $\Gamma(X_T, E \otimes (T^*X)^{\otimes k})$ we have the norms $\| \cdot \|_{L^p(X_T,g,h)}$ defined by

$$\|s \otimes P\|_{L^p(X_T,g,h)} := \left(\int_0^T \int_X (|s \otimes P|_{g,h})^p \, d\mathrm{vol}_g \, dt \right)^{1/p}$$

for s a section of E and P a section of $(T^*X)^{\otimes k}$. Here $| \cdot |_{g,h}$ denotes the pointwise norm on $E \otimes (T^*X)^{\otimes k}$ given by $|s \otimes P|_{g,h} = |s|_h |P|_g$. The Levi–Civita connection ∇^g and the connection ∇^h induce a connection on $E \otimes (T^*X)^{\otimes k}$ denoted by ∇. We define the parabolic Sobolev space $L^p_{1,2}(X_T, g, h)$ as the completion of $\Gamma(X_T, E)$ with respect to the norm

$$\|s\|_{L^p_{1,2}(X_T,g,h)} := \left(\|\partial_t s\|^p_{L^p(X_T,g,h)} + \sum_{k \leqslant 2} \|\nabla^k s\|^p_{L^p(X_T,g,h)} \right)^{1/p} .$$

Remark. The generalisation to higher regularity parabolic Sobolev spaces is straightforward, though mixed derivatives have to be allowed with one time-derivative counting for two spatial ones. We only need $L^p_{1,2}$, however.

Defining parabolic Hölder spaces takes a bit more preparation. We first set the spaces $C^{k,2k}(X_T, g, h)$ to be the completions of $\Gamma(X_T, E)$ with respect to the norms

$$\|s\|_{C^{k,2k}(X_T,g,h)} := \sum_{0 \leqslant 2j+l \leqslant 2k} \sup_{X_T} |\partial_t^j \nabla^l s|_{g,h} \,.$$

The parabolic distance between points $(x,t_x),(y,t_y)\in X_T$ is defined by $d((x,t_x),(y,t_y)) := (d_g(x,y)^2 + |t_x - t_y|)^{1/2}$, where d_g is the geodesic distance on X defined by g. The connection ∇ on $E\otimes(T^*X)^{\otimes k}$ defines a parallel transport maps for paths on X. We denote by $\Xi_{x,y}\colon E_x\otimes(T^*X)^{\otimes k}_x \to E_y\otimes(T^*X)^{\otimes k}_y$ the parallel transport maps along a minimising geodesic joining x to y. Should y lie in the cut locus of x, this might not be well defined, but for the purpose of defining Hölder norms, the only relevant case is that of y being close to x. For $\alpha\in]0,1[$ and s,P sections as above, we can now define the Hölder seminorms as

$$[s\otimes P]_{\alpha,X_T,g,h} := \sup_{(x,t_x)\neq(y,t_y)\in X_T} \frac{|(s\otimes P)_{(x,t_x)} - \Xi^{-1}_{x,y}\cdot(s\otimes P)_{(y,t_y)}|_{g,h}}{d((x,t_x),(y,t_y))^\alpha}$$

and obtain parabolic Hölder spaces $C^{k,2k,\alpha}(X_T,g,h)$ as the completion of $\Gamma(X_T,E)$ with respect to the norms

$$\|s\|_{C^{k,2k,\alpha}(X_T,g,h)} := \|s\|_{C^{k,2k}(X_T,g,h)} + \sum_{2j+l=2k} [\partial_t^j\nabla^l s]_{\alpha,X_T,g,h}\,.$$

Observe that both parabolic Sobolev and Hölder norms makes sense when replacing X by a geodesically convex subset $U\subset X$. The main ingredient in transferring estimates on open sets in $\mathbb{R}^n$ to M is the following lemma:

Lemma A.4.1. *For $k\in\mathbb{N}_0$ there exists a finite collection of points x_i, coordinate charts $\varphi_i\colon U_i\to\Omega_i$ about the points x_i and trivialisations $\Phi_i\colon E|_{U_i}\to\mathbb{R}^m\times U_i$ and a constant K such that the U_i are geodesically convex with smooth boundary and there holds*

$$1/K\|s_{\Phi_i,\varphi_i}\|_{C^{k,2k,\alpha}(\Omega_{iT})} \leqslant \|s\|_{C^{k,2k,\alpha}(U_{iT},g,h)} \leqslant K\|s_{\Phi_i,\varphi_i}\|_{C^{k,2k,\alpha}(\Omega_{iT})}$$

for all $s\in\Gamma(X_T,E)$. Here $s_{\Phi_i,\varphi_i} = \mathrm{pr}_1\circ\Phi_i\circ s\circ\varphi_i^{-1}\colon\Omega_i\to\mathbb{R}^m$ is the section s locally understood as a function on Ω_i with values in $\mathbb{R}^m$ via the chart φ_i and the trivialisation Φ_i. The norms $C^{k,2k,\alpha}(\Omega_{iT})$ are the parabolic Hölder norms on functions from Ω_{iT} to $\mathbb{R}^m$ with respect to the Euclidian metric g_0 on $\mathbb{R}^n$ and the standard inner product h_0 on $\mathbb{R}^m$.

Proof. We only give a detailed outline of the proof and use a less precise, but also less convoluted notation. About each point $x\in X$ construct charts $\varphi\colon U_i\to\Omega$ via geodesic coordinates. These charts have the property that g coincides with g_0 at x and that the connection one-form of the Levi–Civita connection (Christoffel symbols) vanishes at $x\in X$. Making the charts U smaller if necessary, one can then use parallel transport on E along outward geodesics to construct a trivialisation $\Phi\colon E|_U\to\mathbb{R}^m\times U$ such that the connection one-form of ∇^h vanishes at x (i.e. TU corresponds to the horizontal subspace at x). By the continuity of g,h and the connections, given $\delta>0$, one can make the U yet smaller such that on each chart φ with trivialisation Φ of E one has that

1. g_0 and g are δ-close in C^0 (with respect to say g_0),
2. h_0 and h are δ-close in C^0

3. The connection one-forms of the Levi–Civita connection and the connection on E are δ-close to zero in C^0.

Here g and h are to be understood as the structures of X and E transferred to Ω_i via φ_i and Φ_i. By compactness of X we can choose finitely many points x_i such that the corresponding U_i with φ_i and Φ_i satisfy the above property. The properties 1 and 2 guarantee that for sufficiently small δ in each of the (U_i, φ_i, Φ_i), the pointwise norms on time-dependent sections $s \otimes P$ of $E \otimes (T^*M)^{\otimes k}$ defined by g, h and g_0, h_0 are uniformly equivalent on Ω_{iT}, i.e.

$$1/C \sup_{\Omega_{iT}} |s \otimes P|_{g_0,h_0} \leqslant \sup_{\Omega_{iT}} |s \otimes P|_{g,h} \leqslant C \sup_{\Omega_{iT}} |s \otimes P|_{g_0,h_0} \,,$$

where we have identified $s \otimes P$ with a function $\Omega_i \to \mathbb{R}^m \times \mathbb{R}^{kn}$ using φ_i and Φ_i. The local difference between ordinary derivative and covariant derivative $Ds - \nabla s$ is an algebraic expression in s and the connection one-form. It follows by induction that the local expressions for $\nabla^l s - D^l s$ are a sum of derivatives of connection one forms and derivatives of s up to order $l-1$, where the derivatives of s can be expressed either as ordinary or as covariant derivatives. Applying the pointwise estimates then gives

$$\sup_{\Omega_{iT}} |\nabla^l s|_{g,h} \leqslant K \sup_{\Omega_{iT}} \sum_{i=0}^{l} |D^i s|_{g_0,h_0} \,, \quad \sup_{\Omega_{iT}} |D^l s|_{g_0,h_0} \leqslant K \sup_{\Omega_{iT}} \sum_{i=0}^{l} |\nabla^i s|_{g,h} \,,$$

where the constant K arises as a product of the constant C in the pointwise estimates and the supremum over derivatives of the connection one form up to order $l-1$. The same estimates holds true for time-derivatives of s and one obtains

$$1/K \sum_{0 \leqslant 2j+l \leqslant 2k} \sup_{\Omega_{iT}} |\partial_t^j D^l s|_{g_0,h_0} \leqslant \sum_{0 \leqslant 2j+l \leqslant 2k} \sup_{\Omega_{iT}} |\partial_t^j \nabla^l s|_{g,h} \leqslant K \sum_{0 \leqslant 2j+l \leqslant 2k} \sup_{\Omega_{iT}} |\partial_t^j D^l s|_{g_0,h_0} \,.$$

As presented, the constants K depend on i, but taking their supremum over the finitely many indices shows that they can be chosen independently of the i. Comparing $[\partial_t^j \nabla^l s]_{\alpha,g,h}$ to $[\partial_t^j D^l s]_{\alpha,g_0,h_0}$ is slightly more complicated, owing to the involvement of parallel transport and geodesic distances. However, parallel transport is localy defined via an ordinary differential equation whose coefficients are given by the connection one-form of ∇. By property 3 they can be assumed to be arbitrarily small, so parallel transport in the charts (U_i, φ_i, Φ_i) can be uniformly compared to the trivial transport defined by the flat connections on $T\Omega_i$ and $\mathbb{R}^m$. The same argument works for geodesic distances. This can be used to show that

$$1/K [\partial_t^j D^l s]_{\alpha,g_0,h_0} \leqslant [\partial_t^j \nabla^l s]_{\alpha,g,h} \leqslant K [\partial_t^j D^l s]_{\alpha,g_0,h_0}$$

proving the claim. □

An analogous result holds for $L^p_{1,2}$-norms.

Lemma A.4.2. *For $k \in \mathbb{N}_0$ there exists a finite collection of points x_i, coordinate charts $\varphi_i\colon U_i \to \Omega_i$ about the points x_i and trivialisations $\Phi_i\colon E|_{U_i} \to \mathbb{R}^m \times U_i$ and a constant K such that the U_i have smooth boundary and there holds*

$$1/K\|s_{\Phi_i,\varphi_i}\|_{L^p_{1,2}(\Omega_{iT})} \leqslant \|s\|_{L^p_{1,2}(U_{iT},g,h)} \leqslant K\|s_{\Phi_i,\varphi_i}\|_{L^p_{1,2}(\Omega_{iT})}$$

for all $s \in \Gamma(X_T, E)$. Here $L^p_{1,2}(\Omega_{iT})$ is defined with respect to $\mathbb{R}^n$ and $\mathbb{R}^m$ carrying their canonical structures.

Proof. The proof of A.4.1 can be adapted to this case (as there is no parallel transport, it is not even necessary to work with local geodesic coordinates). We merely remark that the volume forms on U_{iT} defined by g and g_0 differ by functions which are uniformly bounded from above and from below by a positive constant. □

We now proceed to show that local parabolic regularity results also apply to compact manifolds.

Proposition A.4.3. *Denote by $\cdot$ contraction in tensor indices. Let $L(x,t) = A_2(x,t)\cdot\nabla^2 + A_1(x,t)\cdot\nabla + A_0(x,t)$ be a strongly elliptic linear second order operator acting on sections of E. Assume the coefficients A_l to be bounded continuous sections of $\mathrm{End}(E)\otimes(TX)^{\otimes l}$ over X_T. Then there exists a constant C such that for all strong solutions $u \in L^p_{1,2}(X_T, g, h)_{loc}$ (the subscript loc indicates that u has the required regularity, but that we do not assume a priori boundedness of the global $L^p_{1,2}$-norm) of the linear parabolic equation $(\partial_t + L)u = f$ on X_T for a continuous section f of E over X_T, one has the estimate*

$$\|u\|_{L^p_{1,2}(X'_T,g,h)} \leqslant C(\|u\|_{L^p(X_T,g,h)} + \|f\|_{L^p(X_T,g,h)})\,,$$

where X'_T is a set of the form $X\times]\varepsilon,T[$ for $\varepsilon > 0$.

