Université Libre de Bruxelles

Thèse de Doctorat

Growth rate of Legendrian contact homology and dynamics of Reeb flows

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Thèse présentée en vue de l'obtention du grade de Docteur en Sciences de l'Université Libre de Bruxelles

au

Département de Mathématique
Faculté des Sciences

Année académique 2014–2015

"Um espírito nobre engrandelhece até o menor dos homens."

Jebediah Springfield

Abstract

In this thesis we study the growth rate of a version of Legendrian contact homology, which we call strip Legendrian contact homology, in 3-dimensional contact manifolds, and its relation to the topological entropy of Reeb flows. We show that if for a pair of Legendrian knots in a contact 3-manifold the strip Legendrian contact homology is defined and has exponential homotopical growth with respect to the action, then every Reeb flow on this contact manifold has positive topological entropy. This has the following dynamical consequence: for all Reeb flows (even degenerate ones) on such a contact manifold, the number of hyperbolic periodic orbits grows exponentially with respect to the period. We show that for an infinite family of distinct 3-manifolds, infinitely many different contact structures exist which present exponential growth rate of the strip Legendrian contact homology for certain pairs of Legendrian knots.

Résumé

L'objectif de cette thèse est d'investiguer la relation entre l'homologie de contact Legendrienne d'une variété de contact de dimension 3, et l'entropie topologique des flots de Reeb associés à cette variété de contact. Une variété de contact est une variété M de dimension impaire munie d'un champ d'hyperplan ξ maximalement non-intégrable. Les champs de Reeb sont une classe speciale de champs de vecteurs sur M qui sont définis en utilisant la structure de contact; ils préservent la structure de contact et ils préservent aussi une forme de volume sur M.

L'entropie topologique h_{top} est un nombre non-négatif qu'on associe à un système dynamique et qui mesure la complexité de ce système. Si un système dynamique est d'entropie topologique positive, on dit que ce système est chaotique.

Comme les champs de Reeb sont construits en utilisant la structure de contact ξ, il est naturel d'attendre que la topologie de (M, ξ) influence la dynamique des champs de Reeb auxquels elle est associée. En particulier, il est naturel de se demander s'il existe des variétés de contact dont tous les champs de Reeb associés ont une entropie topologique positive. Si une varieté de contact a cette propriété, on dira qu'elle est d'entropie positive.

Macarini et Schlenk ont été les premiers à étudier cette question. Ils ont montré qu'il existe un grand ensemble de variétés différentielles Q, telles que le fibré unitaire T_1Q muni de sa structure de contact canonique est d'entropie topologique positive. Plus précisement, ils ont utilisé l'homologie de Floer Lagrangienne, qui est un invariant symplectique, pour montrer que si Q est rationnellement hyperbolique alors (T_1Q, ξ_{can}) est d'entropie positive.

Pour étudier l'entropie topologique dans le cas où M n'est pas un fibré unitaire on substitue à l'homologie de Floer Lagrangienne un invariant plus naturel des variétés de contact: l'homologie de contact Legendrienne à bandes. On demontre dans cette thèse que l'homologie de contact Legendrienne à bandes est bien adaptée pour étudier l'entropie topologique. Plus précisement, on montre que quand l'homologie de contact Legendrienne à bandes est bien définie pour un champ de Reeb associé à (M^3, ξ) et sa croissance est exponentielle, alors (M^3, ξ) est d'entropie positive.

On utilise ce résultat pour trouver des nouveaux exemples de variétés de contact de dimension 3 qui sont d'entropie positive. On montre même qu'il y a des variétés de dimension 3 qui possèdent une infinité de structures de contact différentes qui sont toutes d'entropie positive. Ces résultats et bien d'autres nous permettent de conjecturer que la "plupart" des variétés de contact de dimension 3 sont d'entropie positive.

Acknowledgements

I would like to express my gratitude to my Professors Frédéric Bourgeois and Chris Wendl for their guidance, patience, and support which were crucial in the development of this thesis. Their passion for mathematics and optimism in research have been an inspiration for my studies. I also thank them for their openness in sharing with me their ideas and scientific knowledge.

I want to thank Professor Felix Schlenk for his interest in this work, for suggesting improvements in the thesis and future directions of research, and for accepting to be a member of the defense jury. My thanks to Professors Joel Fine and Mélanie Bertelson for accepting to be members of the defense jury and suggesting corrections in the exposition.

Many thanks to the members of the Differential Geometry group of the ULB mathematics department for providing a rich and pleasant mathematical environment.

I would also like to thank professor Pedro Salomão for many helpful discussions and for explaining me the techniques which made it possible to avoid dealing with transversality problems arising from multiply covered pseudoholomorphic curves. I also thank him for introducing me to the world of pseudoholomorphic curves when I was a Master's student.

My gratitude to Mario Jorge Dias Carneiro and Ilan Grabe for their support and for the many things they shared with me when I was an undergraduate student in Belo Horizonte.

I have learned much from my colleagues and friends over many years. Thanks to André Alves, Romero Solha, Gustavo Souza, Eduardo Mascarenhas, Leandro Leme and Naim Zenaidi for the fun we had studying math together.

My personal thanks to Ana Nechita, André Alves, Hilda Ribeiro and Lucio Alves for their unconditional personal support. We have traversed many things together, and this work would not have been possible without you.

Last but not least, I thank FNRS-Belgium, for the financial support which allowed me to concentrate my efforts on mathematical research.

Contents

To Hilda Ribeiro, in memoriam.

Chapter 0

Introduction and main results

0.1 History of the problem

The objective of this thesis is to study the growth rate of a version of Legendrian contact homology and its implications to the dynamics of Reeb flows on contact 3-manifolds. It is part of a project of the author which aims at understanding the relationship between SFT-invariants of a contact structure and global dynamical invariants of Reeb flows. In this thesis we focus our attention on one invariant, the topological entropy. The topological entropy h_{top} is a non-negative number that one associates to a dynamical system and which attempts at measuring how chaotic the system is. A precise definition of topological entropy is given in section 1.2, but we will stay with the following intuitive picture: positivity of the topological entropy for a dynamical system implies that this system has "chaotic" behaviour. As an example, positivity of topological entropy for a 3-dimensional flow suffices to guarantee the existence of many hyperbolic periodic orbits for this flow; we refer to Corollary 1.2 of Section 1.2 for a precise statement.

Precise computations of the topological entropy for a dynamical system are quite rare, and one can only perform them in very specific cases. For example, the topological entropy for any rigid rotation of S^1 is zero, but for any hyperbolic torus automorphism the topological entropy is positive. Given the difficulty in making precise computations, it is natural to try to obtain estimates in form of inequalities for the topological entropy, and to find topological or geometric conditions on dynamical systems that guarantee at least the positivity of h_{top}. We will begin by recalling some topological conditions that can force positivity of h_{top} in certain dynamical systems.

One of the simplest results in this direction, is the following: if f is a diffeomorphism of the 2-dimensional torus $\mathbb{T}^2$ which is isotopic to a hyperbolic torus automorphism, then $h_{top}(f)$ is positive. This follows from the following theorem due to Manning [34]:

Theorem 0.1. *If f is a diffeomorphism of a compact finite dimensional manifold Q, then:*

$$h_{top}(f) \geq \log(s(f_{*1})) \tag{1}$$

where $s(f_{*1})$ is the supremum of the absolute value of the eigenvalues of the map f_{*1} induced by f on the first homology group of Q.

Notice that this result allows us not only to obtain positivity of the topological entropy for diffeomorphisms in certain isotopy classes, but actually to obtain a positive lower bound for h_{top} of all diffeormorphisms in the class.

A strengthening of this result for diffeomorphisms on surfaces was obtained by Fathi and Shub in [18]. They showed:

Theorem 0.2. *Let $f : S \to S$ be a diffeomorphism of a surface of genus bigger or equal to 2. If f is isotopic to a pseudo-Anosov diffeormorphism then the topological entropy of f is positive.*

A different direction in the study of the topological entropy was taken, independently, by Manning and Dinaburg. They studied conditions on a manifold that would force the topological entropy of geodesic flows to be positive. They discovered that there are actually topological conditions that force **all geodesic flows** associated to Riemannian metrics on a manifold to have positive topological entropy. More precisely:

Theorem 0.3. *If the fundamental group of a manifold Q has exponential growth, then for any Riemannian metric g on Q, the geodesic flow ϕ_g^t of g on T_1Q has positive topological entropy.*

We refer the reader to [38] for the precise definition of the exponential growth of the fundamental group of a manifold, and also for a complete proof of this theorem.

The combined efforts of many mathematicians, such as Gromov, Mañé, Paternain, and others, led to the discovery that the homology of the loop space of manifold Q has a profound impact on the dynamics of geodesic flows in T_1Q. This principle led to the following, very general, result:

Theorem 0.4. *If a manifold Q is energy hyperbolic then for any Riemannian metric g on Q the geodesic flow ϕ_g^t of g on T_1Q has positive topological entropy.*

Energy hyperbolic manifolds include those whose fundamental group has exponential growth as a subset. We refer the reader to chapter 5 of [38] for a description and proof of this result.

The results above motivated Frauenfelder and Schlenk to investigate how symplectic topological invariants could be used to study entropy invariants of symplectomorphisms (see [22] and [23]). For example, these authors used Lagrangian Floer homology to obtain a lower bound for the slow entropy of symplectomorphisms that are in the same symplectic isotopy class of the Seidel-Dehn twist. Essentially the same techniques, were used to obtain lower bounds for the topological entropy of symplectomorphisms of cotangent bundles, having a certain asymptotic behaviour.

Theorem 0.4 and the results in [22] and [23] served as an inspiration for the study of relations between contact topological invariants and topological entropy of Reeb flows. The reason for this is that geodesic flows are a particular case of Reeb flows and the development of symplectic and contact topology showed that many dynamical results obtained for geodesic flows, admit generalisations and adaptations to the world of Reeb flows. This inspired Macarini and Schlenk in [33] to study the topological entropy of Reeb flows on unit tangent bundles. They related the growth rate of Lagrangian Floer homology to the topological entropy of Reeb flows on the unit tangent bundle (T_1Q, ξ_{can}) of a manifold Q (where ξ_{can} is the contact structure associated to geodesic flows). Precisely they obtained the following generalisation of theorem 0.4:

Theorem 0.5. *If a manifold Q is energy hyperbolic then for all contact forms α associated to (T_1Q, ξ_{can}), the Reeb flow $\phi_{X_\alpha}^t$ of α on T_1Q has positive topological entropy.*

It is important to mention that Frauenfelder and Schlenk [24], and Frauenfelder, Labrousse and Schlenk [21] also used Lagrangian Floer homology to study the so called intermediate and slow entropies for Reeb flows in unit tangent bundles.

0.2 Main results

Inspired by the works [22], [23] and [33] we study in this thesis the relationship of other contact topological invariants to the study of topological entropy of Reeb flows. As for more general contact manifolds, Lagrangian Floer homology cannot always be defined, we substitute it by a version of Legendrian contact homology, called strip Legendrian contact homology, which we use to obtain entropy estimates for more general families of contact 3-manifolds. We construct many examples of contact 3-manifolds (most of which are non-symplectically fillable) that have pairs of Legendrian knots with exponential homotopical growth rate of strip Legendrian contact homology and show that this implies positivity of topological entropy for all Reeb flows on this contact manifold. This is the content of the following Theorem which is proved in Chapter 3 of this thesis:

Theorem 4.4: *Let $(Y, \xi = \ker(\lambda_0))$ be a contact 3-manifold with a contact form λ_0 adapted to the pair of disjoint Legendrian knots $(\Lambda, \widehat{\Lambda})$. Assume that $LC\mathbb{H}_{st}(\lambda_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth rate (with respect to the action) with exponential weight $a > 0$. For any contact form λ associated to (Y, ξ), let g_λ be the function such that $\lambda = g_\lambda \lambda_0$. Then, the Reeb flow of X_λ has positive topological entropy, and moreover:*

$$h_{top}(\phi_{X_\lambda}) \geq \frac{a}{\max(g_\lambda)} \tag{2}$$

We explain in Chapter 3 what it means for a contact form to be adapted to a pair of Legendrian curves. The notion of homotopical growth rate is used in theorem 4.4 above to avoid dealing with transversality problems that arise from the appearance of multiply covered pseudohomolomorphic curves. However, if one accepts that these transversality problems can be solved (as it is believed by many, through the use of the Polyfold technology being developed by Hofer, Wysocki and Zehnder) then we can substitute "exponential homotopical growth rate" by "exponential growth rate". It is an interesting fact that in all 3-dimensional examples known to the author where one has exponential growth rate of Legendrian contact homology, one also has exponential homotopical growth rate of Legendrian contact homology. Although I believe that this should indeed always be the case, a proof of this fact seems completely beyond current technology.

The simplest examples of contact manifolds satisfying the hypothesis of theorem 4.4 are unit tangent bundles of surfaces with higher genus. This allows us to re-obtain a result of Macarini and Schlenk [33] on the growth rate of Reeb chords from a unit tangent fiber to another, using the strip Legendrian contact homology in place of the Lagrangian Floer homology. This is the content of:

Theorem 5.4: *$LC\mathbb{H}_{st}(\alpha_{g_{hyp}}, \Lambda_q \to \Lambda_{q'})$ has exponential homotopical growth rate with exponential weight a_S.*

A major part of the present work is dedicated to present new examples which satisfy the hypothesis of theorem 4.4 above, as by the following:

Theorem 6.13: *Let M be a closed oriented connected 3-manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary such that the component M_0 satisfies:*

- *M_0 is the mapping torus of a punctured torus S by a diffeomorphism $h : S \to S$ such that the homology map h^* is a hyperbolic automorphism of $H^1(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$.*

Then M can be given infinitely many different tight contact structures ξ_k, such that there exist disjoint Legendrian knots Λ_k, Λ'_k and contact forms τ_k associated to (M, ξ_k) and adapted to the pair Λ_k, Λ'_k for which $LC\mathbb{H}_{st}(\tau_k, \Lambda_k \to \Lambda'_k)$ has exponential homotopical growth rate.

The contact manifolds of this theorem fall under the umbrella of the examples constructed in [11] and studied in [40], [4] and [5]. In particular, the theorem above implies that the contact 3-manifolds above have the positive topological entropy property, a result that also follows from Theorem 3 in [4].

Another class of examples is studied in the context of a contact surgery introduced by Foulon and Hasselblatt in [20]; the definition and details of this surgery are presented in chapter 7. Denote by α_F the contact form obtained by performing the Foulon-Hasselblatt surgery on the Legendrian lift of a separating geodesic $\mathfrak{r}$ of a hyperbolic surface. We show in chapter 7, that there is a constant $a(\mathfrak{r}) > 0$ such that:

Theorem 7.7: *$LC\mathbb{H}_{st}(\alpha_F, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth rate with exponential weight a.*

In a different line of development we also study a forcing problem in this Thesis. We consider the following question: in a contact 3-manifold does there exist a transverse link whose appearance as a set of Reeb orbits for a Reeb flow in this contact manifold, suffice to force positivity of topological entropy of the Reeb flow?

In order to study this question, we must first find invariants of transverse links that would be appropriate to study such a question. Secondly, using such invariants, we must find contact topological conditions on a transverse link that would imply that any Reeb flow having this link as a set of Reeb orbits must have positive topological entropy.

The invariant that helps us to study the forcing problem is the strip Legendrian contact homology on the complement of Reeb orbits, which we define in chapter 3. This is a relative version of an invariant introduced by Momin in [35]. In chapter 4 we obtain the following structural result:

Theorem 4.7: *Let (Y, ξ) be a contact manifold and Λ and $\widehat{\Lambda}$ be two disjoint Legendrian submanifolds, such that α_0 is associated to (Y, ξ) and adapted to the pair $(\Lambda, \widehat{\Lambda})$ in the complement of $\mathcal{G}$. Suppose that the strip contact homology $LC\mathbb{H}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$. Let α be another contact form associated to (Y, ξ) and having $\mathcal{G}$ as a set of Reeb orbits, and take $g > 0$ to be the unique function such that $\alpha = g_\alpha \alpha_0$. Then, the Reeb flow of X_α has positive topological entropy, and moreover:*

$$h_{top}(\phi_{X_\alpha}) \geq \frac{a}{\max(g_\alpha)} \tag{3}$$

We will show in joint work with Pedro Salomão that there exist many examples of contact manifolds and transverse links satisfying the hypothesis of Theorem 4.7.

0.2.1 Geometric idea of proof of Theorem 4.4

We will give an intuitive idea of the proof of Theorem 4.4, which is the main structural result in this thesis. The basic idea, which is also used in [33], is to use the number of Reeb chords of α from a Legendrian submanifold Λ_0 to other Legendrian submanifolds to estimate the growth rate of the volume of $\phi^t_{X_\alpha}(\Lambda_0)$.

In [33] the authors study Reeb chords from one fixed unit tangent fiber Λ_0 in the unit tangent bundle T_1Q to all other unit tangent fibers Λ_q, where $q \in Q$. Through the use of Lagrangian Floer homology they show that if Q is energy hyperbolic, then there exists $C_0 \geq 0$, $\overline{a}(\alpha) > 0$, and $\overline{d}(\alpha)$ such that:

$$N_C(\alpha, \Lambda_0, \Lambda_q) \geq e^{C\overline{a}+\overline{d}} \text{ for all } C \geq C_0, \tag{4}$$

where $N_C(\alpha, \Lambda_0, \Lambda_q)$ is the number of Reeb chords of α from Λ_0 to Λ_q. They then use the canonical projection from T_1Q to Q to estimate the area of the cylinder $Cyl^C_{X_\alpha}(\Lambda_q) := \{\phi^t_{X_\alpha}(\Lambda_0); t \in [0, C]\}$ through the use the counting functions $N_C(\alpha, \Lambda_0, \Lambda_q)$ (the idea of using counting functions for such area estimates is due to Gabriel Paternain; see [38] and [37]). The result is an inequality of the form:

$$Area(Cyl^C_{X_\alpha}(\Lambda_q)) \geq \int_Q N_C(\alpha, \Lambda_0, \Lambda_q) d\mu_g \geq \mathsf{V}(\mathsf{Q}) e^{C\overline{a}+\overline{d}} \tag{5}$$

where μ_g is the measure induced by a Riemannian metric g on Q, and V is volume of Q in the measure. We point out, that in this last inequality, the fact that T_1Q is a Legendrian fibration is used in a crucial way.

Most contact 3-manifolds do not have the structure of Legendrian fibration; in fact the only contact 3-manifolds with such structures are unit tangent bundles of surfaces and their coverings. However, a sufficiently small neighbourhood of a Legendrian knot $\widehat{\Lambda}$ on a contact 3-manifold (Y, ξ) always has the structure of a Legendrian fibration. This is a consequence of the Weinstein Legendrian neighbourhood Theorem, whose 3-dimensional version asserts that sufficiently small neighbourhoods of Legendrian knots are always contactomorphic; i.e there exists a normal form for small neighbourhoods of Legendrian knots.

Now, in the hypotheses of Theorem 4.4 we have a pair of Legendrian knots $(\Lambda, \widehat{\Lambda})$ in (Y, ξ), and a contact form α_0 for which the Legendrian contact homology $LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopic growth rate with exponential weight $a > 0$. We begin by choosing a small neighbourhood N_ϵ of $\widehat{\Lambda}$ which is contactomorphic $(S^1 \times \mathbb{D}, \ker(\cos(\theta)dx + \sin(\theta)dy))$; where $(\theta, x, y) \in S^1 \times \mathbb{D}$ and $\widehat{\Lambda}$ is identified with $S^1 \times \{0\}$. It is clear that for all $z := (x, y) \in \mathbb{D}$ the curve $\widehat{\Lambda}^z := S^1 \times \{z\}$ is Legendrian in N_ϵ.

A combination of the exponential homotopical growth of $LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ and invariance properties of the Legendrian contact homology imply that given $\delta > 0$, if N_ϵ is a sufficiently small neighbourhood, there is a uniform lower bound on $N_C(\alpha, \Lambda_0, \widehat{\Lambda}^z)$ for all $z \in \mathbb{D}$ which is given in the following inequality:

$$N_C(\alpha, \Lambda_0, \widehat{\Lambda}^z) \geq e^{\frac{aC}{\max g_\alpha (1+4\delta)}} \tag{6}$$

for all $C \geq C_0$, where g_α is the positive function Y such that $\alpha = g_\alpha \alpha_0$.

We now want to use the counting function to estimate the area of the intersection $Cyl^C_{X_\alpha}(\Lambda) \cap N_\epsilon$ between the cylinder $Cyl^C_{X_\alpha}(\Lambda)$ and the neighbourhood N_ϵ (see figure). As N_ϵ has the structure of a Legendrian fibration this is indeed possible, if we choose a metric g_0 in Y which restricts to the "flat" metric $d\theta \otimes d\theta + dx \otimes dx + dy \otimes dy$ on the coodinates (θ, x, y) on N_ϵ.

Because N_ϵ has the structure of a Legendrian fibration, we can apply a "local" version of Paternain's idea to obtain the following inequality:

$$Area(Cyl^C_{X_\alpha}(\Lambda) \cap N_\epsilon) \geq \int_{\mathbb{D}} N_C(\alpha, \Lambda, \widehat{\Lambda}^z) dx dy \geq e^{\frac{Ca}{\max(g_\alpha)(1+4\delta)} + d'} \tag{7}$$

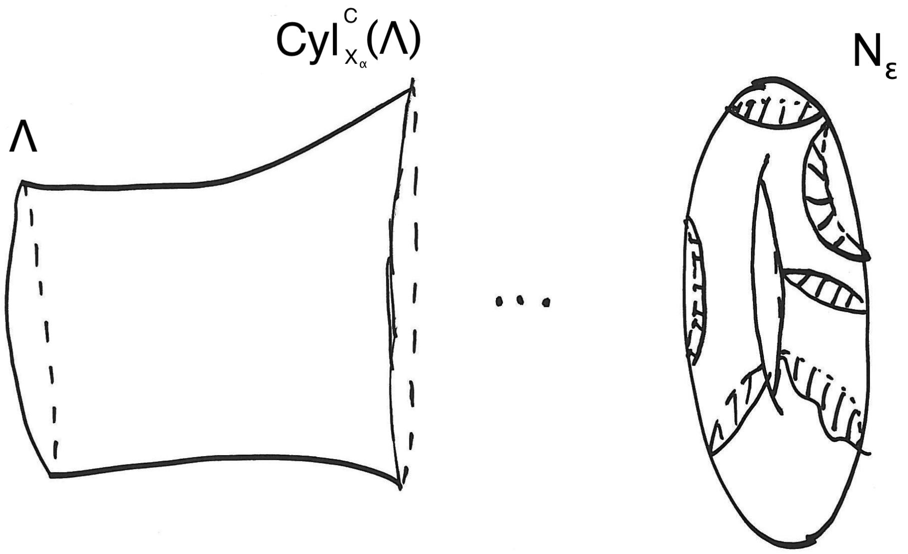

FIGURE 1: The hatched discs on the left side of the picture represent the pieces of the intersection $Cyl^C_{X_\alpha}(\Lambda) \cap N_\epsilon$. Under the hypotheses of Theorem 4.4, the sum of the area of these discs grows exponentially with respect C.

This local picture shows that the area $Area(Cyl^C_{X_\alpha}(\Lambda) \cap N_\epsilon)$ (of the intersection between $Cyl^C_{X_\alpha}(\Lambda)$ and N_ϵ) already grows exponentially fast and allows us to estimate the topological entropy of the Reeb flow of X_α by using Yomdin's theorem.

Lastly, we mention that it is likely that the local picture we used here can also be applied to estimate the intermediate and slow entropies which were studied in [24] and [21] in contact manifolds which are not unit tangent bundles.

Chapter 1

Contact manifolds, Reeb flows and dynamics

In this thesis we study dynamical properties of Reeb flows on 3-dimensional contact manifolds. We start by recalling some basic concepts from contact geometry and dynamical systems which are central for this thesis.

1.1 Basic definitions from contact geometry

We first recall some definitions from contact geometry. A 3-dimensional contact manifold is a pair (Y, ξ), where Y is a compact 3 dimensional manifold and ξ, called the *contact structure*, is a"totally" non-integrable distribution of planes on Y; the total non-integrability condition means that for every locally defined 1-form ζ such that $\xi = \ker(\zeta)$ we have that $\zeta \wedge (d\zeta)^n \neq 0$. When there exists a globally defined 1-form α such that $\ker(\alpha) = \xi$ we call α a *contact form* associated to the contact manifold (Y, ξ), and say that (Y, ξ) is a co-orientable contact manifold. In this thesis we only study co-orientable contact manifolds, and from now on, every time we write contact manifold we actually mean co-orientable contact manifold.

Given a contact manifold (Y, ξ), there are many different contact forms associated to it. To see this, let α be a contact form associated to (Y, ξ). Then for every positive function $f : Y \to \mathbb{R}$, $f\alpha$ is also a contact form associated to (Y, ξ).

To a contact form α, we can associate a vector field X_α, that we call its *Reeb vector field*, and that is completely characterised by the following 2 conditions:

$$i_{X_\alpha} d\alpha = 0, \tag{1.1}$$

$$\alpha(X_\alpha) = 1. \tag{1.2}$$

The *Reeb flow* of α is the flow of the vector field X_α.

Among the submanifolds of a contact manifold a special important class is that of *Legendrian* submanifolds. An isotropic submanifold of (Y, ξ) is a submanifold Λ of Y whose tangent space is contained in ξ for all points of Λ. The Legendrian submanifolds of (Y, ξ) are the isotropic submanifolds of (Y, ξ) which have the maximal possible dimension. It turns out, that for 3-dimensional contact manifolds, that this maximal possible dimension is 1, and therefore the Legendrian submanifolds are the isotropic submanifolds of dimension 1.

There are two special types of trajectories of Reeb flows that have played a central role in the study of contact topology and dynamics of Reeb vector fields. One of them are the periodic orbits of a given Reeb flow, which we call *Reeb orbits*. The other are the trajectories of a Reeb flow which start in a Legendrian submanifold Λ and end in a Legendrian submanifold $\widehat{\Lambda}$ (notice that Λ and $\widehat{\Lambda}$ might coincide); these trajectories are called *Reeb chords* from Λ to $\widehat{\Lambda}$. Given a Reeb orbit γ of the Reeb flow of α, we define its action to be $A(\gamma) := \int_\gamma \alpha$; it follows from equation 1.2 above that $A(\gamma)$ coincides with the period of γ. Analogously, for a Reeb chord c of the Reeb flow of α, we define its action to be $A(c) := \int_c \alpha$; like for Reeb orbits the action of c coincides with the "period" of the trajectory c. Following terminology widely used in the literature, a contact form α is called *hypertight* when it doesn't have any contractible Reeb orbits. Lastly, a Reeb orbit γ is said to be *non-degenerate* when 1 is not an eigenvalue of the linearisation $D\phi_{X_\alpha}^{A(c)} |_\xi$ of the Poincaré return map associated to the γ; and a Reeb chord c is said to be *transverse* if the intersection $\phi_{X_\alpha}^{A(c)}(\Lambda) \cap \widehat{\Lambda}$ is transverse at the endpoint of c.

One important feature of Reeb flows is that they appear in many different models of mathematical physics. For instance, every Reeb flow appears as the restriction to an energy level, of some Hamiltonian flow in a (possibly non-compact) symplectic manifold (see [26]). One important example of this relation between Reeb flows and Hamiltonian flows is seen in the case of geodesic flows. For any Riemannian metric on a compact manifold Q, the restriction of its geodesic flow to the unit tangent bundle T_1Q of Q is a Reeb flow.

More recently, Etnyre and Ghrist showed in [17] that 3-dimensional Reeb flows also appear in the context of hydrodynamical 3-dimensional flows. Beltrami flows are an important special class of hydrodynamical flows. Etnyre and Ghrist showed that every Reeb flow in a 3-dimensional contact manifold is the reparametrization of some Beltrami flow; they also showed that every Beltrami flow is the reparametrization of some Reeb flow. therefore the classes of Beltrami and Reeb flows are equivalent from a dynamical perspective.

These relations to the field of mathematical physics also justifies the study of the dynamical properties of Reeb vector fields, as this study might also have impact in this field.

1.1.1 The Conley-Zehnder index for Reeb chords

In this section we present a geometric definition of the Conley-Zenhder index for Reeb chords in the 3-dimensional case. This index will allow us to associate a $\mathbb{Z}^2$-grading to a Reeb chord. Keeping the notation above we consider a contact form α associated to the 3-dimensional contact manifold (Y, ξ), and a pair $(\Lambda, \widehat{\Lambda})$ of Legendrian knots in (Y, ξ). We denote by $\mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha)$ the set of Reeb chords of the Reeb flow of X_α going from Λ to $\widehat{\Lambda}$.

For the definition, we first fix, once and for all, orientations for Λ and $\widehat{\Lambda}$. Then, for each Reeb chord $c \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\lambda_0)$, let Ψ_c be a nowhere vanishing section of the vector bundle $\xi\mid_c$ that:

- is tangent to Λ on the initial point of c and, furthermore, coincides with the orientation we fixed for Λ at this initial point;
- is tangent to $\widehat{\Lambda}$ on the final point of c and, furthermore, coincides with the orientation we fixed for $\widehat{\Lambda}$ at this final point.

The section Ψ_c induces a (unique up to homotopy) symplectic trivialisation of the symplectic 2-dimensional vector bundle $(\xi\mid_c, d\alpha)$, which we will also denote by Ψ_c.

Using the Reeb flow ϕ_{X_α} we define a path of Lagrangian subspaces $\mathcal{Z}$ of $\xi\mid_c$. We consider the parametrisation $c : [0, T_c] \to Y$ of the Reeb chord c given by the Reeb flow. Letting $D\phi_{X_\alpha}$ denote the linearisation of the Reeb flow, we define $\mathcal{Z}(t)$ to be the unique Lagragian subspace of $(\xi\mid_{c(t)}, d\alpha)$ that contains $D\phi^t_{X_{\lambda_0}}(c(0))(q)$; where $q \in \xi\mid_{c(0)}$ is a vector tangent to Λ and giving the orientation we chose. At the endpoint $c(T_c)$, we complete $\mathcal{Z}$ by making in the time interval $[T_c, T_c+1]$ a continuous left-rotation of $\mathcal{Z}(T_c)$ (inside ξ) till it meets the tangent space to $\widehat{\Lambda}$; this left-rotation is defined with respect to the orientation given by $d\alpha$ on ξ.

With this completion and using our trivialisation Ψ_c, we associate to $\mathcal{Z}([0,T_c+1])$ a loop $\mathcal{L}([0,T_c+1])$ of Lagrangian subspaces of the standard symplectic plane $(\mathbb{R}^2,\omega_{std})$. The Conley-Zehnder index $\mu_{CZ}^{\Psi_c}(c)$ is defined to be the Maslov index of this path.

It is easy to see that, because we fixed the orientations of Λ and $\widehat{\Lambda}$, the parity of $\mu_{CZ}^{\Psi_c}(c)$ is independent of the trivialisation Ψ_c satisfying the "boundary" conditions above. This allows us to define, for each $c \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha)$, its $\mathbb{Z}_2$-grading $\mid c \mid$ by:

$$\mid c \mid := (\mu_{CZ}^{\Psi_c}(c) - 1) \mod 2. \tag{1.3}$$

We call chords with grading 0 even chords, and chords with grading 1 odd chords. Notice that, in the case where Λ and $\widehat{\Lambda}$ are distinct Legendrian knots, there is no canonical grading for Reeb chords in $\mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha)$, since the grading depends on choices of orientations of Λ and $\widehat{\Lambda}$.

1.2 Topological entropy of dynamical systems

The topological entropy is an important invariant of dynamical systems, which was introduced in the 1960's by Adler, Konheim and McAndrew. It codifies, in a single non-negative number, how chaotic a dynamical system is; it is widely accepted that a dynamical system with positive topological entropy presents some kind of chaotic behaviour.

We present the following definition, which is valid for dynamical systems in compact metric spaces and is due to Bowen [9]. Consider a smooth compact manifold V with a non-vanishing vector field X that generates a flow ϕ_X. We endow V with an auxiliary Riemannian metric g, whose associated metric on V we denote by d_g.

Let T and δ be positive real numbers. A set S is said to be T,δ-separated if for all $q_1 \neq q_2 \in S$ we have:

$$\max_{t\in[0,T]} d_g(\phi_X^t(q_1),\phi_X^t(q_2)) > \delta. \tag{1.4}$$

We denote by $n^{T,\delta}$ the maximal cardinality of a T,δ-separated set for the flow ϕ_X. Then we define the δ-entropy $h_\delta(\phi_X)$ to be:

$$h_\delta(\phi_X) = \limsup_{T\to+\infty} \frac{\log(n^{T,\delta})}{T}. \tag{1.5}$$

The topological entropy h_{top} is then defined by the formula:

$$h_{top}(\phi_X) = \lim_{\delta \to 0} h_\delta(\phi_X). \tag{1.6}$$

One can prove that the topological entropy doesn't depend on the metric d_g but only on the topology determined by the metric. For these and other structural results about topological entropy we refer the reader to any standard textbook in dynamics such as [32] and [39].

The definition of topological entropy is quite involved and it is usually quite difficult to compute or even estimate the topological entropy for a given dynamical system. To motivate such difficult attempts to estimate or compute this quantity, we present one consequence of positivity of topological entropy for low-dimensional dynamical systems:

Theorem 1.1. *Katok [31]. Let X be a $C^{1+\delta}$ ($\delta > 0$) vector field on a smooth 3-dimensional M, whose flow ϕ_X has positive topological entropy $h_{top}(\phi_X)$. Then there exists a hyperbolic periodic orbit x of X, whose stable and unstable manifold have a transverse intersection, i.e. a transverse homoclinic intersection. Consequently there is an invariant set Ω for the flow ϕ_X, such that the dynamics of the restriction of ϕ_X to Ω is topologically conjugate to a subshift of finite type and $h_{top}(\phi_X|_\Omega) > 0$.*

The theorem above means that simply the non-vanishing of h_{top} for a given flow ϕ_X on a 3-dimensional manifold implies that this flow has complicated orbit structure. For example, for a given number $T > 0$, let $P_X^{hyp}(T)$ be the number of hyperbolic periodic orbits of ϕ_X with period smaller then T. As a consequence of the Theorem above we have the following corollary also due to Katok:

Corollary 1.2. *If the flow ϕ_X of a vector field X on a 3-manifold M has positive topological entropy, then we have the following lower bound:*

$$\limsup_{T \to +\infty} \frac{\log(P_X^{hyp}(T))}{T} > 0. \tag{1.7}$$

This means that positivity of topological entropy for these flows implies that they have infinitely many isolated periodic orbits.

Only these results suffice, in my opinion, to justify the study of the topological entropy for Reeb flows in 3-dimensional contact manifolds. Among different existent techniques to estimate the topological entropy, we will use a theorem due to Yomdin that gives a geometric criterion that guarantees positivity of topological entropy for smooth flows. Given an immersed submanifold S of dimension l in V, we denote by $Vol_g^l(S)$ the Riemannian l-dimensional volume induced by the Riemannian metric g.

Theorem 1.3. *Yomdin [42] Let X be a non-singular C^∞ vector field on the manifold V. Then for any immersed submanifold S in V we have the following inequality:*

$$\limsup_{t\to+\infty} \frac{\log(Vol_g^l(\phi_X^t(S))}{t} \leq h_{top}(\phi_X). \tag{1.8}$$

We will now prove a corollary of Yomdin's theorem which will be used for our estimates of topological entropy of Reeb flows.

Let by (Y,ξ) be a contact 3-manifold and α a contact form associated to (Y,ξ). Given a knot $\mathfrak{L}$ in (Y,ξ), we let $Cyl^T_{X_\alpha}(\mathfrak{L}) := \{\phi^t_{X_\alpha}(\mathfrak{L}); t \in [0,T]\}$; it is clear that $Cyl^T_{X_\alpha}(\mathfrak{L})$ is an immersed cylinder in Y. It follows directly from the compactness of Y that for any given Riemannian metrics g_1 and g_2, there exist constants $0 < \mathsf{k} < \mathsf{K}$ (depending only on the metrics g_1 and g_2) such that:

$$\mathsf{k}Vol^2_{g_1}(Cyl^T_{X_\alpha}(\mathfrak{L})) \leq Vol^2_{g_2}(Cyl^T_{X_\alpha}(\mathfrak{L})) \leq \mathsf{K}Vol^2_{g_1}(Cyl^T_{X_\alpha}(\mathfrak{L})). \tag{1.9}$$

With this, we are ready to prove:

Corollary 1.4. *Let (Y,ξ) be a contact 3-manifold, and α be a C^∞ contact form associated to (Y,ξ), and g be a Riemannian metric in Y. Then, for any Legendrian knot Λ in (Y,ξ), we have the following inequality:*

$$\limsup_{T\to+\infty} \frac{\log(Vol^2_g(Cyl^T_{X_\alpha}(\Lambda)))}{T} \leq h_{top}(\phi_{X_\alpha}). \tag{1.10}$$

Proof:

We begin by picking a contact metric g_α for α; a contact metric for α is a Riemannian metric in Y, such that the Reeb vector field X_α has norm 1 and is orthogonal to the plane ξ.

It follows from inequality (1.9) that

$$\limsup_{T\to+\infty} \frac{\log(Vol^2_g(Cyl^T_{X_\alpha}(\Lambda)))}{T} = \limsup_{T\to+\infty} \frac{\log(Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda)))}{T}. \tag{1.11}$$

We choose an arc length parametrisation $\mathfrak{r} : S^1 \to \Lambda$ with respect to g_α. Using this parametrisation and the Reeb flow we obtain a parametrisation $\mathfrak{R} : [0,T] \times S^1 \to Cyl^T_{X_\alpha}(\Lambda)$ of $Cyl^T_{X_\alpha}(\Lambda)$, and we consider coordinates $(t,\theta) \in [0,T] \times S^1$. This parametrisation clearly satisfies that: $\partial_t \mathfrak{R}(t,\theta) = X_\alpha(t,\theta)$ and $\partial_\theta \mathfrak{R}(t,\theta)$ is tangent to ξ.

It follows from these properties and the fact that g_α is a contact metric, that for the pullback metric $\mathfrak{R}^* g_\alpha$, we have:

$$\mathfrak{R}^* g_\alpha(\partial_t, \partial_t) = 1, \tag{1.12}$$

$$\mathfrak{R}^* g_\alpha(\partial_t, \partial_\theta) = 0. \tag{1.13}$$

Then, the area $Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda))$ equals the area of $[0,T] \times S_1$ in the metric $\mathfrak{R}^* g_\alpha$. This allows us to obtain the following formula for $Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda))$:

$$\begin{aligned} Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda)) &= \int_{[0,T]\times S_1} (\mathfrak{R}^* g_\alpha(\partial_t, \partial_t))^{\frac{1}{2}} (\mathfrak{R}^* g_\alpha(\partial_\theta, \partial_\theta))^{\frac{1}{2}} |\sin(\angle(\partial_t, \partial_\theta))| d\theta dt = \\ &= \int_{[0,T]} \Big(\int_{S_1} (\mathfrak{R}^* g_\alpha(\partial_\theta, \partial_\theta))^{\frac{1}{2}} d\theta \Big) dt = \int_{[0,T]} (Vol^1_{g_\alpha}(\phi^t_{X_\alpha}(\Lambda))) dt. \end{aligned} \tag{1.14}$$

If $\limsup_{T\to+\infty} \frac{\log(Vol^2_g(Cyl^T_{X_\alpha}(\Lambda)))}{T} \leq 0$, then the conclusion of the corollary is obvious since the topological entropy is always non-negative. Thus, to finish the proof, we treat the case where $\limsup_{T\to+\infty} \frac{\log(Vol^2_g(Cyl^T_{X_\alpha}(\Lambda)))}{T} = \limsup_{T\to+\infty} \frac{\log(Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda)))}{T} = a > 0$. We will argue by contradiction, assuming that $\limsup_{t\to+\infty} \frac{\log(Vol^1_{g_\alpha}(\phi^t_{X_\alpha}(\Lambda))}{t} < a$.

If that was the case then there would be $t_0 > 0$ and $\epsilon < 0$ such that:

$$Vol^1_{g_\alpha}(\phi^t_{X_\alpha}(\Lambda)) < e^{(a-\epsilon)t} \tag{1.15}$$

for all $t \geq t_0$. Integrating both sides of the equation for t between 0 and T and using (1.14) and (1.15), we would conclude that:

$$Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda)) < \frac{e^{T(a-\epsilon)} - e^{t_0(a-\epsilon)}}{a-\epsilon} + \int_0^{t_0} (Vol^1_{g_\alpha}(\phi^t_{X_\alpha}(\Lambda))) dt \tag{1.16}$$

for all $T \geq t_0$. However, the right side of (1.16) becomes a lot smaller than $e^{T(a-\frac{\epsilon}{2})}$ for sufficiently large T. This forces $Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda))$ to remain smaller than $e^{T(a-\frac{\epsilon}{2})}$ for large T and leads to a contradiction, since we have $\limsup_{T\to+\infty} \frac{\log(Vol^2_{g_\alpha}(Cyl^T_{X_\alpha}(\Lambda)))}{T} = a$.

Therefore we have concluded that $\limsup_{t\to+\infty} \frac{\log(Vol^1_g(\phi^t_{X_\alpha}(\Lambda))}{t} \geq a$, and we can apply Theorem 1.3 to conclude that $h_{top}(\phi_{X_\alpha}) \geq a$.

□

Yomdin's theorem allows one to obtains estimates on the topological entropy from purely geometric considerations, and in many cases also from topological considerations. As an example of an application of this type, one can use Yomdin's theorem to prove that if a C^∞ diffeomorphism h of the two-dimensional torus $\mathbb{T}^2$ is isotopic to a hyperbolic torus automorphism, then h has positive topological entropy.

Chapter 2

Pseudoholomorphic curves in symplectizations and symplectic cobordisms

2.1 Almost complex structures in symplectizations and symplectic cobordisms

We start by reviewing the basic facts about pseudoholomorphic curves in symplectizations and symplectic cobordisms.

2.1.1 Cylindrical almost complex structures

Let (Y, ξ) be a contact manifold and α an associated contact form. The symplectization of (Y, ξ) is the product $\mathbb{R} \times Y$ with the symplectic form $d(e^s \alpha)$ (where s denotes the $\mathbb{R}$ coordinate in $\mathbb{R} \times Y$). $d\alpha$ restricts to a symplectic form on the vector bundle ξ and it is well known that the set $\mathfrak{j}(\alpha)$ of $d\alpha$-compatible almost complex structures on the symplectic vector bundle ξ is non-empty and contractible. Notice that as Y is 3-dimensional the set $\mathfrak{j}(\alpha)$ doesn't depend on the contact form α associated to (Y, ξ).

For $\widehat{j} \in \mathfrak{j}(\alpha)$ we can define an $\mathbb{R}$-invariant almost complex structure J on $\mathbb{R} \times Y$ by demanding that:

$$J\partial_s = X_\alpha, \quad J \mid_\xi = \widehat{j}. \tag{2.1}$$

We will denote by $\mathcal{J}(\alpha)$ the set of almost complex structures in $\mathbb{R} \times Y$ that are $\mathbb{R}$-invariant, $d(e^s\alpha)$-compatible and satisfy equation 2.1 above.

2.1.2 Exact symplectic cobordisms

Let $d\varsigma = \varpi$ be an exact symplectic form on $\mathbb{R} \times Y$ for which there exist contact forms α^+ and α^- in Y and real numbers $R^+ > R^-$ such that:

$$\varsigma = (e^{s-R^+}\alpha^+) \text{ in } [R^+, +\infty) \times Y, \tag{2.2}$$

$$\varsigma = (e^{s-R^-}\alpha^-) \text{ in } (-\infty, R^-] \times Y. \tag{2.3}$$

We call $(W = \mathbb{R} \times Y, \varpi)$ an exact symplectic cobordism α^+ to α^-. We divide the cobordism (W, ϖ) in three pieces and denote $W(\alpha^+) = [R^+, +\infty) \times Y$, $W(\alpha^+, \alpha^-) = [R^-, R^+] \times Y$ and $W(\alpha^-) = (-\infty, R^-] \times Y$. In such a cobordism we say that an almost complex structure $\overline{J}$ is cylindrical if there exist positive constants k and K such that:

$$\overline{J} \text{ coincides with } \; J^+ \in \mathcal{J}(\mathsf{K}\alpha^+) \text{ in the region } [R^+, +\infty) \times Y, \tag{2.4}$$

$$\overline{J} \text{ coincides with } \; J^- \in \mathcal{J}(\mathsf{k}\alpha^-) \text{ in the region } (-\infty, R^-] \times Y, \tag{2.5}$$

$$\overline{J} \text{ is compatible with } \varpi \text{ in } [R^-, R^+] \times Y. \tag{2.6}$$

In this case we will say that $\overline{J}$ is positively asymptotic to J^+ and negatively asymptotic to J^-. For fixed J^+ and J^- we denote by $\mathcal{J}(J^-, J^+)$ the set of cylindrical almost complex structures in $(\mathbb{R} \times Y, \varpi)$ positively asymptotic to J^+ and negatively asymptotic to J^-. $\mathcal{J}(J^-, J^+)$ is well known to be contractible.

We will write $\alpha^+ \succ_{ex} \alpha^-$ when there exists an exact symplectic cobordism from α^+ to α^- as above. We notice that $\alpha^+ \succ_{ex} \alpha$ and $\alpha \succ_{ex} \alpha^-$ implies $\alpha^+ \succ_{ex} \alpha^-$, or in other words, that existence of exact symplectic cobordisms is transitive; see [8].

2.1.3 Splitting symplectic cobordisms

Let α^+, α and α^- be contact forms associated to (Y, ξ) such that $\alpha^+ = f^+\alpha$, $\alpha^- = f^-\alpha$ for positive functions $f^+ > 1$ and $f^- < 1$ on Y. Take $\epsilon > 0$ such that $f^+ > 1 + 2\epsilon$ and

$f^- < 1 - 2\epsilon$. We now pick, for each $R > 0$, a function $f_R : \mathbb{R} \times Y \to \mathbb{R}$ with $\partial_s f > 0$ and such that:

$$f_R : [-R, R] \times Y \to [1-\epsilon, 1+\epsilon] \text{ depends only on the real coordinate,}$$
$$f_R(-R) = 1 - \epsilon,\ f_R(R) = 1 + \epsilon,$$
$$f_R \mid_{[R+2,+\infty)\times Y} = e^{s-R-2} f^+ \text{ and } f_R \mid_{(-\infty,-R-2]\times Y} = e^{s+R+2} f^-.$$

We now define $\varpi_R = d(f_R \alpha)$; a simple computation shows that ϖ_R is indeed an exact symplectic form on $W = \mathbb{R} \times Y$. Moreover, it is clear that $(\mathbb{R} \times Y, \varpi_R)$ is an exact symplectic cobordism from α^+ to α^-; notice that in $[-R, R] \times Y$ the conditions above imply that $f_R\alpha$ looks like the symplectization of α. As $(\mathbb{R}\times Y, \varpi_R)$ is an exact symplectic cobordism we consider on it a compatible cylindrical almost complex structure $\widetilde{J}_R$ as in the previous section, but we demand a stronger condition on $\widetilde{J}_R$:

$$\widetilde{J}_R \text{ coincides with } J \in \mathcal{J}(\alpha) \text{ in } [-R, R] \times Y \tag{2.7}$$

Again we divide our manifold W in pieces:

$$W(\alpha^+) = [R + 2, +\infty) \times Y,$$
$$W(\alpha^+, \alpha) = [R, R + 2] \times Y,$$
$$W(\alpha) = [-R, R] \times Y,$$
$$W(\alpha, \alpha^-) = [-R - 2, -R] \times Y,$$
$$W(\alpha^-) = (-\infty, -R - 2] \times Y.$$

Varying $R > 0$, we obtain a family of exact symplectic cobordisms $(\mathbb{R} \times Y, \varpi_R)$ from α^+ to α^-. As $R \to +\infty$ the region $W(\alpha) = [-R, R] \times Y$ where the symplectic form is similar to the symplectization of (Y, α) becomes larger.

To gain an intuition about this construction, one can initially think that in the limit as $R \to +\infty$ the sequence $(\mathbb{R}\times Y, \varpi_R)$ splits into two exact symplectic cobordisms, $V(\alpha^+, \alpha)$ from α^+ to α, followed by $V(\alpha, \alpha^-)$ from α to α^-. Actually, when one studies sequences of pseudoholomorphic curves in such families of cobordisms the limiting object are more complicated then just the pair of two cobordims we mentioned; levels of symplectizations have to be inserted above $V(\alpha^+, \alpha)$, between $V(\alpha^+, \alpha)$ and $V(\alpha, \alpha^-)$, and below $V(\alpha, \alpha^-)$ to complete the picture. We refer the reader to the paper [8] for a complete discussion about this topic.

2.1.4 Exact Lagrangian cobordisms

Let $(\mathbb{R}\times Y,\varpi)$ be an exact symplectic cobordism from α^+ to α^-. We call a Lagrangian submanifold in $(\mathbb{R}\times Y,\varpi)$ a Lagragian cobordism if there exists Legendrian submanifolds $\overline{\Lambda}^+$ in $(Y,\ker(\alpha^+))$ and $\overline{\Lambda}^-$ in $(Y,\ker(\alpha^-)$, and $N>0$ such that:

$$L\cap([N,+\infty)\times Y)=([N,+\infty)\times\overline{\Lambda}^+) \tag{2.8}$$

$$L\cap((-\infty,-N]\times Y)=((-\infty,-N]\times\overline{\Lambda}^-) \tag{2.9}$$

In this case, we say L is a Lagrangian cobordism from $\overline{\Lambda}^+$ to $\overline{\Lambda}^-$. If such an L is an exact Lagrangian submanifold in the exact symplectic manifold (Y,ϖ), we call it an exact Lagrangian cobordism $\overline{\Lambda}^+$ to $\overline{\Lambda}^-$.

Example: If we take a Legendrian submanifold Λ in $(Y,\ker(\alpha^-))$ then $\mathbb{R}\times\Lambda$ is an exact Lagrangian submanifold in the symplectization of (Y,α^-). It is also an exact Lagrangian cobordism (from Λ to itself) inside $(\mathbb{R}\times Y,d(f_R\alpha))$, for every positive R.

2.2 Pseudoholomorphic curves

Let (S,i) be a closed Riemann surface with boundary, with a finite set $\Gamma\subset S$. We denote $\Gamma_\partial=\partial(S)\cap\Gamma$. We consider an auxiliary metric d_S on S.

Let α be a contact form in Y and $J\in\mathcal{J}(\alpha)$. A finite energy pseudoholomorphic curve in the symplectization $(\mathbb{R}\times Y,J)$ with boundary in a Lagrangian submanifold L is a map $\widetilde{w}:(S\setminus\Gamma;\partial(S)\setminus\Gamma_\partial)\to(\mathbb{R}\times Y;L)$ satisfying:

$$d\widetilde{w}\circ i=J\circ d\widetilde{w}, \tag{2.10}$$

and

$$0<E(\widetilde{w})=\sup_{q\in\mathcal{E}}\int_{S\setminus\Gamma}\widetilde{w}^*d(q\alpha) \tag{2.11}$$

where $\mathcal{E}=\{q:\mathbb{R}\to[0,1];q'\geq 0\}$. The quantity $E(\widetilde{w})$ is called the Hofer energy, and was introduced in [28]. We write $\widetilde{w}=(s,w)\in\mathbb{R}\times Y$.

For us, it will be particularly important the case where $(S\setminus\Gamma,i)$ is biholomorphic to $(\mathbb{R}\times[0,1],i_0)$ (where i_0 is the complex structure in $\mathbb{C}$) and $L=(\mathbb{R}\times\Lambda)\cup(\mathbb{R}\times\widehat{\Lambda})$, with $\widetilde{w}(\{0\}\times\mathbb{R})\subset(\mathbb{R}\times\Lambda)$ and $\widetilde{w}(\{1\}\times\mathbb{R})\subset(\mathbb{R}\times\widehat{\Lambda})$. In this case $\widetilde{w}$ is called a pseudoholomorphic strip. By using a bi-holomorphism $\varphi : (\overline{D} \setminus \{-1,1\}, i_0) \to (\mathbb{R} \times [0,1], i_0)$ satisfying $\varphi(H^+) = \{1\} \times \mathbb{R}$ (where $H_+ \subset (S^1 \setminus \{-1,1\})$ is the northern hemisphere of S^1) and $\varphi(H^-) = \{0\} \times \mathbb{R}$ (where $H_- \subset (S^1 \setminus \{-1,1\})$ is the southern hemisphere) we can also view pseudoholomorphic strips as maps having as domain the disc with two punctures on the boundary.

For an exact symplectic cobordism $(W = \mathbb{R} \times Y, \varpi)$ from α^+ to α^-, and $\overline{J} \in \mathcal{J}(J^-, J^+)$, a finite energy pseudoholomorphic curve with boundary in a Lagrangian submanifold L is again a map $\widetilde{w} : (S \setminus \Gamma, \partial(S) \setminus \Gamma_\partial) \to (\mathbb{R} \times Y, L)$ satisfying:

$$d\widetilde{w} \circ i = \overline{J} \circ d\widetilde{w}, \tag{2.12}$$

and

$$0 < E_{\alpha^-}(\widetilde{w}) + E_c(\widetilde{w}) + E_{\alpha^+}(\widetilde{w}) < +\infty, \tag{2.13}$$

where:
$E_{\alpha^-}(\widetilde{w}) = \sup_{q \in \mathcal{E}} \int_{\widetilde{w}^{-1}W(\alpha^-)} \widetilde{w}^* d(q\alpha^-)$,
$E_{\alpha^+}(\widetilde{w}) = \sup_{q \in \mathcal{E}} \int_{\widetilde{w}^{-1}W(\alpha^+)} \widetilde{w}^* d(q\alpha^+)$,
$E_c(\widetilde{w}) = \int_{\widetilde{w}^{-1}W(\alpha^-,\alpha^+)} \widetilde{w}^* \varpi$.
These energies were also introduced in [28].

In a splitting symplectic cobordisms the definition of finite energy pseudoholorphic map is essentially the same, except that we consider a slightly modified version of energy. Instead of demanding $0 < E_-(\widetilde{w}) + E_c(\widetilde{w}) + E_+(\widetilde{w}) < +\infty$ we demand:

$$0 < E_{\alpha^-}(\widetilde{w}) + E_{\alpha^-,\alpha}(\widetilde{w}) + E_\alpha(\widetilde{w}) + E_{\alpha,\alpha^+}(\widetilde{w}) + E_{\alpha^+}(\widetilde{w}) < +\infty \tag{2.14}$$

where:
$E_\alpha(\widetilde{w}) = \sup_{q \in \mathcal{E}} \int_{\widetilde{w}^{-1}W(\alpha)} \widetilde{w}^* d(q\alpha)$,
$E_{\alpha^-,\alpha}(\widetilde{w}) = \int_{\widetilde{w}^{-1}W(\alpha^-,\alpha)} \widetilde{w}^* \varpi_R$,
$E_{\alpha,\alpha^+}(\widetilde{w} = \int_{\widetilde{w}^{-1}W(\alpha,\alpha^+)} \widetilde{w}^* \varpi_R$,
and $E_{\alpha^-}(\widetilde{w})$ and $E_{\alpha^+}(\widetilde{w})$ are as above.

2.2.1 Asymptotic behaviour of pseudoholomorphic curves

The elements of the set $\Gamma \subset S$ are called punctures of the pseudoholomorphic curve $\widetilde{w}$ in a symplectic cobordism $(\mathbb{R} \times Y, \varpi)$ from α^+ to α^-. We first divide Γ in two classes: we call the elements of Γ_∂ boundary punctures and the elements in $\Gamma \setminus \Gamma_\partial$ interior punctures. The work of Hofer [28], Hofer et al. [29] and Abbas [1] allows us do classify the punctures in five different types. The types can be as follows:

- $z \in \Gamma$ is called positive boundary puncture when $z \in \Gamma_\partial$ and $\lim_{z' \to z} s(z') = +\infty$;
- $z \in \Gamma$ is called negative boundary puncture when $z \in \Gamma_\partial$ and $\lim_{z' \to z} s(z') = -\infty$,
- $z \in \Gamma$ is called positive interior puncture when $z \in \Gamma \setminus \Gamma_\partial$ and $\lim_{z' \to z} s(z') = +\infty$,
- $z \in \Gamma$ is called negative interior puncture when $z \in \Gamma \setminus \Gamma_\partial$ and $\lim_{z' \to z} s(z') = -\infty$
- $z \in \Gamma$ is called a removable puncture if $\lim_{z' \to z} s(z')$ is a real number.

The results in [28], [29] and [1] imply that these are indeed the only possibilities for the behaviour of the real coordinate s of $\widetilde{w}$ near a puncture. In these same references, the authors also show that if $z \in \Gamma$ is a removable puncture, then the limit $\lim_{z' \to z} \widetilde{w}(z')$ exists and is unique; and more, that if we extend $\widetilde{w}$ to $(S \setminus \Gamma) \cup \{z\}$ by defining $\widetilde{w}(z) := \lim_{z' \to z} \widetilde{w}(z')$, the extension is still a C^∞ finite energy pseudoholomorphic curve. This implies that one only needs to consider the first four types of punctures.

We proceed to describe in more detail the way $\widetilde{w}$ behaves in a small neighbourhood of the puncture. For this we let $B_\delta(z)$ the ball of radius $\delta(z)$ in S centered at the puncture z, and denote by $\mathfrak{b}_\delta(z)$ the set defined as the closure $\overline{\partial(B_\delta(z)) \cap int(S)}$ of the intersection of the boundary of $B_\delta(z)$ with the interior of S. Notice that $\mathfrak{b}_\delta(z)$ is a circle or an interval, depending on whether z is an interior or a boundary puncture. The following result was also obtained in [28], [29] and [1]:

- if $z \in \Gamma$ is a positive boundary puncture, then there exists a sequence $\delta_n \to 0$ and a Reeb chord c^+ of X_{α^+} from $\overline{\Lambda}^+$ to itself, such that $w(\mathfrak{b}_{\delta_n}(z))$ converges in C^∞ to c^+ as $n \to +\infty$;
- if $z \in \Gamma$ is a negative boundary puncture, then there exists a sequence $\delta_n \to 0$ and Reeb chord c^- of X_{α^-} from $\overline{\Lambda}^-$ to itself, such that $w(\mathfrak{b}_{\delta_n}(z))$ converges in C^∞ to c^- as $n \to +\infty$,
- if $z \in \Gamma$ is a positive interior puncture, then there exists a sequence $\delta_n \to 0$ and Reeb orbit γ^+ of X_{α^+}, such that $w(\mathfrak{b}_{\delta_n}(z))$ converges in C^∞ to γ^+ as $n \to +\infty$,
- if $z \in \Gamma$ is a negative interior puncture, then there exists a sequence $\delta_n \to 0$ and Reeb orbit γ^- of X_{α^-}, such that $w(\mathfrak{b}_{\delta_n}(z))$ converges in C^∞ to γ^- as $n \to +\infty$.

Intuitively, we have that at the punctures the pseudoholomorphic curve $\widetilde{w}$ detects Reeb chords and Reeb orbits. For a boundary (interior) puncture z, if there is a subsequence δ_n such that $w(\mathfrak{b}_{\delta_n}(z))$ converges to a given Reeb chord c (orbit γ), we will say that $\widetilde{w}$ is asymptotic to this Reeb chord c (orbit γ).

We will describe this behaviour in the case where the Reeb vector fields X_{α^+} and X_{α^-} are non-degenerate, and supposing that the X_{α^+} Reeb chords of $\overline{\Lambda}^+$ to itself are transverse, and the X_{α^-} Reeb chords of $\overline{\Lambda}^-$ to itself are transverse.

In order to describe this behaviour for a boundary puncture z, we take a neighbourhood U of z that admits a holomorphic chart $\psi_U : (U \setminus \{z\}) \to \mathbb{R}^+ \times [0,1] \ \subset \mathbb{C}$, such that $\psi_U((U \cap \partial(S)) \setminus \{z\}) = \mathbb{R}^+ \times \{0\} \cup \mathbb{R}^+ \times \{1\}$. In coordinates $(r,t) \in \mathbb{R}^+ \times [0,1]$ we have $r(x) \to +\infty$ when x tends to the puncture z. Then, if z is a positive (negative) boundary puncture, there exists a sequence r_n going to $+\infty$, and a Reeb chord c^+ (c^-) from $\overline{\Lambda}^+$ ($\overline{\Lambda}^-$) to itself, such that the sequence $w^{r_n}(t) := w(r_n, t)$ of paths in Y converge in C^∞ to c^+ (c^-). In case the Reeb chord c^+ (c^-) is transverse, a more precise asymptotic behaviour is obtained. It is shown in [1], that if z is a positive boundary puncture, $\widetilde{w} \circ \psi_u^{-1}(r,t) = (s(r,t), w(r,t))$ satisfies:

- $s^r(t) = s(r,t) \to +\infty$ uniformly as map from $[0,1]$ to $\mathbb{R}$ as $r \to +\infty$,
- $w^r(t) := w(r,t)$ converges uniformly in C^∞ to a Reeb chord c of X_{α^+} from Λ^+ to $\widehat{\Lambda}^+$ as $r \to +\infty$ (where Λ^+ and $\widehat{\Lambda}^+$ denote connected components of $\overline{\Lambda}^+$);

and if z is a negative boundary puncture, $\widetilde{w} \circ \psi_u^{-1}(r,t) = (s(r,t), w(r,t))$ satisfies:

- $s^r(t) = s(r,t) \to -\infty$ uniformly as map from $[0,1]$ to $\mathbb{R}$ as $r \to +\infty$,
- $w^r(t) = w(r,t)$ converges uniformly in C^∞ to the inverse parametrization of Reeb chord c of X_{α^-} from Λ^- to $\widehat{\Lambda}^-$ as $r \to +\infty$(where Λ^- and $\widehat{\Lambda}^-$ denote connected components of $\overline{\Lambda}^-$).

We discuss now the case of z being an interior puncture; we pick a neighbourhood U of z and a holomorphic chart $\psi_U : (U,z) \to (\mathbb{D}, 0)$. Using polar coordinates $(r,t) \in (0,+\infty) \times S^1$ we can write $x \in (\mathbb{D} \setminus 0)$ as $x = e^{-r}t$. With this notation, it is shown in [28] [29], that if z is a positive interior puncture, $\widetilde{w} \circ \psi_U^{-1}(r,t) = (s(r,t), w(r,t))$ satisfies:

- $s^r(t) = s(r,t) \to +\infty$ uniformly as map from $[0,1]$ to $\mathbb{R}$ as $r \to +\infty$,
- $w^r(t) = w(r,t)$ converges uniformly in C^∞ to a Reeb orbit γ of X_{α^+};

and if z is a negative interior puncture, $\widetilde{w} \circ \psi_u^{-1}(r,t) = (s(r,t), w(r,t))$ satisfies:

- $s^r(t) = s(r,t) \to -\infty$ uniformly as map from $[0,1]$ to $\mathbb{R}$ as $r \to +\infty$
- $w^r(t) = w(r,t)$ converges uniformly in C^∞ to a Reeb orbit γ of $-X_{\alpha^-}$ as $r \to +\infty$.

Remark: actually the convergence of pseudoholomorphic curves near punctures to Reeb orbits and Reeb chords is of exponential nature; the asymptotic formulas that describe this convergence were obtained [29] and [1] and we present them in the appendix A. Such formulas are necessary for the Fredholm theory that gives the dimension of the space of pseudoholomorphic curves with fixed asymptotic data.

The discussion above shows that near punctures the finite energy pseudoholomorphic curves detect Reeb orbits and Reeb chords. It is exactly this behaviour that makes these objects useful for the study of dynamics of Reeb vector fields. When a pseudoholomorphic curve approaches a Reeb orbit (or a Reeb chord) near a puncture we say that the pseudoholomorphic curve is asymptotic to the Reeb orbit (or the Reeb chord) at that puncture.

Fact: as a consequence of the exactness of the symplectic cobordisms and the Lagrangian submanifolds considered above we obtain from a simple calculation that the energy $E(\widetilde{w})$ of $\widetilde{w}$ satisfies $E(\widetilde{w}) \leq 5A(\widetilde{w})$ where $A(\widetilde{w})$ is the sum of the action of the Reeb orbits and Reeb chords detected by the punctures of $\widetilde{w}$ counted with multiplicity.

2.3 Moduli spaces of pseudoholomorphic curves

We will consider now different types of moduli spaces of pseudoholomorphic curves in symplectic cobordisms and in symplectizations. Although symplectizations are a particular case of symplectic cobordisms, the fact that we only consider $\mathbb{R}$-invariant almost complex structures on symplectictizations makes this case present particular properties. For this reason we will consider it separately. We keep the notation of the previous sections.

The first type we consider, are the moduli spaces denoted by $\mathcal{M}(\gamma, \gamma_1', ..., \gamma_m'; J)$ (where $J \in \mathcal{J}(\alpha)$) whose elements are equivalence classes of genus 0 finite energy pseudoholomorphic curves without boundary, modulo biholomorphic reparametrisations of the domain, with one positive interior puncture asymptotic to a non-degenerate Reeb orbit γ of X_α and a finite number m of negative interior punctures asymptotic to non-degenerate orbits $\gamma_1', ..., \gamma_m'$ of X_α. It is well known (see for instance [6] and [14]) that the linearization $D\overline{\partial}_J$ at any element $\mathcal{M}(\gamma, \gamma_1', ..., \gamma_m'; J)$ is a Fredholm map; we remark that this property is actually valid for more general moduli spaces of curves with prescribed asymptotic behaviour. In appendix B, we give a short exposition of the linear Fredholm theory of $D\overline{\partial}_J$. Lastly, we denote by $\mathcal{M}^k(\gamma, \gamma_1', ..., \gamma_m'; \overline{J})$ the moduli space of finite energy pseudoholomorphic curves in $\mathcal{M}(\gamma, \gamma_1', ..., \gamma_m'; \overline{J})$ that have Fredholm index equal to k.

Secondly, we consider the moduli spaces denoted by $\mathcal{M}(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})$ (where $J \in \mathcal{J}(\alpha)$ and $\overline{\Lambda}$ is Legendrian in $(Y, \ker \alpha)$) composed by equivalence classes of genus 0 finite energy pseudoholomorphic curves, modulo biholomorphic reparametrisations of the domain, with one positive boundary puncture asymptotic to a transverse Reeb chord c of X_α going from $\overline{\Lambda}$ to itself, a finite number n of negative boundary punctures asymptotic to transverse Reeb chords $c_1, ..., c_n$ of X_α from $\overline{\Lambda}$ to itself, and a finite number m of negative interior punctures asymptotic to non-degenerate orbits $\gamma_1', ..., \gamma_m'$, and whose boundary is in the cylinder $\mathbb{R} \times \overline{\Lambda}$. Again, it is well known that the linearization $D\overline{\partial}_J$ at any element $\mathcal{M}(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})$ is a Fredholm map. We again refer to the appendix B for a short exposition on the linear Fredholm theory of $D\overline{\partial}_J$. Lastly, we denote by $\mathcal{M}^k(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})$ the moduli space of finite energy pseudoholomorphic curves in $\mathcal{M}(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})$ that have Fredholm index equal to k.

In the case of moduli spaces of curves in a symplectization with a $\mathbb{R}$-invariant almost complex structure J we will need to introduce one more class of moduli spaces. The reason for this is that, because of the $\mathbb{R}$-invariance of J, it is impossible to expect that the moduli spaces introduced above can be compact or admit a reasonable compactification similar to the one obtained by Gromov for moduli spaces in compact symplectic manifolds. There is however a natural notion to consider; because of the $\mathbb{R}$-invariance of J there is an $\mathbb{R}$-action on the spaces $\mathcal{M}^k(\gamma, \gamma_1', ..., \gamma_m'; J)$ and $\mathcal{M}^k(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})$. With this, we denote by $\widetilde{\mathcal{M}}^k(\gamma, \gamma_1', ..., \gamma_m'; J)$ and $\widetilde{\mathcal{M}}^k(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})$ the two quotient moduli spaces $\mathcal{M}^k(\gamma, \gamma_1', ..., \gamma_m'; J)/\mathbb{R}$ and $\mathcal{M}^k(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})/\mathbb{R}$, where the quotient is defined using the just mentioned $\mathbb{R}$-action.

We will now treat the case of symplectic cobordisms. Firstly, we consider moduli spaces denoted by $\mathcal{M}(\gamma, \gamma_1', ..., \gamma_m'; \overline{J})$ whose elements are equivalence classes of genus 0 finite energy pseudoholomorphic curves, modulo biholomorphic reparametrisations of the domain, with one positive interior puncture asymptotic to a non-degenerate Reeb orbit γ of X_{α^+} and a finite number m of negative interior punctures asymptotic to non-degenerate orbits $\gamma_1', ..., \gamma_m'$ of X_{α^-} . Again the linearization $D\overline{\partial}_{\overline{J}}$ at any element $\mathcal{M}(\gamma, \gamma_1', ..., \gamma_m'; \overline{J})$ is a Fredholm map; see appendix B. Therefore it makes sense to define

$\mathcal{M}^k(\gamma, \gamma'_1, ..., \gamma'_m; \overline{J})$ as the moduli space of finite energy pseudoholomorphic curves in $\mathcal{M}(\gamma, \gamma'_1, ..., \gamma'_m; \overline{J})$ that have Fredholm index equal to k.

The second type, are the moduli spaces denoted by $\mathcal{M}(c, c_1, ..., c_n, \gamma'_1, ..., \gamma'_m; \overline{J}, \overline{L})$ whose elements are equivalence classes of genus 0 finite energy pseudoholomorphic curves modulo biholomorphic reparametrisations of the domain, with one positive boundary puncture asymptotic to a transverse Reeb chord c of X_{α^+} going from $\overline{\Lambda}^+$ to itself, a finite number n of negative boundary punctures asymptotic to transverse Reeb chords $c_1, ..., c_n$ of X_{α^-} from $\overline{\Lambda}^-$ to itself, and a finite number m of negative interior punctures asymptotic to non-degenerate orbits $\gamma'_1, ..., \gamma'_m$ of X_{α^-}, and whose boundary lies in the Lagrangian cobordism $\overline{L}$ from $\overline{\Lambda}^+$ to $\overline{\Lambda}^-$. The linearization $D\overline{\partial}_{\overline{J}}$ at any element $\mathcal{M}(c, c_1, ..., c_n, \gamma'_1, ..., \gamma'_m; \overline{J})$ is a Fredholm map; see appendix B. Therefore, it makes sense to consider the moduli space $\mathcal{M}^k(c, c_1, ..., c_n, \gamma'_1, ..., \gamma'_m; \overline{J}, \overline{L})$ of finite energy pseudoholomorphic curves in $\mathcal{M}(c, c_1, ..., c_n, \gamma'_1, ..., \gamma'_m; \overline{J}, \overline{L})$ that have Fredholm index equal to k.

One would like to conclude that the dimension of any connected component of a moduli space equals the Fredholm index of an element belonging to this connected components. However, this is not always the case as problems might occur when multiply covered pseudoholomorphic curves appear.

We will lastly describe a particular case of moduli spaces which will be of crucial importance in this thesis, for it is the main ingredient to the construction of the the strip Legendrian contact homology. Let Λ and $\widehat{\Lambda}$ be two different connected components of $\overline{\Lambda}$. Given two Reeb chords $c, c' \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha)$, and an almost complex structure $J \in \mathcal{J}(\alpha)$, we consider the moduli space $\mathcal{M}(c, c'; J, \Lambda, \widehat{\Lambda})$ of equivalence classes finite energy pseudoholomorphic strips $\widetilde{w} : (\overline{D} \setminus \{-1, 1\}, i_0) \to (\mathbb{R} \times Y, J)$ satisfying:

- 1 is a positive boundary puncture, and $\widetilde{w}$ is asymptotic to c at 1,
- -1 is a negative boundary puncture, and $\widetilde{w}$ is asymptotic to c' at -1,
- $\widetilde{w}(H_-) \subset \mathbb{R} \times \Lambda$,
- $\widetilde{w}(H_+) \subset \mathbb{R} \times \widehat{\Lambda}$.

In many cases, we will write $\mathcal{M}(c, c'; J)$ and omit the Lagrangians where the boundary of the curves lie, because this data is already present in the fact that the Reeb chords c and c' are in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha)$.

We will likewise need moduli spaces of strips in cobordisms. We denote Λ^+ and $\widehat{\Lambda^+}$ two different connected components of $\overline{\Lambda}^+$, and Λ^- and $\widehat{\Lambda}^-$ two different connected components of $\overline{\Lambda^-}$. Likewise we let L and $\widehat{L}$ be exact Lagrangian cobordisms from, respectively, Λ^+ to Λ^-, and $\widehat{\Lambda}^+$ to $\widehat{\Lambda}^-$. Given two Reeb chords $c^+ \in \mathcal{T}_{\Lambda^+ \to \widehat{\Lambda^+}}(\alpha^+)$ and $c^- \in \mathcal{T}_{\Lambda^- \to \widehat{\Lambda^-}}(\alpha^-)$, and an almost complex structure $\overline{J} \in \mathcal{J}(J^-, J^+)$, we consider the moduli space $\mathcal{M}(c^+, c^-; \overline{J}, L, \widehat{L})$, modulo reparametrizations, of finite energy pseudo-holomorphic strips $\widetilde{w} : (\overline{D} \setminus \{-1, 1\}, i_0) \to (\mathbb{R} \times Y, \overline{J})$ satisfying:

- 1 is a positive boundary puncture, and $\widetilde{w}$ is asymptotic to c^+ at 1,
- -1 is a negative boundary puncture, and $\widetilde{w}$ is asymptotic to c^- at -1,
- $\widetilde{w}(H_-) \subset L$,
- $\widetilde{w}(H_+) \subset \widehat{L}$.

When there is no possibility of confusion of which are the exact Lagrangian cobordisms L and $\widehat{L}$ we omit them from the notation and write $\mathcal{M}(c^+, c^-; \overline{J})$ instead of $\mathcal{M}(c^+, c^-; \overline{J}, L, \widehat{L})$.

2.4 Compactness of the space of pseudoholomorphic curves

In order to study compactness properties of spaces of pseudoholomorphic curves, we have to introduce pseudoholomorphic buildings. First, for $j \in \{1, ..., l\}$, consider a collection $(\mathbb{R} \times Y, \varpi_j, \overline{J}_j, \alpha_j^+, \alpha_j^-, L_j, \widehat{L}_j, \Lambda_j^+, \Lambda_j^-, \widehat{\Lambda}_j^+, \widehat{\Lambda}_j^-)$, where:

- $(\mathbb{R} \times Y, \varpi_j)$ is an exact symplectic cobordism from α_j^+ to α_j^-,
- L_j is an exact Lagrangian cobordism from the Legendrian curve Λ_j^+ (in $(Y, \ker(\alpha_j^+))$ to the Legendrian curve Λ_j^- (in $(Y, \ker(\alpha_j^-)$)
- $\widehat{L}_j$ is an exact Lagrangian cobordism from the Legendrian curve $\widehat{\Lambda}_j^+$ (in $(Y, \ker(\alpha_j^+))$ to the Legendrian curve $\widehat{\Lambda}_j^-$ (in $(Y, \ker(\alpha_j^-)$,
- $\overline{J}_j$ is positively asymptotic to $J_j^+ \in \mathcal{J}(\alpha_j^+)$ and negatively asymptotic to $J_j^- \in \mathcal{J}(\alpha_j^-)$,
- $\alpha_{j+1}^+ = \alpha_j^-$, $\Lambda_{j+1}^+ = \Lambda_j^-$, $J_{j+1}^+ = J_j^-$, $\widehat{\Lambda}_{j+1}^+ = \widehat{\Lambda}_j^-$

Now, for $j \in \{1, ..., l\}$ consider a collection of, possibly disconnected, (equivalence classes modulo biholomorphic reparametrizations of the domain) finite energy pseudoholomorphic curves $(S_j, i_j, \Gamma_j, \widetilde{w}^j)$ in $(\mathbb{R} \times Y, \overline{J}_j)$ with boundary in the union $L_j \cup \widehat{L}_j$, and a division $\Gamma_j = \Gamma_j^+ \cup \Gamma_j^-$ in positive and negative punctures. We assume that there exist bijections $\mathfrak{G}_j : \Gamma_j^- \to \Gamma_{j+1}^+$ for $1 \leq j \leq l-1$. Some levels $(\mathbb{R} \times Y, \varpi_j, \overline{J}_j, L_j, \widehat{L}_j)$

might be symplectizations with the Lagrangian cobordisms being trivial cylinders over Legendrian submanifolds, and $\mathbb{R}$-invariant almost complex structures: on such levels we demand that there is some connected component of $\widetilde{w}^j$ which is neither a trivial cylinder over a Reeb orbit or a trivial strip over a Reeb chord. In these symplectization levels we will consider the element $\widetilde{w}^j$ as representing an equivalence class of pseudoholomorphic curves, where we consider two curves to be equivalent if one is the $\mathbb{R}$-translation of the other. We will say that the data $(S_j, i_j, \Gamma_j, \widetilde{w}^j, \mathfrak{G}_j)$ defines a pseudoholomorphic building $\mathfrak{B}$, if the punctures identified by the maps $\mathfrak{G}_j$ are asymptotic to the same Reeb orbits or Reeb chords.

The main motivation behind the introduction of pseudoholomorphic buildings is that they are needed in order to compactify moduli spaces of pseudoholomorphic curves. The reason for that, is the fact already observed by Gromov, that sequences of pseudoholomorphic curves might not converge to a single pseudoholomorphic curve, but to a suitable collection pseudoholomorphic curves.

As we will study only limits of pseudoholomorphic strips, we can restrict our attention to a special kind of buildings that we will call *pseudoholomorphic trees with one principal branch*. To define this type of buildings we first need to introduce some preliminary concepts. Given a pseudoholomorphic building $\mathfrak{B}$ formed by curves $(S_j, i_j, \Gamma_j, \widetilde{w}^j, \mathfrak{G}_j)$ where $j \in \{1, ..., l\}$ and building $\mathfrak{B}'$ formed by curves $(S'_j, i'_j, \Gamma'_j, \widetilde{w}'^j, \mathfrak{G}'_j)$ for $l_{min} \leq j \leq l_{max}$ (where $l_{min} \geq 1$ and $l_{max} \leq l$) we say that $\mathfrak{B}'$ is a sub building of $\mathfrak{B}$ if the the Riemman surfaces (S'_j, i'_j) are bi-holomorphic to subsets of the Riemann surfaces (S_j, i_j), the maps $\widetilde{w}'^j$ coincide with the restrictions of $\widetilde{w}^j$ to (S'_j, i'_j), the punctures Γ'_j coincide with the intersection of Γ_j and S'_j, and the maps $\mathfrak{G}'_j$ are restrictions of $\mathfrak{G}_j$ to Γ'_j.

Given a pseudoholomorphic building $\mathfrak{B}$ formed by curves $(S_j, i_j, \Gamma_j, \widetilde{w}^j, \mathfrak{G}_j)$ for $j \in \{1, ..., l\}$ and a puncture $z \in \Gamma^-_{j_0}$ we will define a pseudoholomorphic building $\mathfrak{B}_z$ associated to z formed by a subcollection of curves of $(S_j, i_j, \Gamma_j, \widetilde{w}^j, \mathfrak{G}_j)$ for $k \in \{j_0 + 1, ..., l_z\}$. Let $(\widetilde{w}^{j_0+1}_z, S_{j_0+1,z})$ be the unique connected pseudoholomorphic curve belonging to $\widetilde{w}^{j_0+1}$ that contains the puncture $\mathfrak{G}_{j_0}(z)$, and take $\Gamma^-_{j_0+1}(z)$ to be the collection of negative punctures of $\widetilde{w}^{j_0+1}_z$ in $S_{j_0+1,z}$. Now define $(\widetilde{w}^{j_0+2}_z, S_{j_0+2,z})$ to be the smallest collection of connected pseudoholomorphic curves containing the punctures $\mathfrak{G}_{j_0+1}(\Gamma^-_{j_0}(z))$ and define $\Gamma^-_{j_0+2}(z)$ to be the negative punctures of $\widetilde{w}^{j_0+2}_z$. We can continue following this same recipe until we reach l or until $\widetilde{w}^{l_z}_z$ has no negative punctures, to obtain a pseudoholomorphic building $\mathfrak{B}_z$ formed by the curves $(S_{j,z}, j_j, \widetilde{w}^j_z)$ for $j \in \{j_0, ..., l_z\}$ and their punctures. We will say that $\mathfrak{B}_z$ is a *contractible building* if for l_z the curve $\widetilde{w}^l_z$ has no negative punctures.

We say that a pseudoholomorphic building $\mathfrak{B}$ formed by curves $(S_j, i_j, \Gamma_j, \widetilde{w}^j, \mathfrak{G}_j)$ for $j \in \{1, ..., l\}$ is a tree with one principal branch, if there exists for each $j \in \{1, ..., l-1\}$ one special puncture $z_j \in \Gamma_j^-$ such that:

- for every $z \neq z_j \in \Gamma_j^-$ the building $\mathfrak{B}_z$, as defined above, is contractible,
- Γ_l^- contains only one element, and Γ_1^+ contains only one element.

The general SFT compactness theorem of [8] (see also [3] for a complete exposition for the case of pseudoholomorphic curves with boundary on a Lagrangian submanifold) asserts that a sequence of (equivalence classes modulo reparametrization of) pseudo-holomorphic curves with a global bound on the energy for all elements of the sequence converges to a pseudoholomorphic building. Here we will need to study only the special case where the elements of our sequence are pseudoholomorphic strips; in this case one can describe precisely the possibilities of the limit pseudoholomorphic building. We will therefore state only the following propositions which deal with this particular case, and which follow directly from the SFT compactness theorem. Before stating them we introduce the following notation: we will denote by $\Sigma_{\Lambda \to \widehat{\Lambda}}$ the set of homotopy classes of paths starting at Λ and ending at $\widehat{\Lambda}$. We begin by stating the compactness result in the case of symplectizations.

Proposition 2.1. *Let α be a contact form associated to a contact 3-manifold (Y, ξ), $J \in \mathcal{J}(\alpha)$ and Λ and $\widehat{\Lambda}$ be a pair of disjoint Legendrian knots in (Y, ξ). Assume that all contractible Reeb orbits of α are non-degenerate, that all contractible Reeb chords going from Λ to itself are transverse and that all contractible Reeb chords going from $\widehat{\Lambda}$ to itself are transverse. Let c and c' be Reeb chords in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha)$ both belonging to the same homotopy class $\rho \in \Sigma_{\Lambda \to \widehat{\Lambda}}$, and assume that all Reeb chords in ρ are transverse. Let $\widetilde{w}_n$ be a sequence of elements in the moduli space $\widetilde{\mathcal{M}}(c, c'; J)$. Then there exists a subsequence of $\widetilde{w}_n$ which converges in the SFT sense to a pseudoholomorphic building $\widetilde{w}$ which has the structure of a tree with one principal branch. More precisely, all the levels $\widetilde{w}^j$ for $j \in \{1, ..., l\}$ of the building $\widetilde{w}$ are equivalence classes of finite energy pseudoholomorphic curves, modulo the $\mathbb{R}$ action, in $(\mathbb{R} \times Y, J)$ that satisfy:*

- *the special negative punctures $z_j \in \Gamma_j^-$ are all asymptotic to Reeb chords c_j which are in the homotopy class ρ,*
- *the unique positive puncture in Γ_1^+ is asymptotic to c and the unique negative puncture in Γ_l^- is asymptotic to c',*
- *for all punctures in $z \in \Gamma_j^-$ the action of the corresponding Reeb chord or Reeb orbit detected by z is smaller then the action $A(c)$.*

We now proceed to state a version of the compactness theorem for a special type of symplectic cobordisms.