Proof. We glue local versions of interior parabolic $L^p_{1,2}$-estimates which can be found in e.g. [31]. The results therein concern parabolic systems on domains in $\Omega \subset \mathbb{R}^n$ assumed to have BMO (bounded mean oscillation) coefficients and inhomogeneity in L^p. The bounded continuous coefficients and inhomogeneity satisfy these assumptions and the result in [31] of relevance to us can be phrased as follows: If $A_2^{i,j}$, A_1^i and A_0 are bounded continuous functions from $\Omega_T \to \mathrm{End}(\mathbb{R}^m)$ for $1 \leqslant i,j \leqslant m$, where A_2 satisfies the ellipticity condition

$$(-A_2^{i,j}\xi_i\xi_j v, v) \geqslant \lambda|\xi|^2|v^2|, \quad \forall \xi \in \mathbb{R}^n, v \in \mathbb{R}^n\,,$$

for a fixed $\lambda > 0$ uniformly in Ω_T and $\Omega'_T \subset \Omega_T$ is of the form $\Omega'_T = \Omega'\times]\varepsilon,T[$ for an open subset Ω' such that the closure of Ω' is contained in Ω (what matters here is that Ω'_T is at a positive distance from the parabolic boundary $\Omega\times\{0\}\cup\partial\Omega\times[0,T[$), then for strong solutions u of the equation

$$\left(\partial_t + \sum_{i,j=1}^n A_2^{i,j}\partial_i\partial_j + \sum_{i=1}^n A_1^i\partial_i + A_0\right)u = f \tag{A.2}$$

with $f\colon \Omega_T \to \mathbb{R}^m$ a bounded continuous function, one has the estimate

$$\|u\|_{L^p_{1,2}(\Omega'_T)} \leqslant C(\|u\|_{L^p(\Omega_T)} + \|f\|_{L^p(\Omega_T)})\,. \tag{A.3}$$

Now let $u \in L^p_{1,2}(X_T, g, h)$ be a solution of $(\partial_t + L)u = f$ as in the hypothesis. We cover X by finitely many charts and trivialisations (U_i, φ_i, Φ_i) in which the metric estimates of Lemma A.4.2 hold and find open subsets U'_i with closure contained in U_i which still cover X. Denote the corresponding domains in $\mathbb{R}^n$ by Ω_i and Ω'_i. The sections u_{φ_i,Φ_i} in local charts solve equations of the form (A.2) and the estimates (A.3) hold, where the constant can be taken to be independent of i. Set $U'_{iT} := U'_i \times]\varepsilon, T[$, and $\Omega'_i := \Omega'_i \times]\varepsilon, T[$ and estimate

$$\begin{aligned}
\|u\|_{L^p_{1,2}(X'_T,g,h)} &\leqslant \sum_i \|u\|_{L^p_{1,2}(U'_{iT},g,h)} \\
&\leqslant K \sum_i \|u_{\varphi_i,\Phi_i}\|_{L^p_{1,2}(\Omega'_{iT},g_0,h_0)} \\
&\leqslant CK \sum_i \left(\|u_{\varphi_i,\Phi_i}\|_{L^p(\Omega_{iT},g_0,h_0)} + \|f_{\varphi_i,\Phi_i}\|_{L^p(\Omega_{iT},g_0,h_0)}\right) \\
&\leqslant C' \left(\|u\|_{L^p(X_T,g,h)} + \|f\|_{L^p(X_T,g,h)}\right)
\end{aligned}$$

which proves the claim.

Remark. The author of [31] makes assumptions on the dimension of Ω restricting attention to odd $n \geqslant 3$. These restrictions can be circumvented by adding a phantom direction $z \in]-\varepsilon, \varepsilon[$ and extending u, f to u', f' trivially in that direction in the case of even $n \geqslant 2$. The operator L is extended parabolically to L' by adding $-\partial_z^2$. If u solves $(\partial_t + L)u = f$, then $(\partial_t + L')u' = f'$ and the estimates can be applied to the latter systems. Estimates for the primed quantities translate directly to estimates for the unprimed ones.

□

The analogous estimates hold for parabolic Hölder spaces.

Proposition A.4.4. *Let $L(x,t) = A_2(x,t)\cdot\nabla^2 + A_1(x,t)\cdot\nabla + A_0(x,t)$ be a strongly elliptic linear second order operator taking sections of E to sections of E. Assume that for $k \in \mathbb{N}_0$ the coefficients A_l lie in $C^{k,2k,\alpha}(X_T, g, h)$ (with the natural extension of the definition of parabolic Hölder spaces to sections of $\mathrm{End}(E) \otimes (TX)^{\otimes l}$). Then there exists a constant C such that for all strong solutions $u \in C^{k+1,2k+2,\alpha}(X_T, g, h)_{loc}$ (here the subscript loc indicates that $\partial_t^j \nabla^l u$ for $2j + l \leqslant 2l + 1$ are Hölder continuous with coefficient α, but no boundedness assumption of the global $C^{k+1,2k+2,\alpha}$-norm is made) of the linear parabolic equation $(\partial_t + L)u = f$ on X_T for $f \in C^{k,2k,\alpha}(X_T, g, h)$, one has the estimate*

$$\|u\|_{C^{k+1,2k+2,\alpha}(X'_T,g,h)} \leqslant C\left(\|u\|_{C^{k,2k,\alpha}(X_T,g,h)} + \|f\|_{C^{k,2k,\alpha}(X_T,g,h)}\right)\,,$$

where X'_T is defined as in Proposition A.4.3.

Proof. The statement of this theorem for domains $\Omega \subset \mathbb{R}^n$ can be found in the classical paper [18]. Though stated slightly differently (see the remark below), the results [18] imply that replacing bounded continuity of the coefficients $A_l^{i,j}$ and the inhomogeneity f in the domain estimates in the proof of Proposition A.4.3 by the condition that they be in $C^{k,2k,\alpha}$ (with values in $\mathrm{End}(\mathbb{R}^m)$ and $\mathbb{R}^m$ respectively), on has estimates analogous to the L^p case, i.e.

$$\|u\|_{C^{k+1,2k+2,\alpha}(\Omega'_T)} \leqslant C\left(\|u\|_{C^{k,2k,\alpha}(\Omega_T)} + \|f\|_{C^{k,2k,\alpha}(\Omega_T)}\right) ,$$

where $u \in C^{k+1,2k+2,\alpha}(\Omega_T)$ is a classical solution the parabolic equation on Ω_T defined by the A_l with inhomogeneity f. As in the L^p case, $\mathbb{R}^n$ and $\mathbb{R}^m$ are endowed with their canonical structures. The transfer of the domain estimates to estimates on X now works exactly as in the proof of Proposition A.4.3, with one additional subtlety: the seminorms $[s \otimes P]_{\alpha,g,h}$ are not local in the sense that their definition involves taking a supremum over $X_T \times X_T \backslash \Delta$ (Δ being the diagonal). One has

$$\begin{aligned}
\|u\|_{C^{k+1,2k+2}(X'_T,g,h)} &\leqslant \sum_i \|u\|_{C^{k+1,2k+2}(U_i{}'_T,g,h)} \\
&\leqslant K\sum_i \|u_{\varphi_i,\Phi_i}\|_{C^{k+1,2k+2}(\Omega_i{}'_T,g_0,h_0)} \\
&\leqslant K\sum_i \|u_{\varphi_i,\Phi_i}\|_{C^{k+1,2k+2,\alpha}(\Omega_i{}'_T,g_0,h_0)} \\
&\leqslant CK\sum_i \left(\|u_{\varphi_i,\Phi_i}\|_{C^{k,2k,\alpha}(\Omega_{iT},g_0,h_0)} + \|f_{\varphi_i,\Phi_i}\|_{C^{k,2k,\alpha}(\Omega_{iT},g_0,h_0)}\right) \\
&\leqslant C'\left(\|u\|_{C^{k,2k,\alpha}(X_T,g,h)} + \|f\|_{C^{k,2k,\alpha}(X_T,g,h)}\right) .
\end{aligned} \tag{A.4}$$

For the Hölder seminorms we distinguish two cases. Denote by δ the infimum of the radii r about points in $x \in X$ such that the geodesic distance ball of radius r about x is entirely contained in one of the U'_i. By compactness of X, δ is strictly positive. Now if $(x, t_x), (y, t_y) \in X'_T$ such that $0 < d((x, t_x), (y, t_y)) < \delta$, then we can suppose that both points are contained in the same $U_i{}'_T$ and estimate for $2j + l = 2k + 2$

$$\begin{aligned}
\frac{|(\partial_t^j \nabla^k u)_{(x,t_x)} - \Xi_{x,y}^{-1}(\partial_t^j \nabla^k u)_{(y,t_y)}|_{g,h}}{d((x,t_x),(y,t_y))^\alpha} &\leqslant \|u\|_{C^{k+1,2k+2,\alpha}(U_i{}'_T,g,h)} \\
&\leqslant K\|u\|_{C^{k+1,2k+2,\alpha}(\Omega_i{}'_T,g_0,h_0)} \\
&\leqslant CK\left(\|u\|_{C^{k,2k,\alpha}(\Omega_i{}'_T,g_0,h_0)} + \|f\|_{C^{k,2k,\alpha}(\Omega_i{}'_T,g_0,h_0)}\right) \\
&\leqslant CK^2\left(\|u\|_{C^{k,2k,\alpha}(U_{iT},g,h)} + \|f\|_{C^{k,2k,\alpha}(U_{iT},g,h)}\right) \\
&\leqslant C'\left(\|u\|_{C^{k,2k,\alpha}(X_T,g,h)} + \|f\|_{C^{k,2k,\alpha}(X_T,g,h)}\right) .
\end{aligned}$$

On the other hand, if $d((x,t_x),(y,t_y)) \geqslant \delta$, then using (A.4) yields

$$\begin{aligned}\frac{|(\partial_t^j \nabla^k u)_{(x,t_x)} - \Xi_{x,y}^{-1}(\partial_t^j \nabla^k u)_{(y,t_y)}|_{g,h}}{d((x,t_x),(y,t_y))^\alpha} &\leqslant \|u\|_{C^{k+1,2k+2}(X'_T,g,h)} \cdot \delta^{-\alpha} \\ &\leqslant C'\left(\|u\|_{C^{k,2k,\alpha}(X_T,g,h)} + \|f\|_{C^{k,2k,\alpha}(X_T,g,h)}\right) .\end{aligned}$$

Combing the two cases, taking the supremum over $X'_T \times X'_T \backslash \Delta$, summing over $2j + l = 2k + 21$ and adding (A.4) one obtains

$$\|u\|_{C^{k+1,2k+1,\alpha}(X'_T,g,h)} \leqslant C\left(\|u\|_{C^{k,2k,\alpha}(X_T,g,h)} + \|f\|_{C^{k,2k,\alpha}(X_T,g,h)}\right)$$

as claimed.

Remark. The parabolic Hölder norms in [18] explicitly take into account the distance of a given point to the parabolic boundary. Working with the domains Ω_{iT}' which are a fixed distance away from the parabolic boundary of Ω_{iT} allows translation of Friedman's parabolic Hölder estimates on domains to the ones used in the above proof.