Proposition 2.2. *Let α^+ and α^- be contact forms on a 3-manifold Y, $J^+ \in \mathcal{J}(\alpha^+)$ and $J^- \in \mathcal{J}(\alpha^-)$, Λ^+ and $\widehat{\Lambda}^+$ be a pair of disjoint Legendrian knots in $(Y, \ker \alpha^+)$, and Λ^- and $\widehat{\Lambda}^-$ be a pair of disjoint Legendrian knots in $(Y, \ker \alpha^-)$. Assume that all contractible Reeb orbits of α^+ and α^+ are non-degenerate, that all contractible Reeb chords of α^+ (α^-) going from Λ^+ (Λ^-) to itself are transverse, and that all contractible Reeb chords of α^+ (α^-) going from $\widehat{\Lambda}^+$ ($\widehat{\Lambda}^-$) to itself are transverse. Let $(\mathbb{R} \times Y, \varpi)$ be an exact symplectic cobordism from α^+ to α^-, $\overline{J} \in \mathcal{J}(J^-, J^+)$, L be an exact Lagrangian cobordism from Λ^+ to Λ^-, and $\widehat{L}$ be an exact Lagrangian cobordism from $\widehat{\Lambda}^+$ to $\widehat{\Lambda}^-$, with L and $\widehat{L}$ disjoint. Let $c^+ \in \mathcal{T}_{\Lambda^+ \to \widehat{\Lambda}^+}(\alpha^+)$ and $c^- \in \mathcal{T}_{\Lambda^- \to \widehat{\Lambda}^-}(\alpha^-)$ be Reeb chords. We assume that all the Reeb chords in $\mathcal{T}_{\Lambda^+ \to \widehat{\Lambda}^+}(\alpha^+)$ in same homotopy class of c^+ are transverse, and that the same is valid for all the Reeb chords in $\mathcal{T}_{\Lambda^- \to \widehat{\Lambda}^-}(\alpha^-)$ belonging to same homotopy class of c^-. Let $\widetilde{w}_n$ be a sequence of elements in the moduli space $\mathcal{M}(c^+, c^-; J)$ of strips with one boundary component on L and one boundary component $\widehat{L}$. Then there exists a subsequence of $\widetilde{w}_n$ which converges in the SFT sense to a pseudoholomorphic building $\widetilde{w}$ which has the structure of a tree with one principal branch. More precisely, there exist $l^+ \geq 1$ such that levels $\widetilde{w}^j$ for $j \in \{1, ..., l\}$ of the building $\widetilde{w}$ are finite energy pseudoholomorphic curves that satisfy:*

- *$\widetilde{w}^{l^+}$ is a pseudoholomorphic curve in $(\mathbb{R} \times Y, \overline{J})$ with boundary in L and $\widehat{L}$,*
- *for $j < l^+$, $\widetilde{w}^j$ are pseudoholomorphic curves in $(\mathbb{R} \times Y, J^+)$ with boundary on the union of $\mathbb{R} \times \Lambda^+$ and $\mathbb{R} \times \widehat{\Lambda}^+$,*
- *for $j > l^-$, $\widetilde{w}^j$ are pseudoholomorphic curves in $(\mathbb{R} \times Y, J^-)$ with boundary on the union of $\mathbb{R} \times \Lambda^-$ and $\mathbb{R} \times \widehat{\Lambda}^-$,*
- *the unique positive puncture in Γ_1^+ is asymptotic to c^+ and the unique negative puncture in Γ_l^- is asymptotic to c^-,*
- *for all punctures in $z \in \Gamma_j^-$ the action of the corresponding the Reeb chord or Reeb orbit detected by z is smaller then the action $A(c^+)$.*

Moreover, if L and $\widehat{L}$ are cylinders and we denote by ρ the homotopy class in $\Sigma_{\Lambda \to \widehat{\Lambda}}$ to which c^+ and c^- belong, we have:

- *the special negative punctures $z_j \in \Gamma_j^-$ are all asymptotic to Reeb chords c_j which are in the homotopy class ρ.*

Lastly we define a version of the theorem for a splitting family of pseudo holomorphic curves.

Proposition 2.3. *Let α^+ and α^- be contact forms on the 3-manifold Y, $J^+ \in \mathcal{J}(\alpha^+)$ and $J^- \in \mathcal{J}(\alpha^-)$, Λ and $\widehat{\Lambda}$ be a pair of disjoint Legendrian knots in $(Y, \ker \alpha^+)$, such that Λ and $\widehat{\Lambda}$ are also Legendrian knots in $(Y, \ker \alpha^-)$. Assume that all contractible Reeb orbits of α^+ and α^- are non-degenerate, that all contractible Reeb chords of α^+ and α^- going from Λ to itself are transverse, and that all contractible Reeb chords of α^+ and α^- going from $\widehat{\Lambda}$ to itself are transverse. Let $(\mathbb{R} \times Y, \varpi_R)$, $R \in (0, +\infty)$, be a splitting family of exact symplectic cobordisms, from α^+ to α^- along a contact form α in Y, with $\widetilde{J}_R \in \mathcal{J}(J^-, J^+)$ for all $R \in (0, +\infty)$ and coinciding with $J \in \mathcal{J}(\alpha)$ in the region $W(\alpha)$. We assume that α is non-degenerate, and that all contractible Reeb chords of α going from Λ to itself are transverse, and that all contractible Reeb chords of α going from $\widehat{\Lambda}$ to itself are transverse. Assume that for all $R \in (0, +\infty)$, $L = \mathbb{R} \times \Lambda$ is an exact Lagrangian cobordism from Λ (seen as a Legendrian submanifold in $(Y, \ker \alpha^+)$) to Λ (seen as a Legendrian submanifold in $(Y, \ker \alpha^-)$), and that $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ is an exact Lagrangian cobordism from $\widehat{\Lambda}$ to $\widehat{\Lambda}$. For a sequence $R_n \to +\infty$, let $c_n^+ \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha^+)$ and $c_n^- \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha^-)$ be sequences of Reeb chords belonging to the same homotopy class $\rho \in \Sigma_{\Lambda \to \widehat{\Lambda}}$, and such that there exists a constant $\mathfrak{C}$ with $\mathfrak{C} > A(c_n^+) > A(c_n^-)$; let $\widetilde{w}_n$ be a sequence of elements in the moduli space $\mathcal{M}(c_n^+, c_n^-; J_{R_n})$ of strips with one boundary component on L and one boundary component on $\widehat{L}$. We assume that all elements of $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha)$, $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha^+)$ and $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha^-)$ which are in the homotopy class ρ are transverse. Then there exists a subsequence of $\widetilde{w}_n$ which converges in the SFT sense to a pseudoholomorphic building $\widetilde{w}$ which has the structure of a tree with one principal branch. More precisely, there exist numbers l_α^{min} and l_α^{max}, such that $l > l_\alpha^{max} \geq l_\alpha^{min} > 1$ and such that the levels $\widetilde{w}^j$ for $j \in \{1, ..., l\}$ of the building $\widetilde{w}$ are finite energy pseudoholomorphic curves that satisfy:*

- *for $j \in \{l_\alpha^{min}, ..., l_\alpha^{max}\}$ the curve $\widetilde{w}^j$ is a pseudoholomorphic curve in the symplectization $(\mathbb{R} \times Y, J)$ of α with boundary in $\mathbb{R} \times \Lambda$ and $\mathbb{R} \times \widehat{\Lambda}$,*
- *for $j = l^{min} - 1$, $\widetilde{w}^j$ is a pseudoholomorphic curve in a cobordism $(\mathbb{R} \times Y, d\varsigma^+, \overline{J}^+)$ from α^+ to α with boundary on the union of $\mathbb{R} \times \Lambda$ and $\mathbb{R} \times \widehat{\Lambda}$,*
- *for $j = l^{max} + 1$, $\widetilde{w}^j$ is a pseudoholomorphic curve in a cobordism from $(\mathbb{R} \times Y, d\varsigma^-, \overline{J}^-)$ from α to α^- with boundary on the union of $\mathbb{R} \times \Lambda$ and $\mathbb{R} \times \widehat{\Lambda}$,*
- *for $j < l^{min} - 1$, $\widetilde{w}^j$ are pseudoholomorphic curves in $(\mathbb{R} \times Y, J^+)$ with boundary on the union of $\mathbb{R} \times \Lambda$ and f $\mathbb{R} \times \widehat{\Lambda}$,*
- *for $j > l^{max} + 1$, $\widetilde{w}^j$ are pseudoholomorphic curves in $(\mathbb{R} \times Y, J^-)$ with boundary on the union of $\mathbb{R} \times \Lambda$ and f $\mathbb{R} \times \widehat{\Lambda}$,*

- *the unique positive puncture in* Γ_1^+ *is asymptotic to a Reeb chord* $c^+ \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha^+)$ *and the unique negative puncture in* Γ_l^- *is asymptotic to* $c^- \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha^-)$ *both in the homotopy class* ρ*,*
- *the special negative punctures* $z_j \in \Gamma_j^-$ *are all asymptotic to Reeb chords* c_j *which are in the homotopy class* ρ*,*
- *for all punctures in* $z \in \Gamma_j^-$ *the action of the corresponding Reeb chord or Reeb orbit detected by* z *is smaller then the action* $A(c^+)$*.*

2.4.1 Compactfied moduli spaces

Using the propositions stated above one can compactify the moduli spaces of equivalence classes of pseudoholomorphic curves. These compactified moduli spaces involve pseudoholomorphic curves, but also pseudoholomorphic buildings which appear as limits in (the sense defined in [8]) of pseudoholomorphic curves. We will use this compactification for moduli spaces of curves in symplectizations and in exact symplectic cobordisms. We begin with the case of symplectizations.

Take a non-degenerate contact form α associated to (Y, ξ), $J \in \mathcal{J}(\alpha)$, and a pair of disjoint Legendrians $(\Lambda, \widehat{\Lambda})$ such that all the Reeb chords from Λ to itself, from $\widehat{\Lambda}$ to itself and from Λ to $\widehat{\Lambda}$ are transverse. Consider the moduli spaces $\widetilde{\mathcal{M}}^k(c, c'; J)$, as introduced in section 2.3. With this, we can define the spaces and $\overline{\mathcal{M}}^k(c, c'; J)$, which are the compactifications of $\widetilde{\mathcal{M}}^k(c, c'; J)$, in the sense defined in [8]. This compactification is constructed through use of Proposition 2.1. It contains all the elements of $\widetilde{\mathcal{M}}^k(c, c'; J)$, but also buildings which have the structure of a tree with one principal branch, which are described in Proposition 2.1. In [8], these compactified moduli spaces are topologized and shown to be compact Hausdorff spaces.

Considering now the case of exact symplectic cobordisms we let $(\mathbb{R} \times Y, \varpi)$ be an exact symplectic cobordism from α^+ and α^-, where both α^+ and α^- are non-degenerate contact forms in Y. We assume that all Reeb chords from $\overline{\Lambda}^+$ to itself are transverse, and that the same is true for all Reeb chords from $\overline{\Lambda}^-$ to itself. Taking a cylindrical almost complex structure $\overline{J} \in \mathcal{J}(J^-, J^+)$, we consider moduli spaces $\mathcal{M}^k(c^+, c^-; \overline{J})$ as defined in 2.4. We define the spaces $\overline{\mathcal{M}}^k(c^+, c^-; \overline{J})$, as the compactifications in the sense of [8], of the spaces $\mathcal{M}^k(c^+, c^-; \overline{J})$. In this case, the main tool used to define the compactification is Proposition 2.2. Again, this space contains all the elements of $\mathcal{M}^k(c^+, c^+; \overline{J})$, plus buildings which have the structure of a tree with one principal branch, which are described in Proposition 2.2. Again in [8], these compactified moduli spaces are topologized and shown to be compact Hausdorff spaces.

2.5 The gluing theorem

In this section we recall the gluing theorem for the special kinds of pseudoholomorphic buildings that will appear in this thesis. The general gluing theory needed for all the SFT-invariants, such as the full contact homology, is still subject of intense research. However, in the case treated in this thesis, we do not need this machinery. The reason for that is that all the curves that appear in our construction of the strip Legendrian contact homology and its cobordism maps are somewhere injective pseudoholomorphic curves this gluing theorem can be obtained using, essentially, the same methods needed to prove a similar statement in Lagrangian Floer homology.

The gluing theorem allows us to glue the levels of a holomorphic building to obtain a pseudoholomorphic curve; it can be seen as the reverse of SFT-compactness. This gluing is possible when the levels of the building are Fredholm regular. Like in previous sections, we will deal separately with the case where all the levels of the building are symplectizations and the case where one of the levels sits in a cobordism. We begin with the case of a symplectization.

In conformity with the previous section, we will consider a contact form α associated to (Y, ξ). Let c, $\check{c}$ and c' be transverse Reeb chords in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha)$. Let $J \in \mathcal{J}(\alpha)$, and assume that for every element of the moduli space $\widetilde{\mathcal{M}}^2(c, c'; J, \Lambda, \widehat{\Lambda})$ the linerized Cauchy-Riemman operator $D\partial_J$ over this element is surjective. In this case one can use the infinite dimensional implicit function theorem to conclude that $\widetilde{\mathcal{M}}^2(c, c'; J, \Lambda, \widehat{\Lambda})$ is a one dimensional manifold. Let $\widetilde{w}^1 \in \widetilde{\mathcal{M}}^1(c, \check{c}; J, \Lambda, \widehat{\Lambda})$ and $\widetilde{w}^2 \in \widetilde{\mathcal{M}}^1(\check{c}, c'; J, \Lambda, \widehat{\Lambda})$, and denote by $\widetilde{w}$ the 2-level building which has $\widetilde{w}^1$ as top level and $\widetilde{w}^2$ as bottom level. We then have:

Theorem 2.4. *Assume that the linearized Cauchy-Riemman operator is surjective at both $\widetilde{w}^1$ and $\widetilde{w}^2$. Then, there exists an embedding $\Psi : [0, +\infty) \to \overline{\mathcal{M}}^2(c, c'; J, \Lambda, \widehat{\Lambda})$ such that:*

- $\Psi(0) = \widetilde{w}$,
- $\Psi(t) \in \widetilde{\mathcal{M}}^2(c, c'; J, \Lambda, \widehat{\Lambda})$ *for every* $t \in (0, +\infty)$
- *the map Ψ is a homeomorphism from $[0, +\infty)$ to a neighbourhood of $\widetilde{w}$ in $\overline{\mathcal{M}}^2(c, c'; J, \Lambda, \widehat{\Lambda})$.*

Moreover, if $\widetilde{w}_n$ is a sequence of elements of $\mathcal{M}^2(c, c'; J, \Lambda, \widehat{\Lambda})$ converging to $\widetilde{w}$, then there exists n_0 such that $\widetilde{w}_n \in \Psi([0, 1])$ for all $n \geq n_0$.

In words, the gluing theorem says that provided the levels of the building $\widetilde{w}$ are regular, then $\widetilde{w}$ is in the boundary of $\overline{\mathcal{M}}^2(c, c'; J, \Lambda, \widehat{\Lambda})$.

We now proceed to state a version of the gluing theorem for buildings involving a cobordism. We will keep the notations in Proposition 2.2, with α^+ and α^- being contact forms associated to (Y, ξ). We take c^+ and $\check{c}^+$ to be transverse Reeb chords in $\mathcal{T}_{\Lambda^+ \to \widehat{\Lambda}^+}(\alpha^+)$, and c^- and $\check{c}^-$ to be transverse Reeb chords in $\mathcal{T}_{\Lambda^- \to \widehat{\Lambda}^-}(\alpha^-)$. We assume that all elements of $\mathcal{M}^1(c^+, c^-; \overline{J}, L, \widehat{L})$ are Fredholm regular; i.e, the linerized Cauchy-Riemman operator $D\partial_{\overline{J}}$ is surjective at all elements of $\mathcal{M}^1(c^+, c^-; \overline{J}, L, \widehat{L})$. Let $\widetilde{w}^1_+ \in \widetilde{\mathcal{M}}^1(c^+, \check{c}^+; J^+, \Lambda^+, \widehat{\Lambda}^+)$ and $\widetilde{w}^2_+ \in \mathcal{M}^0(\check{c}^+, c^-; \overline{J}, L, \widehat{L})$, $\widetilde{w}^1_- \in \widetilde{\mathcal{M}}^0(c^+, \check{c}^-; \overline{J}, L, \widehat{L}^+)$ and $\widetilde{w}^2_- \in \widetilde{\mathcal{M}}^1(\check{c}^-, c^-; J^-, \Lambda^-, \widehat{\Lambda}^-)$.

Theorem 2.5. *Assume that the linearized Cauchy-Riemman operator is surjective at both $\widetilde{w}^1_+$ and $\widetilde{w}^2_+$. Then, there exists an embedding $\Psi^+ : [0, +\infty) \to \overline{\mathcal{M}}^1(c^+, c^-; \overline{J}, L, \widehat{L})$ such that:*

- $\Psi^+(0) = \widetilde{w}_+$, *where $\widetilde{w}_+$ is the two level building whose top level is $\widetilde{w}^1_+$ and bottom level is $\widetilde{w}^2_+$,*
- $\Psi^+(t) \in \mathcal{M}^1(c^+, c^-; \overline{J}, L, \widehat{L})$ *for every $t \in (0, +\infty)$,*
- *the map Ψ^+ is a homeomorphism from $[0, +\infty)$ to a neighbourhood of $\widetilde{w}_+$ in $\overline{\mathcal{M}}^1(c^+, c^-; \overline{J}, L, \widehat{L})$.*

Moreover, if $\widetilde{w}_+(n)$ is a sequence of elements of $\mathcal{M}^1(c^+, c^-; \overline{J}, L, \widehat{L})$ converging to $\widetilde{w}_+$, then there exists n_0 such that $\widetilde{w}_+(n) \in \Psi^+([0, 1])$ for all $n \geq n_0$.

Analogously, if the linearized Cauchy-Riemman operator is surjective at both $\widetilde{w}^1_-$ and $\widetilde{w}^2_-$, then there exists an embedding $\Psi^- : [0, +\infty) \to \overline{\mathcal{M}}^1(c^+, c^-; \overline{J}, L, \widehat{L})$ such that:

- $\Psi^-(0) = \widetilde{w}_-$, *where $\widetilde{w}_-$ is the two level building whose top level is $\widetilde{w}^1_-$ and bottom level is $\widetilde{w}^2_-$,*
- $\Psi^-(t) \in \mathcal{M}^1(c^+, c^-; \overline{J}, L, \widehat{L})$ *for every $t \in (0, +\infty)$,*
- *the map Ψ^- is a homeomorphism from $[0, +\infty)$ to a neighbourhood of $\widetilde{w}_-$ in $\overline{\mathcal{M}}^1(c^+, c^-; \overline{J}, L, \widehat{L})$.*

Moreover, if $\widetilde{w}_-(n)$ is a sequence of elements of $\mathcal{M}^1(c^+, c^-; \overline{J}, L, \widehat{L})$ converging to $\widetilde{w}_-$, then there exists n_0 such that $\widetilde{w}_-(n) \in \Psi^-([0, 1])$ for all $n \geq n_0$.

Chapter 3

Strip Legendrian contact homology

3.1 Definition and basic properties of strip Legendrian contact homology

We are now in position to define the strip Legendrian contact homology. We remind the reader that for a given contact form α, $\mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha)$ is the set of Reeb chords of X_α starting at Λ and ending at $\widehat{\Lambda}$. We will also denote by $\mathcal{T}_\Lambda(\alpha)$ the set of Reeb chords from Λ to itself, and by $Per(\alpha)$ the set of Reeb orbits of X_α. Let α_0 be a contact form associated to (Y,ξ) such that its Reeb flow:

- (a) has no contractible periodic orbits,
- (b) there are no Reeb chords in $\mathcal{T}_\Lambda(\alpha_0)$ that vanish in $\pi_1(Y,\Lambda)$,
- (c) there are no Reeb chords in $\mathcal{T}_{\widehat{\Lambda}}(\alpha_0)$ that vanish in $\pi_1(Y,\widehat{\Lambda})$,
- (d) for every Reeb chord $c \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ the image of c in Y does not intersect the image of any Reeb orbit in $Per(\alpha)$, and every $c \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ is transverse.

We will say that a contact form satisfying the conditions above is *adapted* to the pair $(\Lambda,\widehat{\Lambda})$.

Let $LCH_{st}(\alpha_0,\Lambda\to\widehat{\Lambda})$ be the $\mathbb{Z}_2$ vector-space generated by $\mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$. Remember that the Conley-Zehnder index defined in section 1.2 can be used to define a $\mathbb{Z}_2$ grading of the elements of $\mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$; we extend this grading to $LCH_{st}(\alpha_0,\Lambda\to\widehat{\Lambda})$ in the obvious way. We denote by $LCH_{st,odd}(\alpha_0,\Lambda\to\widehat{\Lambda})$ to be the subspace of $LCH_{st}(\alpha_0,\Lambda\to\widehat{\Lambda})$

generated by odd Reeb chords, and $LCH_{st,even}(\alpha_0, \Lambda \to \widehat{\Lambda})$ to be the subspace generated by even chords.

Given two Reeb chords c_1 and c_2, and a cylindrical almost complex structure $J \in \mathcal{J}(\alpha_0)$, we considered in the previous chapter the moduli space $\mathcal{M}(c_1, c_2; J)$, modulo reparametrizations, of finite energy pseudoholomorphic strips $\widetilde{w} : (\overline{D} \setminus \{-1, 1\}, i_0) \to (\mathbb{R} \times Y, J)$ satisfying:

- 1 is a positive boundary puncture, and $\widetilde{w}$ is asymptotic to c_1 at 1,
- -1 is a negative boundary puncture, and $\widetilde{w}$ is asymptotic to c_2 at -1,
- $\widetilde{w}(H_-) \subset \mathbb{R} \times \Lambda$,
- $\widetilde{w}(H_+) \subset \mathbb{R} \times \widehat{\Lambda}$.

We would like to be able to compute the dimension of a connected component of $\mathcal{M}(c_1, c_2; J)$ by computing the Fredholm index of the linearised Cauchy-Riemann operator $D\overline{\partial}_J$ on an element of this connected component. We discuss now that this is indeed the case, if the complex structure J is well chosen; for the proof of this fact we refer the reader to [14] and [2].

It follows from Abbas' asymptotic analysis presented in the previous chapter and condition (d) above, that all the elements of the moduli space $\mathcal{M}(c_1, c_2; J)$ are somewhere injective pseudoholomorphic curves. Moreover, the combination of works of Dragnev [14] and Abbas [2] proves that for a generic set $\mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$ all the elements in $\mathcal{M}(c_1, c_2; J)$ are transverse in the sense that the linearization $D\overline{\partial}_J$ of the Cauchy-Riemann operator $\overline{\partial}_J$ at the elements of $\mathcal{M}(c_1, c_2; J)$ is surjective; this being valid for all Reeb chords c_1 and c_2. Thus, in the case where $J \in \mathcal{J}_{reg}(\alpha_0)$ one can use the implicit function theorem, and obtain that any connected component of the moduli space $\mathcal{M}(c_1, c_2; J)$ is a finite dimensional manifold with boundary, and its dimension is given by the Fredholm index I_F of $D\overline{\partial}_J$ computed at any element of this connected component of $\mathcal{M}(c_1, c_2; J)$. The proof that this generic set $\mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$ exists is an applications of the techniques in [6], [2] and [14].

Before defining the differential operator of the strip Legendrian contact homology complex we make two remarks that will be important in future arguments.

Remark 3.1: it follows from the formula in [2] for the Fredholm index I_F of the linearised $D\overline{\partial}_J$ operator over a strip in $\mathcal{M}(c_1, c_2; J)$, that I_F has the same parity of the sum $|\, c_1 \,| + |\, c_2 \,|$ of the gradings of c_1 and c_2.

Remark 3.2: as $0 < \int_{\overline{D}\setminus\{-1,1\}} \widetilde{w}^*(d\alpha_0) = A(c_1) - A(c_2)$*, the moduli space* $\mathcal{M}(c_1, c_2; J)$ *can only be non-empty if* $A(c_1) \geq A(c_2)$*.*

We are now ready to define a differential d_J in $LCH_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$.

Definition 3.1. Let $c \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ and $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$. We define:

$$d_J(c) = \sum_{c' \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)} [n_{c,c'} \mod 2] c' \tag{3.1}$$

where $n_{c,c'}$ is the cardinality of the moduli space $\widetilde{\mathcal{M}}^1(c, c'; J)$ of pseudoholomorphic strips of Fredholm index 1 modulo the $\mathbb{R}$-action.
The differential is extended to $LCH_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ by linearity.

To complete the construction of the strip Legendrian contact homology, we must prove that d_J is well-defined and that $d_J \circ d_J = 0$. Before proceeding to give proofs of these results we will discuss the intuition behind the definition of this homology theory. The strip Legendrian contact homology can be seen as a relative version of the cylindrical contact homology (see [7] and [16]). For cylindrical contact homology to be well-defined for a contact form, this contact form has to have some special property; for example, for a hypertight contact form (i.e. one that doesn't have contractible periodic orbits) cylindrical contact homology is well-defined. As we will see later, the non-existence of contractible Reeb orbits precludes the "bubbling" of pseudoholomorphic planes. This, together with SFT-compactness, implies that if its asymptotic orbits are in a primitive homotopy class, a sequence of pseudoholomorphic cylinders of Fredholm index 2 can only break in a pseudoholomorphic building of 2 levels, each containing a cylinder of Fredholm index 1; thus only such buildings can appear in the boundary of the compactified moduli space of pseudoholomorphic cylinders of Fredholm index 2. This description of the compactified moduli spaces of pseudoholomorphic cylinders of index 2, is the crucial step that allows us to define cylindrical contact homology with coefficients in $\mathbb{Z}_2$.

The strip Legendrian contact homology is the natural adaptation of cylindrical contact homology to the relative case. This time the differential involves pseudoholomorphic strips with boundary conditions on exact Lagrangian submanifolds. For such a theory to be well-defined we have to preclude not only "bubbling" of planes but also of pseudoholomorphic half planes. The conditions (b) and (c) above serve exactly to make impossible such "bubbling" phenomena, and the condition (d) is a non-degeneracy condition. Under these hypotheses, it is possible to define the strip Legendrian contact homology, and to carry this constructions one uses results on the analytical properties of pseudoholomorphic strips and discs; these results were presented in chapter 2 and in the beginning of this section, and some of the more technical aspects are discussed in the appendices A, B. Using this machinery we proceed to prove:

Lemma 3.2. *For $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$, and d_J defined before we have:*

- *(1) d_J is well defined,*
- *(2) d_J decreases the action of Reeb chords,*
- *(3) for each $c \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$, $d_J(c)$ is a finite sum,*
- *(4) $d_J : LCH_{st,odd}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LCH_{st,even}(\alpha_0, \Lambda \to \widehat{\Lambda})$ and $d_J : LCH_{st,even}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LCH_{st,odd}(\alpha_0, \Lambda \to \widehat{\Lambda})$.*

Proof: in order for d_J to be well-defined we have to prove that $\widetilde{\mathcal{M}}^1(c, c'; J)$ is finite for every c and c'. Because $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$, $\widetilde{\mathcal{M}}^1(c, c'; J)$ is a 0-dimensional manifold. If we show that it is compact then it has to be a finite set. To obtain the compactness of $\widetilde{\mathcal{M}}^1(c, c'; J)$, we will apply the standard "bubbling of" analysis for pseudoholomorphic curves of [28] and the SFT compactness results of [8] that were recalled in chapter 2.

Let $\widetilde{w}_n$ be a sequence of elements of $\widetilde{\mathcal{M}}^1(c, c'; J)$. Because of the assumptions we made on the contact form α_0, the sequence $\widetilde{w}_n$ cannot have interior bubbling points: as we saw previously, an interior bubbling point would imply the existence of a finite energy plane in the symplectization of α_0 and thus of a contractible periodic orbit of X_{α_0}, something which contradicts condition (a) that is satisfied by the contact form α_0. Bubbling points on the boundary are also forbidden: they would give rise to either: a pseudoholomorphic disc with boundary in $\mathbb{R} \times \Lambda$, a pseudoholomorphic disc with boundary on $\mathbb{R} \times \widehat{\Lambda}$, a pseudoholomorphic disc with only one positive boundary puncture that is asymptotic to a Reeb chord from Λ to itself, or a pseudoholomorphic disc with only one positive boundary puncture that is asymptotic to a Reeb chord from $\widehat{\Lambda}$ to itself. The first two possibilities are impossible because $\mathbb{R} \times \Lambda$ and $\mathbb{R} \times \widehat{\Lambda}$ are exact Lagrangian submanifolds; the later two because they would contradict conditions (b) and (c) satisfied by α_0.

Combining this information with the SFT-compactness Proposition 2.1 of section 2.4, we have that a subsequence of $\widetilde{w}_n$ converges in the SFT sense to a pseudoholomorphic building $\widetilde{w}$ with k-levels $\widetilde{w}^l$, where all levels $\widetilde{w}^l$ are pseudoholomorphic strips satisfying:

- 1 is a positive boundary puncture, and $\widetilde{w}^l$ is asymptotic to $c_l \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ at 1,
- -1 is a negative boundary puncture, and $\widetilde{w}^l$ is asymptotic to $c_{l+1} \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ at -1,

- $\widetilde{w}(H_-) \subset \mathbb{R} \times \Lambda$,
- $\widetilde{w}(H_+) \subset \mathbb{R} \times \widehat{\Lambda}$;

where $c_1 = c$, $c_{k+1} = c'$ and $c_l \neq c_{l+1}$. Because all $\widetilde{w}^l$ are somewhere injective pseudoholomorphic curves, and are different from trivial strips over Reeb chords, we obtain that the Fredholm indexes of these strips satisfy $I_F(\widetilde{w}^l) \geq 1$. From the additivity property of the Fredholm indexes, we have $I_F(\widetilde{w}) = \sum (I_F(\widetilde{w}^l) \geq l$; on the other hand as $\widetilde{w}$ is the limit of a sequence of pseudoholomorphic strips of Fredholm index 1, it has to satisfy $I_F(\widetilde{w}) = 1$. Therefore, we conclude that $l = 1$, and $\widetilde{w} \in \widetilde{\mathcal{M}}^1(c, c'; J)$; this implies the desired compactness of $\widetilde{\mathcal{M}}^1(c, c'; J)$. As a consequence, we have proved that $n_{c,c'}$ is finite for every $c, c' \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, and thus that d_J is well defined. This finishes the proofs of item (1) of the lemma.

To verify item (2), we recall the remark made before the definition of d_J, that for $c \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, the number $n_{c,c'}$ can only be non-zero for Reeb chords c' such that $A(c') < A(c)$. This implies that d_J decreases the action of Reeb chords, as stated in item (2).

By the transversality condition (d) above one obtains that the set of Reeb chords in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ with action smaller then $A(c)$ is finite, and so $n_{c,c'}$ is non-zero only for a finite number of c'. This combined with items (1) and (2), implies that $d_J(c)$ is a finite sum finishing the proof of (3).

Item (4) follows easily from the fact mentioned above that the Fredholm index of a strip connecting two chords c and c' has the same parity as $\mid c \mid + \mid c' \mid$, as this implies that $\widetilde{\mathcal{M}}^1(c, c'; J)$ can be non-empty only if c and c' have different parity.

□

Lemma 3.3. *For $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$, and d_J as defined before we have: $d_J \circ d_J = 0$*

Proof: the lemma will be a consequence of the description we will give of the compactified moduli space $\overline{\mathcal{M}}^2(c, c'; J)$ of pseudoholomorphic strips with Fredholm index 2. Because of regularity of J, it will follow that for all $c, c' \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, $\overline{\mathcal{M}}^2(c, c'; J)$ is either empty, or the finite union of disjoint circles and closed intervals. We summarise that in the following claim:

Compactness Claim: suppose $\overline{\mathcal{M}}^2(c, c'; J)$ is non-empty. Then, each connected component I of $\overline{\mathcal{M}}^2(c, c'; J)$ is either a circle or a closed interval. Moreover, when I is diffeomorphic to a closed interval, its boundary is composed by pseudoholomorphic buildings $\widetilde{w}$ with 2 levels $\widetilde{w}_1$ and $\widetilde{w}_2$ satisfying:

$$\widetilde{w} \in \overline{\mathcal{M}}^1(c, \check{c}; J) \text{ and } \widetilde{w_2} \in \overline{\mathcal{M}}^1(\check{c}, c'; J) \text{ for some } \check{c} \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0).$$

Before proving the claim above we will use it to prove the lemma. For this, we write:

$$d_J \circ d_J(c) = \sum_{r' \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)} (m_{c,c'} \mod 2)c' \tag{3.2}$$

It is clear that the lemma will follow if we prove that $m_{c,c'}$ is always even. On one hand, notice that it follows from our definition of d_J, that $m_{c,c'}$ counts the number of two-level pseudoholomorphic buildings whose levels $\widetilde{w}_1$ and $\widetilde{w}_2$ satisfy: $\widetilde{w}_1 \in \overline{\mathcal{M}}^1(c, \check{c}; J)$ and $\widetilde{w_2} \in \overline{\mathcal{M}}^1(\check{c}, c'; J)$ for some $\check{c} \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$. This combined with the compactness claim implies that the number of boundary points of $\overline{\mathcal{M}}^2(c, c'; J)$ is smaller or equal to $m_{c,c'}$.

On the other hand, because of the regularity of $J \in \mathcal{J}_{reg}(\alpha_0)$, we can apply gluing as in Theorem 2.4: this implies that each 2-level pseudoholomorphic building $\widetilde{w}$ whose levels $\widetilde{w}_1$ and $\widetilde{w}_2$ satisfy $\widetilde{w}_1 \in \overline{\mathcal{M}}^1(c, \check{c}; J)$ and $\widetilde{w_2} \in \overline{\mathcal{M}}^1(\check{c}, c'; J)$ for some $\check{c} \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, is in the boundary of $\overline{\mathcal{M}}^2(c, c'; J)$, more precisely in boundary of exactly one connected component $I_{\widetilde{w}} \subset \overline{\mathcal{M}}^2(c, c'; J)$. We thus obtain that $m_{c,c'}$ is bigger or equal to the number of boundary points of $\overline{\mathcal{M}}^2(c, c'; J)$.

Summarising, the combination of the Compactness Claim and gluing allows us to conclude that the number $m_{c,c'}$ is exactly the number of boundary components of the moduli space $\overline{\mathcal{M}}^2(c, c'; J)$. Because $\overline{\mathcal{M}}^2(c, c'; J)$ is a finite union of disjoint intervals and circles, this number is even. This finishes the proof of the lemma modulo the Compactness claim.

Proof of Compactness Claim: suppose $\overline{\mathcal{M}}^2(c, c'; J)$ is non-empty and let I be a connected component. It follows from the regularity of J that the interior $\dot{I}$ of I is a 1-dimensional manifold. If $\dot{I}$ is compact it has to be a circle.

If that is not the case, let $\widetilde{w}_n$ be a sequence of elements of $\dot{I}$ converging to the boundary of I. As we remarked in the proof of the previous lemma no "bubbling" can occur. Thus the SFT compactness theorem of implies that $\widetilde{w}_n$ converges to a pseudo-holomorphic building $\widetilde{w}$ with k-levels $\widetilde{w}^l$, such that all levels $\widetilde{w}^l$ are pseudoholomorphic strips satisfying:

- 1 is a positive boundary puncture, and $\widetilde{w}^l$ is asymptotic to $c_l \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ at 1

- -1 is a negative boundary puncture, and $\widetilde{w}$ is asymptotic to $c_{l+1} \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ at -1
- $\widetilde{w}^l(H^-) \subset \mathbb{R} \times \Lambda$ where $H_- \subset (S^1 \setminus \{-1,1\})$ is the northern hemisphere
- $\widetilde{w}^l(H_+) \subset \mathbb{R} \times \widehat{\Lambda}$ where $H_- \subset (S^1 \setminus \{-1,1\})$ is the southern hemisphere

where $c_1 = c$ and $c_{k+1} = c'$. Again because every $\widetilde{w}^l$ is somewhere injective we have that the Fredholm index $F(\widetilde{w}^l) \geq 1$ and thus $I_F(\widetilde{w}) = \sum(I_F(\widetilde{w}^l) \geq l$. On the other hand as $\widetilde{w}$ is the limit of a sequence of pseudoholomorphic strips of Fredholm index 2, it has to satisfy $I_F(\widetilde{w}) = 2$.

We have then 2 possibilities: either $l = 1$ and $\widetilde{w} \in \widetilde{\mathcal{M}}^2(c,c';J)$; or $l = 2$ which forces $I_F(\widetilde{w}^1) = I_F(\widetilde{w}^1) = 1$, $\widetilde{w}^1 \in \overline{\mathcal{M}}^1(c,c_2;J)$ and $\widetilde{w}^2 \in \overline{\mathcal{M}}^1(c_2,c';J)$. The first case is ruled out because we supposed that $\widetilde{w}_n$ is converging to the boundary of I. We have obtained that all the elements on the boundary of I are 2-level pseudoholomorphic buildings with the properties claimed. This implies that the boundary of $\overline{\mathcal{M}}^2(c,c';J)$ is a compact 0-dimensional manifold.

On the other hand, the gluing theorem gives the description of a neighbourhood of the 2-level pseudoholomorphic buildings appearing in the boundary I. This neighbourhood admits a diffeomorphism to the infinite interval $[0,+\infty)$, that takes 0 to the 2-level building and all other values to pseudoholomorphic strips in $\overline{\mathcal{M}}^2(c,c';J)$.

Summing up, the compactified moduli $\overline{\mathcal{M}}^2(c,c';J)$ has the structure of a 1-dimensional with 0-dimensional boundary; i.e a closed interval. This finishes the proof of the compactness claim.

□

We will denote by $LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ the homology complex associated to the chain-complex $(LCH_{st}(\alpha_0, \Lambda \to \widehat{\Lambda}), d_J)$.

3.2 Strip Legendrian contact homology in special homotopy classes

Just as in the case of cylindrical contact homology, the free homotopy classes of paths starting at Λ and ending at $\widehat{\Lambda}$ generate subcomplexes of $LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$. To formalize this we denote by $\Sigma_{\Lambda\to\widehat{\Lambda}}$ the set of homotopy classes of paths starting at Λ and ending at $\widehat{\Lambda}$. For our contact form α_0 and an element $\rho \in \Sigma_{\Lambda\to\widehat{\Lambda}}$ we denote by $\mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ the set of Reeb chords from Λ to $\widehat{\Lambda}$ that belong to ρ.

It is clear that, for any $c \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$, the moduli space $\overline{\mathcal{M}}^1(c, c'; J)$ can only be non-empty if $c' \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ too. That implies that the terms $[n_{c,c'} \mod 2]$, appearing in the differential $d_J(c) = \sum_{c' \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)}[n_{c,c'} \mod 2]c'$, can be non-zero only when $c' \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$; and allows us to conclude that the vector spaces $LCH^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ are subcomplexes in the chain complex $(LCH_{st}(\alpha_0, \Lambda \to \widehat{\Lambda}), d_J)$. Thus, we have:

$$LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda}) = \bigoplus_{\rho \in \Sigma_{\Lambda\to\widehat{\Lambda}}} LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda}) \tag{3.3}$$

In our applications, we will usually be interested in computing $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ for some homotopy classes $\rho \in \Sigma_{\Lambda\to\widehat{\Lambda}}$.

3.3 Cobordism maps

Symplectic cobordisms play a crucial role in SFT because they induce maps for the SFT invariants, as observed in [16]. Under suitable conditions this is also true for the strip Legendrian contact homology.

Proposition 3.4. *Let $(Y^+, \ker(\alpha^+))$ be a contact manifold, with: α^+ an adapted contact form for a pair of disjoint Legendrian knots Λ^+ and $\widehat{\Lambda}^+$; and $(Y^-, \ker(\alpha^-))$ be a contact manifold, with α^- an adapted contact form for a pair of disjoint Legendrian knots Λ^- and $\widehat{\Lambda}^-$. Suppose that there exist an exact symplectic cobordism (V, ϖ) from (Y^+, α^+) to $(Y^-, \mathsf{k}\alpha^-)$ for some constant $\mathsf{k} > 0$, and exact Lagrangian cylinders: L, from Λ^+ to Λ^-; and $\widehat{L}$ from $\widehat{\Lambda}^+$ to $\widehat{\Lambda}^-$. Then, these cobordisms induce a map $\Phi_{V,\varpi,L,\widehat{L}}$ from $LC\mathbb{H}_{st}(\alpha^+, \Lambda^+ \to \widehat{\Lambda}^+)$ to $LC\mathbb{H}_{st}(\alpha^-, \Lambda^- \to \widehat{\Lambda}^-)$.*

Proof: taking $J^+ \in \mathcal{J}_{reg}(\alpha^+)$ and $J^- \in \mathcal{J}_{reg}(\alpha^-)$, we can define the homologies $LC\mathbb{H}_{st}(\alpha^+, \Lambda^+ \to \widehat{\Lambda}^+)$ and $LC\mathbb{H}_{st}(\alpha^-, \Lambda^- \to \widehat{\Lambda}^-)$. We then take $J_V \in \mathcal{J}(J^+, J^-)$ as the almost complex structure in the cobordism (V, ϖ). The map $\Phi_{V,\varpi,L,\widehat{L}}$ will count pseudoholomorphic strips $\widetilde{w} : (\overline{D} \setminus \{-1, 1\}, i_0) \to (V, J)$ with Fredholm index 0, having 1 as a positive boundary puncture asymptotic to a Reeb chord $c^+ \in \mathcal{T}_{\Lambda^+\to\widehat{\Lambda}^+}(\alpha^+)$ and -1 as a negative boundary puncture asymptotic to a Reeb chord $c^- \in \mathcal{T}_{\Lambda^-\to\widehat{\Lambda}^-}(\alpha^-)$, and having boundary conditions $\widetilde{w}(H^-) \in L$ and $\widetilde{w}(H^+) \in \widehat{L}$; in other words the map $\Phi_{V,\varpi,L,\widehat{L}}$ will count elements in the moduli spaces $\mathcal{M}^0(c^+, c^-; J_V, L, \widehat{L})$, for the Reeb chords as above. As all the strips in these moduli spaces are somewhere injective curves, we have that for a generic set $\mathcal{J}_{reg}(J^+, J^-) \in \mathcal{J}(J^+, J^-)$, all the pseudoholomorphic curves in $\mathcal{M}^0(c^+, c^-; J_V, L, \widehat{L})$, for every c^+ and c^-, are Fredholm regular. We assume from now on that we picked $J_V \in \mathcal{J}_{reg}(J^+, J^-)$.