□

Lastly, we translate a certain parabolic Sobolev embedding to compact manifolds.

Proposition A.4.5. *Let $p > (n+2)/2$ and $0 < \alpha < \min\{1, 2-(n+2)/p\}$. Then $L^p_{1,2}(X_T,g,h)$ embeds continuously into $C^{0,0,\alpha}(X_T,g,h)$.*

Proof. This follows directly from the metric estimates in Lemmas A.4.3, A.4.4 and the corresponding domain result found in e.g. [41] (page 27, theorem 1.4.1). □

B. Supplement to Chapter 2

B.1. Twisted cscK Metrics on Riemann Surfaces

For the sake of completeness we prove the existence and uniqueness of twisted cscK metrics with unit volume on compact Riemann surfaces of positive genus. These twisted cscK metrics serve as limit objects for twisted Calabi flow and a priori knowledge of their existence is assumed in several proofs of Chapter 2 concerning the flow's long-time behaviour. The existence and uniqueness proofs of twisted cscK metrics closely resemble those of the uniformisation theorem and can be found e.g. in the doctoral thesis of J. Fine.

Theorem B.1.1. *Let X be a compact Riemann surface of positive genus with smooth reference Kähler-metric ω_0 and $\alpha \in \Omega^2(X, \mathbb{R})$ a smooth two-form with nonpositive integral. Then there exists a unique smooth Kähler-metric ω in the cohomology class of ω_0 such that $S(\omega) + \Lambda_\omega \alpha = \hat{S}$ is constant (twisted cscK metric).*

The cohomological constant $\hat{S}$ is given by $\hat{S} \cdot \mathrm{Vol} = \int_X \rho(\omega) + \alpha$ and in particular $\hat{S} \leqslant 0$. Parametrising metrics in the class ω_0 via $e^u \omega_0$, the twisted cscK equation is equivalent to

$$\Delta_0 u + |\hat{S}| e^u = -(\mathrm{Sc}_0 + \Lambda_0 \alpha)\,, \tag{B.1}$$

where the subscript 0 indicates that the corresponding quantity is defined by the reference metric ω_0. If $\hat{S} = 0$, then $\int_X \mathrm{Sc}_0 + \Lambda_0 \alpha\, \omega_0 = 0$ and the equation (B.1) can be uniquely solved by inverting the Laplacian. From now on we assume $\hat{S} \neq 0$ and rescale the metrics such that $\hat{S} = -1$. We consider the map

$$F\colon L^2_4(X, g_0) \to L^2_2(X, g_0)\,, \quad F(u) = \Delta_0 u + \exp(u)$$

and need to solve the nonlinear elliptic problem $F(u) = \phi$ for $\phi \in A$, where

$$A := \left\{ \phi \in L^2_2 \,\middle|\, \int_X \phi\, \omega_0 > 0 \right\}.$$

This is done via a continuity argument: We show that $\operatorname{im} F \subset A$ is open, closed and nonempty, so by connectedness of A, $\operatorname{im} F$ then has to be A.

Remark. Sobolev multiplication works in L^2_k for $k \geqslant 2$ and $u^l \in L^2_k$ if $u \in L^2_k$ for any $l \in \mathbb{N}_0$. Moreover, $\sum_{l=0}^\infty u^l / l!$ is absolutely convergent in L^2_k, so $\exp(u) \in L^2_k$ if $u \in L^2_k$. By rescaling the L^2_k-norms one can achieve that $\|e^u\|_{L^2_k} \leqslant e^{\|u\|_{L^2_k}}$.

We first show uniqueness

Lemma B.1.2. *The map $F\colon L^2_4 \to L^2_2$ is injective, i.e. solutions to $F(u) = \phi$ are unique.*

Proof. There are Sobolev embeddings $L^2_4(X) \hookrightarrow C^2(X)$ and $L^2_2(X) \hookrightarrow C^0(X)$, so it suffices to show that classical solutions to $F(u) = \phi$ are unique. Let $\phi_-, \phi_+ \in C^0(X)$ and $u_-, u_+ \in C^2$ solutions $F(u_\pm) = \phi_\pm$. We claim that if $\phi_- \lesssim \phi_+$, then $u_- \lesssim u_+$, where $\lesssim$ means consistently either $<$ or $\leqslant$. Indeed, $\Delta u_\pm + e^{u_\pm} = \phi_\pm$ and $\phi_- \lesssim \phi_+$ imply $\Delta(u_- - u_+) \lesssim e^{u_+} - e^{u_-}$. Now if $(u_- - u_+)$ attains a global maximum at $x_0 \in X$, then $\Delta(u_- - u_+)(x_0) \geqslant 0$ and hence $e^{u_-}(x_0) \lesssim e^{u_+}(x_0)$ which is equivalent to $u_-(x_0) - u_+(x_0) \lesssim 0$. But x_0 was a global maximum of $u_- - u_+$, so $u_- - u_+ \leqslant (u_- - u_+)(x_0) \lesssim 0$ and thus $u_- \lesssim u_+$. The claimed uniqueness follows from this. □

Lemma B.1.3. *The image of F is open in A.*

Proof. This can be proved by an inverse function theorem argument. We show that $F\colon L^2_4 \to L^2_2$ is a submersion and hence locally surjective thus proving the claim. We first observe that the Laplacian as a map

$$\Delta\colon L^2_4 \to L^2_2$$

is bounded linear and hence Fréchet differentiable with derivative $(d\Delta)_u v = \Delta v$ at any $u \in L^2_4$. A small calculation for $u, v \in L^2_4$ shows that

$$\exp(u+v) - \exp(v) = \exp(u) \cdot v + \exp(u) \left(\sum_{k=0}^{\infty} \frac{v^k}{(k+1)!} \right) \cdot v^2 .$$

The linear map $v \mapsto \exp(u) \cdot v$ from L^2_4 to L^2_2 is bounded and the remainder term $\rho_u(v) := \exp(u) \sum_{k=0}^{\infty} v^k/(k+1)! \cdot v^2$ satisfies $\|\rho_u(v)\|_{L^2_2} / \|v\|_{L^2_4} \to 0$ for $v \to 0$ in L^2_4. Hence $u \mapsto \exp(u)$ is Fréchet differentiable with derivative $(d\exp)_u v = \exp(u) \cdot v$. Also observe that $(d\exp)_u\colon L^2_4 \to L^2_2$ is compact since it factors as a bounded linear map $L^2_4 \to L^2_4$ composed with the compact inclusion $L^2_4 \hookrightarrow L^2_2$. We conclude that F is Fréchet differentiable with

$$(dF)_u v = \Delta v + \exp(u) \cdot v$$

being a compact perturbation of the zero-index Fredholm operator $\Delta\colon L^2_4 \to L^2_2$ and thus is itself Fredholm with index zero. In particular $(dF)_u$ is surjective if and only if it is injective, so it remains to show that the equation

$$\Delta v + \exp(u) \cdot v = 0$$

has $v = 0$ as a unique solution. The uniqueness can be established by a slight modification of the maximum principle argument used to prove Lemma B.1.2. □

Lemma B.1.4. *The image of F is closed in A.*

Proof. Let (ϕ_i) be a sequence in $\operatorname{im} F$ converging to $\phi \in A$. The claim is that $\phi \in \operatorname{im} F$. Let (u_i) be a sequence in L^2_4 such that $F(u_i) = \phi_i$. We shall show that a subsequence of the (u_i) converges to a limit u in L^2_4 and that $F(u) = \phi$. This amounts to establishing appropriate a priori bounds on (u_i) and using compactness of Sobolev embeddings to extract a convergent subsequence.

We begin by showing that (u_i) and e^{u_i} are bounded in L^2. Denote by $\widetilde{\phi} := \int_X \phi\, \omega_0/\mathrm{Vol} > 0$ the average of ϕ and by $g \in L^2_4$ the unique solution to $\Delta g = \phi - \widetilde{\phi}$ with $\int_X g\, \omega_0 = 0$. Set

$$\begin{aligned} u_- &:= g - \max_X g - 1 + \log \widetilde{\phi}\,, \\ u_+ &:= \log(\max_X \phi + 1)\,. \end{aligned}$$

The functions $u_-, u_+ \in C^2$ are constructed to satisfy $F(u_-) < \phi$ and $F(u_+) > \phi$. Indeed one has

$$F(u_-) = \Delta g + e^{g - \max_X g - 1 + \log \widetilde{\phi}} = \phi - \widetilde{\phi} + e^{g - \max_X g - 1}\widetilde{\phi} < \phi \quad \text{and}$$

$$F(u_+) = e^{\log(\max_X \phi + 1)} > \max_X \phi \geqslant \phi\,.$$

Note that $\phi_i \to \phi$ in L^2_2 implies convergence in C^0, so for almost all $i \in \mathbb{N}$ one has the estimate $F(u_-) < \phi_i < F(u_+)$. Applying the arguments in the proof of Lemma B.1.2 this implies $u_- < u_i < u_+$, so the sequences u_i and e^{u_i} are bounded in C^0 and in particular in L^2.

From here standard elliptic estimates for the Laplacian can be used to bound u_i in L^2_4. Via

$$\|u_i\|_{L^2_2} \leqslant C(\|\Delta u_i\|_{L^2} + \|u_i\|_{L^2}) \leqslant C(\|\phi_i\|_{L^2} + \|e^{u_i}\|_{L^2} + \|u_i\|_{L^2})$$

one sees that (u_i) is bounded in L^2_2. Since $\|e^{u_i}\|_{L^2_2} \leqslant e^{\|u_i\|_{L^2_2}}$, the same holds true for e^{u_i}. From here one can use

$$\|u_i\|_{L^2_4} \leqslant C(\|\Delta u_i\|_{L^2_2} + \|u_i\|_{L^2}) \leqslant C(\|\phi_i\|_{L^2_2} + \|e^{u_i}\|_{L^2_2} + \|u_i\|_{L^2})$$

to see that (u_i) is bounded in L^2_4. By compactness of the embedding $L^2_4 \hookrightarrow L^2_2$, one can extract from u_i a subsequence (also denoted by u_i) converging in L^2_2 to a limit u. Since $\exp\colon L^2_2 \to L^2_2$ is continuous, e^{u_i} converges to e^u. The estimate

$$\|u_i - u_j\|_{L^2_4} \leqslant C(\|\phi_i - \phi_j\|_{L^2_2} + \|e^{u_i} - e^{u_j}\|_{L^2_2} + \|u_i - u_j\|_{L^2})$$

implies that (u_i) is Cauchy in L^2_4 and hence that $(u_i) \to u$ in L^2_4. Lastly, by continuity of $F\colon L^2_4 \to L^2_2$, one has $F(u) = F(\lim u_i) = \lim F(u_i) = \lim \phi_i = \phi$. □

The Lemmas B.1.2, B.1.3 and B.1.4 imply that the equation $F(u) = \phi$ admits a unique solution $u \in L^2_4$ whenever $\phi \in L^2_2$ has positive integral. With this we can prove Theorem B.1.1.

Proof. By assumption the smooth function $\phi := -(\mathrm{Sc}_0 + \Lambda_\alpha)$ has nonnegative integral and the case of zero integral was already treated. If $\int_X \phi\,\omega_0 > 0$, there exists a unique $u \in L^2_4$, such that

$$\Delta_0 u + |\hat{S}|e^u = \phi$$

and the metric $e^u\omega_0$ is twisted cscK. It remains to show that u is in fact smooth. This is a local property, so we can – at the cost of replacing Δ by a general elliptic second order operator L with smooth coefficients – assume that u and ϕ are functions on an open ball $B_1(0) \subset \mathbb{R}^2$. Let φ_ε be a family of mollifiers with support in the closure of the ball $B_\varepsilon(0)$. We also assume again that $\hat{S} = -1$. For $n \geqslant 2$, the smoothened functions $u_n := u * \varphi_{1/n}$ and $\phi_n := \phi * \varphi_{1/n}$ satisfy $Lu_n + e^{u_n} = \phi_n$ on $B_{1/2}(0)$. Moreover $(u_n) \to u$ in L^2_4 and $(\phi_n) \to \phi$ in L^2_k for any k. Let $U_l \subset \mathbb{R}^2$ be a family of open sets satisfying $B_{1/4}(0) \subset U_l \subset B_{1/2}(0)$ and $\overline{U}_{l+1} \subset U_l$. Interior elliptic estimates for L give

$$\begin{aligned} \|u_n - u_m\|_{L^2_{k+2}(U_{k+2})} &\leqslant C_k(\|L(u_n - u_m)\|_{L^2_k(U_k)} + \|u_n - u_m\|_{L^2(U_k)}) \\ &\leqslant C_k(\|\phi_n - \phi_m\|_{L^2_k(U_k)} + \|e^{u_n} - e^{u_m}\|_{L^2_k(U_k)} + \|u_n - u_m\|_{L^2(U_k)}) \end{aligned}$$

and an induction argument starting at $k = 4$ shows that (u_n) is Cauchy in any $L^2_k(B_{1/4}(0))$. It follows that $u \in L^2_k(B_{1/4}(0))$ for any k. In particular u is smooth. $\square$

C. Supplement to Chapter 3

C.1. Relation Between Yang–Mills Flow and Hermitian Yang–Mills Flow

Hermitian Yang–Mills flow can be seen a gauge fixed version of Yang–Mills flow on Kähler manifolds. We loosely follow the discussion in [10]. Let (X, g) be a compact Riemannian manifold and (E, h) a complex vector bundle with fixed Hermitian structure. One can define a best connection in the space $\mathcal{A}_h$ of h-unitary connections on (E, h) to be a critical point of the Yang–Mills functional

$$YM\colon \mathcal{A}_h \to \mathbb{R}\,, \quad YM(A) := \int_X |F_A|^2_{g,h}\, d\mathrm{vol}_g$$

Recall that $\mathcal{A}_h$ is an affine space over the space $\Omega^1(X, \mathfrak{u})$ of one-forms on X with values in the Lie algebra of the group of unitary gauge transformations $\mathcal{U}$. The variation of the Yang–Mills functional in direction of $a \in \Omega^1(X, \mathfrak{u})$ is given by

$$(dYM)_A \cdot a = 2 \int_X (d_A^* F_A, a)_h\, d\mathrm{vol}_g\,,$$

where $(\cdot,\cdot)_h$ denotes the real inner product on $\mathrm{End}(E)$-valued forms induced by h, g and d_A^* the adjoint of the extension d_A of the connection to $\Omega^\bullet(X, \mathrm{End}(E))$. It follows that critical points A of the Yang–Mills functional are characterised by the Yang–Mills equation $d_A^* F_A = 0$. In order to find critical points, one can consider Yang–Mills flow, the downward gradient flow of YM, which (up to a factor) is given by

$$\partial_t A + d_A^* F_A = 0\,. \tag{C.1}$$

If $u \in \mathcal{U}$ is a unitary gauge transformation of (E, h), then u acts on unitary connection by pullback, i.e. $u(A) = uAu^{-1}$ and it turns out that the Yang–Mills functional and thus the Yang–Mills equation are invariant under this action. In particular, the Yang–Mills equation is not elliptic and the Yang–Mills flow not parabolic. The gauge invariance, however, is the only obstruction to parabolicity of the Yang–Mills flow and one can construct parabolic flows on $\mathcal{A}_h$ that induce the same flow on gauge equivalence classes $\mathcal{A}_h/\mathcal{U}$ via gauge fixing. The classical way of doing this is to write $A(t) = A_0 + a(t)$ for a path $a(t)$ in $\Omega^1(X, \mathfrak{u})$ and consider the equation

$$\partial_t A + d_A^* F_A + d_A d_A^* a = 0 \tag{C.2}$$

which can be shown to be parabolic and to be gauge equivalent to (C.1) (The tangent space of gauge orbits is given by $T_A \mathcal{U} A = d_A \Omega^0(X, \mathfrak{u})$). In addition, it is possible to pass from a solution $B(t)$ of (C.2) to a solution $A(t)$ of (C.1) by endowing the $\mathcal{U}$-principal fibration $\mathcal{A} \to \mathcal{A}_h/\mathcal{U}$ (some additional technical assumptions are required for the action to be free) with the connection given by defining the horizontal space in $T_A\mathcal{A}_h$ to be the L^2-orthogonal complement to $T_A\mathcal{U}A$ and setting $A(t)$ to be the horizontal lift of the path $[B(t)]$ in $\mathcal{A}_h/\mathcal{U}$.

In the Kähler case, this gauge fixing can be achieved by considering Hermitian Yang–Mills flow which in a suitable sense is gauge equivalent to Yang–Mills flow. To illustrate this, let (X, ω) be compact Kähler and denote by $\mathcal{A}_h^{1,1}$ the set of h-unitary connections whose curvature lies in $\Omega^{1,1}(X, \mathfrak{u})$. Observe that $\overline{\partial}_A := A^{0,1}$ for $A \in \mathcal{A}_h^{1,1}$ defines a holomorphic structure on E and that $A = A_{h,\overline{\partial}_A}$ is the Chern connection with respect to that structure and h. Conversely, a holomorphic structure $\overline{\partial}$ on E defines a Chern connection $A_{h,\overline{\partial}}$, so $\mathcal{A}_h^{1,1}$ is in one-to-one correspondence with holomorphic structures on E. The gauge group $\mathcal{U}$ acts by pullback on $\mathcal{A}_h^{1,1}$ and since $uA_{h,\overline{\partial}}u^{-1} = A_{u \cdot h, u\overline{\partial}u^{-1}} = A_{h,u\overline{\partial}u^{-1}}$, the induced action on holomorphic structures is given by $u(\overline{\partial}) = u\overline{\partial}u^{-1}$. The complexified gauge group $\mathcal{G}$ of all invertible endomorphisms of E covering the identity on X does not act by pullback on $\mathcal{A}_h^{1,1}$, since a $g \in \mathcal{G}$ need not preserve h. It does however act by pullback on holomorphic structures on E and we define the action of $\mathcal{G}$ on $\mathcal{A}_h^{1,1}$ by setting $g(A_{h,\overline{\partial}}) := A_{h,g(\overline{\partial})}$. We remark that Yang–Mills flow leaves $\mathcal{A}_h^{1,1}$-invariant and a more detailed analysis shows that if $A(t)$ solves (C.1), with $A(0) = A_0$ then $A(t) = g(t)(A_0)$ for a suitable path $g(t)$ in $\mathcal{G}$ starting at the identity. The equation for Yang–Mills flow in terms of $g(t)$ (using that X is Kähler) is given by $(\partial_t g)g^{-1} = -i\Lambda_\omega F_{g(A)}$, which after conjugating by g and expressing $A = A_{h,\overline{\partial}}$ can be written as

$$g\partial_t g = -i\Lambda_\omega F_{A_{g^{-1}\cdot h,\overline{\partial}}}\,.$$

The action of g^{-1} on h is given by $g^{-1}\cdot h = h(g\cdot, g\cdot) = h(g^*g\cdot, \cdot)$. Setting $h(t) = h(g^*g\cdot, \cdot)$, one finds that $h(t)$ evolves according to

$$h^{-1}(t)(\partial_t h)(t) = -2i\Lambda_\omega F_{A_{h(t)},\overline{\partial}}\,,$$

which up to a scalar factor and the normalising term $\lambda\, \mathrm{id}_E$ is precisely Hermitian Yang–Mills flow. Conversely, if $h(t)$ satisfies the Hermitian Yang–Mills equation in its above form, then $h(0)^{-1}h(t)$ should correspond to g^*g for a path $g(t)$ defining a solution to Yang–Mills flow via $A(t) = g(t)(A_0)$. Indeed, $\widetilde{g}(t) := (h(0)^{-1}h(t))^{\frac{1}{2}}$ defines a path of connections $B(t) := \widetilde{g}(t)(A_0)$ which is gauge equivalent the solution $A(t)$ of (C.1) and $[B(t)]$ can be lifted horizontally to $A(t)$. This permits to vary the Hermitian metric whilst keeping the complex structure fixed instead varying a Chern connection via the complex structure on E for fixed h.

C.2. Construction of the Donaldson functional

In [9] Donaldson constructs the functional M using Chern–Weil theory. The essence of this construction is explained in the following. Given a holomorphic rank r bundle E over a complex manifold X and an $\mathrm{Ad}_{\mathrm{Gl}_{\mathbb{C}}(r)}$-invariant p-linear totally symmetric function φ on $\mathfrak{gl}(r,\mathbb{C})$, one can define a characteristic class representative for any Hermitian metric h on E:

$$\varphi(iF_h) := \varphi(iF_h, ..., iF_h) \in \Omega^{p,p}(X)\,.$$

It is possible to construct a primitive of $\varphi(iF_h)$ in the following sense.

Proposition C.2.1. *Denote by k, h any pair of Hermitian metrics on E. There exists*

$$R(h,k) \in \Omega^{p-1,p-1}(X)/(\operatorname{Im}\partial + \operatorname{Im}\overline{\partial})$$

with the following properties:

1. *For any three Hermitian metrics h, k, l on E there holds*
$$R(h,l) + R(l,k) = R(h,k)\,.$$
2. *If $h(t)$ is a smooth one-parameter family of metrics and k is another metric on E, then*
$$\partial_t R(k, h(t)) = -p\varphi(h^{-1}\dot{h}, iF_h, ..., iF_h)\,.$$
3. *There holds*
$$i\overline{\partial}\partial R(k,h) = \varphi(iF_k) - \varphi(iF_h) \in \Omega^{p,p}(X)\,.$$

Proof. We just give an outline. If $\mathcal{H}$ is the space of Hermitian metrics on E, one defines a one-form on $\mathcal{H}$ (with values in $\Omega^{p-1,p-1}(X)$) by

$$\theta_h(\eta) = -p\varphi(h^{-1}\eta, iF_h, ..., iF_h)\,.$$

The idea is to pick a reference point $k \in \mathcal{H}$, set $R(k,k) = 0$ and define $R(h,k)$ to be the integral of θ along a piecewise smooth path joining k and h. For this to be well defined modulo $\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}$, the integral of θ along closed loops needs to be in $\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}$. By Stokes' theorem, this is case if $d\theta$ (this is a two-form with values in $\Omega^{p-1,p-1}(X)$) evaluated on any two tangent vectors η, ξ (an element of $\Omega^{p-1,p-1}(X)$) lies in $\partial(\Omega^{p-2,p-1}(X)) + \overline{\partial}(\Omega^{p-1,p-2}(X))$. Using the invariant formula for the exterior derivative, the Bianchi identity for iF_h and the invariance of φ one checks that this is indeed the case. This establishes the existence of R with properties 1 and 2. Using $\partial_t iF_h = i\overline{\partial}\partial_h(h^{-1}\dot{h})$ one checks

$$\begin{aligned}
\partial_t\varphi(iF_h, ..., iF_h) &= p\varphi(i\overline{\partial}\partial_h(h^{-1}\dot{h}), iF_h, ..., iF_h) \\
&= i\overline{\partial}\partial p\varphi(h^{-1}\dot{h}, iF_h, ..., iF_h) \\
&= -\partial_t(i\overline{\partial}\partial R(k, h(t)))
\end{aligned}$$

and then integrates along a path from k to h in $\mathcal{H}$ to obtain property 3. □