Initially, the map $\Phi_{V,\varpi,L,\widehat{L}}$ is obtained by counting elements in $\mathcal{M}^0(c^+,c^-;J_V,L,\widehat{L})$, and is the defined from $LCH_{st}(\alpha^+,\Lambda^+\to\widehat{\Lambda}^+)$ to $LCH_{st}(\alpha^-,\Lambda^-\to\widehat{\Lambda}^-)$. More precisely for each $c^+\in\mathcal{T}_{\Lambda^+\to\widehat{\Lambda}^+}(\alpha^-)$:

$$\Phi_{V,\varpi,L,\widehat{L}}(c^+)=\sum_{c^-\in\mathcal{T}_{\Lambda^-\to\widehat{\Lambda}^-}(\alpha^-)}\#(\mathcal{M}^0(c^+,c^-;J_V,L,\widehat{L}))c^- \tag{3.4}$$

A compactness argument identical to the one in Lemma 3.2 shows that $\Phi_{V,\varpi,L,\widehat{L}}$ is well defined and is a finite sum.

To see that it actually induces a map on the homology level one has to check that $d_{J^-}\circ\Phi_{V,\varpi,L,\widehat{L}}=\Phi_{V,\varpi,L,\widehat{L}}\circ d_{J^+}$. The proof of this fact consists of a combination of compactness and gluing results, and is identical to a similar statement for cylindrical contact homology (see [7]).

Because of the regularity for all pseudoholomorphic curves involved in d_{J^+}, d_{J^-} and $\Phi_{V,\varpi,L,\widehat{L}}$, it is possible to perform gluing for the pseudoholomorphic curves involved in these maps. More precisely the map $d_{J^-}\circ\Phi_{V,\varpi,L,\widehat{L}}(c^+)$ counts the number of 2-level pseudoholomorphic buildings $(\widetilde{w}^1_-,\widetilde{w}^2_-)$ where $\widetilde{w}^1_-\in\mathcal{M}^0(c^+,\check{c}^-;J_V)$ and $\widetilde{w}^2_-\in\mathcal{M}^1(\check{c}^-,c^-;J^-)$ for $\check{c}^-$ and c^- belonging to $\mathcal{T}_{\Lambda^-\to\widehat{\Lambda}^-}(\alpha^-)$. Analogously, $\Phi_{V,\varpi,L,\widehat{L}}\circ d_{J^+}(c^+)$ counts the number of 2-level pseudoholomorphic buildings $(\widetilde{w}^1_+,\widetilde{w}^2_+)$ where $\widetilde{w}^1_+\in\mathcal{M}^0(c^+,\check{c}^+;J^+)$ and $\widetilde{w}^2_+\in\mathcal{M}^1(\check{c}^+,c^-;J^-)$ for $\check{c}^+\in\mathcal{T}_{\Lambda^+\to\widehat{\Lambda}^+}(\alpha^+)$ and $c^-\in\mathcal{T}_{\Lambda^-\to\widehat{\Lambda}^-}(\alpha^-)$. The gluing theorem implies that the union of all such 2-level building belong to the boundary of the compactified moduli space $\overline{\mathcal{M}}^1(c^+,c^-;J_V,L,\widehat{L})$.

On the other hand, because of the exactness of (V,ϖ), and of the Lagrangians L and $\widehat{L}$, and using that α^+ is an adapted contact form for the pairs of Legendrian curves Λ^+ and $\widehat{\Lambda}^+$ and α^- is an adapted contact form for the pairs of Legendrian curves Λ^+ and $\widehat{\Lambda}^+$, we have that a sequence of elements in $\mathcal{M}^1(c^+,c^-;J_V,L,\widehat{L})$ can only break in 2-level buildings that are appear in the maps $d_{J^-}\circ\Phi_{V,\varpi,L,\widehat{L}}$ and $\Phi_{V,\varpi,L,\widehat{L}}\circ d_{J^+}$. The complete proof of this compactness fact uses just the additivity of the Fredholm index for buildings and the Fredholm regularity of the pseudoholomorphic buildings involved, and is identical to a similar arguments we used in Lemma 3.3; for this reason we will not repeat the argument here. Notice, that all the elements in the boundary of $\overline{\mathcal{M}}^1(c^+,c^-;J_V,L,\widehat{L})$ are Fredholm regular buildings .

From this combination of compactness and gluing we obtain that the moduli space $\overline{\mathcal{M}}^1(c^+,c^-;J_V,L,\widehat{L})$ is a one dimensional manifold with boundary, whose boundary is composed exactly by the 2-level buildings that appear in the definition of the maps $d_{J^-}\circ\Phi_{V,\varpi,L,\widehat{L}}$ and $\Phi_{V,\varpi,L,\widehat{L}}\circ d_{J^+}$.

Summarising, we obtain for each $c^+ \in \mathcal{T}_{\Lambda^+ \to \widehat{\Lambda}^+}(\alpha^+)$:

$$(d_{J^+} \circ \Phi_{V,\varpi,L,\widehat{L}} - \Phi_{V,\varpi,L,\widehat{L}} \circ d_{J^-})(c^+) = \sum_{c^- \in \mathcal{T}_{\Lambda^- \to \widehat{\Lambda}^-}(\alpha^-)} [a(c^+, c^-) \mod 2] c^- \tag{3.5}$$

where $a(c^+, c^-)$ is the number of pseudoholomorphic buildings appearing on the boundary of $\overline{\mathcal{M}}^1(c^+, c^-; J_V, L, \widehat{L})$. As $\overline{\mathcal{M}}^1(c^+, c^-; J_V, L, \widehat{L})$ is a 1-dimensional manifold the number of its boundary components is even which implies that $[a(c^-) \mod 2] = 0$, and finishes the proof of the proposition.

$\square$

We point, that in the case (V, ϖ) is the symplectization of a contact manifold with an $\mathbb{R}$-invariant regular almost complex structure and the Lagrangians L and $\widehat{L}$ are just the cylinders over, respectively, Λ and $\widehat{\Lambda}$, then the induced cobordism map is the identity. This is so, because in this situation, the only pseudoholomorphic strips with Fredholm index 0 are the trivial strips over Reeb chords.

3.3.1 A special type of cobordisms

For us, the case of most importance is when the contact forms α^+ and α^- are of the form $\alpha^+ = \alpha_0$ and $\alpha^- = \mathsf{k}\alpha_0$ for a certain contact form α_0 and constant $0 < \mathsf{k} < 1$, the Legendrian curves $\Lambda^+ = \Lambda^- := \Lambda$ and $\widehat{\Lambda}^+ = \widehat{\Lambda}^- := \widehat{\Lambda}$, and the exact symplectic cobordism (V, ϖ) and the exact Lagrangian cobordisms L and $\widehat{L}$ can be deformed, respectively, to the symplectization of (Y, α_0), and the Lagrangian cylinders over Λ and $\widehat{\Lambda}$. In this case the cobordism actually induces a map that respect the homotopy classes of the chords; so we can define (for each $\rho \in \Sigma_{\Lambda \to \widehat{\Lambda}}$) a map $\Phi_{V,\varpi,L,\widehat{L}}$ from $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ to $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$. In order to define the cobordism maps for these subcomplexes $LC\mathbb{H}^{\rho}_{st}(\alpha^+, \Lambda^+ \to \widehat{\Lambda}^+)$ the assumptions of regularity on the almost complex structure J_V are slightly weaker.

Fix $\rho \in \Sigma_{\Lambda \to \widehat{\Lambda}}$. For any pair c and c' of Reeb chords in $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, we consider the moduli space $\mathcal{M}^k(c, c'; J_V, L, \widehat{L})$ of pseudoholomorphic strips with Fredholm index k. As all the strips in these moduli spaces are somewhere injective curves, we have for a generic set $\mathcal{J}_{reg,\rho}(J, J) \in \mathcal{J}(J, J)$, all the pseudoholomorphic curves in all the moduli spaces $\mathcal{M}^k(c, c'; J_V, L, \widehat{L})$, for every c and c' in $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, are Fredholm regular. An argument identical to the proof of the previous proposition shows that for $J_V \in \mathcal{J}_{reg,\rho}(J, J)$ we have a map $\Phi_{V,\varpi,L,\widehat{L}}$ from $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ to $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$; notice that $\mathcal{J}_{reg,\rho}(J, J)$ may contain elements that are not in $\mathcal{J}_{reg}(J, J)$.

We study the cobordism map in the following situation: let $(V = \mathbb{R} \times Y, \varpi)$ be an exact symplectic cobordism from (Y, α_0) to $(Y, \mathsf{k}\alpha_0)$ where $0 < \mathsf{k} < 1$, and assume that α_0 is adapted to a pair of Legendrians Λ and $\widehat{\Lambda}$ and there exists exact Lagrangian cobordisms $L \subset V$ from Λ to Λ, and $\widehat{L} \subset V$ from $\widehat{\Lambda}$ to $\widehat{\Lambda}$ diffeomorphic to cylinders. Suppose that one can make an isotopy $(\mathbb{R} \times Y, \varpi_t)$ of exact symplectic cobordisms from (Y, α_0) to $(Y, \mathsf{k}\alpha_0)$, where ϖ_t satisfies $\varpi_0 = \varpi$ and $\varpi_1 = d(e^s\alpha_0)$, and that:

- (A) $L_t \in (\mathbb{R} \times Y, \varpi_t)$ is an exact Lagrangian cobordisms from Λ to Λ, with $L_1 = \mathbb{R} \times \Lambda$;
- (B) $\widehat{L}_t \in (V, \varpi_t)$ is exact Lagrangian cobordisms from $\widehat{\Lambda}$ to $\widehat{\Lambda}$, with $\widehat{L}_1 = \mathbb{R} \times \widehat{\Lambda}$.

We consider the space $\widetilde{\mathcal{J}}(J, J)$ of smooth homotopies:

$$t \in [0,1]; J_t \in \mathcal{J}(J, J) \tag{3.6}$$

with $J_0 = J_V \in \mathcal{J}_{reg,\rho}(J, J)$, $J_1 \in \mathcal{J}_{reg}(\alpha_0)$, and J_t is compatible with ϖ_t for all t. For Reeb chords $c, c' \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ we consider the moduli space:

$$\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t) = \{(t, \widetilde{w}) \mid t \in [0,1] \text{ and } \widetilde{w} \in \mathcal{M}(c, c'; J_t, L_t, \widehat{L}_t)\} \tag{3.7}$$

By the combination of the techniques in [14], [2] and [6] we know that there is a generic subset $\widetilde{\mathcal{J}}_{reg}(J^+, J^-) = \widetilde{\mathcal{J}}(J^+, J^-)$ such that $\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$ is a smooth manifold of dimension $I_F(\widetilde{w})+1$, where $(t, \widetilde{w}) \in \widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$ and such that for all J_t, the elements of $\mathcal{M}(c, c'; J_t, L_t, \widehat{L}_t)$ have Fredholm index bigger or equal to -1. The crucial condition that makes this possible is again the fact that the all the pseudoholomorphic curves belonging to this moduli space are somewhere injective.

Proposition 3.5. *Let $(V = \mathbb{R} \times Y, \varpi)$ be an exact symplectic cobordism from (Y, α_0) to $(Y, \mathsf{k}\alpha_0)$ where $0 < \mathsf{k} < 1$, and α_0 is adapted to a pair of Legendrians Λ and $\widehat{\Lambda}$. Assume also that there exists in $(\mathbb{R} \times Y, \varpi)$ exact Lagrangian cobordisms $L \subset \mathbb{R} \times Y$ from Λ to Λ, and $\widehat{L} \subset V$ from $\widehat{\Lambda}$ to $\widehat{\Lambda}$. Suppose that one can make an isotopy of exact symplectic cobordisms (Y, α_0) to $(Y, \mathsf{k}\alpha_0)$, where ϖ_t satisfies $\varpi_0 = \varpi$ and $\varpi_1 = d(e^s\alpha_0)$, and such that $L_t \in (\mathbb{R} \times Y, \varpi_t)$ and $\widehat{L}_t \in (\mathbb{R} \times Y, \varpi_t)$ are exact Lagrangian cobordisms satisfying conditions (A) and (B) above. Then for all $J_V \in \mathcal{J}_{reg,\rho}(J, J)$ the map $\Phi_{V,\varpi,L,\widehat{L}}$ from $LC\mathbb{H}^{\rho}_{st}(\alpha, \Lambda \to \widehat{\Lambda})$ to itself is chain homotopic to the identity.*

Proof: the proof is an standard argument in SFT, and we refer the reader to the original source [16] for an exposition of this argument for general SFT invariants and

[7], [6] where the very similar case of cylindrical contact homology is treated. We first take an almost complex structure $J \in \mathcal{J}_{reg}(\alpha_0)$ and choose an almost complex structure $J_V \in \mathcal{J}_{reg}(J, J)$ compatible with ϖ. The map $\Phi_{V,\varpi,L,\widehat{L}}$ will count pseudoholomorphic strips in (V, J_V) satisfying boundary conditions as stated in the previous proposition. For the deformation ϖ_t we can take an homotopy J_t of almost complex structures in $\widetilde{\mathcal{J}}_{reg}(J, J)$.

The crucial ingredient of the proof will be the description of the compactification $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$ of the moduli space $\widehat{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$. We want to understand the boundary of $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$. Take a sequence $(t_n, \widetilde{w}_n)$ of elements in $\widehat{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$ converging to the boundary of $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$; there are three possibilities: t_n goes to 0, t_n goes to 1 or the limit t_∞ of t_n belongs to $(0, 1)$. In all three possibilities we know that the conditions (a), (b), (c) and (d) satisfied by α_0 prevent any "bubbling" in the sequence. In the first possibility, regularity of J_V and J imply that the $\widetilde{w}_n$ has to converge to an element $\widetilde{w}$ of $\mathcal{M}^0(c, c'; J_V, L, \widehat{L})$. Analogously because of the regularity of J, in the second possibility $\widetilde{w}_n$ has to converge to $\widetilde{w}$ of $\mathcal{M}^0(c, c'; J)$. In the third possibility $\widetilde{w}_n$ converges to a pseudoholomorphic building $\widetilde{w}$ whose levels $\widetilde{w}^l$ are all strips; the fact that for J_{t_∞} all the strips appearing have Fredholm index ≥ -1 allows us to limit the possibilities. The building $\widetilde{w}$ has Fredholm index 0 and has at least 2-levels since it is a boundary element of our moduli space. On the other hand, of all its levels there is one in the cobordism $(\mathbb{R} \times Y, \varpi_{t_\infty})$ and all the others are in symplectizations of α_0. The levels in the symplectizations have Fredholm index ≥ 1 and the one on $(\mathbb{R} \times Y, \varpi_{t_\infty})$ has Fredholm index ≥ -1. Combining this with the fact that $\widetilde{w}$ has Fredholm index 0 it is easy to see that $\widetilde{w}$ has exactly 2-levels: one with Fredholm index -1 in $(\mathbb{R} \times Y, \varpi_{t_\infty})$ and one in the symplectization of α_0. More precisely in this third possibility, there are two possible cases: either the building $\widetilde{w}$ has the top level $\widetilde{w}^1 \in \widetilde{\mathcal{M}}^1(c, \check{c}; J)$ and the lower level $\widetilde{w}^2 \in \widetilde{\mathcal{M}}^{-1}(\check{c}, c'; J_{t_\infty}, L_{t_\infty}, \widehat{L}_{t_\infty})$ (for some $\check{c} \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$), or the building $\widetilde{w}$ has the top level $\widetilde{w}^1 \in \widetilde{\mathcal{M}}^{-1}(c, \check{c}; J_{t_\infty}, L_{t_\infty}, \widehat{L}_{t_\infty})$ and the lower level $\widetilde{w}^2 \in \widetilde{\mathcal{M}}^1(\check{c}, c'; J)$ (for some $\check{c} \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$).

On the other hand, an appropriate version of the gluing theorem (see section 3.5) imply that: if $\widetilde{w}$ is a 2-level building whose top level $\widetilde{w}^1 \in \widetilde{\mathcal{M}}^1(c, \check{c}; J)$ and whose lower level $\widetilde{w}^2 \in \widetilde{\mathcal{M}}^{-1}(\check{c}, c'; J_{t_\infty}, L_{t_\infty}, \widehat{L}_{t_\infty})$, then $\widetilde{w}$ is an element of the boundary of $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$. The same is valid if the building $\widetilde{w}$ has the top level $\widetilde{w}^1 \in \widetilde{\mathcal{M}}^{-1}(c, \check{c}; J_{t_\infty}, L_{t_\infty}, \widehat{L}_{t_\infty})$ and the lower level $\widetilde{w}^2 \in \widetilde{\mathcal{M}}^1(\check{c}, c'; J)$. Finally if $\widetilde{w} \in \mathcal{M}^0(c, c'; J_V, L, \widehat{L})$ the regularity of the homotopy J_t implies that $\widetilde{w}$ is in the boundary of $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$; and the same is valid if $\widetilde{w} \in \mathcal{M}^0(c, c'; J)$. We have thus obtained a complete description of the boundary of $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$.

We now define a map $\mathcal{K} : LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ that counts finite energy, Fredholm index -1 pseudoholomorphic strips in (V, ϖ_t), with one boundary component in L_t and one in $\widehat{L_t}$. Because of the regularity of our homotopy, the set of index -1 strips whose positive puncture detects a fixed chord c is finite, and therefore the map $\mathcal{K}$ is really well defined.

Consider the map $Id + \Phi_{V,\varpi,L,\widehat{L}} + \mathcal{K} \circ d_J + d_J \circ \mathcal{K}(c)$. It is clear from our discussion so far, that the pseudoholomorphic curves which are counted in the definition of this sum, are exactly the ones that appear in the boundary of the moduli spaces $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$. As the compactified moduli space $\overline{\mathcal{M}}^0(c, c'; J_t, L_t, \widehat{L}_t)$ is a finite union of compact intervals it has an even number of boundary components. Therefore we obtain $Id + \Phi_{V,\varpi,L,\widehat{L}} + \mathcal{K} \circ d_J + d_J \circ \mathcal{K}(c) = 0$. As this is valid for all Reeb chords $c \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ we have proved that $\Phi_{V,\varpi,L,\widehat{L}}$ is chain-homotopic to the identity, as claimed.

□

One very important consequence of the above proposition is that the strip Legendrian contact homology $LC\mathbb{H}^{\rho}_{st}(\alpha, \Lambda \to \widehat{\Lambda})$ doesn't depend on the almost complex structure $J \in \mathcal{J}_{reg}(\alpha_0)$ used to define d_J, something which is not at all obvious from the definition of $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$.

3.4 Strip Legendrian contact homology on the complement of Reeb orbits

In this section we adapt our construction of the strip Legendrian contact homology to study implied existence problems; the idea of using SFT invariants to study this type of question comes from the works [35] and [30]. The idea is briefly described as follows: on a contact manifold (Y, ξ) we consider an oriented transverse link which we denote by $\mathcal{G}$; we want to study dynamical properties of contact forms associated to (Y, ξ) which have the connected components link $\mathcal{G}$ as Reeb orbits. Another way of phrasing it is the following: if $\mathcal{G}$ appears as a set of Reeb orbits of a contact form α associated to (Y, ξ), does this have non-trivial implications for the dynamics of ϕ^t_α?

The works [35] and [30] have answered this question positively for certain particular examples; in these works, a version of cylindrical contact homology is used to study how the condition of having $\mathcal{G}$ as a set of periodic orbits can force the appearance of other Reeb orbits. We will follow a similar approach but will be interested in the appearance of Reeb chords.

We begin presenting the setup on which we will work: let $\mathcal{G}$ be an oriented transverse link in the contact manifold (Y, ξ), and α_0 be a contact form for which $\mathcal{G}$ is the union of positively oriented Reeb orbits of ϕ_{α_0}; let also Λ and $\widehat{\Lambda}$ be a pair of Legendrian submanifolds which do not intersect $\mathcal{G}$. Suppose that the Reeb flow of X_{α_0} satisfies:

- (a') all Reeb orbits of X_{α_0} disjoint from $\mathcal{G}$ are not contractible in $Y \setminus \mathcal{G}$, and for every Reeb orbit $\gamma_{\mathcal{G}} \in \mathcal{G}$ either $\gamma_{\mathcal{G}}$ is non-contractible or for any disc $D_{\gamma_{\mathcal{G}}}$ in Y having $\gamma_{\mathcal{G}}$ as boundary, the interior of $D_{\gamma_{\mathcal{G}}}$ has to intersect $\mathcal{G}$,
- (b') all the Reeb chords of X_{α_0} going from Λ to itself do not vanish in $\pi_1(Y \setminus \mathcal{G}, \Lambda)$,
- (c') all the Reeb chords of X_{α_0} going from $\widehat{\Lambda}$ to itself do not vanish in $\pi_1(Y \setminus \mathcal{G}, \widehat{\Lambda})$,
- (d') for every Reeb chord $c \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ the image of c in Y does not intersect the image of any Reeb orbit in $Per(\alpha)$, and every $c \in \mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ is transverse.

We will say that a contact form satisfying the conditions above is *adapted* to the pair $(\Lambda, \widehat{\Lambda})$ in the complement of $\mathcal{G}$.

Let $\Sigma_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}$ denote the set of homotopy classes of curves in $Y \setminus \mathcal{G}$ having their starting point at Λ and their end point at $\widehat{\Lambda}$. For $\rho \in \Sigma_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}$ we denote by $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ the set of Reeb chords from Λ to $\widehat{\Lambda}$ that belong to ρ. Analogous to what we did previously, we will define $LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ as the $\mathbb{Z}_2$ vector-space generated by $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$. The $\mathbb{Z}_2$ grading in $LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ is defined in the same way we did in section 3.1, and we denote by $LCH^{\rho}_{st,even|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ the subspace of even elements of $LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ and by $LCH^{\rho}_{st,odd|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ the subspace of odd elements of $LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$

In order to define the differential adapted to our goals we have to consider another class of moduli spaces. For $J \in \mathcal{J}(\alpha_0)$, we notice that the set $\mathbb{R} \times \mathcal{G}$ is the union of a finite number of pseudoholomorphic cylinders in the symplectization of α_0 endowed with the almost complex structure J. We define $\mathcal{M}^k_{\mathcal{G}}(c, c'; J)$ to be the subset of $\mathcal{M}^k(c, c'; J)$ formed by pseudoholomorphic curves whose image in $\mathbb{R} \times Y$ do not intersect $\mathbb{R} \times \mathcal{G}$. Analogously, we define $\widetilde{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$ as the quotient of $\mathcal{M}^k_{\mathcal{G}}(c, c'; J)$ by the $\mathbb{R}$ action that comes from the fact that J is $\mathbb{R}$-invariant, and we let $\overline{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$ be the SFT-compactification of $\widetilde{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$.

Suppose we choose $J \in \mathcal{J}_{reg}(\alpha_0)$, then we will show that $\widetilde{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$ is a manifold of dimension $k-1$. First introduce a metric d in $\mathbb{R} \times Y$ which is $\mathbb{R}$ invariant. Taking $\widetilde{w} \in \widehat{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$ we claim that the infimum of the distance between points of the image of $\widetilde{w}$ and points of $\mathbb{R} \times \mathcal{G}$ is bigger then 0, i.e:

$$\inf_{x \in Rg(\widetilde{w})\ y \in \mathbb{R} \times \mathcal{G}} d(x, y) > 0 \tag{3.8}$$

where $Rg(\widetilde{w})$ denotes the image of the curve $\widetilde{w}$. To see that, notice that because its asymptotic behaviour, in a neighbourhood U_1 (U_{-1}) of the puncture 1 (-1) the curve $\widetilde{w}$ stays very close to the strip $\mathbb{R} \times c$ ($\mathbb{R} \times c'$). As both these strips have positive distance to $\mathbb{R} \times \mathcal{G}$ we conclude that there if there was a sequence x_n of points in $(\mathbb{D} \setminus \{1, -1\})$ such that the distance of $\widetilde{w}(x_n)$ to $\mathbb{R} \times \mathcal{G}$ converged to 0, then this sequence would have to have infinitely many points in the compact set $\mathbb{D} \setminus (U_1 \cup U_{-1})$; extracting then from x_n a convergent subsequence inside $\mathbb{D} \setminus (U_1 \cup U_{-1})$ we would have that for the limit point x of this subsequence the distance $d(\widetilde{w}(x), \mathbb{R} \times \mathcal{G})$ would be 0. This would imply that $\widetilde{w}(x)$ belongs to $\mathbb{R} \times \mathcal{G}$, contradicting the fact that $\widetilde{w} \in \widehat{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$. Let then $\epsilon > 0$ be such that $\inf_{x \in Rg(\widetilde{w})\ y \in \mathbb{R} \times \mathcal{G}} d(x, y) > \epsilon$

Now, there is an neighbourhood $\mathcal{U}_{\widetilde{w}}$ of $\widetilde{w}$ inside $\widetilde{\mathcal{M}}^k(c, c'; J)$ such that for every curve $\widetilde{w}' \in \mathcal{U}_{\widetilde{w}}$ the distance between the image of $\widetilde{w}'$ and the image of $\widetilde{w}$ is smaller than $\frac{\epsilon}{100}$; this implies that for such a $\widetilde{w}'$ distance between $\mathbb{R} \times \mathcal{G}$ and the image of $\widetilde{w}'$ is bigger than $\frac{\epsilon}{2}$ and we conclude that $\widetilde{w}'$ is also in $\widetilde{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$. Summing up, we have shown that for every $\widetilde{w} \in \widetilde{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$ there is a neighbourhood $\mathcal{U}_{\widetilde{w}}$ of $\widetilde{w}$ inside $\widetilde{\mathcal{M}}^k(c, c'; J)$ which is also inside $\widetilde{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$; as $\widetilde{\mathcal{M}}^k(c, c'; J)$ is a manifold of dimension $k-1$ it follows that so is $\widetilde{\mathcal{M}}^k_{\mathcal{G}}(c, c'; J)$.

We are now ready to define a differential $d^{\rho,\mathcal{G}}_J$ in $LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$.

Definition 3.6. Let $c \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ and $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$. We define:

$$d^{\rho,\mathcal{G}}_J(c) = \sum_{c' \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)} [n_{c,c'} \mod 2] c' \tag{3.9}$$

where $n_{c,c'}$ is the cardinality of the moduli space $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$ of pseudoholomorphic strips of Fredholm index 1 modulo the $\mathbb{R}$-action.

Similarly to what we did previously we have to prove that the differential is well-defined.

Lemma 3.7. *For $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$, $\rho \in \Sigma_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}$ and $d^{\rho,\mathcal{G}}_J$ defined before we have:*

- *(1') $d^{\rho,\mathcal{G}}_J$ is well defined,*

- *(2') $d_J^{\rho;\mathcal{G}}$ decreases the action of Reeb chords,*
- *(3') for each $c \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda},\mathcal{G}}(\alpha_0)$, $d_J^{\rho;\mathcal{G}}$ is a finite sum,*
- *(4') $d_J : LCH^{\rho}_{st,odd|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LCH^{\rho}_{st,even|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$*
  and $d_J : LCH^{\rho}_{st,even|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LCH^{\rho}_{st,odd|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$

Proof: we start proving that $d_J^{\rho;\mathcal{G}}$ is well-defined. For that, we will show that $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$ is finite for every c and c' in $\mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda},\mathcal{G}}(\alpha_0)$. Because of $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$, $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$ is a 0-dimensional manifold. We will show that it is compact, and therefore it has to be a finite set. To obtain the compactness of $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$, we will apply the standard "bubbling of" analysis for pseudoholomorphic curves of [28] and the SFT compactness results of [8] that we recalled in chapter 2.

Let $\widetilde{w}_n$ be a sequence of elements of $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$. We first argue that there are no "bubbling" points. Suppose there exists an interior "bubbling" point for the sequence $\widetilde{w}_n$; then a subsequence $\widetilde{w}_n$ converges to to a pseudoholomorphic building $\widetilde{w}$ with a tree structure as described in section 2.4.1, and one of the elements of this tree is a pseudoholomorphic plane $\widetilde{w}_{pl}$. We claim that the image of $\widetilde{w}_{pl}$ must intersect the set $\mathbb{R} \times \mathcal{G}$. There are two possibilities for the plane $\widetilde{u}$: its asymptotic limit is either a Reeb orbit in $\mathcal{G}$ or a Reeb orbit outside of $\mathcal{G}$; in both cases the condition (1') we imposed on α_0 implies that the projection w_{pl} of $\widetilde{w}_{pl}$ to Y must have an interior intersection point with $\mathcal{G}$. Because of positivity of intersections for pairs of pseudoholomorphic curves we know that the intersection point between $\widetilde{w}_{pl}$ and $\mathbb{R}\times\mathcal{G}$ (which is a collection of pseudoholomorphic cylinders) has to survive if we make a small perturbation of $\widetilde{w}_{pl}$ restricted to a compact neighbourhood Ko_{pl} of the intersection point. For an n belonging to our subsequence and sufficiently large, it follows from the definition of SFT-convergence, that there is a compact set Ko such that the restriction $\widetilde{w}_n \mid_{Ko}$ is a small perturbation of $\widetilde{w}_{pl} \mid_{Ko_{pl}}$, something that forces $\widetilde{w}_n \mid_{Ko}$ to have an intersection point with $\mathbb{R} \times \mathcal{G}$. But this is impossible, since we had assumed all $\widetilde{w}_n$ were elements of $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$.

To see that the sequence $\widetilde{w}_n$ has no boundary "bubbling" points we proceed in a similar manner. Assuming that such a "bubbling" point does exist we know that there is a subsequence of $\widetilde{w}_n$ converging to a pseudoholomorphic building $\widetilde{w}$ with a tree structure as described in section 2.4.1, such that one of the elements of this tree is a pseudoholomorphic half-plane $\widetilde{w}_{hp}$. The half-plane $\widetilde{w}_{hp}$ has its boundary entirely contained on either $\mathbb{R} \times \Lambda$ or $\mathbb{R} \times \widehat{\Lambda}$ and his only positive puncture is asymptotic to a Reeb chord c_{hp} going either from Λ to itself or from $\widehat{\Lambda}$ to itself. In both situations this c_{hp} is a contractible chord on $\pi_1(Y\setminus, \Lambda)$ or in $\pi_1(Y, \widehat{\Lambda})$. From assumptions (b') and (c') we placed on α_0 we can conclude that in both these possibilities the plane $\widetilde{w}_{hp}$ has to have an interior intersection point with the set $\mathbb{R} \times \mathcal{G}$. Reasoning exactly as above, it follows from positivity of intersections and the definition of SFT convergence that for an n belonging to our subsequence and sufficiently large, the map $\widetilde{w}_n$ would intersect $\mathbb{R} \times \mathcal{G}$, leading to a contradiction.

The rest of the proof of item (1') is very similar to the final part of the proof of item (1) of lemma 3.2. Basically, we apply SFT-compactness theorem, to obtain that a subsequence of $\widetilde{w}_n$ converges in the SFT sense to a pseudoholomorphic building $\widetilde{w}$ with k-levels $\widetilde{w}^l$, where all levels $\widetilde{w}^l$ are pseudoholomorphic strips with:

- 1 is a positive boundary puncture, and $\widetilde{w}^l$ is asymptotic to $c_l \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ at 1,
- -1 is a negative boundary puncture, and $\widetilde{w}$ is asymptotic to $c_{l+1} \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ at -1,
- $\widetilde{w}(H_-) \subset \mathbb{R} \times \Lambda$,
- $\widetilde{w}(H_+) \subset \mathbb{R} \times \widehat{\Lambda}$;

where $c_1 = c$, $c_{k+1} = c'$ and $c_l \neq c_{l+1}$. It follows then, from the regularity of J and fact that all $\widetilde{w}^l$ are somewhere injective pseudoholomorphic strips different from trivial strips over Reeb chords, that the Fredholm indexes of each of these strips satisfies $F(\widetilde{w}^l) \geq 1$. As the Fredholm index is the building $\widetilde{w}$ is the sum of the indexes of its levels we have $I_F(\widetilde{w}) = \sum (I_F(\widetilde{w}^l) \geq l$; on the other hand as $\widetilde{w}$ is the limit of a sequence of pseudoholomorphic strips of Fredholm index 1, it has to satisfy $I_F(\widetilde{w}) = 1$; as a consequence we must have $l = 1$, and $\widetilde{w} \in \widetilde{\mathcal{M}}^1(c, c'; J)$. To see that $\widetilde{w}$ is actually an element of $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$ we reason by contradiction. If this was not the case then there would be an interior point of $\widetilde{w}$ intersecting $\mathbb{R} \times \mathcal{G}$. Again a combination of positivity of intersection and SFT convergence would imply that in this case elements of the sequence $\widetilde{w}_n$ would have to intersect $\mathbb{R} \times \mathcal{G}$ forcing a contradiction.

As a conclusion, we have shown that a sequence of elements in $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$ has a subsequence that converges to an element of $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$, obtaining the desired compactness of $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$. As $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$ is a 0-dimensional manifold, it follows that it is a finite set, which implies (1').

Item (2') follows from the fact that $n_{c,c'}$ can only be non-zero for Reeb chords c' with $A(c') < A(c)$. This implies that d_J decreases the action of Reeb chords.

The condition (d') placed on α_0 implies that the set of Reeb chords in $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ with action smaller then $A(c)$ is finite, and therefore $n_{c,c'}$ is non-zero only for a finite number of c'. This plus items (1') and (2') gives us that $d_J(c)$ is a finite sum as claimed in (3').

Item (4') follows easily from the fact that the Fredholm index of a strip connecting two chords c and c' has the same parity as $\mid c \mid + \mid c' \mid$. Thus, $\widetilde{\mathcal{M}}^1_{\mathcal{G}}(c, c'; J)$ can be non-empty only if c and c' have different parity.

□

Having established these properties of $d^{\rho,\mathcal{G}}_J$ we proceed to prove that $d^{\rho,\mathcal{G}}_J \circ d^{\rho,\mathcal{G}}_J = 0$.

Lemma 3.8. *For $J \in \mathcal{J}_{reg}(\alpha_0) \subset \mathcal{J}(\alpha_0)$ and $d^{\rho,\mathcal{G}}_J$ as defined above we have:*

$$d^{\rho,\mathcal{G}}_J \circ d^{\rho,\mathcal{G}}_J = 0 \tag{3.10}$$

Proof: the lemma will be a consequence of the description we will give of the compactified moduli space $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ of pseudoholomorphic strips with Fredholm index 2. Because of regularity of J, it will follow that for all $c, c' \in \mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$, $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ is either empty, or the finite union of disjoint circles and closed intervals. We summarise that in the following claim:

Compactness Claim: suppose $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ is non-empty. Then, each connected component I of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ is either a circle or a closed interval. Moreover, when I is diffeomorphic to a closed interval, its boundary is composed by pseudoholomorphic buildings $\widetilde{w}$ with 2 levels $\widetilde{w}_1$ and $\widetilde{w}_2$ satisfying:

$$\widetilde{w} \in \overline{\mathcal{M}}^1_{\mathcal{G}}(c, \check{c}; J) \textit{ and } \widetilde{w_2} \in \overline{\mathcal{M}}^1_{\mathcal{G}}(\check{c}, c'; J) \textit{ for some } \check{c} \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda},\mathcal{G}}(\alpha_0).$$

As we did in the proof of Lemma 3.3 we will first show how to use the claim to prove the Lemma, and only after we will provide the proof of the Compactness Claim.

$$d^{\rho,\mathcal{G}}_J \circ d^{\rho,\mathcal{G}}_J(c) = \sum_{r' \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda},\mathcal{G}}(\alpha_0)} (m_{c,c'} \mod 2)c' \tag{3.11}$$

The lemma will follow if we prove that $m_{c,c'}$ is always even; this will be achieved by showing that $m_{c,c'}$ is exactly the number of boundary components of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$.

Firstly, notice that it follows from our definition of $d^{\rho,\mathcal{G}}_J$, that $m_{c,c'}$ counts the number of two-level pseudoholomorphic buildings whose levels $\widetilde{w}_1$ and $\widetilde{w}_2$ satisfy: $\widetilde{w}_1 \in \overline{\mathcal{M}}^1_{\mathcal{G}}(c, \check{c}; J)$ and $\widetilde{w_2} \in \overline{\mathcal{M}}^1_{\mathcal{G}}(\check{c}, c'; J)$ for some $\check{c} \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda},\mathcal{G}}(\alpha_0)$. The compactness claim implies that only these types of buildings can appear in the boundary of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$. We thus have that the number of boundary components of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ is smaller or equal to $m_{c,c'}$.

On the other hand, because of the regularity of $J \in \mathcal{J}_{reg}(\alpha_0)$, we can apply the gluing theorem: this theorem implies that if $\widetilde{w}$ is a 2-level pseudoholomorphic building whose levels $\widetilde{w}_1$ and $\widetilde{w}_2$ satisfy $\widetilde{w}_1 \in \overline{\mathcal{M}}^1_{\mathcal{G}}(c, \check{c}; J)$ and $\widetilde{w_2} \in \overline{\mathcal{M}}^1_{\mathcal{G}}(\check{c}, c'; J)$ for some $\check{c} \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$, then $\widetilde{w}$ is in the boundary of a connected component of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$. We then conclude that $m_{c,c'}$ is bigger or equal to the number of boundary points of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$.

Summarising, the combination of the Compactness Claim and the gluing allows us to conclude that the number $m_{c,c'}$ is exactly the number of boundary components of the moduli space $\overline{\mathcal{M}}^2(c, c'; J)$. Because $\overline{\mathcal{M}}^2(c, c'; J)$ is a finite union of disjoint intervals and circles, this number is even. This finishes the proof of the lemma modulo the Compactness claim.

Proof of Compactness Claim: suppose $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ is non-empty. It follows from the regularity of J that each connected component I of $\widetilde{\mathcal{M}}^2(c, c'; J)$ is a 1-dimensional manifold. We have now two possibilities, either I is compact or not. If I is compact then it is a circle.

If that is not the case, let $\widetilde{w}_n$ be a sequence of elements of $\widetilde{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ converging to the boundary of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$. Reasoning as in the proof of lemma 3.7 one obtains that no "bubbling" can occur. Thus the SFT compactness theorem of implies that $\widetilde{w}_n$ converges to a pseudoholomorphic building $\widetilde{w}$ with k-levels $\widetilde{w}^l$, such that all levels $\widetilde{w}^l$ are pseudoholomorphic strips satisfying:

- 1 is a positive boundary puncture, and $\widetilde{w}^l$ is asymptotic to $c_l \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ at 1
- -1 is a negative boundary puncture, and $\widetilde{w}$ is asymptotic to $c_{l+1} \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ at -1
- $\widetilde{w}(H^-) \subset \mathbb{R} \times \Lambda$,
- $\widetilde{w}(H_+) \subset \mathbb{R} \times \widehat{\Lambda}$,

where $c_1 = c$ and $c_{k+1} = c'$. Again because every $\widetilde{w}^l$ is somewhere injective we have that the Fredholm index $F(\widetilde{w}^l) \geq 1$ and thus $I_F(\widetilde{w}) = \sum(I_F(\widetilde{w}^l) \geq l$. On the other hand as $\widetilde{w}$ is the limit of a sequence of pseudoholomorphic strips of Fredholm index 2, it has to satisfy $I_F(\widetilde{w}) = 2$.

We have then 2 possibilities: either $l = 1$ and $\widetilde{w} \in \overline{\mathcal{M}}^2(c, c'; J)$; or $l = 2$ which forces $I_F(\widetilde{w}^1) = I_F(\widetilde{w}^1) = 1$, $\widetilde{w}^1 \in \overline{\mathcal{M}}^1(c, c_2; J)$ and $\widetilde{w}^2 \in \overline{\mathcal{M}}^1(c_2, c'; J)$. In the first possibility, again a applying positivity intersections and the notion of SFT-convergence we can guarantee that $\widetilde{w} \in \widetilde{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$; but this is can be ruled out since we assumed that the sequence we took converged to the boundary of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$.

Fot the second possibility, we can prove (again through use of positivity of intersections) that $\widetilde{w}^1 \in \overline{\mathcal{M}}^1_{\mathcal{G}}(c, c_2; J)$ and $\widetilde{w}^2 \in \overline{\mathcal{M}}^1_{\mathcal{G}}(c_2, c'; J)$.

We have obtained that all the elements on the boundary of I are 2-level pseudoholomorphic buildings with the properties claimed. This implies that the boundary of $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ is 0-dimensional manifold.

On the other hand, the gluing theorem gives the description of a neighbourhood of the 2-level pseudoholomorphic buildings appearing in the boundary $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$. This neighbourhood admits a diffeomorphism to the infinite interval $[0, +\infty)$, that takes 0 to the 2-level building and all other values to pseudoholomorphic strips in $\widetilde{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$.