We can now define the Donaldson functional for a holomorphic vector bundle E on a Riemann surface (X,ω). Let $\varphi_1(A) := \operatorname{tr}(A)$ and $\varphi_2(A,B) = \operatorname{tr}(AB)$ and let R_1, R_2 be the primitives associated to φ_1 and φ_2 as constructed above. Set

$$M(k,h) := \int_X -\frac{1}{2}R_2(k,h) + \lambda R_1(k,h) \wedge \omega\,.$$

To see that this is well defined (recall that R was only defined up to $\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}$), we use Stokes' theorem and the fact that ω is closed. Obviously $M(k,l) + M(l,h) = M(k,h)$ holds. To check the variation property, we compute

$$\begin{aligned}\partial_t M(k,h(t)) &= \int_X -\frac{1}{2}\partial_t R_2(k,h(t)) + \partial_t \lambda R_1(k,h(t)) \wedge \omega \\ &= \int_X \varphi_2(h^{-1}\dot{h}, iF_h) - \lambda\varphi_1(h^{-1}\dot{h})) \wedge \omega \\ &= \int_X \operatorname{tr}(iF_h h^{-1}\dot{h}) - \lambda \operatorname{tr}(h^{-1}\dot{h}) \wedge \omega \\ &= \int_X \operatorname{tr}[(\Lambda_\omega iF_h - \lambda\,\mathrm{id}_E)h^{-1}\dot{h}]\,\omega\,.\end{aligned}$$

It remains to show that for a fixed reference Hermitian metric k, the functional $M(k,\cdot)$ is bounded from below provided that the bundle $E \to X$ is stable. This is a consequence of a convexity property of M. Set $h(t) = h_0\exp(t\eta)$ for self-adjoint section η. By the variational property one then has $\partial_t M(k,h(t)) = \int_X \operatorname{tr}([iF_{h(t)} - \lambda\,\mathrm{id}_E\,\omega)\eta]$. Differentiating again yields

$$\partial_t^2 M(k,h(t)) = \int_X i\operatorname{tr}(\eta\overline{\partial}\partial_{h(t)}\eta) = \int_X i\operatorname{tr}(\overline{\partial}\eta \wedge \partial_{h(t)}\eta) = \frac{1}{2}\cdot \|d_{h(t)}\eta\|^2_{h(t),\omega} \geqslant 0\,,$$

where we have used that $\partial_t F_h = \overline{\partial}\partial_h(h^{-1}\partial_t h)$. This implies that choosing h_0 to be a critical point of $M(k,\cdot)$ (which exists by assumption of stability) minimises the Donaldson functional. Indeed, if h is any Hermitian metric, one can pick $\eta = \log h_0^{-1}h$ and finds $M(k,h_0) \leqslant M(k,h)$ by integration.

Remark. If $d_{h(t)}\eta = 0$, then η is a holomorphic section of $\operatorname{End}(E)$ and hence a multiple of the identity. Consequently, $\partial_t M(k,h(t)) > 0$ unless $h(t)$ is a rescaling of h_0. In particular, this shows that Hermite–Einstein metrics are unique up to scale.

C.3. Blowup Argument

Proposition C.3.1. *Let X be a compact n-dimensional Kähler manifold and $I \subset \mathbb{R}$ an interval of the form $[0,T[$ for $0 < T \leqslant \infty$. Let $h(t)$ a smooth family of Hermitian metrics on $E \to X$ and $g(t)$ a smooth convergent family of uniformly equivalent Kähler metrics with Kähler forms $\omega(t)$ on X for $t \in I$. Then if $h(t)$ and $\Lambda_{\omega(t)}F_{h(t)}$ are uniformly bounded in C^0 for $t \in I$, $h(t)$ is uniformly bounded in C^1 for $t \in I$.*

Proof. This proof is a more detailed version of Donaldson's in [9]. We start by clarifying a technicality.

Remark. There is no natural C^1-structure on the space of Hermitian metrics, so we pick finitely many holomorphic trivialisations (local finiteness would suffice) $E|_{U_i}$ of E with the U_i covering X. In each of the trivialisations $h(t)$ is a family of Hermitian matrices $H(t)$ smoothly varying in U_i and we can take the gradient ∇H_i with respect to the metrics $g(t)$ on X. We then set $|\nabla h(t)|_{g(t)}(x) := \sup\{|\nabla H_i(t)|_{g(t)}(x) \mid U_i \ni x\}$. A different choice of holomorphic trivialisations gives other norms, but the same topology.

The C^0-topology for the Hermitian matrices $H(t)$ locally representing $h(t)$ used in the following proof is that induced by the Euclidian inner product on the space of square matrices over $\mathbb{C}$. The arguments in the main text show that $H(t)$ has uniformly bounded distance from a reference point in the symmetric space metric on Hermitian matrices. We check that symmetric boundedness implies Euclidean boundedness. Using the triangle inequality, we can assume that the reference point is the identity matrix $\mathbf{1}$. Denote by d_e the Euclidean and by d_s the symmetric distance on the space of Hermitian matrices. If λ_i are the eigenvalues of H, then one finds $d_s(\mathbf{1}, H)^2 = \operatorname{tr}(\log H)^2 = \sum_i (\log \lambda_i)^2$ and $d_e(\mathbf{1}, H)^2 = \operatorname{tr}(H - \mathbf{1})^2 = \sum_i (\lambda_i - 1)^2$. The claim then follows from comparing $x \mapsto (\log x)^2$ to $x \mapsto (x-1)^2$. It is also true that convergence with respect to d_s implies convergence with respect to d_e.

Suppose the claim of the proposition is false. Then one can find an increasing sequence (t_i) in I and a sequence (x_i) of points in X such that for

$$m_i := \sup_{\substack{x \in X \\ t \leqslant t_i}} |\nabla h(t)|_{g(t)}(x) = |\nabla h(t_i)|_{g(t_i)}(x_i)$$

one has $m_i \to \infty$. In addition, the compactness of X allows us to assume that (x_i) converges to an $x \in X$. Now define a sequence of metrics on X by

$$g_i := m_i^2 g(t_i) \,.$$

With the rescaling, we have

$$\sup_{\substack{x \in X \\ t \leqslant t_i}} |\nabla h(t)|_{g_i}(x) = |\nabla h(t_i)|_{g_i}(x_i) = 1$$

Set g_0 to be the limit metric of the family $g(t)$. We would like to work with open sets in $\mathbb{C}^n$. In order to do so, fix a $\rho > 0$ sufficiently small such that

1. on the geodesic ball $B_\rho(x, g_0)$ the vector bundle E is holomorphically trivial and
2. there is a holomorphic chart $\mu\colon B_\rho(x, g_0) \to V \subset \mathbb{C}^n$ such that $\mu(x) = 0$ and $d\mu_x\colon (T_x X, g_{0x}) \to (\mathbb{C}^n, g_{std})$ is an isometry.

Set $\mu_i := \mu - \mu(x_i)$ and let $S_i := m_i \operatorname{id}$ be a rescaling of $\mathbb{C}^n$. Since the $\mu(x_i)$ converge to 0, there exists a neighbourhood D of $0 \in \mathbb{C}^n$ that is contained in the images of all μ_i for sufficiently large i. By changing the chart μ we can assume that D is the Euclidean ball $\{z \in \mathbb{C}^n \mid |z|_{std} < 1\}$. Now define $\phi_i := (S_i \circ \mu_i)^{-1}|_D$ (ϕ_i can also be defined on $m_i D$, but we assume $m_i \geqslant 1$ and restrict to D). A computation yields $(\phi_i^* g_i)_z = (\mu^{-1*} g(t_i))_{z/m_i + \mu(x_i)}$ for $z \in D$ and it is convenient to consider $\phi_i^* g_i$ as a composition of $b_i \colon z \mapsto m_i^{-1} z + \mu(x_i)$ with $\mu^{-1*} g \colon w \mapsto (\mu^{-1*} g(t_i))_w$. Making i large, the maps b_i send D to arbitrarily small neighbourhoods of $0 \in \mathbb{C}^n$ in which g_{std} is increasingly well approximated by $\mu^{-1*} g(t_i)$ in C^0. We see that $\phi_i^* g_i - g_{std}$ converges to 0 in C^0 uniformly on D. Taking spatial derivatives of $\mu^{-1*} g(t_i) \circ b_i$ gives

$$\partial^\alpha (\mu^{-1*} g(t_i) \circ b_i) = m_i^{-|\alpha|} (\partial^\alpha \mu^{-1*} g(t_i)) \circ b_i \,,$$

where α is a multiindex. The derivative $\partial^\alpha \mu^{-1*} g(t_i)$ is bounded on a neighbourhood of $0 \in \mathbb{C}^n$, so if $\alpha \neq 0$ then $\partial^\alpha \phi_i^* g_i \to 0$ uniformly on D. This means that $\phi_i^* g_i \to g_{std}$ in C^∞ on D.

Next, fix a holomorphic trivialisation

$$\Psi \colon E|_{B_\rho(x,g_0)} \to \mathbb{C}^r \times B_\rho(x, g_0)$$

of E over $B_\rho(x, g_0)$ and let $\Psi_y \colon E_y \to \mathbb{C}^r$ be the associated isomorphisms of vector spaces for $y \in B_\rho(x, g_0)$. Now define

$$\Phi_i := \Psi^{-1} \circ (\mathrm{id} \times \phi_i) \colon \mathbb{C}^r \times D \to E|_{\phi_i(D)} \,.$$

For $z \in D$ we compute

$$(\Phi_i^* h(t_i))_z = (\Psi^{-1*} h(t_i))_{\phi_i(z)} = ((\mathrm{id} \times \mu \circ \Psi)^{-1*} h(t_i))_{z/m_i + \mu(x_i)}$$

and express this as a composition of the maps b_i and $(\mathrm{id} \times \mu \circ \Psi)^{-1*} h(t_i) \colon w \mapsto ((\mathrm{id} \times \mu \circ \Psi)^{-1*} h(t_i))_w$. Since $h(t_i)$ is by assumption bounded in C^0, we can argue that the C^0-distances between $\Phi_i^* h(t_i)$ and the constant Hermitian metrics $((\mathrm{id} \times \mu \circ \Psi)^{-1*} h(t_i))_0$ become arbitrarily small uniformly on D. As tacitly done before, given $\varepsilon > 0$ find a neighbourhood V of $0 \in \mathbb{C}^n$ on which $|((\mathrm{id} \times \mu \circ \Psi)^{-1*} h(t_i))_w - ((\mathrm{id} \times \mu \circ \Psi)^{-1*} h(t_i))_0| < \varepsilon$ independently of i (using any norm on the vector space of Hermitian matrices). Then find N big enough such that for $i > N$ the maps b_i map D into V.

Continuing preparations, fix a background Hermitian structure h_0 on E, write $\eta = h_0^{-1} h$ and recall

$$\Delta_{\partial, h_0, \omega} \eta = \eta \Lambda_\omega i F_h + i \Lambda_\omega (\bar{\partial} \eta) \eta^{-1} (\partial_{h_0} \eta) - \eta \Lambda_\omega i F_{h_0} \,.$$

Setting $H_i := \Phi_i^* h(t_i)$ and choosing h_0 to correspond to the standard Hermitian metric on $\mathbb{C}^r$ in the trivialisation Ψ we get

$$\Delta_{\partial, H_0, \phi_i^* \omega_i} H_i = H_i \Lambda_{\phi_i^* \omega_i} i F_{H_i} + i \Lambda_{\phi_i^* \omega_i} [(\bar{\partial} H_i) H_i^{-1} (\partial H_i)]$$

over $D \subset \mathbb{C}^n$. Using that H_i and $\phi_i^* \omega_i$ are uniformly bounded in i and that by construction $|\nabla H_i|_{\phi_i^* g_i} = 1$, we can infer

$$|\Delta_{\partial, H_0, \phi_i^* g_i} H_i| \leqslant C \,,$$

since in the rescaled metrics, $|\nabla H_i|_{\phi_i^*\omega_i} = 1$. Because H_0 was chosen to be standard, $\Delta_{\partial,H_0,\phi_i^*\omega_i}$ on $\mathbb{C}^r$-valued functions on D is just 1/2 times the componentwise d-Laplacian $\Delta_{\phi_i^* g_i}$ defined by the metric $\phi_i^* g_i$. The lowest eigenvalue of the $\phi_i^* g_i$ is bounded from below uniformly in i and $x \in D$, so the same holds true for the constants of ellipticity of the associated Laplacians. In addition, the coefficients of these Laplacians are bounded uniformly in C^∞, so in particular the highest order coefficients are bounded in C^1 and the lower order ones in L^∞. One can then use the interior elliptic estimates for the operators $\Delta_{\phi_i^* g_i}$ with a constant independent of i (see e.g. [19] p. 235, Theorem 9.11. Note that a common modulus of continuity of the highest order coefficients can be found since their uniform boundedness in C^1 guarantees equicontinuity):

$$\|u\|_{L_2^p(D',g_{std})} \leqslant C(\|\Delta_{\phi_i^* g_i(t)} u\|_{L^p(D,g_{std})} + \|u\|_{L^p(D,g_{std})}),$$

where $D' \subset D$ is a slightly smaller disc. Applying this to $u = H_i$ and the fact that the H_i are uniformly bounded in C^0 we get that they are in fact uniformly bounded in $L_2^p(D', g_{std})$.

With these preparatory considerations we derive a contradiction as follows. For sufficiently high p there is a compact embedding $L_2^p \hookrightarrow C^1$, so a subsequence of the H_i converges in C^1(on a slightly smaller set) and in particular in C^0. By the previous considerations, the C^0-limit is necessarily constant, but the C^1-limit cannot be constant since $|\nabla H_i|_{\phi_i^* g_i} = 1$. $\square$

D. Supplement to Chapter 4

D.1. Direct Computation of the Curvature of $\mathcal{O}(1) \to \mathbb{P}E$

Consider a complex manifold X with a holomorphic vector bundle $\pi\colon E \to X$ of rank $k+1$ equipped with a Hermitian metric h. Let $U \subset X$ be an open set over which E is holomorphically trivialised as $E|_U \cong U \times \mathbb{C}^{k+1}$. The covariant derivative associated to the Chern connection on (E, h) locally takes the form $\nabla|_U = d + A$ with $A = h^{-1}\partial h$, where by abuse of notation we also denote by h the family of Hermitian matrices defined by the Hermitian metric in the local trivialisation. One can describe the Chern connection in terms of a decomposition of TE into horizontal and vertical parts as follows.

A tangent vector $\zeta = (\xi, v) \in TE_{(x,z)} \cong T_xU \times T_z\mathbb{C}^{k+1} \cong T_xU \times \mathbb{C}^{k+1}$ is vertical precisely if it lies in the kernel of $d\pi$. This is the case if $\xi = 0$. A tangent vector (ξ, v) at (x, z) is horizontal if it can be realised geometrically as the derivative of a path $(x(t), z(t))$ through x, z at $t = 0$, such that if we interpret $z(t)$ as a section of $E|_U$ over $x(t)$, then $\nabla_{x'(t)}z(t)|_{t=0} = z'(0) + A_{x'(t)}z(t)|_{t=0} = v + A_\xi z = 0$. This is equivalent to $v = -A_\xi z$. We recapitulate: $(\xi, v) \in TE_{(x,z)}$ is

- vertical if and only if $\xi = 0$ and
- horizontal if and only if $v = -h^{-1}(\partial_\xi h)z$.

Next, denote by 0 the zero section of E. One has the commutative diagram

$$\begin{tikzcd} E\backslash 0 \arrow[r, "\pi"] \arrow[d, "p"] & X \\ \mathbb{P}E \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

Consider the relative hyperplane bundle $\mathcal{O}(1) \to \mathbb{P}E$ which inherits a Hermitian metric $\widetilde{h}^{-1}$. We are interested in its curvature $F_{\mathcal{O}(1),\tilde{h}^{-1}}$. Instead, we compute $F_{\mathcal{O}(-1),\tilde{h}}$ which differs from $F_{\mathcal{O}(1),\tilde{h}^{-1}}$ only by a sign. For computational convenience we pull back $(\mathcal{O}(-1), \widetilde{h})$ via $p\colon E\backslash 0 \to \mathbb{P}E$. In the trivialisation $E|_U \cong U \times \mathbb{C}^{k+1}$, the fibre of $p^*(\mathcal{O}(-1), \widetilde{h})$ over $(x, z), z \neq 0$ is the line in $\mathbb{C}^{k+1}$ containing z with Hermitian metric given by the restriction of h to that line. We trivialise $p^*\mathcal{O}(-1)$ over $E\backslash 0|_U$ by the section $(x, z) \mapsto z$, i.e.

$$p^*\mathcal{O}(-1)|_U \cong U \times \mathbb{C}^{k+1}\backslash 0 \times \mathbb{C}\,,$$

where the point in the line defined by z is given by λz. In this holomorphic trivialisation the Hermitian metric $\widetilde{h}$ is a family of Hermitian $1{\times}1$-matrices indexed by $U \times \mathbb{C}^{k+1}\backslash 0$

given by the function $h(z,z)$ (where h depends on the U-variable x). The curvature of $p^*(\mathcal{O}(-1),\widetilde{h})$ is given by

$$F_{p^*(\mathcal{O}(-1),\widetilde{h})} = \overline{\partial}\partial \log h(z,z)\,.$$

We write this as $d\alpha$ for $\alpha = \partial \log h(z,z)$ and use the invariant formula for the exterior derivative to express the curvature as

$$(d\alpha)(\zeta_1,\zeta_2) = d(\alpha(\zeta_2))(\zeta_1) - d(\alpha(\zeta_1))(\zeta_2) - \alpha([\zeta_1,\zeta_2])\,,$$

where $\zeta_1,\zeta_2 \in TE_{(x,z)}$ are locally extended to smooth vector fields. We want to consider the cases where both arguments lie in V or both in H or one in V and one in H. If $\zeta = (0,v)$ is vertical, we extend ζ to a local vector field by demanding v be constant in (x,z). If $\zeta = (\xi, -h^{-1}(\partial_\xi h)z)$ is horizontal, we extend ζ to a local vector field by demanding ξ be constant in (x,z). Note that the extension remains horizontal and that quantities involving h depend on x. It is helpful to observe that $H \subset \ker\alpha$. To see this, simply compute

$$\alpha(\xi,v) = \frac{\partial_\xi h(z,z) + h(v,z)}{h(z,z)}$$

where we have used $\partial_v h(z,z) = h(v,z)$ (this is since $h(z,z) = h_{ij}z^i\bar{z}^j$). Clearly this vanishes if $v = -h^{-1}(\partial_\xi h)z$. We now individually examine the three aforementioned cases.

- First case: $\zeta_1 = (0,v) \in V$ and $\zeta_2 = (\xi, -h^{-1}(\partial_\xi h)z) \in H$.

  We have already seen that $d(\alpha(\zeta_2))(\zeta_1) = 0$. In addition, it is $\alpha(\zeta_1) = \frac{h(v,z)}{h(z,z)}$, so

$$\begin{aligned} d(\alpha(\zeta_1))(\zeta_2) &= \frac{\partial_{\zeta_2} h(v,z)}{h(z,z)} - \frac{h(v,z)}{h(z,z)^2}\partial_{\zeta_2} h(z,z) + \frac{\overline{\partial}_{\zeta_2} h(v,z)}{h(z,z)} - \frac{h(v,z)}{h(z,z)^2}\overline{\partial}_{\zeta_2} h(z,z) \\ &= \frac{\partial_\xi h(v,z)}{h(z,z)} + \frac{\overline{\partial_{\zeta_2} h(z,v)}}{h(z,z)} - \frac{h(v,z)}{h(z,z)^2}\overline{\partial_{\zeta_2} h(z,z)} \\ &= \frac{\partial_\xi h(v,z)}{h(z,z)}\,, \end{aligned}$$

  where we used that $\partial_{\zeta_2} h(z,v) = \partial_{\zeta_2} h(z,z) = 0$ (for the same reason that $\alpha(\zeta_2) = 0$). Finally, $[\zeta_1,\zeta_2] = (0, -h^{-1}(\partial_\xi h)v)$, so

$$\alpha([\zeta_1,\zeta_2]) = -\frac{\partial_\xi h(v,z)}{h(z,z)}\,,$$

  from which we see that $(d\alpha)(\zeta_1,\zeta_2) = 0$, i.e. the horizontal-vertical component of $F_{p^*(\mathcal{O}(-1),\widetilde{h})}$ vanishes.

  Remark. Since the map $p\colon E\backslash 0 \to \mathbb{P}E$ is a submersion, we obtain that $iF_{\mathcal{O}(1),\widetilde{h}^{-1}}$ has no horizontal-vertical component for the connection on $\mathbb{P}(E)$ induced by the Chern connection on E.

- Second case: $\zeta_1 = (\xi_1, -h^{-1}(\partial_{\xi_1} h)z) \in H$ and $\zeta_2 = (\xi_2, -h^{-1}(\partial_{\xi_2} h)z) \in H$.