Summing up, the compactification of the component I in $\overline{\mathcal{M}}^2_{\mathcal{G}}(c, c'; J)$ has the structure of a manifold with boundary, and in our particular case I it must be a 1-dimensional with 0-dimensional boundary; i.e a closed interval. This finishes the proof of the compactness claim.

□

We have thus obtained that under appropriate conditions the strip Legendrian contact homology on the complement of Reeb orbits can be well defined. We will now proceed to show how to construct cobordism maps for the strip Legendrian contact homology in appropriate conditions.

We consider the following situation. Let $(\mathbb{R} \times Y, d\varsigma)$ be an exact symplectic cobordism from the contact form α_0 to $\mathsf{k}\alpha_0$ where $1 > \mathsf{k} > 0$ is a constant, and let L and $\widehat{L}$ be exact Lagrangian cobordism diffeomorphic to cylinders from, respectively, Λ to itself, and $\widehat{\Lambda}$ to itself. On $(\mathbb{R} \times Y, d\varsigma)$ we consider the space $\mathcal{J}^{\mathcal{G}}(J, J)$ of almost complex structures compatible with $d\varsigma$ which are positively and negatively asymptotic to J, and for which the set $\mathbb{R} \times \mathcal{G}$ is a union of pseudoholomorphic cylinders. For $\overline{J} \in \mathcal{J}^{\mathcal{G}}(J, J)$, we consider the moduli spaces $\mathcal{M}_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})$ of pseudoholomorphic strips in $\mathcal{M}(c, c'; \overline{J}, L, \widehat{L})$ which do not intersect $\mathbb{R} \times \mathcal{G}$, where the Reeb chords $c, c' \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}$. Using perturbation techniques as in [30] and [35] one can prove that there is a generic set $\mathcal{J}^{\mathcal{G}}_{reg}(J, J) \subset \mathcal{J}^{\mathcal{G}}(J, J)$ such that for all $\overline{J} \in \mathcal{J}^{\mathcal{G}}_{reg}(J, J)$ the moduli spaces $\mathcal{M}_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})$ are Fredholm regular for every pair $c, c' \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}$.

Taking $\overline{J} \in \mathcal{J}^{\mathcal{G}}_{reg}(J, J)$ we define the map $\Phi_{\overline{J}, \varpi, L, \widehat{L}} : LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$, defined by counting elements in $\mathcal{M}^0_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})$. Precisely, for $c \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}$ we define:

$$\Phi_{\overline{J}, \varpi, L, \widehat{L}}(c) = \sum_{c' \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}} [\#(\mathcal{M}^0_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})) \mod 2] c' \tag{3.12}$$

We then have the following proposition, which is a analogous of proposition 3.4 for the present setup:

Proposition 3.9. *The map* $\Phi_{\overline{J},\varpi,L,\widehat{L}} : LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ *induces a map* $\Phi_{\overline{J},\varpi,L,\widehat{L}} : LC\mathbb{H}^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LC\mathbb{H}^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ *on the homology level.*

Proof: the proof is completely analogous to the one of Proposition 3.4. The main step one has to prove is that $d^{\rho,\mathcal{G}}_J \circ \Phi_{\overline{J},\varpi,L,\widehat{L}} = \Phi_{\overline{J},\varpi,L,\widehat{L}} \circ d^{\rho,\mathcal{G}}_J$.

This identity follows from the description of the the compactfied moduli space $\overline{\mathcal{M}}^1_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})$. Similarly to what was done in the proof of proposition 3.4 what we want is to show that the boundary of the moduli space $\overline{\mathcal{M}}^1_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})$ is equal to the space of 2-level pseudoholomorphic buildings $\widetilde{w}$ whose levels $\widetilde{w}^1$ and $\widetilde{w}^2$ satisfy one of the following conditions:

- there exists $\check{c} \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}}$ such that $\widetilde{w}^1 \in \overline{\mathcal{M}}^0_{\mathcal{G}}(c, \check{c}; \overline{J}, L, \widehat{L})$ and $\widetilde{w}^2 \in \overline{\mathcal{M}}^1_{\mathcal{G}}(\check{c}, c'; J)$,
- there exists $\check{c} \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}}$ such that $\widetilde{w}^1 \in \overline{\mathcal{M}}^1_{\mathcal{G}}(c, \check{c}; J)$ and $\widetilde{w}^2 \in \overline{\mathcal{M}}^0_{\mathcal{G}}(\check{c}, c; \overline{J}, L, \widehat{L})$.

The proof of this fact involves two steps: gluing and compactness. To show that the 2-level buildings of this type appear in the boundary of $\overline{\mathcal{M}}^1_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})$ is just an application of the gluing theorem of section 2.4.

The compactness part is to show that any sequence of elements in $\overline{\mathcal{M}}^1_{\mathcal{G}}(c, c'; \overline{J}, L, \widehat{L})$ converging to the boundary can only converge to a 2-level building of the type mentioned above. To see that this is the case we first notice that no bubbling can occur: this follows from the fact that the contact form α_0 satisfies the conditions (a'), (b'), (c') and (d') from the beginning of the section combined with the fact that $\mathbb{R} \times \mathcal{G}$ is a union of pseudoholomorphic cylinders. Thus such a sequence can only converge to a building formed only by strips. An argument using positivity of intersections identical to the ones in lemmas 3.7 and 3.8, implies that no level of this building can intersect $\mathbb{R} \times \mathcal{G}$. That one can only have two levels and that they are of the type above follows from the regularity of J and $\overline{J}$.

□

Lastly, we prove for the present setup a proposition which analogous to Proposition 3.5. Let $(\mathbb{R} \times Y, d\varsigma)$ be an exact symplectic cobordism from the contact form α_0 to $\mathsf{k}\alpha_0$ where $1 > \mathsf{k} > 0$ is a constant, and let L and $\widehat{L}$ be exact Lagrangian cobordisms from, respectively, Λ to itself, and $\widehat{\Lambda}$ to itself. We assume there exists an isotopy $(\mathbb{R} \times Y, d\varsigma_t)$ (for $t \in [0, 1]$) of exact symplectic cobordisms, together with isotopies L_t $\widehat{L}_t$ of exact Lagrangian cobordisms from, respectively, Λ to itself, and $\widehat{\Lambda}$ to itself, satisfying:

- $d\varsigma_0 = d\varsigma$ and $d\varsigma_1 = d(e^s\alpha_0)$,
- $L_0 = L$ and $L_1 = \mathbb{R} \times \Lambda$,
- $\widehat{L}_0 = \widehat{L}$ and $\widehat{L}_1 = \mathbb{R} \times \widehat{\Lambda}$.

Letting $\overline{J} \in \mathcal{J}^{\mathcal{G}}_{reg}(J, J)$ we have the following:

Proposition 3.10. *Under the conditions above, the induced map* $\Phi_{\overline{J},\varpi,L,\widehat{L}} : LC\mathbb{H}^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LC\mathbb{H}^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ *on the homology level is the identity.*

Proof: the idea is to mimic the proof of Proposition 3.5. For that the first thing to do is to introduce a space of homotopy of almost complex structures. We denote by $\widehat{\mathcal{J}}^{\mathcal{G}}(J, J)$ the space of smooth homotopies:

$$\overline{J}_t \in \mathcal{J}^{\mathcal{G}}(J, J) : t \in [0, 1] \tag{3.13}$$

such that $\overline{J}_0 = \overline{J}$ and $\overline{J}_1 = J$. Then, for all Reeb chords $c, c' \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}$ we consider the moduli spaces:

$$\widehat{\mathcal{M}}_{\mathcal{G}}(c, c'; \overline{J}_t, L_t, \widehat{L}_t) = \{(t, \widetilde{w}) : t \in [0, 1] \text{ and } \widetilde{w} \in \mathcal{M}_{\mathcal{G}}(c, c'; \overline{J}_t, L_t, \widehat{L}_t)\} \tag{3.14}$$

Using the techniques of [14], [2] and [35] one obtains that for a generic subset $\widehat{\mathcal{J}}^{\mathcal{G}}_{reg}(J, J) \subset \widehat{\mathcal{J}}^{\mathcal{G}}(J, J)$ the moduli spaces $\widehat{\mathcal{M}}_{\mathcal{G}}(c, c'; \overline{J}_t, L_t, \widehat{L}_t)$ are Fredholm regular for all $c, c' \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}$.

With this in hand we define a map $\mathcal{K}_{\mathcal{G}} : LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \to LCH^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ that counts finite energy, Fredholm index -1 pseudoholomorphic strips in $(\mathbb{R} \times Y, d\varsigma_t,)$, with one boundary component in L_t and one in $\widehat{L_t}$. Because of the Fredholm regularity of $\widehat{\mathcal{M}}_{\mathcal{G}}(c, c'; \overline{J}_t, L_t, \widehat{L}_t)$, this set of index -1 strips whose positive puncture detects a fixed chord c is finite, and therefore the map $\mathcal{K}$ is really well defined.

We now want to show that $Id + \Phi_{\overline{J},\varpi,L,\widehat{L}} + \mathcal{K}_{\mathcal{G}} \circ d^{\rho,\mathcal{G}}_{J} + d^{\rho,\mathcal{G}}_{J} \circ \mathcal{K}_{\mathcal{G}} = 0$. The proof of this fact is an adaptation of the analogous result that was proved in proposition 3.5 and we leave it to the reader to complete it. We just mention that it is again a combination of compactness and gluing to show that the pseudoholomorphic buildings involved in the map$Id + \Phi_{\overline{J},\varpi,L,\widehat{L}} + \mathcal{K}_{\mathcal{G}} \circ d^{\rho,\mathcal{G}}_{J} + d^{\rho,\mathcal{G}}_{J} \circ \mathcal{K}_{\mathcal{G}} = 0$ form the boundary of the 1-dimensional space with boundary $\widehat{\mathcal{M}}^1_{\mathcal{G}}(c, c'; \overline{J}_t, L_t, \widehat{L}_t)$. It is in the compactness part that one uses the fact that α_0 satisfies the conditions (a'), (b'), (c') and (d') combined with the fact that at all times t, the set $\mathbb{R} \times \mathcal{G}$ is a union of pseudoholomorphic cylinders for the almost complex structure $\overline{J}_t$ in $\mathbb{R} \times Y$.

□

3.5 Gluing revisited

For propositions 3.5 and 3.10 we need a slightly different version of the gluing theorem that we now proceed to describe. We remember the reader that the SFT-compactness theorem allows us to compactly the moduli spaces $\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$. We notice that the tangent space of $\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$ is not expected to be the kernel of the Cauchy-Riemann operator, but of a modified version $\mathcal{D}_{J_t}$ of it that takes into account the parameter t of deformation of almost complex structures. This modified operator whose kernel is supposed to describe the tangent space $\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$ has all the"good" properties of the Cauchy-Riemann operators, for example it is a Fredholm operator in an appropriate functional setting. This is why, one can use essentially the same techniques used to show that moduli spaces of the type $\widetilde{\mathcal{M}}(c, c'; J)$ are Fredholm regular to show that spaces of type $\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$ are Fredholm regular. The idea for this last case is to obtain conditions under which one can guarantee that $\mathcal{D}_{J_t}$ is surjective. We refer to [14] and [15] for a discussion of such ideas.

For gluing one has a similar picture in the sense that to prove a gluing theorem that helps us to describe the compactification of $\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$ one does not need to introduce new techniques, but only to modify the ones used for proving theorems 2.4 and 2.5.

We will use the notation of section 3.3.1. Let $J_t \in \widetilde{\mathcal{J}}_{reg}(J, J)$ be a regular homotopy of almost complex structures in $(\mathbb{R} \times Y, \varpi_t)$, where $J \in \mathcal{J}_{reg}(\alpha_0)$. We will denote by $\widehat{\mathcal{M}}^k(c, c'; J_t, L_t, \widehat{L}_t)$ the set of elements of $\widehat{\mathcal{M}}(c, c'; J_t, L_t, \widehat{L}_t)$ such that the Cauchy-Riemann operator $\overline{\partial}_{J_t}$ has Fredholm index k. Because of the regularity of J_t and from the discussion above, we have that the dimension of $\widehat{\mathcal{M}}^k(c, c'; J_t, L_t, \widehat{L}_t)$ is $k+1$, which coincides with Fredholm index of $\mathcal{D}_{J_t}$.

We let now c, c', $\check{c}^+$ and $\check{c}^-$ be transverse Reeb chords in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$. For a fixed t_0, let $\widetilde{w}_+$ be a 2-level building such that its top level $\widetilde{w}_+^1$ is an element of $\widetilde{\mathcal{M}}^1(c, \check{c}^+; J_{t_0})$ and $\widetilde{w}_+^2$ is an element of $\widehat{\mathcal{M}}^{-1}(\check{c}^+, c'; J_{t_0})$. Analogously, let $\widetilde{w}_-$ be a 2-level building such that its top level $\widetilde{w}_-^1$ is an element of $\widehat{\mathcal{M}}^{-1}(c, \check{c}^-; J_{t_0})$ and $\widetilde{w}_-^2$ is an element of $\widetilde{\mathcal{M}}^1(\check{c}^-, c'; J_{t_0})$.

Theorem 3.11. *If there is a building $\widetilde{w}_+$ as described above, then there exists an embedding $\Psi^+ : [0, +\infty) \to \overline{\mathcal{M}}^1(c, c'; J_t, L_t, \widehat{L}_t)$ such that:*

- $\Psi^+(0) = \widetilde{w}_+$, *where $\widetilde{w}_+$ is the two level building whose top level is $\widetilde{w}_+^1$ and bottom level is $\widetilde{w}_+^2$,*
- $\Psi^+(s) \in \widehat{\mathcal{M}}^1(c, c'; J_t, L_t, \widehat{L}_t)$ *for every $s \in (0, +\infty)$,*

- *the map* Ψ^+ *is a homeomorphism from* $[0,+\infty)$ *to a neighbourhood of* $\widetilde{w}_+$ *in* $\overline{\mathcal{M}}^1(c,c';J_t,L_t,\widehat{L}_t)$.

Moreover, if $\widetilde{w}_+(n)$ *is a sequence of elements of* $\widehat{\mathcal{M}}^1(c,c';J_{t_0})$ *converging to* $\widetilde{w}_+$*, then there exists* n_0 *such that* $\widetilde{w}_+(n) \in \Psi^+([0,1])$ *for all* $n \geq n_0$.

Analogously, if there is a building $\widetilde{w}_-$ *as described previously, then there exists an embedding* $\Psi^- : [0,+\infty) \to \overline{\mathcal{M}}^1(c,c';J_t,L_t,\widehat{L}_t)$ *such that:*

- $\Psi^-(0) = \widetilde{w}_-$*, where* $\widetilde{w}_-$ *is the two level building whose top level is* $\widetilde{w}_-^1$ *and bottom level is* $\widetilde{w}_-^2$*,*
- $\Psi^-(s) \in \widehat{\mathcal{M}}^1(c,c';J_t,L_t,\widehat{L}_t)$ *for every* $s \in (0,+\infty)$*,*
- *the map* Ψ^- *is a homeomorphism from* $[0,+\infty)$ *to a neighbourhood of* $\widetilde{w}_-$ *in* $\overline{\mathcal{M}}^1(c,c';J_t,L_t,\widehat{L}_t)$.

Moreover, if $\widetilde{w}_-(n)$ *is a sequence of elements of* $\widehat{\mathcal{M}}^1(c,c';J_{t_0})$ *converging to* $\widetilde{w}_-$*, then there exists* n_0 *such that* $\widetilde{w}_-(n) \in \Psi^-([0,1])$ *for all* $n \geq n_0$.

It is this gluing theorem that we used in the proofs of propositions 3.5 and 3.10.

Chapter 4

Homotopical growth rate of Legendrian contact homology and topological entropy

Given a contact 3-manifold (Y, ξ), for an associated contact form α_0 adapted to a pair of disjoint Legendrian knots Λ and $\widehat{\Lambda}$ we define the exponential homotopical growth of the strip Legendrian contact homology $LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ with respect to the action. We then use it to estimate the growth of the number of Reeb chords from Λ to Λ' for other contact forms associated to (Y, ξ) when Λ' is sufficiently close to $\widehat{\Lambda}$.

We define for each number $C > 0$ the set $\Sigma^C_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ of homotopy classes ρ satisfying:

- all the chords in $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ have action smaller then C
- $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda}) \neq 0$

Definition 4.1. We say that a hypertight contact form α_0 (associated to a contact manifold (Y, ξ)) presents exponential homotopical growth of strip Legendrian contact homology if α_0 is adapted to a Legendrian link Λ, $\widehat{\Lambda}$, and there are constants $C_0 > 0$, $a > 0$ and d such that:

$$\#(\Sigma^C_{\Lambda \to \widehat{\Lambda}}(\alpha_0)) > e^{aC+d} \tag{4.1}$$

for all $C > C_0$.

In this case we will say that $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$.

4.1 Growth of the number of Reeb chords

Let Λ and $\widehat{\Lambda}$ be Legendrian submanifolds of $(Y, \xi = \ker(\alpha))$: we will say that $\widehat{\Lambda}$ is (α, Λ)-transverse if all the Reeb chords in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha)$ are transverse. We will denote by $N_C(\alpha, \Lambda, \widehat{\Lambda})$ the number of Reeb chords in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha)$ with action $\leq C$. We will start by presenting a result which shows how to estimate the growth of number of Reeb chords using the strip Legendrian contact homology.

We will consider the space of $C^k_{diff}(M)$ of C^k-diffeomorphisms of M. We will fix in this space a metric which generates the canonical topology on $C^k_{diff}(M)$. From now on, when we say that two diffeomorphisms of M are ϵ-close, we mean close with respect to this fixed metric. Likewise, we will fix a metric on the space of loops in M.

By the Weinstein tubular neighbourhood theorem the Legendrian knot $\widehat{\Lambda}$ has a tubular neighbourhood N_ϵ contactomorphic to the local model given by:

$$\ker(\cos(\theta)dx + \sin(\theta)dy) \tag{4.2}$$

where $(\theta, z) \in S^1 \times \mathbb{D}$, $D = \{z \in \mathbb{C}; \mid z \mid \leq 1\}$ and $\widehat{\Lambda}$ is identified with the circle $(\theta, 0, 0)$. We denote by $\widehat{\Lambda}^z$ the Legendrian submanifolds (θ, z) obtained by fixing z. As the Λ^z form a fibration of N_ϵ by Legendrian knots we will refer to them as Legendrian fibers. The neighbourhood N_ϵ is chosen sufficiently small so that all Legendrian fibers Λ^z are ϵ-close to $\widehat{\Lambda}$.

We will now show how to construct an appropriate Lagrangian cobordism from $\widehat{\Lambda}$ to the Legendrian fibers $\widehat{\Lambda}^z$. It is clear that $\widehat{\Lambda}^z$ is Legendrian isotopic to $\widehat{\Lambda}$, this can be done explicitly by taking a path of Legendrian fibers which starts at $\widehat{\Lambda}^z$ and finishes at $\widehat{\Lambda}$. Clearly for every $z \in \mathbb{D}$, by a smoothing process we can then obtain a path $\widehat{\Lambda}_t$ $(t \in \mathbb{R})$ with $\widehat{\Lambda}_t = \widehat{\Lambda}^z; t \leq 0$ and $\Lambda_t = \widehat{\Lambda}; t \geq 1$. Consider now a path of diffeomorphisms $F^z_t : Y \to Y$, which equals the identity outside a small neighbourhood of $\widehat{\Lambda}$ disjoint from Λ and such that $F^z_t(\widehat{\Lambda}_t) = \widehat{\Lambda}$, $F^z_t = Id$ for $t \geq 1$ and F^z_t is independent of t for $t \leq 0$. The main observation is that, if we take the neighbourhood N_ϵ to be sufficiently small, we can demand that our F^z_t satisfies, $d_{C^9}(F^z_t, Id) < \mu(\epsilon)$, for all $z \in \mathbb{D}$. In other words, we can have an uniform control on the distance of F^z_t from the identity map Id if we restrict our attention to a sufficiently small neighbourhood of $\widehat{\Lambda}$.

We will use the maps F^z_t to construct an exact Lagrangian cobordism from $\widehat{\Lambda}$ to the Legendrian fiber $\widehat{\Lambda}^z$. For this we define the smooth family of contact forms $\alpha^t_z = (F^z_t)^*\alpha_0$. Notice that the Reeb vector fields of these contact forms have the same dynamics of the Reeb vector field X_{α_0}; however for the contact form α^t_z the curve $\widehat{\Lambda}^z$ is represented by $\widehat{\Lambda}$. Precisely, we have that the Reeb chords in $\mathcal{T}_{\Lambda \to \widehat{\Lambda}}(\alpha^0_z)$ are in bijective correspondence with the Reeb chords in $\mathcal{T}_{\Lambda\to\widehat{\Lambda}^z}(\alpha_0)$. This follows from the relation $\phi^t_{X_{\alpha^0_z}} \circ (F^z_0)^{-1} = (F^z_0)^{-1} \circ \phi^t_{X_{\alpha_0}}$ between the Reeb flows of α^0_z and α_0. Let now $h_\delta : \mathbb{R} \to \mathbb{R}$, be an increasing function satisfying:

$$h_\delta(0) = (1-\delta) \text{ and } h(1) = 1, \tag{4.3}$$

$$h_\delta(t) = (1-2\delta)e^{t+1} \text{ for } t \leq -1 \text{ and } h_\delta(t) = (1+2\delta)e^{t-2} \text{ for } t \geq 2, \tag{4.4}$$

$$h'_\delta(t) > 0 \text{ for all } t \in \mathbb{R} \text{ and } h'_\delta(t) = \delta \text{ for } t \in [0,1]. \tag{4.5}$$

If we choose ϵ (and consequently the neighbourhood N_ϵ) sufficiently small, we have that $d(h_\delta(t)\alpha^t_z)$ is an exact symplectic form in the manifold $\mathbb{R} \times Y$. This defines an exact symplectic cobordism from α^t_z to $(1-2\delta)\alpha_0$, with the property that the $\widehat{L} = \mathbb{R}\times\widehat{\Lambda}$ is an exact Lagrangian submanifold. Performing an identical construction, we can produce an exact symplectic cobordism from $(1-2\delta)\alpha_0$ to α^t_z, such that $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ is an exact Lagrangian submanifold.

Summing up the discussion above, we have shown that, given $\delta > 0$, there is $\epsilon > 0$, so that for every Legendrian fiber $\widehat{\Lambda}^z$ in N_ϵ, we can construct an exact symplectic cobordisms from $(1+2\delta)\alpha_0$ to $(1-2\delta)\alpha_0$ that coincide with a piece of the symplectization of α^0_z in $[0,1] \times Y$, and such that $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ and $L = \mathbb{R} \times \Lambda$ are exact Lagrangian submanifolds.

Proposition 4.2. *Let (Y,ξ) be a contact manifold and Λ and $\widehat{\Lambda}$ be two disjoint Legendrian submanifolds, such that α_0 is associated to (Y,ξ) and adapted to the pair $(\Lambda,\widehat{\Lambda})$. Suppose that the strip contact homology $LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$. Let α be another contact form associated to (Y,ξ), and take $g > 0$ to be the function such that $\alpha = g_\alpha \alpha_0$. Then given $\delta > 0$ there exists $\epsilon > 0$ and $C_0 \geq 0$ such that, for every Legendrian fiber $\widehat{\Lambda}^z$ in N_ϵ which is (α,Λ) transverse, the numbers $N_C(\alpha,\Lambda,\widehat{\Lambda}^z)$ satisfy*

$$e^{\frac{aC}{(1+4\delta)\max(g_\alpha)}} < N_C(\alpha,\Lambda,\widehat{\Lambda}^z) \tag{4.6}$$

for all $C \geq C_0$.

Proof: we divide the proof in steps.

Step 1: Reduction to the non-degenerate case. We first show that it suffices to prove the estimate for non-degenerate contact forms; as the general case follows from this one by an approximation procedure.

Let j be a natural number. As $\widehat{\Lambda}^z$ is (α, Λ) transverse, it is possible to make a C^∞ small perturbation of the contact form α to a non-degenerate contact form $\alpha(j)$ which generates the same contact structure as α and such that:

- $N_C(\alpha, \Lambda, \widehat{\Lambda}^z) = N_C(\alpha(j), \Lambda, \widehat{\Lambda}^z)$, for all $C \leq j$,
- $\widehat{\Lambda}^z$ is $(\alpha(j), \Lambda)$ transverse.

We demand that the perturbations $\alpha(j)$ are taken to be close enough to α to guarantee that there are exact symplectic cobordisms from α to $\frac{1}{1+j}\alpha(j)$ and from $\alpha(j)$ to $\frac{1}{1+j}\alpha$. Supposing now that the proposition is true for all $\alpha(j)$, we get, because of the conditions on the $\alpha(j)$, that for a given $C > 0$ we have for all $j \geq C$:

$$N_C(\alpha', \Lambda, \widehat{\Lambda}^z) = N_C(\alpha'_j, \Lambda, \widehat{\Lambda}^z) > e^{\frac{aC}{(1+4\delta)\max(g_j)}} \tag{4.7}$$

where g_j is the function such that $g_j \alpha = \alpha(j)$.

Letting $j \to +\infty$ we have that $g_j \to g_\alpha$, what implies the desired inequality for α.

Step 2: We construct a sequence of different symplectic cobordisms.

First, we construct a symplectic cobordism from $(Y, (\max(g_\alpha) + 2\mu)\alpha_0)$ to (Y, α); for any $\mu > 0$ we pick a function $f_\mu : \mathbb{R} \times Y \to \mathbb{R}$ such that $\frac{\partial f_\mu}{\partial s} > 0$ if $s \in [0, 1]$, $f(s, x) = e^s g_\alpha(x)$ if $s \leq 0$ and $f(s, x) = e^{s-1}(\max(g_\alpha) + 2\mu)$ for $s \geq 1$; notice that as $\max(g_\alpha)$ is a positive constant, the Reeb flow of $\max(g_\alpha)\alpha_0$ is just a reparametrization of the Reeb flow of α_0. $(\mathbb{R} \times Y, d(f_\mu)\alpha)$ is the desired cobordism.

By an analogous construction we can define an exact symplectic cobordism from (Y, α) to $(Y, b\alpha_0)$ for a sufficiently small constant $b > 0$. Notice that in both constructions $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ and $L = \mathbb{R} \times \Lambda$ are exact Lagrangian submanifolds.

Using the map F_0^z, we can modify the above construction to obtain an exact symplectic cobordism from $(Y, (\max(g_\alpha) + 2\mu)\alpha_z^0)$ to $(Y, (F_0^z)^*\alpha)$, and and exact symplectic cobordism from $(Y, (F_0^z)^*\alpha)$ to $(Y, b\alpha_z^0)$. Again, the cylinders $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ and $L = \mathbb{R} \times \Lambda$ are exact Lagrangian submanifolds in both these cobordisms.

We enumerate the exact symplectic cobordisms we have constructed so far, all of them diffeomorphic to $\mathbb{R} \times Y$.

- By the construction we made before of the proposition, we have an exact simplectic cobordism V_1 from $(\max(g_\alpha) + 2\mu)(1 + 4\delta)\alpha_0$ to $(\max(g_\alpha) + 2\mu)\alpha_z^0$.
- By the construction in this step we have an exact simplectic cobordism V_2 from $(\max(g_\alpha) + 2\mu)\alpha_z^0$ to $(1 + \mu)(F_0^z)^*\alpha$.
- Again, by the construction in this step we have an exact simplectic cobordism V_3 from $(1 - \mu)(F_0^z)^*\alpha$ to $b(1 - \mu)\alpha_z^0$.
- Finally, by the construction before the proposition, we know that there is an exact simplectic cobordism V_4 from $b\alpha_z^0$ to $b(1 - 4\delta)\alpha_0$

From our discussion in this session, it is clear that such cobordisms can be constructed in such a way that for all of them, the cylinders $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ and $L = \mathbb{R} \times \Lambda$ are exact Lagrangian submanifolds.

Step 3: The exact Lagrangian cobordism and the chain map on the strip Legendrian contact homology of the ends.

We can glue these cobordisms to obtain a single exact symplectic cobordism, as is done in [16]. The result is that we can produce an exact symplectic cobordism $(V = \mathbb{R} \times Y, \omega)$ such that:

- $\omega = d(e^{t-5}(\max(g_\alpha)+2\mu)(1+4\delta)\alpha_0)$ in $[5, +\infty)\times Y$ and $\omega = d(\frac{be^t}{2}\alpha_0)$ in $(-\infty, -5]\times Y$
- $\omega = d(1 + t)(F_0^z)^*\alpha$ in $[-\mu, \mu] \times Y$
- $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ and $L = \mathbb{R} \times \Lambda$ are exact Lagrangian submanifolds

Following section 2.1.3 we can produce a splitting family (V, ω_R) ,for $R > 0$, of exact symplectic cobordisms from $(\max(g_\alpha) + 2\mu)(1 + 4\delta)\alpha_0$ to $\frac{b}{2}\alpha_0$ along $(F_0^z)^*\alpha$. It is clear from our construction that for all R the symplectic cobordims (V, ω_R) can be deformed through exact symplectic cobordisms to the symplectization of α_0, in such a way that $L = \mathbb{R}\times\Lambda$ and $\widehat{L} = \mathbb{R}\times\widehat{\Lambda}$ are exact Lagrangian submanifolds at every stage of the isotopy. As a consequence, we can apply Proposition 3.4, to obtain that for regular almost complex structures $\breve{J} \in \mathcal{J}_{reg,\rho}(J, J)$ in the cobordisms (V, ω_R) we get an induced isomorphism $\Phi_{V,\omega_R,L,\widehat{L}}$ from $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ to itself.

Step 4: Proof of the theorem for α non-degenerate.

We first pick for (V, ω_R) an almost complex structure J_V as in section 2.1.3 and take $\rho \in \Sigma^C_{\Lambda \to \widehat{\Lambda}}$. We claim that for such an almost complex structure, there exist chords $c, c' \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ such that $\mathcal{M}(c, c'; J_V)$ is non-empty.

We argue by contradiction: If no such strip existed we have that $J_V \in\in \mathcal{J}_{reg,\rho}(J, J)$ and therefore induces an isomorphism on $LC\mathbb{H}^{\rho}_{st}(\alpha, \Lambda \to \widehat{\Lambda})$. Because $\mathcal{M}(c, c'; J_V)$ is empty we know that the cobordism map $\Phi_{V,\varpi,L,\widehat{L}}$ is the zero map; on the other hand, as $LC\mathbb{H}^{\rho}_{st}(\alpha, \Lambda \to \widehat{\Lambda}) \neq 0$ and $\Phi_{V,\varpi,L,\widehat{L}}$ is an isomorphism it cannot be the zero map. Thus we have reached a contradiction and therefore $\mathcal{M}(c, c'; J_V)$ is non-empty.

We now pick a sequence $R_n \to +\infty$ and take a sequence of elements $\widetilde{w}_{R_n} \in \mathcal{M}(c, c'; J_V)$ and invoke the SFT compactness results of [8]. Because there is a global bound on the energy of all elements of $\mathcal{M}(c, c'; J_V)$, the results in [8] imply that $\widetilde{w}_{R_n}$ converges to a holomorphic building $\widetilde{w}$. Because of the stretching the neck process, we have that one of the levels of this building lives in the symplectization of $(F^z_0)^*\alpha$.

It follows from the properties of the splitting family we are considering we can apply Proposition 2.3 in order to describe the limiting building which has the structure of a tree with one principal branch. For topological reasons one of the punctures of this level has to detect a Reeb chord $\widehat{c} \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}((F^z_0)^*\alpha)$; with action smaller than $(1 + 4\delta)(1 + 2\mu)\max(g_\alpha)C$. Let $\widetilde{w}^j$ for $j \in \{1, ..., m\}$ be the levels of the pseudoholomorphic building $\widetilde{w}$. From Proposition 2.3 we obtain the following picture:

- the upper level $\widetilde{w}^1$ is composed of one pseudoholomorphic disc, with has one positive puncture, which is asymptotic to a Reeb chord $c_0 \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, and several negative boundary and interior punctures. All of the negative punctures detect contractible Reeb orbits or contractible self Reeb chords of either Λ or $\widehat{\Lambda}$, excepting one negative boundary puncture that detects a Reeb chord c_1 in either $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ in case this level lives in the symplectization of α_0 or in $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha^0_z)$ in case this level lives in a cobordism from α to α^0_z;

- on every other level $\widetilde{w}^k$ there is a special curve which has one positive puncture, which is asymptotic to a Reeb chord c_{k-1} in ρ and possibly several interior and boundary negative punctures. Of the negative boundary punctures there is one that is asymptotic to an orbit c_k in ρ and all the others are contractible.

As a consequence we obtain that the level $\widetilde{w}^k$ living in the symplectization of $(F^z_0)^*\alpha$ contains a curve with one positive puncture asymptotic to a Reeb chord $\widehat{c} \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}((F^z_0)^*\alpha)$. Because all punctures in the building detect Reeb orbits and chords with action smaller than the action $A(c_0)$, we conclude that $A(\widehat{c}) \leq (1 + 4\delta)(1 + \mu)\max(g_\alpha)C$. As seen previously, this implies that $(F_0^z)^{-1}(\widehat{c})$ is a Reeb chord in $\mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}^z}(\alpha)$ with action $\leq (1+4\delta)(1+2\mu)\max(g_\alpha)C$.

As a consequence of this existence result for all our homotopy classes $\rho \in \Sigma^C_{\Lambda\to\widehat{\Lambda}}$, we have obtained that $e^{\frac{aC}{(1+2\mu)(1+4\delta)\max(g_\alpha)}} < N_C(\alpha,\Lambda,\widehat{\Lambda}^z)$. As μ can be chosen arbitrarily small, we obtain the promised estimate for α.

□

4.2 Positivity of the topological entropy

Using the results of the previous subsection, we will now prove that if a contact 3-manifold (Y,ξ), admits an associated contact form α_0 adapted a pair of disjoint Legendrian knots Λ and $\widehat{\Lambda}$ for which the strip Legendrian contact homology $LC\mathbb{H}_{st}(\alpha,\Lambda\to\widehat{\Lambda})$ has exponential homotopical growth rate with respect to the action, then every Reeb flow for this contact manifold has positive topological entropy.

We now study the Reeb chords from a fixed Legendrian knot Λ (disjoint from N_ϵ) to the Legendrian fibers Λ^z of the neighbourhood N_ϵ constructed previously.

In the disc $\mathbb{D}$ we will consider the Lebesgue measure which we obtain as considering this disc embedded in the plane. Therefore, when we say for almost every $z\in\mathbb{D}$, we mean almost every with respect to the Lebesgue measure.

Lemma 4.3. *Let α be a contact form associated to a contact 3-manifold (Y,ξ), with Λ and $\widehat{\Lambda}$ being Legendrian curves. We consider a neighbourhood N_ϵ of $\widehat{\Lambda}$ with coordinates (θ,z) as constructed in the previous section. Then, for almost every $z\in\mathbb{D}$, all chords from Λ to Λ^z are (α,Λ) transverse.*

Proof: Taking a parametrization $\rho : S^1 \to \Lambda$, we use the flow to define the following map on the cylinder $S^1\times\mathbb{R}$:

$$F(s,t) = \phi^t_\alpha(\rho(s))$$

The set $U_\epsilon = F^{-1}(N_\epsilon)$ is an open subset of $S^1\times\mathbb{R}$. Let $\pi : N_\epsilon \to \mathbb{D}$ be the projection to the two last coordinates. The restriction $F\mid_{U_\epsilon}$ can be composed with π to obtain a differentiable map:

$$\pi\circ F\mid_{U_\epsilon}: U_\epsilon \to \mathbb{D}.$$

We remark that the critical points of $\pi \circ F\mid_{U_\epsilon}$ are exactly points (s,t) where the curve $\{\phi^t(\rho(s); s \in S^1\}$ is tangent to some fiber Λ^z. To see this, note that by construction $\partial_t(F\mid_{U_\epsilon}) = X_\alpha$ and $\partial_s(F\mid_{U_\epsilon}) \neq 0$. This implies that $\partial_t(\pi \circ F\mid_{U_\epsilon})$ is always non-zero, because the vector field X_α is never tangent to the Legendrian fibers Λ^z; and $\partial_s(\pi \circ F\mid_{U_\epsilon}))$ is zero if, and only if, $\partial_s(F\mid_{U_\epsilon})$ is tangent to a Legendrian fiber. As a consequence, we obtain that the regular values z are of $\pi \circ F\mid_{U_\epsilon}$ in $\mathbb{D}$ are in bijective correspondence with the set of Legendrian fibers Λ^z satisfying that every Reeb chord from Λ to Λ_z is transverse.

Applying the finite dimensional Sard's theorem to $\pi \circ F\mid_{U_\epsilon}$ we have that almost every element of $\mathbb{D}$ is a regular value of $\pi \circ F\mid_{U_\epsilon}$, completing the proof of the lemma.

□

With this lemma at hand, we are ready to prove the main theorem of this chapter.

Theorem 4.4. *Let $(Y, \xi = \ker(\alpha_0))$ be a contact 3-manifold with a contact form α_0 adapted to the pair of disjoint Legendrian knots $(\Lambda, \widehat{\Lambda})$. Assume that $LC\mathbb{H}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth rate (with respect to the action) with exponential weight $a > 0$. For any contact form α associated to (Y, ξ), let g_α be the function such that $\alpha = g_\alpha \alpha_0$. Then, the Reeb flow of X_α has positive topological entropy, and moreover:*

$$h_{top}(\phi_{X_\alpha}) \geq \frac{a}{\max(g_\alpha)} \tag{4.8}$$

Proof: Our idea is to mimic the use of Yomdin's theorem which is done for geodesic flows and Reeb flows in spherizations (see [33] and [38]).

Step 1:

Let α_0 be our contact form satisfying the hypothesis of the theorem and fix $\delta > 0$. Take $\epsilon > 0$, so that the tubular neighbourhood N_ϵ satisfies the hypothesis of Proposition 4.2. Then, combining Proposition 4.2 and Lemma 4.3, we obtain that for almost every $z \in \mathbb{D}$, the number $N_C(\alpha, \Lambda, \Lambda^z)$ of Reeb chords of X_α from Λ to Λ^z satisfies:

$$e^{\frac{aC}{(1+4\delta)\max(g_\alpha)}} \leq N_C(\alpha, \Lambda, \Lambda^z) \tag{4.9}$$

for all $C \geq C_0$.

Step 2:

We first introduce a Riemannian metric on the manifold Y which restricted to N_ϵ is just the euclidean metric for the coordinates (θ, z). This metric induces a measure of area $Area(\Sigma)$ for all surfaces Σ immersed in Y. We want to estimate the area $Area^C(\Lambda)$ of the immersed surface $\{\phi^t_{X_\alpha}(\Lambda), t \in [0, C]\}$, by the Reeb flow $\phi^t_{X_\alpha}$ of α. The surface $\{\phi^t_{X_\alpha}(\Lambda), t \in [0, C]\}$, can be seen as the image of the map $F_{C,\Lambda} : \Lambda \times [0, C] \to Y$, where $F_{C,\Lambda}(p, t) = \phi^t_{X_\alpha}(p)$. Denoting $U_{\epsilon,C} = F^{-1}_{C,\Lambda}(N_\epsilon)$, we have:

$$Area^C(\Lambda) \geq Area(F_{C,\Lambda}(U_{\epsilon,C})) \geq Area(\pi(F_{C,\Lambda}(U_{\epsilon,C}))) \tag{4.10}$$

where the last area is taken with multiplicities with respect to the Lebesgue measure in $\mathbb{D}$. To obtain the inequality on the right side, one uses the fact that the area measure on N_ϵ coincides with the area measure induced by the euclidean metric.

Using the estimate made in step 1 for the counting function $N_C(\alpha, \Lambda, \Lambda^z)$ for almost every $z = (x, y) \in \mathbb{D}$, we get the inequality:

$$Area(\pi(F_{C,\Lambda}(U_{\epsilon,C}))) = \int_{\mathbb{D}} N_C(\alpha, \Lambda, \widehat{\Lambda}^z) dx \wedge dy \geq \int_{\mathbb{D}} e^{\frac{aC}{(1+4\delta)\max(g_\alpha)}} dx \wedge dy \tag{4.11}$$

for all $C \geq C_0$. As a result, we obtain that:

$$Area^C(\Lambda) \geq Area(\pi(F_{C,\Lambda}(U_{\epsilon,C}))) \geq e^{\frac{aC}{(1+4\delta)\max(g_\alpha)}}. \tag{4.12}$$

for $C \geq C_0$.

It follows then from (4.12) that:

$$\limsup_{C \to +\infty} \frac{1}{C} \log(Area^C(\Lambda)) \geq \frac{a}{(1 + 4\delta)\max(g_\alpha)}. \tag{4.13}$$

Corollary 1.4 of Yomdin's theorem now implies that $h_{top}(\phi_{X_\alpha}) \geq \frac{a}{(1+4\delta)\max(g_\alpha)}$. As the constant $\delta > 0$ in Proposition 4.2 can be taken arbitrarily small, we obtain the claimed estimate.

□

Remark: it is expected that by using the Polyfold technology which is being developed by Hofer, Wysocki and Zehnder one will be able to replace the condition "exponential homotopical growth rate" by the weaker condition "exponential growth rate" on the statement Theorem 1 above. It is also expected that by unpublished work of Bourgeois, Ekholm and Eliashberg one could obtain a similar estimate on the topological entropy *from the exponential growth of the linearized Legendrian contact homology, which includes the strip Legendrian contact homology as a special case. As these results would depend on technologies which are still being developed we opted for the use of our less general versions which are, however, sufficient do deal for all the examples that we are aware of.*

4.3 Implied positivity of topological entropy

In the previous sections of this chapter we described properties of a contact 3-manifold (Y, ξ) which guarantee that all Reeb flows associated to (Y, ξ) have positive topological entropy. In this section we turn our attention to the problem of how the existence of periodic orbits of certain types might force positivity of topological entropy.