  All terms, but the commutator term vanish. A direct computation yields

$$
\begin{aligned}
[\zeta_1, \zeta_2] &= (0, d_{\xi_1}(-h^{-1}\partial_{\xi_2} h)z + d_{-h^{-1}(\partial_{\xi_1} h)z}(-h^{-1}(\partial_{\xi_2} h)z)) \\
&\quad -(1 \leftrightarrow 2) \\
&= (0, -\overline{\partial}_{\xi_1}(h^{-1}(\partial_{\xi_2} h)z) - \partial_{\xi_1}(h^{-1}(\partial_{\xi_2} h)z) + h^{-1}(\partial_{\xi_2} h)h^{-1}(\partial_{\xi_1} h)z) \\
&\quad -(1 \leftrightarrow 2) \\
&= (0, -[\overline{\partial}_{\xi_1}(h^{-1}(\partial_{\xi_2} h))]z) + [h^{-1}(\partial_{\xi_1} h)h^{-1}(\partial_{\xi_2} h) + h^{-1}(\partial_{\xi_2} h)h^{-1}(\partial_{\xi_1} h)]z) \\
&\quad -(1 \leftrightarrow 2) \\
&= (0, -[\overline{\partial}_{\xi_1}(h^{-1}(\partial_{\xi_2} h)) - \overline{\partial}_{\xi_2}(h^{-1}(\partial_{\xi_1} h))]z) \\
&= (0, -F_{(E,h)}(\xi_1, \xi_2)z)
\end{aligned}
$$

  and hence

$$
F_{p^*(\mathcal{O}(-1),\tilde{h})}(\zeta_1, \zeta_2) = -\alpha([\zeta_1, \zeta_2]) = \frac{h(F_{E,h}(\xi_1, \xi_2)z, z)}{h(z, z)} .
$$

  Recalling that the standard fibrewise moment map on $\mathbb{P}E$ was given by

$$
M(h, A)([z]) = \frac{i}{2\pi} \frac{h(Az, z)}{h(z, z)}
$$

  one obtains

$$
i/(2\pi) \cdot F_{\mathcal{O}(1),\tilde{h}^{-1}}(\xi_1^{\#}, \xi_2^{\#}) = -M(h, F_{E,h}(\xi_1, \xi_2)) ,
$$

  where $\xi^{\#}$ denotes the horizontal lift of $\xi \in TX$ to $T\mathbb{P}E$ (a tangent vector of the base at $x \in X$ gives a horizontal vector field on the fibre of $\mathbb{P}E_x$).

- Third case: $\zeta_1 = (0, v_1) \in V$ and $\zeta_2 = (0, v_2) \in V$.

  The commutator term vanishes. For the rest we obtain

$$
\begin{aligned}
(d\alpha)(\zeta_1, \zeta_2) &= d(\alpha(\zeta_2))(\zeta_1) - d(\alpha(\zeta_1))(\zeta_2) \\
&= d_{v_1} \frac{h(v_2, z)}{h(z, z)} - (1 \leftrightarrow 2) \\
&= \frac{h(v_2, v_1)}{h(z, z)} - \frac{h(v_2, z)h(v_1, z)}{h(z, z)^2} - \frac{h(v_2, z)h(z, v_1)}{h(z, z)^2} - (1 \leftrightarrow 2) \\
&= \frac{h(v_2, v_1) - h(v_1, v_2)}{h(z, z)} + \frac{h(v_1, z)h(z, v_2) - h(v_2, z)h(z, v_1)}{h(z, z)^2} \\
&= 2i \frac{-\operatorname{Im} h(v_1, v_2)}{h(z, z)} + 2i \frac{\operatorname{Im} h(v_1, z)h(v_2, z)}{h(z, z)^2} \\
&= 2i \frac{\Omega_h(v_1, v_2)}{|z|_h^2} - 2i \frac{\Omega_h(v_1, z)g_h(v_2, z) + \Omega_h(v_2, z)g_h(v_1, z)}{|z|_h^4}
\end{aligned}
$$

and hence

$$p^*F_{\mathcal{O}(1),\widetilde{h}^{-1}}(\zeta_1,\zeta_2) = -2i\frac{\Omega_h(v_1,v_2)}{|z|_h^2} + 2i\frac{\Omega_h(v_1,z)g_h(v_2,z)+\Omega_h(v_2,z)g_h(v_1,z)}{|z|_h^4}\,.$$

Compare this to the characterising relation $\hat{p}^*\sigma_h = 1/\pi\cdot\iota^*\Omega_h$ for the collection of Fubini–Study metrics on the fibres of $\mathbb{P}E$, where $\hat{p}$ is the projection from the unit sphere bundle of (E,h) to $\mathbb{P}E$ and ι is the inclusion of the unit sphere bundle of (E,h) into E. By restricting the right hand side to points on the h-unit sphere (i.e. $|z|_h^2 = 1$) and tangent vectors v_1, v_2 tangent to the unit sphere (this implies $\Omega_h(v_i,z) = 0$), one obtains

$$\hat{p}^*F_{\mathcal{O}(1),\widetilde{h}^{-1}}(\zeta_1,\zeta_2) = -2i\Omega_h(v_1,v_2) = -2\pi i\,\hat{p}^*\sigma_h(\zeta_1,\zeta_2)$$

which then implies that $i/(2\pi)\cdot F_{\mathcal{O}(1),\widetilde{h}^{-1}}$ evaluated on two vertical vectors (with respect to the connection on $\mathbb{P}E$ induced by the Chern connection on (E,h)) is the Fubini–Study metric on that fibre.

D.2. Eigenvalues of the Laplacian on S^n and $\mathbb{CP}^k$

We need the eigenvalues of the Fubini–Study Laplacian on $\mathbb{CP}^k$ for the adiabatic expression of the scalar curvature of ω_r on $\mathbb{P}E$. As a preparation we perform the analysis for the round metric Laplacian on S^n.

D.2.1. Eigenvalues and Eigenfunctions of the Laplacian of the Round Metric on S^n

Proposition D.2.1. *The eigenfunctions of Δ_{S^n} on S^n equipped with the radius 1 round metric are in one-to-one correspondence with harmonic homogeneous polynomials of degree l on $\mathbb{R}^{n+1}$. If P is a harmonic homogeneous degree l polynomial on $\mathbb{R}^{n+1}$, then $\Delta_{S^n}P|_{S^n} = l(l+n-1)P|_{S^n}$. Conversely, if $\Delta_{S^n}f = \lambda f$, then f extends homogeneously to a harmonic polynomial on $\mathbb{R}^{n+1}$. In particular, one has* $\operatorname{spec}(\Delta_{S^n}) = \{l(l+n-1) | l\in\mathbb{N}_0\}$.

Proof. We view $S^n\subset\mathbb{R}^{n+1}$ as the unit sphere with respect to the standard inner product. In polar coordinates (r,Θ) the Laplacian on $\mathbb{R}^{n+1}$ is given by

$$\Delta_{\mathbb{R}^{n+1}} = -\partial_r^2 - \frac{n}{r}\partial_r + \frac{1}{r^2}\Delta_{S^n}\,, \tag{D.1}$$

so if P is a harmonic degree l polynomial on $\mathbb{R}^{n+1}$, then

$$0 = (r^2\Delta_{\mathbb{R}^{n+1}}P)|_{S^n} = (-l(l-1)-nl)P|_{S^n} + \Delta_{S^n}P\,,$$

i.e. $\Delta_{S^n}P|_{S^n} = l(l+n-1)P|_{S^n}$.

Conversely, for $\lambda\in\mathbb{R}$, suppose that f is a solution to $\Delta_{S^n}f = \lambda f$. By ellipticity of $\Delta_{S^n} - \lambda\,\mathrm{id}_{S^n}$ the solution f is automatically smooth, so there is a unique and smooth solution u to the Dirichlet problem on the unit ball $\Delta_{\mathbb{R}^n} u = 0$, $u|_{S^n} = f$. We explicitly construct u using the product ansatz $u(r, \Theta) = R(r) f(\Theta)$. First remark that by self-adjointness of Δ_{S^n} the eigenvalue λ is nonnegative. If $\lambda = 0$, then $f = 0$. From now on assume $\lambda > 0$. Equation (D.1) and the eigenfunction equation imply that the radial factor of a product solution has to satisfy the Euler type equation

$$[r^2\partial_r + nr\partial_r - \lambda]R(r) = 0$$

provided f is not 0 everywhere, which cannot happen since $\lambda > 0$. Substituting $q = \ln r$ we find that

$$[\partial_q^2 + (n-1)\partial_q - \lambda]R(q) = 0$$

for which the general solution is

$$R(q) = a_+ \cdot e^{\mu_+ q} + a_- \cdot e^{\mu_- q}\,, \quad \mu_\pm = \pm\sqrt{\lambda + ((n-1)/2)^2} - (n-1)/2$$

or equivalently

$$R(r) = a_+ r^{\mu_+} + a_- r^{\mu_-}\,.$$

Since μ_- is always negative, the continuity of u at the origin forces $a_- = 0$. Furthermore $a_+ = 1$, since u has to agree with f if $r = 1$, so

$$u(r, \Theta) = r^{\sqrt{\lambda + ((n-1)/2)^2} - (n-1)/2} f(\Theta)\,. \tag{D.2}$$

One verifies that u is continuous on $\mathbb{R}^n$ with $u(0) = 0$ and smooth away from 0 where it satisfies $\Delta_{\mathbb{R}^n} u = 0$. Singularity lifting for harmonic functions then implies that u is harmonic on all of $\mathbb{R}^n$ and is indeed the solution to the Dirichlet problem. A generalised version of Liouville's theorem states that polynomially bounded harmonic functions on $\mathbb{R}^{n+1}$ are themselves polynomials (cf. e.g. [12], p. 342f for a proof). Equation (D.2) also shows that u is polynomially bounded, so u has to be a homogeneous polynomial of degree $l := \mu_+$. In particular l has to be an integer, i.e. $\lambda = l(n + l - 1)$. □

D.2.2. Eigenvalues and Eigenfunctions of the Fubini–Study Laplacian on $\mathbb{CP}^k$

Proposition D.2.2. *The eigenfunctions of $\Delta_{\mathbb{CP}^k}$ on $\mathbb{CP}^k$ equipped with the Fubini–Study metric are in one-to-one correspondence with harmonic homogeneous S^1-invariant real polynomials on $\mathbb{C}^{k+1} = \mathbb{R}^{2(k+1)}$. The spectrum of $\Delta_{\mathbb{CP}^k}$ is* $\operatorname{spec}(\Delta_{\mathbb{CP}^n}) = \{4l(k+l) | l \in \mathbb{N}_0\}$.

Proof. We view $\mathbb{CP}^k$ as the quotient of S^{2k+1} with the radius 1 round metric by the isometric S^1-action $\theta \cdot z = e^{i\theta} z$. The Fubini–Study metric on $\mathbb{CP}^k$ is the quotient metric induced by the round metric on S^{2k+1} (Reminder: Define $g_{\mathbb{CP}^k}(X, Y)_{[z]}$ by choosing a $z \in [z] \cap S^{2k+1}$ and lifting X, Y to $\widetilde{X}, \widetilde{Y} \in (T_z S^1)^\perp \subset T_z S^{2k+1}$. The lifts are unique. Then set $g_{\mathbb{CP}^k}(X, Y)_{[z]} := g_{S^{2k+1}}(\widetilde{X}, \widetilde{Y})_z$. Since S^1 acts isometrically, this is independent of the choice of $z \in [z] \cap S^{2k+1}$). Functions on $\mathbb{CP}^k$ are in one-to-one correspondence with

S^1-invariant functions on S^{2k+1} (functions pulled back from $\mathbb{CP}^k$ via $\pi\colon S^{2k+1} \to \mathbb{CP}^k$). One can check that for $f \in C^\infty(\mathbb{CP}^k)$ one has $\pi^*\Delta_{\mathbb{CP}^k} f = \Delta_{S^{2k+1}}\pi^* f$ (e.g. by using the local expression $\Delta f = -|g|^{-1/2}\partial_i(|g|^{1/2}g^{ij}\partial_j f)$ and choosing local coordinates on S^{2k+1} such that the S^1-direction corresponds to ∂_1). This shows that $\Delta_{\mathbb{CP}^k}$-eigenfunctions on $\mathbb{CP}^k$ are in one-to-one correspondence with S^1-invariant eigenfunctions of $\Delta_{S^{2k+1}}$ on S^{2k+1}.