We explain more precisely our aim. Let (Y, ξ) be a contact 3-manifold which admit associated Reeb flows with zero topological entropy. We study the following question: is there an oriented transverse link $\mathcal{G}$ in (Y, ξ) such that, every Reeb flow associated to (Y, ξ) possessing $\mathcal{G}$ as a set of Reeb orbits have positive topological entropy?

In this section we give a first step to answer this: we show that there exist conditions on the triple $(Y, \xi, \mathcal{G})$ that imply positivity of topological for all Reeb flows associated to (Y, ξ) having $\mathcal{G}$ as a set of Reeb orbits has positive topological entropy. We call this an implied positivity of entropy result: having $\mathcal{G}$ as a set of Reeb orbits *implies* positivity of topological entropy.

For our contact 3-manifold (Y, ξ), for a contact form α_0 adapted to a pair of disjoint Legendrian knots Λ and $\widehat{\Lambda}$ in the complement of $\mathcal{G}$, we define the exponential homotopical growth of the linearized Legendrian contact homology $LC\mathbb{H}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ with respect to the action.

We define for each number $C > 0$ the set $\Sigma^C_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}(\alpha_0)$ of homotopy classes $\rho \in \Sigma_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}$ satisfying:

- all the chords in $\mathcal{T}^{\rho, \mathcal{G}}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ have action smaller then C
- $LC\mathbb{H}^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \neq 0$

Definition 4.5. With the notation above, we say that $LC\mathbb{H}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth if there exist $a > 0$ and $d \in \mathbb{R}$ such that

$$\#(\Sigma^C_{\Lambda \to \widehat{\Lambda}}(\alpha_0)) > e^{aC+d} \tag{4.14}$$

for all $C > 0$.

More precisely, if this is satisfied we say that $LC\mathbb{H}^{\rho}_{st}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$. Analogous to Proposition 4.2, and keeping the notation of N_ϵ for a neighbourhood of $\widehat{\Lambda}$ as in section 4.1, we have the following:

Proposition 4.6. *Let (Y, ξ) be a contact manifold and Λ and $\widehat{\Lambda}$ be two disjoint Legendrian submanifolds, such that α_0 is associated to (Y, ξ) and adapted to the pair $(\Lambda, \widehat{\Lambda})$ in the complement of $\mathcal{G}$. Suppose that the strip contact homology $LC\mathbb{H}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$. Let α be another contact form associated to (Y, ξ) and having $\mathcal{G}$ as a set of Reeb orbits, and take $g_\alpha > 0$ to be the function such that $\alpha = g_\alpha \alpha_0$. Then given $\delta > 0$ there exists $\epsilon > 0$ such that, for every Legendrian fiber $\widehat{\Lambda}^z$ in N_ϵ which is (α, Λ) transverse, the numbers $N_C(\alpha, \Lambda, \widehat{\Lambda}^z)$ satisfy*

$$e^{\frac{aC}{(1+4\delta)\max(g_\alpha)}} \leq N_C(\alpha, \Lambda, \widehat{\Lambda}^z). \tag{4.15}$$

Proof: The strategy of the proof is identical to the one of proposition 4.2. We will follow this strategy, pointing out the necessary modifications but referring many times to the proof of proposition 4.2 in order to avoid repetitions.

Firstly, we consider the neighbourhood N_ϵ and the diffeomorphisms F^z_t as in section 4.1; the only extra requirement we make is that both N_ϵ and the sets where diffeomorphisms F^z_t are different to the identity are disjoint from the link $\mathcal{G}$. Then, following the recipe of section 4.1, we have that, given $\delta > 0$, there is $\epsilon > 0$, so that for every Legendrian finer $\widehat{\Lambda}^z$ in N_ϵ, we can construct exact symplectic cobordisms from $(1+2\delta)\alpha_0$ to $(1-2\delta)\alpha_0$, that coincides with piece of the symplectization of α^0_z in $[0,1]\times Y$, such that $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ and $L = \mathbb{R} \times \Lambda$ are exact Lagrangian submanifolds. Notice that in these exact symplectic cobordisms, for every $s \in \mathbb{R}$ the contact submanifold $\{s\} \times Y$ has $\mathcal{G}$ as a set of Reeb orbits.

Step 1: the reduction to the non-degenerate case is identical to the one performed in *Step 1* of the proof of Proposition 4.2, with the extra assumption that the perturbations $\alpha(j)$ of α also have the property that the link $\mathcal{G}$ is a set of Reeb orbits of $\alpha(j)$.

Step 2: The construction of a series of symplectic cobordisms.

The cobordisms considered here are again identical to the ones in *Step 2* of Proposition 4.2. Again it is clear that performing that construction with the present setup, gives us that in all the four cobordisms, for every for every $s \in \mathbb{R}$, the contact submanifold $\{s\} \times Y$ has $\mathcal{G}$ as a set of Reeb orbits.

Step 3: The exact Lagrangian cobordism and the chain map on the strip Legendrian contact homology on the complement of Reeb orbits.

We simply glue the cobordisms exactly as it is done in *Step 3* of proposition 4.2. The result is that we can produce an exact symplectic cobordism $(V = \mathbb{R} \times Y, \omega)$ such that:

- $\omega = d(e^{t-5}(\max(g_\alpha)+2\mu)(1+4\delta)\alpha_0)$ in $[5, +\infty)\times Y$ and $\omega = d(\frac{be^t}{2}\alpha_0)$ in $(-\infty, -5]\times Y$
- $\omega = d(1+t)(F_0^z)^*\alpha$ in $[-\mu, \mu] \times Y$
- $\widehat{L} = \mathbb{R} \times \widehat{\Lambda}$ and $L = \mathbb{R} \times \Lambda$ are exact Lagrangian submanifolds
- for every $s \in \mathbb{R}$, the contact submanifold $\{s\} \times Y$ has $\mathcal{G}$ as a set of Reeb orbits.

It is immediate from the construction that this cobordism satisfies the hypothesis of Proposition 3.10. As a result for a generic choice of almost complex structure in $(V = \mathbb{R} \times Y, \omega)$ the induced cobordism map on the homology level is defined and equals the identity.

Again following the recipe of section 2.1.3 we can produce a splitting family (V, ω_R) ,for $R > 0$, of exact symplectic cobordisms from $(\max(g_\alpha) + 2\mu)(1 + 4\delta)\alpha_0$ to $\frac{b}{2}\alpha_0$ along $(F_0^z)^*\alpha$. For every $R > 0$, the cobordisms $(V = \mathbb{R} \times Y, \omega)$ satisfies the hypothesis of Proposition 3.10. This implies that for Fredholm regular almost complex structures $\breve{J} \in \mathcal{J}^{\mathcal{G}}_{reg,\rho}(J, J)$ in the cobordisms (V, ω_R) the induced isomorphism $\Phi_{V,\omega_R,L,\widehat{L}}$ from $LC\mathbb{H}^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ to itself is the identity.

Step 4: Proof of the proposition for α non-degenerate.

We start like in *Step 4* of Proposition 4.2. We pick for (V, ω_R) an almost complex structure J_V as in section 2.1.3 and take $\rho \in \Sigma^C_{\Lambda\to\widehat{\Lambda}}$. We claim that for such an almost complex structure, there exist chords $c, c' \in \mathcal{T}^{\rho}_{\Lambda\to\widehat{\Lambda}}(\alpha_0)$ such that $\mathcal{M}_{\mathcal{G}}(c, c'; J_V)$ is non-empty. The reasoning is that if this was not the case we would have that J_V is Fredholm regular in (V, ω_R) and therefore the induced map in $\Phi_{V,\omega_R,L,\widehat{L}}$ would be the identity. If $\mathcal{M}_{\mathcal{G}}(c, c'; J_V)$ was empty, this map would also take every element to 0; however as $LC\mathbb{H}^{\rho}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda}) \neq 0$ we arrive at a contradiction. Notice that $\mathbb{R} \times \mathcal{G}$ is a union of pseudoholomorphic cylinders for the almost complex structure J_V.

With this in hand we take a sequence $R_n \to +\infty$ and a sequence of elements $\widetilde{w}_{R_n} \in \mathcal{M}_{\mathcal{G}}(c, c'; J_V)$ invoke the SFT compactness results of [8]. Because there is a global bound on the energy of all elements of $\mathcal{M}_{\mathcal{G}}(c, c'; J_V)$, the results in [8] imply that a subsequence of $\widetilde{w}_{R_n}$ converges to a holomorphic building $\widetilde{w}$. Because of the stretching the neck process, we have that one of the levels of this building lives in the symplectization of $(F_0^z)^*\alpha$.

Because of the properties of the splitting family we are considering we can apply Proposition 2.3 in order to describe the limiting building which has the structure of a tree with one principal branch. We start by pointing out that no connected component of $\mathcal{G}$ can appear as asymptotic limit for the levels of the building $\widetilde{w}$. We will prove this by contradicton; suppose that a Reeb orbit $\gamma_{\mathcal{G}}$ is an asymptotic orbit for a puncture z of a level $\widetilde{w}^l$ of $\widetilde{w}$. Then the building $\mathfrak{B}_z$ defined in section 2.4 and associated to this puncture z is contractible; it projects in Y to a disc $D_{\gamma_{\mathcal{G}}}$ which has $\gamma_{\mathcal{G}}$ as boundary and therefore its interior must intersect $\mathcal{G}$. It follows then from positivity of intersection that this interior intersection had to exist in $\widetilde{w}_{R_n}$ for R_n sufficiently large; but this contradicts the fact that $\widetilde{w}_{R_n} \in \mathcal{M}_{\mathcal{G}}(c, c'; J_V)$. For similar reasons we can also rule out that any level of $\widetilde{w}$ intersects $\mathbb{R} \times \mathcal{G}$.

Using that the curve $\widetilde{w}$ does not intersect $\mathbb{R} \times \mathcal{G}$ we will see that one of the punctures of this $\widetilde{w}$ has to detect a Reeb chord $\widehat{c} \in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}((F_0^z)^*\alpha)$; with action smaller than $(1+4\delta)(1+\mu)\max(g_\alpha)C$. Too see this we denote by $\widetilde{w}^j$ for $j \in \{1, ..., m\}$ the levels of the pseudoholomorphic building $\widetilde{w}$, and obtain from Proposition 2.3 the following picture:

- the upper level $\widetilde{w}^1$ is composed of one pseudoholomorphic disc, with has one positive puncture, which is asymptotic to a Reeb chord $c_0 =\in \mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$, and several negative boundary and interior punctures. All of the negative punctures detect contractible Reeb orbits or contractible self Reeb chords of either Λ or $\widehat{\Lambda}$, excepting one negative boundary puncture that detects a Reeb chord c_1 in either $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_0)$ in case this level lives in the symplectization of α_0 or in $\mathcal{T}^{\rho}_{\Lambda \to \widehat{\Lambda}}(\alpha_z^0)$ in case this level lives in a cobordism from α to α_z^0;
- on every other level $\widetilde{w}^k$ there is a special curve which has one positive puncture, which is asymptotic to a Reeb chord c_{k-1} in ρ and possibly several interior and boundary negative punctures. Of the negative boundary punctures there is one that is asymptotic to an orbit c_k in ρ and all the others are contractible.

The idea is that as $\widetilde{w}$ does not intersect $\mathbb{R} \times \mathcal{G}$ the special Reeb chord c_{k-1} of each level has to be in the same homotopy class in $\Sigma_{\Lambda \to \widehat{\Lambda}, \mathcal{G}}$ as the orbit c which is ρ. From here one can proceed as in the last two paragraphs of the proof of proposition 4.2.

□

The idea now is that Proposition 4.5, can be used to prove a version of theorem 1 which is adapted to the current situation.

Theorem 4.7. *Let (Y, ξ) be a contact manifold and Λ and $\widehat{\Lambda}$ be two disjoint Legendrian submanifolds, such that α_0 is associated to (Y, ξ) and adapted to the pair $(\Lambda, \widehat{\Lambda})$ in the complement of $\mathcal{G}$. Suppose that the strip contact homology $LC\mathbb{H}_{st|\mathcal{G}}(\alpha_0, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$. Let α be another contact form associated to (Y, ξ) and having $\mathcal{G}$ as a set of Reeb orbits, and take $g_\alpha > 0$ to be the function such that $\alpha = g_\alpha \alpha_0$. Then, the Reeb flow of X_α has positive topological entropy, and moreover:*

$$h_{top}(\phi_{X_\alpha}) \geq \frac{a}{\max(g_\alpha)} \tag{4.16}$$

Proof: the proof of Theorem 1 carries verbatim to the present theorem, only by using proposition 4.5 in the places where we previously invoked Proposition 4.2.

□

Chapter 5

Unit tangent bundle of surfaces of genus ≥ 2

5.1 Contact forms for geodesic flows

The first class of examples we will study is of unit tangent bundles of orientable surfaces of genus greater or equal to 2. Given a manifold Q its unit tangent bundle T_1Q can be given a canonical contact structure which we will denote by ξ_{can}; this contact structure is associated to geodesic flows. We begin by recalling how this can be done; our reference for this construction is [38].

Given a manifold Q of dimension n, let g be any Riemannian metric on Q. This metric induces a unique distribution of n planes in the tangent bundle TQ, the so called horizontal distribution H_g; see section 1.3 in [38]. This distribution is always transverse to the vertical distribution V in TQ, which is the unique distribution of n-planes always tangent to the fibres of TQ; this implies that for every $y \in TQ$ we have the following $T_yTQ := H_y \bigoplus V_y$. Let $\pi : TQ \to Q$ be the canonical projection. Because H is transversal to the fibers we have that at each point $y \in TQ$ the restriction of the the differential $D\pi$ to H_y is an isomorphism between H_y and $T_{\pi(y)}Q$. With this in hand, we can use the map π to pull back g to an inner product in the distribution H. As the metric g also induces an inner product on the distribution V; using these two inner products we have as a result a metric $\hat{g}$ induced by g on the bundle TQ; this metric is usually called the Sasaki metric.

We will now introduce an almost complex structure J_g on TQ associated to the metric g. Let $v \in T_qQ$ and $y \in \pi^{-1}(q)$; from our previous discussion we know that there are unique vectors $v_H \in H_y$ and $v_V \in H_y$ which are associated to v; v_H is the unique vector in $H_{(q,v)}$ that is in the pre-image of the restriction of $D\pi$ to $H_{(q,v)}$, and v_V is the vector on the fiber T_qQ of TQ canonically identified with v. Now, for each vector $\check{v} \in H_y$ we define $J_g(\check{v}) := (\pi(\check{v}))_V \in V_y$, and for each $\check{v} \in V_y$ we define $J_g(\check{v}) := -(\pi(\check{v}))_H \in H_y$. We extend J_g linearly to an almost complex structure in TQ; it is easy to see that there is a unique way to do that.

With this in hand we can define a symplectic form ϖ^g in TQ; for vectors $v_1, v_2 \in T_yTQ$ we define:

$$\varpi^g_y(v_1, v_2) := \hat{g}(J_g(v_1), v_2). \tag{5.1}$$

We will not prove here that ϖ^g is indeed a symplectic form, but refer to [38] for the proof. Let $H(q,v) = g_q(v,v)$; in this same reference, it is proven that the Hamiltonian vector field X_H associated to H via the symplectic form ϖ is the geodesic vector field G_g of the metric g, and the unit tangent bundle T_1Q is diffeomorphic to set $H^{-1}(1)$.

Lastly we have the following definition:

$$\alpha_g := i_{G_g}\hat{g} \tag{5.2}$$

With these definitions, we can state the following proposition, which one can find demonstrated in page 16 of [38]:

Proposition 5.1. *The restriction $\alpha_g\mid_{H^{-1}(1)}$ of α_g is a contact form on T_1Q. Moreover, its Reeb vector field X_α is the restriction of geodesic vector field G_g to $H^{-1}(1)$.*

This proposition justifies what we claimed previously about the relation between geodesic flows and Reeb flows; it shows that any geodesic flow is a Reeb flow for some contact form. As the contact form α_g on T_1Q varies continuously as we very g continuously in the contractible space $\mathcal{MET}$ of Riemannian metrics on Q, we can apply Gray's stability ([38]) theorem to conclude that all α_g are associated to a unique (up to diffeomorphism) contact structure ξ_{can} on the unit tangent bundle T_1Q. This contact structure ξ_{can} is therefore related to all geodesic flows on T_1Q as it contains all of them among its Reeb flows.

There are two classes of Legendrian submanifolds which are relatively easy to construct and will be important for us. For a point $q \in Q$ let Λ_q be the unit fiber over q, i.e the set of vectors in T_qQ with g norm equal to 1. Then Λ_q is Legendrian in (T_1Q, ξ_{can}). This follows easily from the fact, which one can also find proved in [38], that the geodesic vector field is horizontal. Secondly, for an embedded closed geodesic ν in Q we consider the set Λ_ν which consists of all unit vectors normal to the geodesic ν. By using the definition we gave of α_g it is also not hard to prove that Λ_ν is a Legendrian submanfold.

5.2 The case of hyperbolic surfaces

We now specialise our discussion to the case where of a compact surface S of genus ≥ 2. In this case, let g_{hyp} be a hyperbolic metric on the surface and $\alpha_{g_{hyp}}$ be the contact form on T_1S constructed as in section 4.1

We will show that given a point $q \in S$, then for almost every point $q' \in S$ we have that $\alpha_{g_{hyp}}$ is adapted to the pair $(\Lambda_q, \Lambda_{q'})$.

Proposition 5.2. *Fix $q \in S$, then, for almost every $q' \neq q$ in S, the contact form $\alpha_{g_{hyp}}$ is adapted to the pair $(\Lambda_q, \Lambda_{q'})$.*

Proof: as a first step we consider the universal cover of the hyperbolic surface (S, g_{hyp}) by the Poincaré disc $(\mathbb{D}, g_{hyp})$ and denote by π_{hyp}, the associated covering map from $(\mathbb{D}, g_{hyp})$ to (S, g_{hyp}) which is locally an isometry.

The fact that $\alpha_{g_{hyp}}$ has no contractible geodesics is a classic result in the study of geodesic flows for hyperbolic surfaces and follows directly from the fact that there are no closed geodesics in the Poincaré disc $(\mathbb{D}, g_{hyp})$.

Secondly, let $\check{q}$ be any point in S. The Reeb chords of $\alpha_{g_{hyp}}$ going from $\Lambda_{\check{q}}$ to itself are in to one to one correspondence with hyperbolic geodesic trajectories starting and ending at $\check{q}$. Therefore, if there was Reeb chord from $\Lambda_{\check{q}}$ to itself which was contractible in $\pi_1(T_1Q, \Lambda_{\check{q}})$, this would force the existence of a contractible geodesic starting and ending at $\check{q}$. However, this cannot exist since there are no hyperbolic geodesic trajectories in $(\mathbb{D}, g_{hyp})$, starting and ending at a same point. Taking $\check{q}$ to be q or q' gives us that $\alpha_{g_{hyp}}$ conditions (b) and (c) os section 3.1, with respect to the pair of Legendrian $(\Lambda_q, \Lambda_{q'})$.

Therefore to finish the proof all we must do is to show that for almost every q' in S, the Reeb chords starting at Λ_q and ending at $\Lambda_{q'}$ are all transverse. However, this is a general fact which doesn't depend at all on the particular metric we chose and follows directly from the finite dimensional Sard's theorem. For the proof of this fact we refer the reader to Proposition 3.1 in page 53 of [38]. This finishes the proof of the proposition.

□

5.3 Exponetial homotopical growth rate of $LC\mathbb{H}_{st}(\alpha_{g_{hyp}}, \Lambda_q \to \Lambda_{q'}))$

It follows from the previous lemma that, if all the Reeb chords from Λ_q to $\Lambda_{q'}$ are transverse, we can define $LC\mathbb{H}_{st}(\alpha_{g_{hyp}}, \Lambda_q \to \Lambda_{q'}))$. It is easy to see, using Sard's theorem lemma4.3, that for a generic choice of points $q \neq q'$ this transversality condition is indeed satisfied and we can therefore define $LC\mathbb{H}_{st}(\alpha_{g_{hyp}}, \Lambda_q \to \Lambda_{q'}))$.

In order to estimate the growth rate of $LC\mathbb{H}_{st}(\alpha_{g_{hyp}}, \Lambda_q \to \Lambda_{q'}))$, we begin with the following observation.

Lemma 5.3. *Each element in $\Sigma_{\Lambda \to \widehat{\Lambda}}(T_1S)$ can contain at most one Reeb chord.*

Proof: suppose there were two distinct Reeb chords c and c' belonging to same homotopy class ρ in $\Sigma_{\Lambda_q \to \Lambda_{q'}}(T_1S)$. Then they would project to two different hyperbolic geodesics l and l' in S. Taking appropriate lifts of l and l' to $(\mathbb{D}, g_{hyp})$, this would imply that there are lifts $\mathfrak{q}$ and $\mathfrak{q}'$ of q and q', for which there exist two different hyperbolic geodesics $\widetilde{l}$ and $\widetilde{l'}$ both starting at point $\mathfrak{q}$ and ending $\mathfrak{q}'$, something that is well known to be impossible.

□

From the lemma 5.3, one can deduce immediately that for every element $\rho \in \Sigma_{\Lambda \to \Lambda_{q'}}(T_1S)$ containing a Reeb chord one has $LC\mathbb{H}^{\rho}_{st}(\alpha_{g_{hyp}}, \Lambda_q \to \Lambda_{q'})) \neq 0$. More precisely we can conclude that $\Sigma^{C}_{\Lambda_q \to \Lambda_{q'}}(\alpha_{g_{hyp}})$ equals the number $N_C(\alpha_{g_{hyp}}, \Lambda_q, \Lambda_{q'})$. As the fundamental group of S^1 has exponential growth, we know that there are constants C_0, $a_S > 0$ and d_S depending only on S such that:

$$N_C(\alpha_{g_{hyp}}, \Lambda_q, \Lambda_{q'}) \geq e^{Ca_S + d} \tag{5.3}$$

for all $C \geq C_0$. Combining all this information, we have proven the following result:

Theorem 5.4. *$LC\mathbb{H}^{\rho}_{st}(\alpha_{g_{hyp}}, \Lambda_q \to \Lambda_{q'}))$ has exponential homotopical growth rate with exponential weight a_S.*

Chapter 6

3-manifolds with a special hyperbolic component that fibers over S^1

In this section we will construct more examples of contact manifolds which have pairs of Legendrians with exponential homotopical growth of the linearized Legendrian contact homology.

We denote by S the surface with boundary obtained by taking the two-dimensional torus and cutting out a small open disc and ω a symplectic form on S. The first homology group $H_1(S)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let h be a symplectomorphism of (S, ω) to itself, such that:

- the map h is the identity in a small neighbourhood V of the boundary circle ∂S,
- all the periodic points of h contained in $S \setminus \overline{V}$ are non-degenerate,
- the induced map $h^* : H^1(S) \to H^1(S)$ is given by a hyperbolic automorphism of $\mathbb{Z} \oplus \mathbb{Z}$.

We follow the recipe of [13] to construct a special contact form on the mapping torus of (S, h). The following lemma of Eliashberg (see [13]) will be important for our construction:

Lemma 6.1. *Let h be a diffeomorphism of a surface S with nonempty boundary which preserves a symplectic form ω. If 1 is not an eigenvalue of the h^*, then there exists a primitive β of ω such that $[h^*\beta - \beta] = 0$ in $H^1(S; R)$ and such that the characteristic vector ζ_β field of β is transverse to ∂S.*

Remark: We reproduce here the proof of Lemma 3.4 from [13] because we want to obtain a primitive β of ω with the extra property that the characteristic vector field ζ_β of the 1-form β (defined by the equation $i_{\zeta_\beta}\omega = \beta$) is transverse to ∂S. Notice that, in this case, the characteristic vector field ζ_β is also a Liouville vector field for the symplectic form ω, and must, therefore, point outward direction along ∂S.

Proof: As ω is an area-form on a surface with non-empty boundary, we know that there exists a primitive β_0 of ω such that its characteristic vector field ζ_{β_0} is transverse to ∂S. As h is a symplectomorphism of (S, ω), we have the identity $d(h^*\beta_0 - \beta_0) = 0$ that implies that the 1-form $(h^*\beta_0 - \beta_0)$ represents a cohomology class. As by hypothesis, the map $(h^* - id) : H^1(S, \mathbb{R}) \to H^1(S, \mathbb{R})$ is surjective, one can find $\theta \in H^1(S; \mathbb{R})$ such that $[h^*\beta_0 - \beta_0] = (h^* - id)[\theta]$.

As ∂S is null-homologous, $[\theta]$ evaluates to 0 over ∂S. It is easy to see that we can choose a representative θ of the cohomology class $[\theta]$ that vanishes on an open neighbourhood of $\overline{V}$. Setting $\beta = \beta_0 - \theta$ gives the desired primitive. The primitive β satisfies that ζ_β is transverse to ∂S, as β coincides with β_0 at V.

$\square$

Remark: we can parametrize V using coordinates $(r, \vartheta) \in [-\delta_0, 0] \times S^1$. We can take these coordinates so that the base $(\partial_r, \partial_\vartheta)$ is positively oriented with respect to ω. We will assume without loss of generality that $\beta_0 = H(r)d\vartheta$ in V, where $H > 0$ and $H' > 0$.

Over the manifold $\mathbb{R} \times S$ (with coordinates $(t, p) \in \mathbb{R} \times S$ consider the 1-form $\alpha = dt + \beta$. It follows easily from the fact that $d\beta = \omega$ is a symplectic form, that α is a contact form. Moreover, the Reeb vector field of α is ∂_t. The following construction of Giroux (presented in [13] Lemma 2.3) gives us a special contact form on the mapping torus of (S, h). From lemma 6.1, we know that $[h^*\beta - \beta] = 0$ and consequently we can find a positive function f, which is constant in V, and that satisfies $df = h^*\beta - \beta$. It is a direct computation to check that α is invariant by the diffeomorphism:

$$F : (t, p) \to (t - f(p), h(p)) \tag{6.1}$$

from $\mathbb{R} \times S$ to itself, and therefore it induces a contact form $\widetilde{\alpha}$ on the mapping torus $\Omega(S, h) = (\mathbb{R} \times S)/((t, p) \sim F(t, p))$. We denote by $p_H : \mathbb{R} \times S \to \Omega(S, h)$ the covering map associated to the above construction.

In the covering $\mathbb{R} \times S$ the Reeb vector field of $X_{\widetilde{\alpha}}$ lifts to the simple form $X_\alpha = \partial_t$. This property will be useful in making some of our subsequent arguments simpler.

6.1 A special Legendrian knot in $\Omega(S, h)$

As the title suggests, in this section we will construct a Legendrian knot in the interior of the mapping torus $\Omega(S, h)$. We begin with the following lemma.

Lemma 6.2. *There is an embedded curve η in $\{0\} \times S$ such that $\int_\eta \beta = 0$*

Proof: Since the characteristic vector field ζ_{β_0} is transverse to the boundary of S, Peixoto's theorem [26, page 172] is valid for the vector field ζ_{β_0} on $\{0\} \times S$. We can thus apply the arguments of [26, Proposition 4.6.1] to make a C^∞ small perturbation S' of $\{0\} \times S$, that makes the characteristic vector field ζ' (defined by $i_{\zeta'}(d\widetilde{\alpha}) \mid_{S'} = \widetilde{\alpha} \mid_{S'}$) induced by the contact structure $\ker(\alpha \mid_{S'})$ a Morse-Smale vector field, and so that S' is a graph of S on which $d\widetilde{\alpha} \mid_{S'}$ is an area form in S' (see [36] and [39] for properties Morse-Smale vector fields).[1] Since S and S' are C^∞-close, it follows that v' points outward on the boundary of S'. Notice that as S' comes with an area form $d\alpha \mid_{S'}$, it comes endowed with an orientation.

As $d\widetilde{\alpha} \mid_{S'}$ is an area form, the condition $i_{\zeta'}(d\widetilde{\alpha}) \mid_{S'} = \widetilde{\alpha} \mid_{S'}$ means that the vector field $-\zeta'$ contracts the area form $d\widetilde{\alpha} \mid_{S'}$. This implies that $-\zeta'$ has no singularities of source type.

Because of the Morse-Smale condition for the flow generated by $-\zeta'$, its ω-limit is the union of periodic orbits and singularities of the flow. If this flow has a periodic orbit P we take our η to be the projection of P on $\{0\} \times S$ and we are done.

If this is not the case, let the 1-skeleton Δ be the union of the singularities of the flow of $-\zeta'$ and the unstable manifolds of its saddle singularities. Because $-\zeta'$ is of Morse-Smale type it has no source singularities, and as it is directed inward along $\partial(S')$ the flow of $-v'$ retracts the surface S' to the the 1-skeleton Δ, as the time goes to $+\infty$,. This means that Δ is a deformation retract of S' by the flow of $-\zeta'$. The topology of S' forces Δ to contain a piecewise smooth simple curve γ tangent to the $\ker(\alpha)$. The vertices of γ are located at sink singularities of the characteristic foliation of S'. Observe that because $\int_\gamma \alpha = 0$ and $d\alpha \mid_{S'}$ is an area-form, γ cannot be null-homologous in S'. We pick an orientation for γ.

Fix a point p_0 in γ which is not a vertex. We can then smoothen the vertices of γ in a small neighbourhood of the vertices disjoint from p_0 and produce a smooth embedded curve γ' on S'. As γ' coincides with γ in a neighbourhood of p_0, we pick the orientation in γ' which coincide with the orientation chosen for γ in the region the two

[1] In the remark after the proof of the Lemma, we give another way of constructing the perturbation S'.

curves overlap. Given $\delta > 0$, one can ensure, by doing the smoothing in a sufficiently small neighbourhood of the vertices, that $q = \int_{\gamma'} \alpha$ satisfies $q \in (-\delta, \delta)$.

Using that near the point p_0 the curve γ' is tangent to $\ker(\alpha)$, it is possible, for sufficiently small δ, to make a small perturbation $\widetilde{\gamma}$ of γ' supported near p_0 such that, for the small region U bounded by $\widetilde{\gamma}$ of γ', and oriented such that $\partial U = \gamma' - \widetilde{\gamma}$, we have:

$$\int_U d\alpha = q \tag{6.2}$$

Now Stokes' theorem and $q = \int_{\gamma'} \alpha$ implies that $\int_{\widetilde{\gamma}} \alpha = 0$. This perturbation $\widetilde{\gamma}$ which aims at correcting the change in the integral after the smoothing of the corners, can be made explicitly if one uses Darboux coordinates in a neighbourhood of p_0.

Since S' is a graph of $\{0\} \times S$ and $\widetilde{\gamma}$ is an embedded curve in S', by projecting $\widetilde{\gamma}$ on $\{0\} \times S$ one obtains an embedded curve η on $\{0\} \times S$. Notice that $\int_{\widetilde{\gamma}} \alpha = \int_{\widetilde{\gamma}} (dt + \beta) = \int_{\widetilde{\gamma}} \beta$ and thus $\int_\eta \beta = 0$.

$\square$

Remark: Maybe the easiest way to obtain the surface S' used in the proof above is to construct a contact manifold (M, λ) which contains $(\Omega(S, h), \widetilde{\alpha})$ as a component (see the next subsection) and where $\{0\} \times S$ can be extended to an embedded surface $\widehat{S}$ in M (this is the case, for example, if all the other components M are also mapping tori glued in an appropriate manner to $\Omega(S, h)$). In this case one can apply [26][Proposition 4.6.1] to $\widehat{S}$ in (M, λ) to obtain a C^∞ perturbation $\widehat{S}'$ of $\widehat{S}$ with a Morse-Smale characteristic foliation. The restriction of $\widehat{S}'$ to $\Omega(S, h)$ is the desired S'; observe that the C^∞ proximity of S' with $\{0\} \times S$ implies that the characteristic foliation of S' is transverse to $\partial(S')$ and that S' is a graph of $\{0\} \times S$ in $\mathbb{R} \times S$.

Because $\int_\eta \beta = 0$ we have that the curve η obtained in the lemma is the Lagrangian projection of a Legendrian curve Λ_0 in $(\Omega(S, h), \alpha)$. Using that $X_\alpha = \partial_t$, one sees that a Legendrian knot which is the graph of an embedded curve in $\{0\} \times S$ has the remarkable property that they have no Reeb chords for the Reeb flow of X_α. This property is valid for Λ_0 and also for all Legendrians close to Λ_0 in the C^∞ topology do not have any Reeb chords for the Reeb flow of X_α.

By making a vertical translation of Λ_0 if necessary, there exists $N > 0$ such that $\Lambda_0 \subset [1, N-1] \times S$. By summing a sufficiently large constant, we can pick the function f used to construct the mapping torus satisfying $f > 2N$. Remember that f is constant in V, and let $K = f(p), \forall p \in V$. We denote by Λ the Legendrian submanifold of $\Omega(S, h)$, which is the image by p_H of Λ_0.

Lemma 6.3. *For the Reeb vector field $X_{\widetilde{\alpha}}$ there is no Reeb chord c from Λ to itself such that $[c]$ is the trivial element in $\pi_1(\Omega(S,h),\Lambda)$. The same is true for any Legendrian Λ' which are sufficiently close to Λ in the C^∞ topology.*

Proof: The lemma follows from the remark above about the non-existence of Reeb chords from Λ_0' to itself for the Reeb vector field X_α in $\mathbb{R}\times S$ (which is the lift of $X_{\widetilde{\alpha}}$) for Λ_0' equal or sufficiently close to Λ_0.

□

6.2 Contact 3-manifolds containing $(\Omega(S,h),\widetilde{\alpha})$ as a component

Our objective now is to construct on closed 3-manifolds M that contain $\Omega(S,h)$ as a component, hypertight contact forms that coincide $\widetilde{\alpha}$ on $\Omega(S,h)$. We will begin by presenting some very explicit examples of how this can be done in Section 6.2.1; in Section 6.2.2, we will present a general theorem of Colin and Honda that says that if M is a 3-manifold whose JSJ-decomposition has $\Omega(S,h)$ as a component, the there exists a hypertight contact form on M that coincides with $\widetilde{\alpha}$ on $\Omega(S,h)$.

6.2.1 Some explicit examples

First we will construct a special contact form on $T=(\mathbb{R}/K\mathbb{Z})\times[0,1]\times S^1$. Let g_1 and g_2 be functions from $[0,1]$ to $\mathbb{R}$ satisfying:

- the curve defined by $(g_1,g_2):[0,1]\to\mathbb{R}^2$ starts on $(1,H(0))$ and ends at $(-1,-1)$, and intersects the real axis in only one point,
- $(g_1g_2'-g_1'g_2)(s)>0$ for all $r'\in[0,n]$
- the function $\widehat{g}_1:[-\delta_0,+\delta_0]\to\mathbb{R}$ defined by $\widehat{g}_1(s)=1$ if $s\le 0$ and $\widehat{g}_1(s)=g_1(s)$ if $s\ge 0$ is smooth,
- the function $\widehat{g}_2:[-\delta_0,+\delta_0]\to\mathbb{R}$ defined by $\widehat{g}_2(s)=H(s)$ if $s\le 0$ and $\widehat{g}_2(s)=g_2(s)$ if $s\ge 0$ is smooth,
- $g_1(s)=-1$ in a neighbourhood of 1 and $g_2(s)=-s$.

It is easy to check that $\nu=g_1(r')dt+g_2(r')d\vartheta$ is a contact form in T, where $(t',r',\vartheta')\in(\mathbb{R}/K\mathbb{Z})\times[0,1]\times S^1$ are coordinates for T.

Let now $\mathcal{S}_k$ be the orientable surface with genus k and whose boundary is composed by exactly one circle. We parametrize a neighbourhood V^k of $\partial\mathcal{S}_k$ with coordinates $(\widehat{r}, \widehat{\vartheta}) \in [1, 1+\delta_0]$ and consider in $\mathcal{S}_k$ a symplectic form ω' that coincides with $d\widehat{r} \wedge d\widehat{\vartheta}$ in V^k. The we choose β' to be primitive of ω' that coincides with $\widehat{r} \wedge d\widehat{\vartheta}$. Finally, consider $\alpha_{\mathcal{S}_k}$ to be the contact form given by $-dt - \beta'$ on the manifold $\mathbb{R} \times \mathcal{S}_k$, which clearly induces a contact form on the mapping torus $\Omega(\mathcal{S}_k, id) := (\mathbb{R} \times \mathcal{S}_k) \setminus (t, p) \sim (t - K, p)$.

To obtain a closed 3-manifold M we identify the boundary component $(\mathbb{R}/K\mathbb{Z}) \times \{0\} \times S^1$ of T with $\partial\Omega(S, h)$; the points in these two-dimensional tori are identified by the diffeormorphims that is the identity map for the coordinate systems we constructed for these tori. The other component $(\mathbb{R}/K\mathbb{Z}) \times \{1\} \times S^1$ of T is identified with $\partial\Omega(\mathcal{S}_k, id)$; again the identification map is the identity for the coordinates we have constructed. These identifications are then used to glue the 3 pieces, $\Omega(S, h)$, T and $\Omega(\mathcal{S}_k, id)$ obtaining a closed 3-manifold that we will denote by $M_{\mathcal{S}_k}$. Because of the properties satisfied by the pair (g_1, g_2) we see that the 3 contact forms on the pieces are also glued to produce a contact form $\tau_{\mathcal{S}_k}$ on $M_{\mathcal{S}_k}$. All the 3 contact forms are hypertight in each piece and their Reeb flow is tangent to the boundaries; combining this with the fact and that the boundaries of the 3 pieces are incompressible tori in the glued manifold $M_{\mathcal{S}_k}$ implies that $\tau_{\mathcal{S}_k}$ is hypertight.

This argument already suffices to construct infinitely many different 3-manifolds admitting a hypertight contact form containing $(\Omega(S, h), \widetilde{\alpha})$ as a component. We proceed to show one way in which the construction can be generalised.

If we maintain all the conditions we demanded of (g_1, g_2) but change the first one to:

- the curve defined by $(g_1, g_2) : [0, 1] \to \mathbb{R}^2$ starts on $(1, H(0))$ and ends at $(-1, -1)$, and intersects the real axis in exactly $2i + 1$ points (where $i \geq 0$),

we can still proceed as above to obtain a hypertight contact form on $M_{\mathcal{S}_k}$. However, as we can choose i to be arbitrarily large the contact structure obtained can have arbitrarily large Giroux torsion. The Giroux torsion is an invariant of contact structures which associates to it either a non-negative number or $+\infty$; we will give a precise definition in the next session. For now, it suffices to say that if we perform the construction in this section with (g_1, g_2) satisfying the modified condition we just mentioned, the resulting contact structure has Giroux torsion at least $2i + 1$. This implies that the modified construction can be used to produce contact structures with arbitrarily large Giroux torsion.

Summing up we have shown that $M_{\mathcal{S}_k}$ admits an infinite number of distinct contact structures ξ_j, such that for each ξ_j there exists a hypertight contact form τ_j associated to $(M_{\mathcal{S}_k}, \xi_j)$ which coincides with $\widetilde{\alpha})$ in the component $\Omega(S, h)$.

6.2.2 The Colin-Honda construction

Let $\mathcal{W}$ be a compact, oriented, irreducible 3-manifold such that $\partial(\mathcal{W})$ is a union of incompressible tori. The following theorem of Colin and Honda [12] tells us that $\mathcal{W}$ admits a hypertight contact form tangent to the boundary:

Theorem 6.4. *(Colin-Honda [12]) Let $\mathcal{W}$ be a compact, oriented, irreducible 3-manifold such that $\partial(\mathcal{W})$ is a union of incompressible tori. Then there exists a hypertight contact form ς such that, in a neighbourhood $(\mathbb{R}/K\mathbb{Z}) \times S^1 \times I$ with coordinates $(\widetilde{t}, \widetilde{\vartheta}), \widetilde{r}$ of each component of $\partial(\mathcal{W})$, $\varsigma = \cos(\widetilde{r})d\widetilde{t} - \sin(\widetilde{r})d\widetilde{\vartheta}$.*

Now suppose we are given a finite collection compact oriented irreducible 3-manifolds $\{\mathcal{W}_i, 0 \leq i \leq N\}$, such that $\mathcal{W}_0 = \Omega(S, h)$, and can be glued along their boundaries to give an oriented 3-manifold M. This means that $\{\mathcal{W}_i, 0 \leq i \leq N\}$ is the JSJ decomposition of the 3-manifold M. Using the above theorem of Colin and Honda we put hypertight contact forms tangent to the boundary on the manifolds $\mathcal{W}_i$ for $i \geq 1$; on the special piece $\mathcal{W}_0$ we consider again the the contact form $\widetilde{\alpha}$ constructed above (which is also tangent to the boundary). Our objective now is to glue these contact forms to obtain a contact form on M. The details on how to make the gluing process are presented in [40] and [12]; we sketch it here for the convenience of the reader . From the remark following the proof of Lemma 6.1, we know that in a neighbourhood of $\partial(\Omega(S, h)$ diffeomorphic to $(\mathbb{R}/K\mathbb{Z}) \times V$, with coordinates (t, r, ϑ), we have $\widetilde{\alpha} = dt + H(r)d\vartheta$ where $H > 0$ and $H' > 0$.