We can write any homogeneous polynomial on $\mathbb{R}^{2(k+1)}$ as $P = \sum_{|\alpha|+|\beta|=l} A_{\alpha,\beta} z^\alpha \bar{z}^\beta$, where the sum is over multiindices $\alpha = (\alpha_0, ..., \alpha_n)$ and $z^\alpha = z_0^{\alpha_0} \cdots z_n^{\alpha_n}$ (similarly for $\bar{z}$ and β). In this form it is manifest that the S^1-invariant homogeneous polynomials are those for which one only has summands with $|\alpha| = |\beta| = l/2$. In particular l needs to be even and one obtains that $\operatorname{spec}(\Delta_{\mathbb{CP}^k}) \subset \{2j(2j + (2k+1) - 1) | j \in \mathbb{N}_0\} = \{4j(k+j) | j \in \mathbb{N}_0\}$. To show that this is in fact already the entire spectrum, we remark that for every $k \in \mathbb{N}, l \in \mathbb{N}_0$ one can find S^1-invariant homogeneous harmonic real polynomials on $\mathbb{C}^{k+1} = \mathbb{R}^{2(k+1)}$. For instance, consider

$$P = \sum_{i=0}^{k} a_i z^{(k-i,i,0,\dots,0)} \bar{z}^{(k-i,i,0,\dots,0)}\,, \quad a_{i+1} = -\left(\frac{k-i}{i+1}\right)^2 a_i \quad \text{for} \quad i = 1,..,k-1\,, \quad a_0 = 1\,.$$

$\square$

Remark. The computations here use the full Riemannian d-Laplacian of the Fubini–Study metric with volume $\pi^k/k!$. In the main text we mainly use the $\overline{\partial}$-Laplacian which is half of the d-Laplacian and the integral Fubini–Study metric in $c_1(\mathcal{O}(1))$ with volume $1/k!$. The spectrum of the $\overline{\partial}$-Laplacian of the integral Fubini–Study metric is

$$\operatorname{spec}_{\Delta_{\mathbb{CP}^k}} = \{2\pi j(k+j) \,|\, j \in \mathbb{N}_0\}\,.$$

We now explicitly describe the eigenspace of the first nonzero eigenvalue in detail.

Proposition D.2.3. *The eigenspace of the first nonzero eigenvalue of $\Delta_{\mathbb{CP}^k}$ are precisely zero integral Hamiltonians for the $\mathrm{SU}(k+1)$-action on $\mathbb{CP}^k$.*

Proof. We have seen that the eigenspace of the first nonzero eigenvalue can naturally be identified with S^1-invariant degree 2 homogeneous harmonic real polynomials on $\mathbb{C}^{k+1}$ by pulling the eigenfunction up to S^{2k+1} and extending harmonically. These polynomials are of the form

$$P = \frac{i}{2}\sum_{p,q=0}^{k} A_{qp} z_p \bar{z}_q\,,$$

where harmonicity forces $\operatorname{tr} A = \sum_{p=0}^k A_{pp} = 0$ and real-valuedness $\overline{A}_{pq} = -A_{qp}$. This means that $A \in \mathfrak{su}(k+1)$. Next we see that P as a function on $\mathbb{C}^{k+1}$ is a Hamiltonian for the action of A on $\mathbb{C}^{k+1}$. This is a simple computation (rescaling the metric rescales the moment map by the same factor):

$$
\begin{aligned}
\langle \mu_{\mathbb{C}^{k+1}}, A\rangle(z) &= \frac{1}{2}\Omega(Az, z) \\
&= \frac{i}{4}\sum_{q=0}^{k} dz_q \wedge d\bar{z}_q(Az, z) \\
&= \frac{i}{4}\sum_{q,p=0}^{k} A_{qp} z_p \bar{z}_q - \overline{A}_{qp}\bar{z}_p z_q \\
&= \frac{i}{2}\sum_{p,q=0}^{k} A_{qp} z_p \bar{z}_q \\
&= P(z)\,.
\end{aligned}
$$

By construction, P restricted to S^{2k+1} descends to the Hamiltonian $\langle \mu_{\mathbb{CP}^k}, A\rangle$ on $\mathbb{CP}^k$. Since A is traceless, that Hamiltonian integrates to 0. This shows that each eigenfunction of $\Delta_{\mathbb{CP}^k}$ to the first nonzero eigenvalue is a Hamiltonian for the infinitesimal action of $A \in \mathfrak{su}(k+1)$. Conversely, such a Hamiltonian is of the form $\langle \mu_{\mathbb{CP}^k}, A\rangle([z]) = \frac{i}{2}\sum_{p,q=0}^{k} A_{qp} z_p \bar{z}_q / |z|^2$ which is induced by the S^1-invariant homogeneous harmonic real polynomial $\frac{i}{2}\sum_{p,q=0}^{k} A_{qp} z_p \bar{z}_q$, so $\langle \mu_{\mathbb{CP}^k}, A\rangle$ is an eigenfunction for the first nonzero eigenvalue of $\Delta_{\mathbb{CP}^k}$. $\square$

Bibliography

[1] T. Aubin, *Nonlinear Analysis on Manifolds, Monge–Ampère Equations.* Grundlehren, Springer, 1982.

[2] S. Boucksom, P. Eyssidieux and V. Guedj (Eds.), *An Introduction to the Kähler–Ricci Flow.* Lecture Notes in Mathematics, vol. 2086, Springer, 2013.

[3] T. Brönnle, *Extremal Kähler Metrics on Projectivised Vector Bundles.* `arXiv:1301.7045`, 2013.

[4] S.-C. Chang, *The 2-Dimensional Calabi Flow.* Nagoya Math. J., vol. 181, p. 63–73, 2006.

[5] X.X. Chen, *Calabi Flow in Riemann Surfaces Revisited: A New Point of View.* Int. Math. Res. Not., vol. 6, p. 275–297, 2001.

[6] X.X. Chen, *The Space of Kähler Metrics.* J. Differential Geom., vol. 56, p. 189–234, 2000.

[7] X.X. Chen and G. Tian, *Geometry of Kähler Metrics and Holomorphic Foliations by Discs.* `arXiv:0409433`, 2004.

[8] P. Chruściel, *Semi-global Existence and Convergence of Solutions of the Robinson-Trautman (2-Dimensional Calabi) Equation.* Comm. Math. Phys., vol. 137, p. 289–313, 1991.

[9] S. K. Donaldson, *Anti Self-Dual Yang–Mills Connections Over Complex Algebraic Surfaces and Stable Vector Bundles.* Proc. London Math. Soc., vol. s3-50 (1), p. 1–26, 1985.

[10] S. Donaldson, P. Kronheimer, *The Geometry of Four-Manifolds.* Oxford University Press, 1990.

[11] J. Eells, S. Salamon, *Twistorial Construction of Harmonic Maps of Surfaces Into Four-Manifolds.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4, vol. 12 (4), p. 589–640, 1985.

[12] H. Fischer, H. Maul, *Mathematik für Physiker II* (German). Teubner, 2004.

[13] J. Fine, *Constant Scalar Curvature Kähler Metrics on Fibred Complex Surfaces.* J. Differential Geom., vol. 68 (3), p. 397–432, 2004.

[14] J. Fine, *Fibrations with Constant Scalar Curvature Kähler Metrics and the CM-Line Bundle*. Math. Res. Lett., vol. 14 (2), p. 239–247, 2007.

[15] J. Fine, D. Panov, *Hyperbolic Geometry and non-Kähler Manifolds with Trivial Canonical Bundle*. Geom. Topol., vol 14 (3), p. 1723–1763, 2010.

[16] J. Fine, D. Panov, *Symplectic Calabi–Yau Manifolds, Minimal Surfaces and the Hyperbolic Geometry of the Conifold*. J. Differential Geom., vol. 82 (1), p. 155–205, 2009.

[17] J. Fine, D. Panov, *The Diversity of Symplectic Calabi–Yau Six-Manifolds*. J. Topology, vol. 6 (1), p. 644–658, 2013.

[18] A. Friedman, *Interior Estimates for Parabolic Systems of Partial Differential Equations*. J Math. and Mech., p. 393–417, 1958.

[19] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Grundlehren, Springer, 1984.

[20] K. Hasegawa, *Minimal Models of Nilmanifolds*. Proc. Amer. Math. Soc., vol. 106, p. 65–71, 1989.

[21] Y.-J. Hong, *Constant Hermitian Scalar Curvature Equations on Ruled Manifolds*. J. Differential Geom., vol. 53 (3), p. 465–516, 1999.

[22] Y.-J. Hong, *Ruled Manifolds with Constant Hermitian Scalar Curvature*. Math. Res. Lett., vol. 5 (5), p. 657–673, 1998.

[23] D. Huybrechts, *Complex Geometry: An Introduction*. Universitext, Springer, 2004.

[24] B. Kleiner and J. Lott, *Notes on Perelman's Papers*. `arXiv:math/0605667v5`, 2006.

[25] H. Le, G. Wang, *Anti-Complexified Ricci Flow on Compact Symplectic Manifolds*. J. Reine Angew. Math., vol. 530, p. 17–31, 2001.

[26] U. Mayer, *Gradient Flows on Nonpositively Curved Metric Spaces and Harmonic Maps*. Analysis and Geometry, vol. 6 (2), p. 199–253, 1998.

[27] A. I. Mal'cev, *On a Class of Homogeneous Spaces* (Russian). Izvestiya Akad. Nauk. SSSR. Ser. Mat., vol. 13, p. 9–32, 1949.

[28] J. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture*. Clay Mathematics Monographs, vol. 3, AMS, 2007.

[29] J. Morgan and G. Tian, *Completion of the proof of the Geometrization Conjecture*. `arXiv:0809.4040v1`, 2008.

[30] D. Phong, J. Sturm, *Lectures on Stability and Constant Scalar Curvature*. `arXiv:0801.4179v2`, 2008.

[31] M.A. Ragusa, *Parabolic Systems with Non-Continuous Coefficients*. Proceedings of the Fourth International Conference on Dynamical Systems and Differential Equations, p. 727–733, 2002.

[32] A. Reznikov, *Symplectic Twistor Spaces*. Ann. Global Anal. Geom., vol. 11 (2), p. 109–118, 1993.

[33] S. Rollenkse, *Geometry of Nilmanifolds with Left-Invariant Complex Structure and Deformations in the Large*. Proc. London Math. Soc., vol. 99, p. 425–460, 2009.

[34] C. Simpson, *Constructing Variations of Hodge Structure Using Yang-Mills Theory and Applications to Uniformization*. J. Amer. Math. Soc., vol. 1 (4), p. 867–918, 1988.

[35] J. Song, G. Tian, *The Kähler–Ricci Flow on Surfaces of Positive Kodaira Dimension*. Invent. Math., vol. 170, p. 609–653, 2007.

[36] J. Streets, *Long Time Existence of Minimizing Movement Solutions of Calabi Flow*. `arXiv:1208.2718v3`, 2012.

[37] J. Streets, G. Tian, *Symplectic Curvature Flow*. J. Reine Angew. Math., De Gruyter, to appear.

[38] M. Struwe, *Curvature Flow on Surfaces*. Ann. Scuola Norm. Sup. Pisa Cl. Sci., vol. 1 (5), p. 247–274, 2002.

[39] D. Toledo, *Rigidity Theorems in Kähler Geometry and Fundamental Groups of Varieties*. Cambridge Univ. Press, 1999.

[40] P. Topping, *Lectures on the Ricci Flow*. L.M.S. Lecture note series 325, Cambridge Univ. Press, 2006.

[41] C. Wang, Z. Wu, J. Yin, , *Elliptic and Parabolic Equations*. Word Scientific Publishing, 2006.