For a natural number $n \geq 1$ we consider a neck T_n of the form $(\mathbb{R}/K\mathbb{Z}) \times [0, 1] \times S^1$ with coordinates (t', r', ϑ'). Let g_1^n and g_2^n be the functions:

- $g_1^n(r') = \cos(2n\pi r')$ and $g_2(r') = \sin(2n\pi r')$ if $r' \in [0, n]$

from $\times[0, 1]$ to $\mathbb{R}$. Then $\nu_n = g_1^n(r')dt + g_2^n(r')d\vartheta$ is a contact form in T_n; we will call its associated contact structure ξ_{T_n}. The idea is that we can glue one boundary of the neck ν_n to the boundary of a component W_i for $i > 0$ and that this gluing can be done in such a way that the contact structures considered in the two pieces are glued smoothly. For the component W_0 we make a small modification of $g_1(r')$ and $g_2(r')$ on a neighbourhood of 0 so that when we identify the boundary $(\mathbb{R}/K\mathbb{Z}) \times \{0\} \times S^1$ with $\partial(\Omega(S, h)$ in the gluing process, the contact forms on these two pieces are glued smoothly .

Thus by introducing the necks T_n we can interpolate the contact forms in the boundaries of the components $\mathcal{W}_i$ to obtain a contact form τ on M. The hypertightness of τ comes from the hypertightness of $\widetilde{\alpha}$ and of the contact forms on the components $\mathcal{W}_i$ for $i \geq 1$, combined with the fact that all the periodic orbits in the neck T_n represent non-trivial homology classes in the incompressible tori; these boundary tori of the components $\mathcal{W}_i$ remain incompressible in M.

Definition 6.5. We define $Tor(Y, \xi)$ to be the supremum of the integers $n \geq 1$ for which there is a contact embedding of (T_n, ξ_{T_n}) into (Y, ξ). We say that $Tor(Y, \xi) = 0$ if no such embedding exists.

A contact 3-manifold (M, ξ) is said to have positive Giroux torsion if there is a contact embedding (T_1, ξ_{T_1}) in (M, ξ).

It is thus clear that the above construction includes manifolds with positive Giroux torsion. By a theorem of Gay [25] (see also [41]) manifolds with positive Giroux torsion are not strongly fillable. An interesting feature of these examples is that they are not strongly fillable, while the unit tangent bundles studied in [33] are. The examples constructed above coincide with the ones studied by Colin in [11] and Colin and Honda [12] for the manifold M with the JSJ decomposition given by $\{\mathcal{W}_i, 0 \leq i \leq N\}$. By the recipe above we can obtain contact structures on M with arbitrarily large Giroux torsion which admit an associated contact form that coincides with $\widetilde{\alpha}$ on the component $\mathcal{W}_0$. Thus, one obtains that M admits an infinite number of distinct contact structures ξ_j, such that for each ξ_j there exists a hypertight contact form τ_j associated to (M, ξ_j) which coincides with $\widetilde{\alpha}$) in the component $\Omega(S, h)$.

6.3 Exponential homotopical growth of $LC\mathbb{H}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$

In this section we will study the homotopical growth rate of the strip Legendrian contact in the contact 3-manifolds constructed in the previous section. We will keep essentially the notation from the previous section which we now recall: M is three manifold whose JSJ-decomposition is $\{\mathcal{W}_i, 0 \leq i \leq N\}$ where $\mathcal{W}_0 = \Omega(S, h)$. We consider on M a hypertight contact form τ that coincides with $\widetilde{\alpha}$ on the component $\mathcal{W}_0$.

Proposition 6.6. *For the Reeb vector field X_τ there is no Reeb chord c from Λ to itself such that $[c]$ is the trivial element in $\pi_1(M, \Lambda)$.*

Proof: We will show that Lemma 6.3 implies the proposition.

By contradiction suppose there is a smooth disc D such that ∂D is the concatenation of a Reeb chord c with a path $\gamma \subset \Lambda$. By genericity, we can suppose that D

intersects $\partial\mathcal{W}_0$ transversely. This implies that $D \cap \partial\mathcal{W}_0$ is a collection of embedded circles $w_1, ..., w_n$ in $\partial\mathcal{W}_0$; these circles need not be disjoint, they might intersect each other. As all w_i are contractible in M and $\partial\mathcal{W}_0$ is an incompressible torus in M, this implies that the w_i are also contractible in $\partial\mathcal{W}_0$.

Let u_i be the disc in $\partial\mathcal{W}_0$ whose boundary is w_i, and v_i be the disc in D whose boundary w_i. Select from the set $\{v_i, 1 \leq i \leq n\}$ a subset $K = \{v_{i_1}, ..., v_{i_k}\}$ such that each v_i is contained in at least one v_{i_j}, and such that no v_{i_j} is contained in a v_{i_l} for $l \neq j$. Then by cutting of the discs v_{i_j} and gluing in their place the discs u_i we get a disc D' in $\Omega(S, h)$ whose boundary is the concatenation of the Reeb chord c with the path $\gamma \subset \Lambda$. The existence of such a disc contradicts Lemma 6.3; this finishes the proof of the proposition.

$\square$

It is clear that, for $\epsilon > 0$ sufficiently small, the above proposition is valid also for Legendrians ϵ close to Λ in the C^∞ topology. As a consequence of this, we have the following corollary:

Corollary 6.7. *Let $\widehat{\Lambda}$ be a generic Legendrian ϵ-close to Λ in the C^∞ topology and disjoint from Λ. Then, the strip Legendrian contact homology $LC\mathbb{H}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ is defined.*

It is clear that the contact form τ and the pair of disjoint Legendrian knots $(\Lambda, \widehat{\Lambda})$ satisfy conditions (a), (b) and (c) from Chapter 3. By genericity $\widehat{\Lambda}$, we can also guarantee that the triple $(\tau, \Lambda, \widehat{\Lambda})$ also satisfies condition (d) from Chapter 3. Notice that Λ and $\widehat{\Lambda}$ are graphs of embedded curves η and $\widehat{\eta}$.

We now have that τ is adapted to $(\Lambda, \widehat{\Lambda})$ and therefore $LC\mathbb{H}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ is well-defined. We can then proceed to show that the homotopical growth rate of $LC\mathbb{H}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ is exponential. To study the growth rate of $LC\mathbb{H}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ we will consider some special relative homotopy classes of paths from Λ to $\widehat{\Lambda}$.

Definition 6.8. Let c_1 and c_2 be Reeb chords from Λ to $\widehat{\Lambda}$. We say that c_1 and c_2 are in the same **Relative Nielsen class** if, and only if, there exists a smooth strip $u : [0, 1] \times [0, 1] \to \Omega(S, h)$ such that:

- $u(0 \times [0, 1])$ is a path in Λ and $u(1 \times [0, 1])$ is a path in $\widehat{\Lambda}$,
- $u([0, 1] \times 0) = c_1$ and $u([0, 1] \times 1) = c_2$.

It is immediate to check that the relative Nielsen classes are equivalence classes, because relative Nielsen classes are just homotopy classes of paths from Λ to $\widehat{\Lambda}$ in the mapping torus $\Omega(S,h)$. Our first step is to prove that the Relative Nielsen classes generate a partition of $LCH_{st}(M,\tau,\Lambda\to\widehat{\Lambda})$ in subcomplexes because they can be regarded as elements in the set $\Sigma_{\Lambda\to\widehat{\Lambda}}$ of homotopy classes of paths from Λ to $\widehat{\Lambda}$ in M.

Lemma 6.9. *Let c_1 and c_2 be Reeb chords from Λ to $\widehat{\Lambda}$, and $u:[0,1]\times[0,1]\to M$ such that:*

- *$u(0\times[0,1])$ is a path in Λ and $u(1\times[0,1])$ is a path in $\widehat{\Lambda}$*
- *$u([0,1]\times 0)=c_1$ and $u([0,1]\times 1)=c_2$*

Then, there exists a strip $u':[0,1]\times[0,1]\to\Omega(S,h)$ such that $u'(\partial([0,1]\times[0,1]))=u(\partial([0,1]\times[0,1]))$.

Proof: the proof is very similar to the one of proposition 6.6 above, so we will only give an outline of it.

By genericity we can assume that the image of u intersects $\partial(\Omega(S,h))$ transversely. The intersection consists of a finite collection of circles w_1,...,w_k which are contractible in M. The assumption that $\partial(\Omega(S,h))$ is incompressible implies that w_1,...,w_k are also contractible in $\partial(\Omega(S,h))$. The intersection of $u([0,1]\times[0,1])$ with $\mathcal{W}$ is composed by discs d_i with boundary w_i. We can cut out these discs and replace them by discs d_i' contained in $\partial(\Omega(S,h))$ and whose boundary is w_i. This cut and paste procedure gives the desired u'.

$\square$

As seen in section 3.1, that the differential ∂_{st} of the strip Legendrian contact homology $LC\mathbb{H}_{st}(M,\tau,\Lambda\to\widehat{\Lambda})$ count index 1 holomorphic strips $\widetilde{u}:\mathbb{R}\times[0,1]\to\mathbb{R}\times M$ in the symplectization of (M,τ) with the boundary conditions:

- $\widetilde{u}(\mathbb{R}\times\{0\})\subset\mathbb{R}\times\Lambda$,
- $\widetilde{u}(\mathbb{R}\times\{1\})\subset\mathbb{R}\times\widehat{\Lambda}$.

As mentioned earlier, it is a consequence of the Lemma 6.9 that relative Nielsen classes can be seen as elements in $\Sigma_{\Lambda\to\widehat{\Lambda}}$. More precisely, denoting by $\mathfrak{R}$ the set of relative Nielsen classes, we have a map $\mathcal{I}:\mathfrak{R}\to\Sigma_{\Lambda\to\widehat{\Lambda}}$, defined as follows: given a relative Nielsen class ρ, we pick a Reeb chord $c\in\rho$ and define $\mathcal{I}(\rho)$ to be the class of $[c]\in\Sigma_{\Lambda\to\widehat{\Lambda}}$. It is easy to see that $\mathcal{I}$ is well defined and the above Lemma 6.9 implies that $\mathcal{I}$ is injective.

Remark: notice that because of the way we constructed the contact form τ, we have that all the Reeb chords in $\mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\tau)$ are contained in the component $M_0 = \partial(\Omega(S,h))$, and therefore belong to elements in $\Sigma_{\Lambda\to\widehat{\Lambda}}$ which are in the image of our map $\mathcal{I}$.

It is therefore possible to write $LC\mathbb{H}_{st}(M,\tau,\Lambda\to\widehat{\Lambda})$ as a direct sum:

$$LC\mathbb{H}_{st}(M,\tau,\Lambda\to\widehat{\Lambda}) = \bigoplus_{\varrho\in\mathfrak{R}} LC\mathbb{H}_{st}^{\mathcal{I}(\varrho)}(M,\tau,\Lambda\to\widehat{\Lambda})_{\varrho} \tag{6.3}$$

6.3.1 The Relative Nielsen classes

We will use the covering $(\mathbb{R}\times S,\alpha)$ of $(\Omega(S,h),\widetilde{\alpha})$ to obtain information about the Relative Nielsen classes. We begin by fixing in $(\mathbb{R}\times S,\alpha)$ the lift Λ_0 of Λ that is contained in $[0,N]\times S$. The lifts of $\widehat{\Lambda}$ to $(\mathbb{R}\times S,\alpha)$ can be ordered in the following way: letting $\widehat{\Lambda_0}$ be the lift of $\widehat{\Lambda}$ contained in $[0,N]\times S$, $\widehat{\Lambda_n} = F^{-n}(\widehat{\Lambda})$.

Given a Reeb chord c from Λ to $\widehat{\Lambda}$ we take the lift $\widetilde{c}$ which has its starting point in Λ_0. It is not difficult to see that if c_1 and c_2 are Reeb chords from Λ to $\widehat{\Lambda}$ that are in the the same Relative Nielsen class, then $\widetilde{c_1}$ and $\widetilde{c_2}$ have to have endpoints in the same lift $\widehat{\Lambda_n}$ of $\widehat{\Lambda}$. We will see, however, that this condition of $\widetilde{c_1}$ and $\widetilde{c_2}$ having the endpoints in the same lift $\widehat{\Lambda_n}$ is far from sufficient to guarantee that c_1 and c_2 are in the same Relative Nielsen class.

Let $\pi_S : \mathbb{R}\times S \to S$ be the projection in the second coordinate. Remembering our construction in section 6.3, we know that $\eta = \pi_S(\Lambda_0)$ and $\widehat{\eta} := \pi_S(\widehat{\Lambda_0})$ are embedded curves in S. From the definition of the map F, we have that $\pi_S(\widehat{\Lambda_n}) = \pi_S \circ F^{-n}(\widehat{\Lambda}) = h^{-n}(\widehat{\eta})$, and as h is a diffeomorphism, $\pi_S(\widehat{\Lambda_n})$ is an embedded curve in S. Observe that the Reeb chords from Λ_0 to $\widehat{\Lambda_n}$ are in one-to-one correspondence with the intersection points of η and $h^{-n}(\widehat{\eta})$. Notice that because ∂_t is the pull-back of the Reeb vector field in this covering space, the transversality of all the Reeb chords from Λ to $\widehat{\Lambda}$ is equivalent to the transversality of η and $h^{-n}(\widehat{\eta})$ for every natural number n. We now proceed for the following characterization of the Relative Nielsen classes.

Proposition 6.10. *Let c_1 and c_2 be Reeb chords in $\mathcal{T}_{\Lambda\to\widehat{\Lambda}}(\widetilde{\alpha})$ with $p_1 := \pi_S(c_1)$ and $p_2 := \pi_S(c_2)$. Then c_1 and c_2 are in the same Relative Nielsen class if, and only if, $\widetilde{c_1}$ and $\widetilde{c_2}$ have end points in the same $\widehat{\Lambda_n}$, and there exists a map $v : [0,1]\times[0,1] \to S$,* *such that:*

$$v([0,1] \times \{0\}) = p_1, \tag{6.4}$$

$$v([0,1] \times \{1\}) = p_2, \tag{6.5}$$

$$v(\{0\} \times [0,1]) \subset \eta, \tag{6.6}$$

$$v(\{1\} \times [0,1]) \subset h^{-n}(\widehat{\eta}) \tag{6.7}$$

Proof: suppose c_1 and c_2 are in the same relative Nielsen class. We take the map $u : [0,1] \times [0,1] \to \Omega(S,h)$ given in Definition 6.8, and consider its lift $\widehat{u} : [0,1] \times [0,1] \to \mathbb{R} \times S$, such that $\widehat{u}([0,1] \times \{0\}) = \widetilde{c_1}$ and $\widehat{u}([0,1] \times \{1\}) = \widetilde{c_2}$. It is easy to see that taking $v = \pi_S \circ \widehat{u}$ gives a strip in S satisfying the conditions in the statement proposition; this finishes one implication.

To prove the reverse implication take a v satisfying the conditions in the statement proposition. By taking the path $v(\{0\} \times [0,1]) \subset \eta$ there exists a unique function $g_0 : [0,1] \to \mathbb{R}$ such that the path $\gamma_0(s)$ of the form: $\gamma_0(s) = (v(0,s), g_0(s))$ is a path in Λ_0. Analogously there exists a function $g_1 : [0,1] \to \mathbb{R}$ such that $\gamma_1(s) = (v(1,s), g_1(s))$ is a path in $\widehat{\Lambda_n}$. Take $f : [0,1] \times [0,1] \to \mathbb{R}$ to be an homotopy between g_0 and g_1 that is, $f(0,s) = g_0(s)$ and $f(1,s) = g_1(s)$. Now we can define the strip $u(r,s) = (v(r,s), f(r,s))$ in $\mathbb{R} \times S$, and considering $p_H \circ u$ we get a strip in $\Omega(S,h)$ which satisfies the conditions of the definition of Relative Nielsen classes for $\widetilde{c_1}$ and $\widetilde{c_2}$; this finishes the reverse implication and the proof of the proposition.

□

Proposition 6.10 gives a complete description of the Relative Nielsen classes. It also shows how to identify different Relative Nielsen classes of Reeb chords by looking at properties of intersection points of the curves $h^{-n}(\widehat{\eta})$ and η. This is the crucial link that will allow us to use the hyperbolicity of h^* to estimate the growth of the number of Relative Nielsen classes. Among the relative Nielsen classes, the subset of relative Nielsen classes with an odd number of chords will be of special importance to us: we will call them fundamental Relative Nielsen classes and denote their set by $\mathfrak{R}^f$.

From the discussion above we can partition the set $\mathfrak{R}^f$, in subsets $\mathfrak{R}^f_n$ defined by: an element $\varrho \in \mathfrak{R}^f_n$ if, and only if, for every Reeb chord $c \in \varrho$, the lift $\widetilde{c}$ has its endpoint in $\widehat{\Lambda_n}$. Our next step will be to estimate the cardinality of $\mathfrak{R}^f_n$. As η is an embedded closed curve in the oriented surface S it is possible to take another embedded curve ν such that $\{[\eta], [\nu]\}$ is an oriented basis of $H_1(S)$. It is well known that the intersection number of the pair $\{\eta, \nu\}$ is 1.

By the assumption made on the map $h : S \to S$ on the beginning of the chapter, the matrix $P \in PSL(2,\mathbb{Z})$ representing $h^* : H_1(S) \to H_1(S)$ in the basis $\{\eta, \nu\}$ is hyperbolic. The homology class of curve $h^{-n}(\eta)$ can be written as a linear combination of $[\eta]$ and $[\nu]$; let (a_n, b_n) be the unique pair of integers such that $[h^{-n}(\eta)] = a_n[\eta]+b_n[\nu]$. It is immediate from the well known description of the dynamics of hyperbolic linear automorphisms of the 2-torus (see [39]), that the hyperbolicity of the matrix P implies that there exist a constant $d > 0$ such that:

$$\mid a_n \mid, \mid b_n \mid > e^{dn} \tag{6.8}$$

or, in other words, a_n and b_n grow exponentially. We remind the reader b_n equals the homological intersection number of $h^{-n}(\eta)$ and η. We are now ready to prove the main result of this subsection:

Theorem 6.11. *$\sharp(\mathfrak{R}^f_n) \geq b_n$, and consequently $\sharp(\mathfrak{R}^f_n)$ grows exponentially with respect to n.*

Proof: we endow S with a hyperbolic metric g having ∂S as a geodesic boundary. Notice that as η and ν are simple closed curves, and the number of intersection of η and ν equals the intersection number $i([\eta], [\nu])$ then Lemma 2.6 on page 28 of [10], implies that there is a homeomorphism $\psi : S \to S$, homotopic to the identity and such that $\psi(\eta)$ and $\psi(\nu)$ are geodesics of the metric the hyperbolic metric g.

As $\psi(h^{-n}(\widehat{\eta}))$ is an embedded closed curve in S it is possible to isotopy it to an embedded hyperbolic geodesic γ. Such a geodesic γ has intersection number b_n with $\psi(\eta)$. We denote by $\{p^n_1, ..., p^n_{z_n}\}$ the set of the intersection points of γ and $\psi(\eta)$, and it is clear that $z_n \geq b_n$.

We consider the Poincaré disc $(\mathbb{D}, g_{-1})$ as the universal cover of (S, g), and denote $\pi : \mathbb{D} \to S$ the covering map. Given an embedded closed curve q in S which is not homologous to $\psi(\eta)$, let $\overline{q}$ be a lift of q in $\mathbb{D}$ and take a closed subinterval I_q of $\overline{q}$ such that $(\pi(\partial I)) = p_0 \notin \psi(\eta)$ and that covers every point $x \neq p_0$ of γ exactly once (the intersection of $\pi^{-1}(x)$ and I has one element). We call I_q a fundamental interval of q.

From now on we suppose $n \geq 1$ so that γ and $\psi(h^{-n}(\widehat{\eta}))$ are not homologous to $\psi(\eta)$. Consider a lift $\overline{\gamma}$ in $\mathbb{D}$, and take a fundamental interval I_γ of $\overline{\gamma}$. Because of the convexity of the hyperbolic metric, we know that a $\overline{\gamma}$ cannot intersect one lift of $\psi(\eta)$ more than once. Therefore, I_γ intersects exactly z_n different lifts $\{\kappa_1, ..., \kappa_{z_n}\}$ of $\psi(\eta)$.

Denote by γ_t isotopy for $t \in [0,1]$ between γ and $\psi(h^{-n}(\widehat{\eta}))$. Because of the hyperbolicity of h^* (and as $n \geq 1$), the curves $\psi(h^{-n}(\widehat{\eta}))$ and γ_{t_0} (obtained by fixing the coordinate t of the homotopy above) are not homologous to $\psi(\eta)$. Under these conditions we can use the isotopy γ_t, to construct a path I_t of fundamental intervals of γ_t. This generates an isotopy of I_γ to a fundamental interval $I_{\psi(h^{-n}(\widehat{\eta}))}$ of $\psi(h^{-n}(\widehat{\eta}))$ through fundamental intervals of γ_t. From the properties of fundamental intervals we have that $\pi(\partial(I_t)$ is disjoint from $\psi(\eta)$ for all $t \in [0,1]$. It is then clear that $I_{\psi(h^{-n}(\widehat{\eta}))}$ must also intersect the same z_n different lifts $\{\kappa_1, ..., \kappa_{z_n}\}$ of $\psi(\eta)$ intersected by I_γ, though it can in theory intersect also others lifts of $\psi(h^{-n}(\widehat{\eta}))$.

The set A of intersection points of η and $h^{-n}(\widehat{\eta})$ is in bijective correspondence with the set O of intersection points of $\psi(\eta)$ and $\psi(h^{-n}(\widehat{\eta}))$. Because of the properties of a fundamental interval, there also exists a bijection between the set O of intersection points of $\psi(\eta)$ and $\psi(h^{-n}(\widehat{\eta}))$, and the set B of intersection points of $I_{\psi(h^{-n}(\widehat{\eta}))}$ with the geodesics $\{\kappa_1, ..., \kappa_{z_n}\}$. There exists then a bijection map $\varphi : A \to B$. We remind the reader that as we mentioned above, A is in bijective correspondence with the set of Reeb chords from Λ_0 to $\widehat{\Lambda}_n$.

Taking now $p_1, p_2 \in A$, we claim that there is a strip v satisfying the four conditions of Proposition 6.10 above if, and only if, $\varphi(p_1)$ and $\varphi(p_2)$ lie in the same κ_j. To prove one direction of the claim notice that if there exists such a strip v then we can take a lift $\overline{v}$ of v in the universal cover $\mathbb{D}$. By looking at the boundary conditions that are satisfied by $\overline{v}$ and using that $\psi(\eta)$ and $\psi(h^{-n}(\widehat{\eta}))$ are embedded in S, it is easy to see that $\varphi(p_1)$ and $\varphi(p_2)$ have to lie in the same κ_j. For the other direction if $\varphi(p_1)$ and $\varphi(p_2)$ lie in the same κ_j we can construct a strip $\overline{v}$ satisfying $\overline{v}([0,1] \times \{0\}) = \varphi(p_1)$, $\overline{v}([0,1] \times \{1\}) = \varphi(p_2)$, $\overline{v}(\{0\} \times [0,1]) \subset \kappa_j$ and $\overline{v}(\{1\} \times [0,1]) \subset \iota$ (ι being the lift of $\psi(h^{-n}(\widehat{\eta}))$ that contains $I_{\psi(h^{-n}(\widehat{\eta}))}$), and taking $v = \pi(\overline{v})$ we obtain the desired strip satisfying the conditions of Proposition 6.10.

As a consequence of the previous claim and Proposition 6.10, we have that to each different lift κ_j is associated a different Relative Nielsen class ϱ_j in $\mathfrak{R}_n$. Moreover, the intersections between $I_{\psi(h^{-n}(\widehat{\eta}))}$ and κ_j are in bijective correspondence with the Reeb chords in ϱ_j. An immediate consequence is that there are at least z_n different Relative Nielsen classes in $\mathfrak{R}_n$.

To conclude the proof of the theorem, we have to prove that ϱ_j is a fundamental Relative Nielsen class. To see this we observe that I_γ intersects each κ_j an odd number of times, and the isotopy I_t between I_γ and $I_{\psi(h^{-n}(\widehat{\eta}))}$ is such that $\partial(I_t)$ never intersects κ_j. As I_γ and $I_{\psi(h^{-n}(\widehat{\eta}))}$ are both transversal to κ_j, we conclude that $I_{\psi(h^{-n}(\widehat{\eta}))}$ also has to intersect κ_j an odd number of times, which proves that ϱ_j is a fundamental Relative

Nielsen class. Thus, there are in fact at least z_n different fundamental relative Nielsen classes in $\mathfrak{R}_n$, and the theorem is proved.

□

Before proving the main result of this section we need one last ingredient. The inverse of the diffeomorphism $F : \mathbb{R} \times S \to \mathbb{R} \times S$ is $F^{-1}(t, p) = (t + f(h^{-1}), h^{-1}(p))$. Let $K > 0$ be a constant such that $\max(f) < K$. Then $\widehat{\Lambda_n} = F^{-1}(\widehat{\Lambda_0}) \subset [0, (n+1)K] \times S$. This implies that for all Relative Nielsen classes $\varrho \in \mathfrak{R}^f_k$ where $0 \leq k \leq n$, the Reeb chords $c \in \varrho$ satisfy $A(c) \leq (n+1)K$.

We have now, all the ingredients needed to obtain the exponential homotopical growth rate of $LC\mathbb{H}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$.

Theorem 6.12. *The linearized Legendrian contact homology $LC\mathbb{H}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth rate with exponential weight $\frac{d}{K}$.*

Proof: the strategy is to use the growth rate of the number of different fundamental relative Nielsen classes, to estimate the set $\#(\Sigma^{K(n+1)}_{\Lambda \to \widehat{\Lambda}}(\tau))$ defined in section 4.1.

Step 1: for every $\varrho \in \mathfrak{R}^f_k$ with $0 \leq k \leq n$, we have $\mathcal{I}(\varrho) \in \Sigma^{K(n+1)}_{\Lambda \to \widehat{\Lambda}}(\tau)$ (for the constant $K > 0$ as above).

From the defining property of fundamental relative Nielsen classes we know that: $\dim(LCH^{\mathcal{I}(\varrho)}_{st}(M, \tau, \Lambda \to \widehat{\Lambda}) = \dim(Im(\partial_{st}) + \dim(\ker(\partial_{st}))$ is odd for every fundamental relative Nielsen class ϱ. Now:

$$\dim(LC\mathbb{H}^{\mathcal{I}(\varrho)}_{st}(M, \tau, \Lambda \to \widehat{\Lambda}) = \dim(\ker(\partial_{st})) - \dim(Im(\partial_{st})) =$$
$$= \dim(LCH^{\mathcal{I}(\varrho)}_{st}(M, \tau, \Lambda \to \widehat{\Lambda}) - 2(\dim(Im(\partial_{st})))$$

implies that the numbers $\dim(LC\mathbb{H}^{\mathcal{I}(\varrho)}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ and $\dim(LCH^{\mathcal{I}(\varrho)}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ have the same parity. Therefore $\dim(LC\mathbb{H}^{\mathcal{I}(\varrho)}_{st}(M, \tau, \Lambda \to \widehat{\Lambda})$ cannot be zero, and has to be a positive number.

This combined with the fact that for all Relative Nielsen classes $\varrho \in \mathfrak{R}^f_k$ with $0 \leq k \leq n$, all the Reeb chords $c \in \varrho$ satisfy $A(c) \leq (n+1)C$, imply that:

for all $\varrho \in \mathfrak{R}^f_k$ with $0 \leq k \leq n$, we have $\mathcal{I}(\varrho) \in \Sigma^{K(n+1)}_{\Lambda \to \widehat{\Lambda}}(\tau)$.

Step 2:

From step 1 above we know that $\#(\Sigma^{K(n+1)}_{\Lambda\to\widehat{\Lambda}}(\tau)) \geq b_n$. Taking $d' = \frac{d}{K}$ and $a = e^{-d}$ it follows that:

$$ae^{d'(n+1)K} = e^{dn} \leq b_n \leq \#(\Sigma^{K(n+1)}_{\Lambda\to\widehat{\Lambda}}(\tau)) \tag{6.9}$$

which establishes the theorem.

□

Theorem 6.13. *Let M be a closed oriented connected 3-manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary such that the component M_0 satisfies:*

- *M_0 is the mapping torus of a punctured torus S by a diffeomorphism $h : S \to S$ such that the homology map h^* is a hyperbolic automorphism of $H_1(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$.*

Then M can be given infinitely many different tight contact structures ξ_k, such that there exist disjoint Legendrian knots Λ_k, Λ'_k and contact forms τ_k associated to (M, ξ_k) and adapted to the pair Λ_k, Λ'_k for which $LC\mathbb{H}_{st}(\tau_k, \Lambda_k \to \Lambda'_k)$ has exponential homotopical growth rate.

Proof :

As we mentioned before, Colin showed in [11] that the recipe we used to produce the contact form τ in M can generate infinitely many different contact structures in M; this is because the construction of τ depended on the necks T_n we glued to make the interpolation of the forms, and depending on the T_n used one gets different contact structures. With this, we have finished the proof of theorem .

□

As a consequence of this theorem and Theorem 4.4, we have that for every contact form τ' associated to $(M, \xi = \ker(\tau))$, the Reeb flow of $X_{\tau'}$ have positive topological entropy.

Chapter 7

Graph manifolds and Foulon-Hasselblatt surgery

In [27] Handel and Thurston used Dehn surgery to obtain non-algebraic Anosov flows in 3-manifolds. Their surgery was adapted to the contact setting by Foulon and Hasselblatt in [20], who interpreted it as a Legendrian surgery and used it to produce non-algebraic Anosov Reeb flows on 3-manifolds. We consider here a surgery that includes the Foulon-Hasselblatt one as a particular case: they restrict their attention to Dehn surgeries with positive integer coefficients while we consider the case of any integer coefficient.

7.1 The surgery

We start by fixing some notation. Let (S, g) be an oriented hyperbolic surface and $\mathfrak{c} : S^1 \to S$ an embedded oriented separating geodesic of g. We denote by $\pi : (\mathbb{D}, g) \to (S, g)$ the locally isometric universal covering of (S, g) by the the hyperbolic disc $(\mathbb{D}, g)$ with the property that $(-1, 1) \times \{0\} \subset \pi^{-1}(\mathfrak{c}(S^1))$; such a covering always exist, since the segment $(-1, 1)\times\{0\}$ of the real axis is a geodesic in $(\mathbb{D}, g)$. We denote by $v(\theta)$ the unique unitary vector field over $\mathfrak{c}(\theta)$ satisfying $\angle(\mathfrak{c}'(\theta), v(\theta)) = -\frac{\pi}{2}$. Our orientation convention is chosen, so that for coordinates $z = x + iy \in \mathbb{D}$, the lift of $v(\theta)$ to $(-1, 1) \times \{0\}$ is a positive multiple of the vector field $-\partial_y$ over $(-1, 1) \times \{0\}$. Also, let $\Pi : T_1S \to S$ denote the base point projection.

Because $\mathfrak{c}$ is a separating geodesic, we can cut S along c to obtain two oriented hyperbolic surfaces with boundary which we denote by S_1 and S_2; our labelling is chosen so that the vector field $v(\theta)$ points inside S_2 and outside S_1. This decomposition of S

induces a decomposition of T_1S in T_1S_1 and T_1S_2. Both T_1S_1 and T_1S_2 are 3-manifolds whose boundary is the torus formed by the the unit fibers over $\mathfrak{r}$.

Denote by $V_{\mathfrak{r},\delta}$ the closed $\delta-$neighbourhood of the the geodesic $\mathfrak{r}$ for the hyperbolic metric g. For $\delta > 0$ sufficiently small we have that $V_{\mathfrak{r},\delta}$ is an annulus such that the only closed geodesics contained in $V_{\mathfrak{r},\delta}$ are the covers of $\mathfrak{r}$, and that satisfies the following convexity property: if $\breve{V}$ is the connected component of $\pi^{-1}(V_{\mathfrak{r},\delta})$ containing $(-1,1) \times \{0\}$, then every segment of a hyperbolic geodesic starting and ending in $\breve{V}$ is completely contained in $\breve{V}$. It also follows from the conventions adopted above, that if we denote by U^+ the upper hemisphere of the $\mathbb{D}$ composed of points with positive imaginary component and by U^+ the lower hemisphere of the $\mathbb{D}$ composed of points with negative imaginary component, we have:

$$\breve{V} \cap U^+ \subset \pi^{-1}(S_1) \text{ and } \breve{V} \cap U^- \subset \pi^{-1}(S_2). \tag{7.1}$$

This fact has the following important consequence: if $\nu([0,K])$ is a hyperbolic geodesic segment starting and ending at $V_{\mathfrak{r},\delta}$ and contained in one of the S_i, then $[\nu]$ is a non-trivial homotopy class in the relative fundamental group $\pi_1(S_i, V_{\mathfrak{r},\delta})$.

On the unit tangent bundle T_1S, we consider the contact form α_g whose Reeb vector field is the geodesic vector field for the hyperbolic metric g. It is well known that the lifted curve $(c(\theta), v(\theta))$ in T_1S is Legendrian on the contact manifold $(T_1S, \ker(\alpha_g))$. The geodesic vector field X_{α_g} over the Legendrian curve the geodesic vector field coincides with the horizontal lift of v (see section 1.3 of [38]), and therefore points inward T_1S_2 and outwards T_1S_1, and is normal to $\partial(T_1S_2)$ for the Sasaki metric.

Moreover if $\delta > 0$ is small enough we know that for every $\vartheta \in L_{\mathfrak{r}}$ there exists numbers $t_1 < 0$ and $t_2 > 0$ such that:

$$\phi^t_{\alpha_g}(\vartheta) \in T_1S_1 \setminus \Pi^{-1}(V_{\mathfrak{r},\delta}) \tag{7.2}$$

$$\phi^t_{\alpha_g}(\vartheta) \in T_1S_2 \setminus \Pi^{-1}(V_{\mathfrak{r},\delta}) \tag{7.3}$$

Following [20], we know that there exists a neighbourhood $B^{3\eta}_{2\epsilon}$ of $L_{\mathfrak{r}}$ on which we can find coordinates $(t,s,w) \in (-3\eta, 3\eta) \times S^1 \times (-2\epsilon, 2\epsilon)$ such that:

$$\alpha_g = dt + wds, \tag{7.4}$$

$$L_{\mathfrak{r}} = \{0\} \times S^1 \times \{0\}, \tag{7.5}$$

where $\{0\} \times \{\theta\} \times (-2\epsilon, 2\epsilon)$ is a local parametrization of the unitary fiber over $\theta \in L_{\mathfrak{r}}$, and $\epsilon < \frac{\eta}{4|q|\pi}$, with q being a fixed non-zero integer. Let $\mathcal{W}^- = \{-3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon)$ and $\mathcal{W}^+ = \{+3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon)$. It is clear that $\Pi(\mathcal{W}^-) \subset S_1$ and $\Pi(\mathcal{W}^+) \subset S_2$. Because on $\overline{B}^{3\eta}_{2\epsilon}$ the Reeb vector field X_{α_g} is given by ∂_t, it is clear that for every point $p \in B^{3\eta}_{2\epsilon}$ there are $p^- \in \mathcal{W}^-$, $p^+ \in \mathcal{W}^+$, $t^- \in (-6\eta, 0)$ and $t^+ \in (0, 6\eta)$ for which:

$$\phi^{t^-}_{X_{\alpha_g}}(p) = p^- \text{ and } \phi^{t^+}_{X_{\alpha_g}}(p) = p^+ \tag{7.6}$$

This means that trajectories of the flow of X_{α_g} that enter the box $B^{3\eta}_{2\epsilon}$ enter through $\mathcal{W}^-$ and exit through $\mathcal{W}^+$; they cannot stay inside $B^{3\eta}_{2\epsilon}$ for positive or negative time. We can say even more about these trajectories.

For $\sigma = (\mathfrak{p}, \dot{\mathfrak{p}}) \in (T_1 S)$ (where $\mathfrak{p} \in S$ and $\dot{\mathfrak{p}} \in T_{\mathfrak{p}} S$) in $\mathcal{W}^+ \cup \mathcal{W}^-$ let $\widetilde{\sigma} = (\widetilde{\mathfrak{p}}, \dot{\widetilde{\mathfrak{p}}})$ be a lift of σ to the unit tangent bundle $T_1\mathbb{D}$ such that $\widetilde{\mathfrak{p}} \in \check{V}$. The geodesic vector field X_{α_g} in $\widetilde{\sigma}$ coincides with the horizontal lift of $\dot{\mathfrak{p}}$ ([38][section 1.3]). For δ, $\eta > 0$ and $\epsilon < \frac{\eta}{4|q|\pi}$ sufficiently small we can guarantee that:

- $B^{3\eta}_{2\epsilon}$ is contained in $V_{\mathfrak{r},\delta}$
- for the lifts $\widetilde{\sigma} = (\widetilde{\mathfrak{p}}, \dot{\widetilde{\mathfrak{p}}})$ of points in $\mathcal{W}^+ \cup \mathcal{W}^-$ as above, the vector $\dot{\widetilde{\mathfrak{p}}}$ (which is the projection of the geodesic vector field $X_{\alpha_g}(\widetilde{\sigma})$ satisfies $\angle(\dot{\widetilde{\mathfrak{p}}}, -\partial_y) < \delta$

Shrinking $\delta > 0$, $\eta > 0$ and $0 < \epsilon < \frac{\eta}{4|q|\pi}$ if necessary this implies that for every $\sigma^+ \in \mathcal{W}^+$ there exists $t_{\sigma^+} > 0$ and for every $\sigma^- \in \mathcal{W}^-$ there exists $t_{\sigma^-} < 0$ such that:

$$\phi^{t_{\sigma^+}}_{X_{\alpha_g}}(\sigma^+) \in (T_1 S_2) \setminus V_{\mathfrak{r},\delta} \text{ and } \forall t \in [0, t_{\sigma^+}]\ \phi^{t}_{X_{\alpha_g}}(\sigma^+) \notin B^{3\eta}_{2\epsilon} \tag{7.7}$$

$$\phi^{t_{\sigma^-}}_{X_{\alpha_g}}(\sigma^-) \in (T_1 S_1) \setminus V_{\mathfrak{r},\delta} \text{ and } \forall t \in [t_{\sigma^-}, 0]\ \phi^{t}_{X_{\alpha_g}}(\sigma^+) \notin B^{3\eta}_{2\epsilon} \tag{7.8}$$

To prove this last condition above one uses the fact that $\angle(\dot{\widetilde{p}}, -\partial_y) < \delta$ is small and studies the behavior of geodesics in $(\mathbb{D}, g)$ starting at points close to the real axis and with initial velocity close to $-\partial_y$. It is easy to see that such geodesics have to cut through the region $V_{\mathfrak{r},\delta}$ and visit the interior of both $S_1 \setminus V_{\mathfrak{r},\delta}$ and $S_2 \setminus V_{\mathfrak{r},\delta}$ From now on we will assume that $\delta > 0$, $\eta > 0$ and $0 < \epsilon < \frac{\eta}{4|q|\pi}$ are such that the all the above mentioned properties described for them being sufficiently small hold simultaneously.

Consider the following map $F : B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon} \to B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}$:

$$F(t, s, w) = (t, s + f(w), w) \text{ for } (t, s, w) \in (\eta, 2\eta) \times S^1 \times (-2\epsilon, 2\epsilon) \tag{7.9}$$

where $f(w) = -q\mathcal{R}(\frac{w}{\epsilon})$ (for our previously chosen integer q) and $\mathcal{R} : [-1, 1] \to [0, 2\pi]$ satisfies $\mathcal{R} = 0$ on a neighbourhood of -1, $\mathcal{R} = 2\pi$ on a neighbourhood of 1, $0 \leq \mathcal{R}' \leq 4$ and $\mathcal{R}'$ is an even function; and F is the identity otherwise.

Our new 3-manifold M is obtained by gluing $T_1S \setminus \overline{B}^{\eta}_{\epsilon}$ and $B^{2\eta}_{2\epsilon}$ using the map F:

$$M = (T_1S \setminus \overline{B}^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon} \Big/ (x \in B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \sim (F(x) \in T_1S \setminus \overline{B}^{\eta}_{\epsilon}) \tag{7.10}$$

Notice that $T_1S = (T_1S \setminus \overline{B}^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon} \Big/ (x \in B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \sim (x \in T_1S \setminus \overline{B}^{\eta}_{\epsilon})$. This clarifies our construction of M and shows that M is obtained from T_1S via a Dehn surgery on $L_{\mathfrak{r}}$. We follow [20] to endow M with a contact form which coincides α_g outside of $B^{2\eta}_{2\epsilon}$. As a preparation we define the function $\beta : (-3\eta, 3\eta) \to \mathbb{R}$:

- β is equal to 1 in an open neighbourhood of $[-2\eta, 2\eta]$,
- $|\beta'| \leq \frac{\pi}{\eta}$ and $supp\beta$ is contained in $[-3\eta, 3\eta]$.

Using β we define:

$$r(t, w) = \beta(t) \int_{-2\epsilon}^{w} x f'(x) dx \tag{7.11}$$

We point out to the reader that $supp(r)$ is contained in $B^{3\eta}_{\epsilon}$ and therefore so is $supp(dr)$. Notice also, that in $B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}$ one has $dr = \frac{w}{2} f'(w) dw$.

Again following [20] we define in $T_1S \setminus \overline{B}^{\eta}_{\epsilon}$ the 1-form:

$$A_r = dt + wds + dr \text{ for } (-3\eta, -\eta), \tag{7.12}$$
$$A_r = dt + wds - dr \text{ for } (\eta, 3\eta), \tag{7.13}$$
$$A_r = \alpha_g \text{ otherwise}. \tag{7.14}$$

Notice that because $supp(dr)$ is contained in $B^{3\eta}_{\epsilon}$ the 1-form A_r is well-defined.

On the box $B^{2\eta}_{2\epsilon}$ we define:

$$\widetilde{A} = dt + wds + dr \tag{7.15}$$

Computing, we obtain $F^*(A_r) = \widetilde{A}$ which means that the gluing map F allows us to glue the 1-forms A_r and $\widetilde{A}$. We denote by α_F the 1-form in M obtained by gluing $\widetilde{A}$ and A_r. We will denote by $\widetilde{B}$ the following region:

$$\widetilde{B} = ((B^{3\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \ \subset M) \cup B^{2\eta}_{2\epsilon} \Big/ (x \in B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \sim (F(x) \in (B^{3\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \tag{7.16}$$

The importance of this region lies in the fact that in $M \setminus \widetilde{B} = T_1S \setminus B^{3\eta}_{2\epsilon}$, the contact form α_F coincides with α_g.

Following [20] one shows through a direct computation that $(dt + wds \pm dr) \wedge (dw \wedge ds) = (1 \pm \frac{\partial r}{\partial t})dt \wedge dw \wedge ds$. Using the fact that $\epsilon < \frac{\eta}{8\pi|q|}$ one gets that $|\frac{\partial r}{\partial t}| < 1$, thus obtaining that $(dt + wds \pm dr)$ is a contact form. It follows from this that A_r and $\widetilde{A}$ are contact forms in their respective domains and therefore α_F is a contact form in M. More strongly, Foulon and Hasselblatt proceed to show that if q is non-negative the Reeb flow of α_F is an Anosov Reeb flow.

7.2 Hypertightness of α_F

For $q \in \mathbb{N}$ the hypertightness of α_F follows from the fact that its Reeb flow is Anosov; this is a consequence of Novikov's theorem as mentioned in [19]. In this subsection we give an independent and completely geometrical proof of hypertightness of α_F, which is valid for every $q \in \mathbb{Z}$.

To understand the topology of Reeb orbits of α_F we will study trajectories that enter the surgery region $\widetilde{B}$. We start by studying trajectories in $B^{2\eta}_{2\epsilon}$. On this region we have:

$$X_{\alpha_F} = \frac{\partial_t}{1 + \partial_t r} \tag{7.17}$$

This implies, similarly to what happens for α_g, that for points $p \in B^{2\eta}_{2\epsilon}$ the trajectory $\phi^t_{X_{\alpha_F}}(p)$ leaves the box $B^{2\eta}_{2\epsilon}$ in forward and backward time. More precisely, there exists a constant $\widetilde{a} > 0$ depending only on α_F, such that for $p \in B^{2\eta}_{2\epsilon}$ there are $\breve{p}^- \in \breve{\mathcal{W}}^- = \{-2\eta\} \times S^1 \times [-2\epsilon, 2\epsilon]$, $\breve{p}^+ \in \breve{\mathcal{W}}^+ = \{+2\eta\} \times S^1 \times [-2\epsilon, 2\epsilon]$, $\breve{t}^- \in (-\widetilde{a}, 0]$ and $\breve{t}^+ \in [0, \widetilde{a})$ such that:

$$\phi^{t^-}_{X_{\alpha_F}}(\breve{p}) = \breve{p}^- \text{ and } \phi^{t^+}_{X_{\alpha_g}}(\breve{p}) = \breve{p}^+ \tag{7.18}$$

We now analyse the trajectories of points $\breve{p}^- \in \breve{\mathcal{W}}^-$ and $\breve{p}^+ \in \breve{\mathcal{W}}^+$. For this, we first notice that on $\widetilde{B} \setminus B^{2\eta}_{2\epsilon}$ the contact form α_F is given by: $dt + wds \pm dr$ and therefore we have in this region:

$$X_{\alpha_F} = \frac{\partial_t}{1 \pm \partial_t r} \tag{7.19}$$

which is still a positive multiple of ∂_t.

This implies that for every $\breve{p}^- \in \breve{\mathcal{W}}^-$ and $\breve{p}^+ \in \breve{\mathcal{W}}^+$ there exist $t^{\breve{p}^-} < 0$ and $t^{\breve{p}^+} < 0$ such that

$$\phi_{X_{\alpha_F}}^{t^{\breve{p}^-}}(\breve{p}^-) \in \mathcal{W}^- \text{ and } \phi_{X_{\alpha_g}}^{t^{\breve{p}^-}}(\breve{p}^+) \in \mathcal{W}^+ \tag{7.20}$$

Again using that X_{α_F} is a positive multiple of ∂_t on $\widetilde{B} \setminus B_{2\epsilon}^{2\eta}$ we have that for every point p in $\widetilde{B} \setminus B_{2\epsilon}^{2\eta}$ whose t coordinate is in $[2\eta, 3\eta]$ the trajectory of the flow $\phi_{X_{\alpha_F}}^t$ going through p is a straight line with fixed coordinates s and w, that goes from $\breve{\mathcal{W}}^+$ to $\mathcal{W}^+$. Analogously, for every point p in $\widetilde{B} \setminus B_{2\epsilon}^{2\eta}$ whose t coordinate is in $[-3\eta, -2\eta]$ the trajectory of the backward flow of $\phi_{X_{\alpha_F}}^t$ going through p is a straight line $\breve{\mathcal{W}}^-$ to $\mathcal{W}^-$.

Summing up, with all the cases considered above we have showed that for every point $p \in \widetilde{B}$ the trajectory of the flow $\phi_{X_{\alpha_F}}^t$ going through p for $t = 0$ intersects $\mathcal{W}^-$ for non-positive time and $\mathcal{W}^+$ for for non-negative time. In other, all trajectories that intersect $\widetilde{B}$ enter through $\mathcal{W}^-$ and leave through $\mathcal{W}^+$, which means that for all $\breve{p} \in \widetilde{B}$ there exist times $t_{\breve{p}}^- \leq 0$ and $t_{\breve{p}}^+ \geq 0$ such that:

$$\phi_{X_{\alpha_F}}^{t_{\breve{p}}^+}(\breve{p}) \in \mathcal{W}^+ \tag{7.21}$$

$$\phi_{X_{\alpha_F}}^{t_{\breve{p}}^-}(\breve{p}) \in \mathcal{W}^- \tag{7.22}$$

$$\phi_{X_{\alpha_F}}^{t}(\breve{p}) \in \widetilde{B} \text{ for all } t \in [t_{\breve{p}}^-, t_{\breve{p}}^+] \tag{7.23}$$

Now, because on $M \setminus \widetilde{B} = T_1S \setminus B_{2\epsilon}^{3\eta}$ the contact form α_F coincides with α_g we have that trajectories of X_{α_F} starting at $\mathcal{W}^-$ at the time $t = 0$ have to leave $M \setminus N$ for negative time before reentering on $\widetilde{B}$; similarly the trajectories starting at $\mathcal{W}^+$ have to leave $M \setminus N$ for positive time before reentering on $\widetilde{B}$. More precisely, one can use equations (7.7) and (7.8) to see that for $p^- \in \mathcal{W}^-$ and $p^+ \in \mathcal{W}^+$ there exist $t_{p^-} < 0$ and $t_{p^+} > 0$ such that:

$$\phi_{X_{\alpha_F}}^{t_{p^+}}(p^+) \in M_2 \setminus N \text{ and } \forall t \in [0, t_{p^+}]\ \phi_{X_{\alpha_F}}^{t}(p^+) \notin \widetilde{B} \tag{7.24}$$

$$\phi_{X_{\alpha_F}}^{t_{p^-}}(p^-) \in M_1 \setminus N \text{ and } \forall t \in [t_{p^-}, 0]\ \phi_{X_{\alpha_F}}^{t}(p^-) \notin \widetilde{B} \tag{7.25}$$

where

$$M_1 = (T_1S_1 \setminus B_\epsilon^\eta) \cup B_{2\epsilon}^{2\eta}(-) \Big/ (x \in B_{2\epsilon}^{2\eta}(-) \setminus \overline{B}_\epsilon^\eta) \sim (F(x) \in ((B_{2\epsilon}^{3\eta} \cap T_1S_1) \setminus \overline{B}_\epsilon^\eta) \tag{7.26}$$

$$M_2 = (T_1S_2 \setminus B_\epsilon^\eta) \cup B_{2\epsilon}^{2\eta}(+) \Big/ (x \in B_{2\epsilon}^{2\eta}(+) \setminus \overline{B}_\epsilon^\eta) \sim (F(x) \in ((B_{2\epsilon}^{2\eta} \cap T_1S_2) \setminus \overline{B}_\epsilon^\eta) \tag{7.27}$$

$$N = \Pi^{-1}(V_{\mathfrak{r},\delta} \setminus) \cup B_{2\epsilon}^{2\eta}(-) \Big/ (x \in B_{2\epsilon}^{2\eta}(-) \setminus \overline{B}_\epsilon^\eta) \sim (F(x) \in ((B_{2\epsilon}^{3\eta} \cap T_1S^1) \setminus \overline{B}_\epsilon^\eta) \tag{7.28}$$

for $B^{2\eta}_{2\epsilon}(-) = [-2\eta, 0] \times S^1 \times (-2\epsilon, 2\epsilon)$ and $B^{2\eta}_{2\epsilon}(+) = [0, 2\eta] \times S^1 \times (-2\epsilon, 2\epsilon)$.

Remark: it is not hard to see that $M = {M_1 \cup M_2} \big/ (x \in \partial M_1) \sim (\widetilde{F}(x) \in \partial M_2)$, where $\widetilde{F}$ is a Dehn twist which coincides with $(s + f(w), w)$ for $w \in [-2\epsilon, 2\epsilon]$ and is the identity elsewhere. This picture of M is closer to the one in the paper [27] and shows that M is a graph manifold (a graph manifold is one whose JSJ decomposition consists of Seifert S^1 bundles). By using this description of M and applying Van-Kampen's theorem to analyse the fundamental group of M, Handel and Thurston show that, for q not belonging to a finite subset of $\mathbb{Z}$, no finite cover of M is a Seifert manifold thus obtaining that M is an "exotic" graph manifold.

From their definitions, one sees that as manifolds $M_1 \cong T_1S_1$ and $M_2 \cong T_1S_2$, and it follows that ∂M_1 and ∂M_2 are incompressible tori. By looking at M_1 and M_2 as submanifolds of M, one obtains easily that their boundary $\mathbb{T}$ coincides and remains incompressible in M. We remark that $M_i \setminus N$ is diffeomorphic to $T_1S_i \setminus \Pi^{-1}(V_{\mathfrak{r},\delta})$ which is diffeomorphic to T_1S_i for $i = 1, 2$.

In a similar way we can describe the topology of N. Let $N_i = M_i \cap N$; reasoning identically as one does to show that M_i is diffeomorphic to T_1S_i one shows that N_i is diffeomorphic to a thickned two torus $\mathcal{T}^2 \times [-1, 1]$. As N is obtained from N_1 and N_2 by gluing them along $\mathbb{T}$ (which is a boundary component of both of them) we have that N is also diffeomorphic to the product $\mathcal{T}^2 \times [-1, 1]$.

The discussion above proves the following:

Lemma 7.1. *For all $\check{p} \in \widetilde{B}$ the trajectory $\{\phi^t_{X_{\alpha_F}}(\check{p}); t \in \mathbb{R}\}$ intersects $M_1 \setminus N$ and $M_2 \setminus N$.*

Proof: we have already established that for $\check{p} \in \widetilde{B}$ its trajectory intersect $\mathcal{W}^+$ for some non-negative time and $\mathcal{W}^-$ for some non-positive time, as it is show in equation (7.21) and (7.22). One now applies equations (7.24) and (7.25) to finish the proof of the lemma. □

Notice that trajectories can only enter in $\widetilde{B}$ through the wall $\mathcal{W}^-$ which is completely contained in M_1 and can only exit $\widetilde{B}$ through the wall $\mathcal{W}^+$ which is completely contained in M_2. We also point out that all trajectories of the flow $\phi^t_{X_{\alpha_F}}$ are transversal to $\mathbb{T}$, with the exception of the two Reeb orbits which correspond to the hyperbolic geodesic c (they continue to exist as periodic orbits after the surgery because they are distant from the surgery region).

We will deduce from the previous discussion the following important lemma:

Lemma 7.2. *Let $\gamma([0,T])$ be a trajectory of X_{α_F} such that $\gamma(0) \in \mathbb{T}$, $\gamma(T') \in \mathbb{T}$ and for all $t \in (0,T')$ we have $\gamma(t) \notin \mathbb{T}$ (in such a situation $\gamma([0,T]) \subset M_i$ for some i equals to 1 or 2). Then $\gamma([0,T]) \cap (M_i \setminus N)$ is non-empty.*

Proof: we divide the proof in 3 possible scenarios.

First case: suppose that $\gamma([0,T]) \cap \widetilde{B}$ is empty. In this case $\gamma([0,T])$ also exists as a hyperbolic geodesic with endpoints in the closed geodesic c. It follows from the convexity of the hyperbolic metric that $[\gamma([0,T])] \in \pi_1(T_1S_i, \mathbb{T})$ is non-trivial. This implies that $[\gamma([0,T])] \in \pi_1(M_i, \mathbb{T})$ is non-trivial which can be true only if $\gamma([0,T]) \cap (M_i \setminus N)$ is non-empty since N is a tubular neighbourhood of $\mathbb{T}$.

Second case: suppose that $\gamma([0,T]) \cap \widetilde{B}$ is non-empty and $\gamma([0,T]) \subset M_2$. Take $\widehat{t} \in [0,T']$ such that $\gamma(\widehat{t}) \in \widetilde{B}$. We know from our previous discussion that there are $\widehat{t}_1 \leq \widehat{t} \leq \widehat{t}_2$ such that $\gamma([\widehat{t}_1, \widehat{t}_2]) \subset \widetilde{B}$, $\gamma(\widehat{t}_1) \in (\mathbb{T} \cap \widetilde{B})$ and $\gamma(\widehat{t}_1) \in \mathcal{W}^+$; notice that in coordinates (t,s,w), $\mathbb{T} \cap \widetilde{B}$ is the annulus $\{0\} \times S^1 \times (-2\epsilon, 2\epsilon)$. From this picture it is clear that for t smaller that $\widehat{t}_1$ the trajectory enters in M_1; therefore we have that $\widehat{t}_1 = 0$ and $\gamma([0,\widehat{t}_2]) \subset \widetilde{B}$. Notice also that for all t slightly bigger than $\widehat{t}_2$ the trajectory is outside $\widetilde{B}$. Because trajectories of X_{α_F} can only enter $\widetilde{B}$ in M_1 we obtain that $\gamma([\widehat{t}_2, T'])$ does not intersect the interior of $\widetilde{B}$ and therefore exists as a hyperbolic geodesic in T_1S_2. Now, using equations (7.7) and (7.8) we obtain that, because $\gamma(\widehat{t}_2) \in \mathcal{W}^+$, the trajectory $\gamma : [\widehat{t}_2, T'] \to M_2$ has to intersect $M_2 \setminus N$ before hitting $\mathbb{T}$ at $t = T'$. Thus there is some $t \in (\widehat{t}_2, T')$ for which $\gamma(t) \in M_2 \setminus N$.

Third case: the proof in the case where $\gamma([0,T]) \cap \widetilde{B}$ is non-empty and $\gamma([0,T]) \subset M_1$ is analogous to the one of the Second case.

This three cases exhaust all possibilities and therefore prove the lemma.

□

Our reason for introducing the above decomposition of M into M_1 and M_2 and for proving Lemmas 7.1 and 7.2 above is to introduce the following representation of Reeb orbits of α_F. Let (γ, T) be a Reeb orbit of α_g which intersects both $M_1 \setminus N$ and $M_2 \setminus N$. We can assume that the chosen parametrization of the Reeb orbit satisfies: $\gamma(0) \in \partial N$, and that there are $t_+ > 0$ and $t_- < 0$ such that:

$$\gamma(t_+) \in M_1 \setminus N \text{ and } \gamma([0,t_+]) \in M_1 \cup N \tag{7.29}$$

$$\gamma(t_-) \in M_2 \setminus N \text{ and } \gamma([t_-,0]) \in M_2 \cup N \tag{7.30}$$

This means that in an interval of the origin γ is coming from $M_2 \setminus N$ and going to $M_1 \setminus N$. It follows from Lemma 7.2 that there exists a unique sequence $0 = t_0 < t_{\frac{1}{2}} < t_1 < t_{\frac{3}{2}} < ... < t_n = T$ such that $\forall k \in \{0, ..., n-1\}$:

- $\gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_i$ for i equals to1 or2
- $\gamma([t_{k+\frac{1}{2}}, t_{k+1}]) \in N$ and there is a unique $\widetilde{t_k} \in [t_{k+\frac{1}{2}}, t_{k+1}]$ such that $\gamma(\widetilde{t_k}) \in \mathbb{T}$
- if $\gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_i$ then $\gamma([t_{k+1}, t_{k+\frac{3}{2}}]) \subset M_j$ for $j \neq i$

Notice that $\gamma([t_0, t_{\frac{1}{2}}]) \subset M_1$ and $\gamma([t_{n-1}, t_{n-\frac{1}{2}}]) \subset M_2$. This implies that n is even so that we write $n = 2n'$, and that $\gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_1$ for k even, and $\gamma([t'_k, t_{k'+\frac{1}{2}}]) \subset M_2$ for k' odd. For each $k \in \{0, ..., 2n'-1\}$ the existence of the unique $\widetilde{t_k}$ in the interval $[t_{k+\frac{1}{2}}, t_{k+1}]$ for which $\gamma(\widetilde{t_k}) \in \mathbb{T}$ is guaranteed from Lemma 7.2 and the fact that $\mathbb{T}$ is the hypersurface that separates M_1 and M_2.

In order to obtain information on the free homotopy class of (γ, T) we observe that for $\gamma([t_k, t_{k+\frac{1}{2}}])$ coincides with a hyperbolic geodesic segment in T_1S_i starting and ending $V_{\mathfrak{r},\delta}$. Therefore, as we have previously seen the homotopy class $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ in $\pi_1(T_1S_i, V_{\mathfrak{r},\delta})$ is non-trivial which implies that $\gamma([t_k, t_{k+\frac{1}{2}}])$ is a non-trivial relative homotopy class in $\pi_1(M_i, N)$. We consider now the curve $\gamma([\widetilde{t}_k, \widetilde{t}_{k+1}])$: it is the concatenation of 3 curves, the first and the third ones being completely contained in N and the middle one being $\gamma([t_k, t_{k+\frac{1}{2}}])$; from this description and the fact that $\gamma([t_k, t_{k+\frac{1}{2}}])$ is a non-trivial relative homotopy class in $\pi_1(M_i, N)$ it is clear that $\gamma([\widetilde{t}_k, \widetilde{t}_{k+1}])$ is also non-trivial in $\pi_1(M_i, N)$ (and also non-trivial in $\pi_1(M_i, \mathbb{T})$).

We now denote by $\widetilde{M}$ the universal cover of M with $\widehat{\pi} : \widetilde{M} \to M$ being the covering map. From the incompressibility of $\mathbb{T}$ it follows that every lift of $\mathbb{T}$ is an embedded plane in $\widetilde{M}$. We denote by $\widetilde{N}^0$ a lift of N; because N is a thickened neighbourhood of an incompressible torus it follows that $\widetilde{N}^0$ is diffeomorphic to $\mathbb{R}^2 \times [-1, 1]$, i.e. it is a thickened neighbourhood of an embedded plane in $\widetilde{M}$. Because N separates M in two components, it follows that $\widetilde{N}^0$ separates $\widetilde{M}$ is two connected components. $\partial(\widetilde{N}^0)$ is the union of two embedded planes P_+^0 and P_-^0 which are characterized by the fact that there are neighbourhoods V_+ and V_- of, respectively, P_+^0 and P_-^0 such that $\widehat{\pi}(V_+) \subset M_1$ and $\widehat{\pi}(V_-) \subset M_2$. We will denote by C_+^0 the connected component of $\widetilde{M} \setminus \widetilde{N}^0$ which intersects V_+, and by C_-^0 the connected component of $\widetilde{M} \setminus \widetilde{N}^0$ which intersects V_-.

As we saw earlier, the trajectory $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ is a non-trivial relative homotopy class in $\pi_1(M_i, N)$. It is not hard to see that it remains as such in $\pi_1(M, N)$. Let $\mathbb{T}_i = \partial(N) \cap M_i$; because N is a tubular neighbourhood of $\mathbb{T}_i$ it is clear that $[\gamma([t_k, t_{k+\frac{1}{2}}])$ would be trivial in $\pi_1(M_i, \mathbb{T}_i)$ if and only if it is trival in $\pi_1(M_i, N)$, and we know it is not. As $\mathbb{T}_i$ is isotopic to $\mathbb{T}$ it is also an incompressible torus that divide M in 2 components. Now, $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ would be trivial in $\pi_1((M_i \setminus int(N)), \mathbb{T}_i)$ if, and only if, there is a curve c in $\mathbb{T}_i$ which endpoints $\gamma(t_k)$ and $\gamma(t_{k+\frac{1}{2}})$ such that the concatenation $\gamma * c$ is contractible in $(M_i \setminus int(N))$. Because of the incompressibility of $\mathbb{T}_i$ such a curve $\gamma * c$ is contractible in $(M_i \setminus int(N))$ if, and only if, it is contractible in M. This implies that $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ would be trivial in $\pi_1(M, \mathbb{T}_i)$ if, and only if, it was trivial in $\pi_1((M_i \setminus int(N)), \mathbb{T}_i)$ which we know it is not the case. Lastly, because N is a tubular neighbourhood of $\mathbb{T}_i$ it is clear that as $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ is not trivial $\pi_1(M, \mathbb{T}_i)$ it cannot be trivial in $\pi_1(M, N)$, as we wished to show.

Let now $\widetilde{\gamma}$ be a lift of γ such that $\widetilde{\gamma}(0) \in \widetilde{N}^0$. We know that $\widetilde{\gamma}([t_{2n'-\frac{1}{2}} - T, t_{\frac{1}{2}}]) \subset \widetilde{N}^0$. It will be useful to us to define the following sequence:

$$\widetilde{t_i} = q_i T + t_{r_i}, \tag{7.31}$$

where q_i and $r_i < 2n'$ are the unique integers such that $i = q_i(2n') + r_i$. Associated to $\widetilde{t_i}$ we associate the lift $\widetilde{N}^i$ of N, which is determined by the property that $\widetilde{\gamma}(\widetilde{t_i}) \in \widetilde{N}^i$. It is clear that the sequence $\widetilde{N}^i$ contains all lifts of N which are intersected by the curve $\widetilde{\gamma}(\mathbb{R})$. For the lifts $\widetilde{N}^i$ we define the connected components C_+^i and C_-^i of $\widetilde{M} \setminus \widetilde{N}^i$, and the planes P_+^i and P_-^i analogously as how we defined them for $\widetilde{N}^0$. A priori it could be that for $i \neq j$ we had $\widetilde{N}^i = \widetilde{N}^j$. We will show however, that this cannot happen.

Firstly, $\widetilde{N}^0 \neq \widetilde{N}^1$because $\gamma([\widetilde{t_0}, \widetilde{t_1}])$ is non-trivial in $\pi_1(M, N)$. Also, we have that $\widetilde{N}^1 \subset C_+^0$ because $\gamma([t_0, t_{\frac{1}{2}}]) \subset M_1$. An identical reasoning shows that $\widetilde{N}^2 \neq \widetilde{N}^1$ and:

$$\widetilde{N}^2 \subset C_-^1. \tag{7.32}$$

On the other hand we have that $\widetilde{N}^0 \subset C_+^1$, because $\widetilde{\gamma}([\widetilde{t_0}, t_{\frac{1}{2}}])$ gives a path totally contained in $\widetilde{M} \setminus \widetilde{N}^1$ connecting $\widetilde{N}^0$ and P_+^1. As $\widetilde{N}^2 \subset C_-^1$ and $\widetilde{N}^0 \subset C_+^1$, we must have $\widetilde{N}^2 \neq \widetilde{N}^0$. In an identical way, one shows that $\widetilde{N}^3 \neq \widetilde{N}^1$, and more generally that $\widetilde{N}^{i+2} \neq \widetilde{N}^i$ and $\widetilde{N}^{i+1} \neq \widetilde{N}^i$.

Now for $\widetilde{N}^3$, we have that $\widetilde{N}^3 \subset C_+^2$. As $\widetilde{\gamma}([\widetilde{t_0}, t_{\frac{3}{2}}])$ is a path completely contained in $\widetilde{M} \setminus \widetilde{N}^2$ connecting $\widetilde{N}^0$ and P_-^2 we obtain that $\widetilde{N}^0 \subset C_-^2$, and therefore $\widetilde{N}^3 \neq \widetilde{N}^0$.

Proceeding inductively along this line one obtains that $\widetilde{N}^i \neq \widetilde{N}^0$ for all $i \neq 0$, and more generally, $\widetilde{N}^i \neq \widetilde{N}^j$for all $i \neq j$. As a consequence of this, we obtain that the curve $\widetilde{\gamma}(\mathbb{R})$ cannot be homeomorphic to a circle and therefore $\gamma(\mathbb{R})$ cannot be contractible.

We are now ready to prove for the main result of this subsection:

Proposition 7.3. *α_F is hypertight.*

Proof: there are two possibilities for Reeb orbits.

Possibility 1: the Reeb orbit γ visits both $M_1 \setminus N$ and $M_2 \setminus N$.
In this case, we have just showed above that γ is not contractible.

Possibility 2: the Reeb orbit γ is totally contained in M_i for i equal to 1 or 2. In this case, the Reeb orbit does not visit the surgery region $\widetilde{B}$. Therefore it existed also before the surgery as a closed hyperbolic geodesic in $M_i \setminus \widetilde{B} = T_1 S_i \setminus B^{3\eta}_{2\epsilon}$. Such a closed geodesic is non-contractible in $T_1 S_i$ which is diffeomorphic to M_i. We have thus obtained that $\gamma \subset M_i$ is non-contractible in M_i.

Looking now at M_i as a submanifold with boundary of M, we remind the reader that ∂M_i is an incompressible torus in M. This implies that every non-contractible closed curve in M_i remains non-contractible in M; therefore γ is also a non-contractible Reeb orbit for this case. □

7.3 Special Legendrians in M

Our objective now is to show that there are disjoint Legendrian knots Λ and $\widehat{\Lambda}$ such that the contact form α_F is adapted to the pair $(\Lambda, \widehat{\Lambda})$. We choose the Legendrians in the following way: as in the piece $M_1 \subset N$ of M is α_F coincides with α_g, we pick Λ and $\widehat{\Lambda}$ to be unitary fibers of $T_1 S_1$. By choosing these unitary fibers generically we can guarantee that the triple $(\alpha_F, \Lambda, \widehat{\Lambda})$ satisfies condition (d) from section 3.1, and Proposition 7.3 implies that α_F satisfies condition (a) from section 3.1.

Our objective now is to prove that there are no Reeb chords from Λ to itself which are trivial in $\pi_1(M, \Lambda)$, and that the same is true for $\widehat{\Lambda}$. We will show that this is true for any Legendrian in $M_1 \subset N$ which is a unit tangent fiber of $T_1 S_1$.

Let then $\widetilde{\Lambda}$ be a "unit tangent fiber" in $M_1 \subset N$. Then there are two types of Reeb chords of $\widetilde{\Lambda}$ to itself: those which are completely contained in M_1 and those that visit both components.

For a Reeb chord c from $\widetilde{\Lambda}$ which visits both components we can introduce a certain decomposition as we did for Reeb orbits in a similar situation. We consider the natural parametrisation $c : [0, T_c] \to M$. We consider the unique sequence $0 < t_1(c) < t_2(c) < .... < t_{n(c)}(c) < T_c$ such that $c(t_i(c)) \in \mathbb{T}$ for all $1 \leq i \leq n_c$, and that contains all times $t \in [0, T_c]$ for which $c(t) \in \mathbb{T}$. It is then clear that n_c is an even number, and that $c([t_k(c), t_{k+1}(c)]) \subset M_2$ for k odd, and $c([t_k(c), t_{k+1}(c)]) \subset M_1$ for k even.

Now considering the lift $\widetilde{c}$ of c to the universal cover of M we consider for $1 \leq i \leq n_c$ the unique lift $\mathfrak{P}_i$ of $\mathbb{T}$ containing $\widetilde{c}(t_i(c))$. Making an analysis identical to the one done before Proposition 7.3 one shows that for $i \neq j$ we have $\mathfrak{P}_i \neq \mathfrak{P}_j$ and uses this to show that $\widetilde{c}(0)$ and $\widetilde{c}(T_c)$ are in different lifts of $\widetilde{\Lambda}$ to the universal cover. This implies that c is not trivial in $\pi_1(M, \widetilde{\Lambda})$. We now have:

Lemma 7.4. *Let $\widetilde{\Lambda}$ be a "unit tangent fiber" contained in $M_1 \subset N$. Then there are no Reeb chords from $\widetilde{\Lambda}$ to itself that are trivial in $\pi_1(M, \widetilde{\Lambda})$.*

Proof: for a Reeb chords c from $\widetilde{\Lambda}$ to itself there are two possibilities.

Possibility 1: c visits both components. We have just argued that in this case c cannot be trivial in $\pi_1(M, \widetilde{\Lambda})$.

Possibility 2: c is completely contained in M_1. In this case c also existed as a hyperbolic geodesic for c starting and ending at $\widetilde{\Lambda}$ and contained in T_1S_1. As a hyperbolic geodesic starting and ending at a unit tangent fiber, c is non-trivial in $\pi_1(M_1, \widetilde{\Lambda}) = \pi_1(T_1S_1, \widetilde{\Lambda})$. Using the incompressibility of $\mathbb{T}$, it is easy to see that c remain non-trivial in $\pi_1(M, \widetilde{\Lambda})$.

As these are the only 2 possibilities, we have finished the proof of the lemma.

$\square$

Applying this lemma to our pair Λ and $\widehat{\Lambda}$, we have shown the following:

Proposition 7.5. *The contact form α_F is adapted to the pair $(\Lambda, \widehat{\Lambda})$.*

7.4 Exponential homotopical growth rate of $LC\mathbb{H}_{st}(\alpha_F, \Lambda \rightarrow \widehat{\Lambda})$

In this section we prove that $LC\mathbb{H}_{st}(\alpha_F, \Lambda \rightarrow \widehat{\Lambda})$ has exponential homotopical growth rate.

We begin by introducing a special class of elements of $\Sigma_{\Lambda \rightarrow \widehat{\Lambda}}(M)$. Let p and $\widehat{p}$ be the points in S_1 such that Λ is the unit tangent fiber over p and $\widehat{p}$. We consider the set $\mathfrak{H}$ of elements in $\Sigma_{\Lambda \rightarrow \widehat{\Lambda}}(T_1S_1)$ that contain hyperbolic geodesics connecting p and $\widehat{p}$. In other words, the elements of $\mathfrak{H}$ contain Reeb chords of α_g from Λ to $\widehat{\Lambda}$ completely contained in S_1.

Let $\rho \in \mathfrak{H}$ and $c_\rho \in \rho$ be the unique Reeb chord corresponding to the unique hyperbolic geodesic from p to $\widehat{p}$ in the ρ. The main observation is that c_ρ also exists as a

Reeb chord of α_F from Λ to $\widehat{\Lambda}$. The reason for that is that c_ρ is contained in a region of $T_1S_1 = M_1$ which away from the neighbourhood of $L_{\mathfrak{r}}$ where we performed the surgery: the reason for that is that, as we saw in section 7.1 (specifically equations (7.2), (7.3), (7.7) and (7.8)), only trajectories that only geodesics that visit both components of S intersect the neighbourhood of $L_{\mathfrak{r}}$ where we perform the surgery. As consequence, the trajectory c_ρ is not altered by the surgery and it exists also as a Reeb chord for α_F. N

We define $\mathfrak{H}^C$ as the set of elements $\rho \in \mathfrak{H}^C$ such that the Reeb chord $A(c_\rho) \leq C$. It is well known that the fundamental group of S_1 has exponential growth, and this implies that there exists real numbers $a > 0$, d and C_0 (which depend only on the geodesic $\mathfrak{r}$) such that $\#(\mathfrak{H}^C) \geq e^{aC+d}$ for all $C \geq C_0$.

From now on we will consider $\mathcal{I}(\mathfrak{H}^C)$ also as a subset $\Sigma_{\Lambda\to\widehat{\Lambda}}(M)$ coming from the inclusion $\mathcal{I} : \Sigma_{\Lambda\to\widehat{\Lambda}}(T_1S_1 = M_1) \to \Sigma_{\Lambda\to\widehat{\Lambda}}(M)$. Coherent with this, we will denote by $c_{\mathcal{I}(\rho)}$ the Reeb chord that we previously denoted as c_ρ in M_1.

Lemma 7.6. *For every $\rho \in \mathfrak{H}^C$, the unique Reeb chord of $\mathcal{I}(\rho)$ in $\mathcal{T}^{\mathcal{I}(\rho)}_{\Lambda\to\widehat{\Lambda}}(\alpha_F)$ is the $c_{\mathcal{I}(\rho)}$ considered above.*

Proof: let $\check{c}$ be a Reeb chord different from $c_{\mathcal{I}(\rho)}$. There are two possibilities for $c_{\mathcal{I}(\rho)}$.

Possibility 1: $\check{c}$ is completely contained in M_1.
In this case, we know that $\check{c}$ also existed as a hyperbolic geodesic connecting p and $\widehat{p}$. From classical properties of geodesics in hyperbolic surfaces, we deduce that $c_{\mathcal{I}(\rho)}$ and $\check{c}$ belong to different elements of $\Sigma_{\Lambda\to\widehat{\Lambda}}(T_1S_1) = \Sigma_{\Lambda\to\widehat{\Lambda}}(M_1)$. As the boundary of M_1 is an incompressible, we have that if $c_{\mathcal{I}(\rho)}$ and $\check{c}$ belong to different elements of $\Sigma_{\Lambda\to\widehat{\Lambda}}(M_1)$, they also belong to different elements of $\Sigma_{\Lambda\to\widehat{\Lambda}}(M)$. Therefore we conclude that that $\check{c} \notin \mathcal{I}(\rho)$.

Possibility 2: $\check{c}$ intersects both M_1 and M_2.
In this case, we consider a lift $\check{c}$ of $\check{c}$ to the universal cover $\widetilde{M}$ of M and let P_0 be a lift of $\mathbb{T}$ intersected by $\check{c}$. Reasoning identically as we did in the proof of Lemma 7.2, we can show that $\check{c}$ intersects P_0 at only one point. This allows us to conclude that the intersection number of $\check{c}$ and P_0 equals to 1.

Let now, Λ_0 and $\widehat{\Lambda}_0$ be the lifts of, respectively, Λ and $\widehat{\Lambda}$ such that: the initial point of $\check{c}$ is in Λ_0 and the final point of $\check{c}$ is in $\widehat{\Lambda}_0$. Clearly both Λ_0 and $\widehat{\Lambda}_0$ are disjoint from P_0.

For any Reeb chord c from Λ to $\widehat{\Lambda}$ that is in the same class of $\check{c}$ in $\Sigma_{\Lambda\to\widehat{\Lambda}}(M)$, we can consider a lift $\widetilde{c}$ which starts at Λ_0 and ends $\widehat{\Lambda}_0$: clearly the algebraic intersection number of $\widetilde{c}$ and P_0 must be the same as the algebraic intersection number of $\check{c}$ and P_0, which is 1.

As $c_{\mathcal{I}(\rho)}$ does not intersect $\mathbb{T}$, none of its lifts will intersect P_0. From the above we conclude that $\check{c} \notin \mathcal{I}(\rho)$ also in this case.

$\square$

From the previous lemma we conclude that $LC\mathbb{H}_{st}^{\mathcal{I}(\rho)}(\alpha_F, \Lambda \to \widehat{\Lambda}) \neq 0$ for all $\rho \in \mathfrak{H})$. More precisely, Lemma 7.6 shows that $\#(\mathfrak{H}^C) \geq \#(\Sigma^C_{\Lambda \to \widehat{\Lambda}}(\alpha_F))$. We have thus proved the following:

Theorem 7.7. *$LC\mathbb{H}_{st}(\alpha_F, \Lambda \to \widehat{\Lambda})$ has exponential homotopical growth rate with exponential weight a.*

Appendix A

Asymptotic behaviour near punctures

In this appendix we present the precise asymptotic formulas obtained by Abbas [1] which describe the behaviour of a pseudoholomorphic curve near a boundary puncture.

Let $\widetilde{v} : (\dot{S}, j) \to (V, J)$ be a pseudoholomorphic curve in an exact symplectic cobordism and z_0 a boundary puncture of $\widetilde{v}$. Then we can pick a neighbourhood N_0 of z_0 and a biholomorphism ψ $\mathbb{R}^+ \times [0,1]$ to N_0. By considering coordinates $(r,t) \in \mathbb{R}^+ \times [0,1]$, we assume that the biholomorphism was chosen so that when $r \to +\infty$, $\psi(r,t)$ goes to z_0. We denote by c_0 be the transverse Reeb chord detected by $\widetilde{v}$ at z_0.

In [1], the author introduces an operator $A_\infty^{c_0}$ which is a self-adjoint operator of $L^2([0,1], \mathbb{R}^2)$. Because this Reeb chord is transverse, he shows that the kernel of $A_\infty^{c_0}$ does not contain $\delta_0 > 0$ such that $[-\delta_0, \delta_0]$ does not contain eigenvalues of $A_\infty^{c_0}$.

Supposing z_0 is a positive puncture, we assume that N_0 was chosen so that $\widetilde{v}(N_0)$ is contained in the positive end of V which is exact symplectomorphic to $[\nvdash, +\infty) \times \mathbb{Y}$. Abbas shows that there exist a coordinate system $(\theta, x, y) \in [0,1] \times \mathbb{D}$ in the neighbourhood of c_0, which identifies c_0 with $[0,1] \times \{0\}$ such that the $\widetilde{v} \circ \psi(r,t) = (\theta, x, y)(r,t)$ satisfies:

- $sup_{t \in [0,1]} (\partial^\beta s(r,t) - b_0 - r) \leq a_1 e^{-\delta_0 r}$
- $\sup_{t \in [0,1]} (\partial^\beta \theta(r,t) - Tr) \leq a_2 e^{-\delta_0 r}$ for some constant a,
- either $(x,y)(r,t) = e^{\int_{r_0}^{r} \lambda(u) du} [e(t) + R(r,t)]$ or $(x,y)(r,t)$ vanishes,

where $\partial^\beta = \partial_r^{\beta_1} \partial_t^{\beta_2}$, a_1 and a_2 are positive constants, $e(t)$ is an eigenvector of $A_\infty^{c_0}$, and $\lambda(r)$ goes to an eigevalue $\lambda < 0$ of $A_\infty^{c_0}$ as r goes to $+\infty$.

A similar statement is valid when z_0 is a negative puncture, with $\widetilde{v} \circ \psi(r,t) = (\theta, x, y)(r,t)$ satisfying:

- $sup_{t\in[0,1]}(\partial^\beta s(r,t) - b_0 + r) \leq a_1 e^{-\delta_0 r}$
- $\sup_{t\in[0,1]}(\partial^\beta \theta(r,t) + Tr) \leq a_2 e^{-\delta_0 r}$ for some constant a,
- either $(x,y)(r,t) = e^{\int_{r_0}^{r} \lambda(u)du}[e(t) + R(r,t)]$ or $(x,y)(r,t)$ vanishes,

where $\partial^\beta = \partial_r^{\beta_1}\partial_t^{\beta_2}$, a_1 and a_2 are positive constants, $e(t)$ is an eigenvector of $A_\infty^{c_0}$, and $\lambda(r)$ goes to an eigevalue $\lambda > 0$ of $A_\infty^{c_0}$ as r goes to $+\infty$.

A similar statement is valid for interior punctures, for which we refer the reader to [6] and [29].

Appendix B

Fredholm theory

In this appendix we review a bit of the Fredholm theory involved in the study of finite energy pseudoholomorphic curves in symplectizations. In order to obtain that the linearisation of the Cauchy-Riemann operator is a Fredholm operator we have to introduce the appropriate function spaces on which the Cauchy-Riemann operator will act.

Let $(\mathbb{R} \times Y^3, d\varpi)$ be an exact symplectic cobordism from α^+ to α^-. We assume that:

- for Legendrian knots Λ^+ in $(Y, \ker \alpha^+)$ and Λ^- in $(Y, \ker \alpha^-)$, there exists an exact Lagrangian cobordism L in $(\mathbb{R} \times Y^3, d\varpi)$ diffeomorphic to a cylinder,
- for Legendrian knots $\widehat{\Lambda}^+$ in $(Y, \ker \alpha^+)$ and $\widehat{\Lambda}^-$ in $(Y, \ker \alpha^-)$, there exists an exact Lagrangian cobordism $\widehat{L}$ in $(\mathbb{R} \times Y^3, d\varpi)$ diffeomorphic to a cylinder.

We consider Reeb chords $c^+ \in \mathcal{T}_{\Lambda^+ \to \widehat{\Lambda}^+}(\alpha^+)$ and $c^- \in \mathcal{T}_{\Lambda^- \to \widehat{\Lambda}^-}(\alpha^-)$. Again we construct coordinate systems $(\theta^+, x^+, y^+) \in [0,1] \times \mathbb{D}$ in the neighbourhood N^+ of c^+, which identifies c^+ with $[0,1] \times \{0\}$ and $(\theta^-, x^-, y^-) \in [0,1] \times \mathbb{D}$ in the neighbourhood $N^-$of c^-, which identifies c^- with $[0,1] \times \{0\}$. We consider the space $\mathfrak{B}_k^{a,p}(c^+, c^-)$ of maps f from $\mathbb{R} \times [0,1]$ to $\mathbb{R} \times Y$ that:

- take $\mathbb{R} \times \{0\}$ to L and $\mathbb{R} \times \{1\}$ to $\widehat{L}$,
- f is locally in L^a_k,
- there is $R^+ > 0$ such that $f([R^+, +\infty) \times [0,1]) \subset \mathbb{R} \times N^+$ and writing $f(r,t) = (\theta^+, x^+, y^+)(r,t)$ for $(r,t) \in [R^+, +\infty) \times [0,1]$ we have $(s(r,t) - s - b_0, \theta^+(r,t) - T^+t, x^+(r,t), y^+(r,t)) \in L_k^{a,p} = \{g(r,t); g(r,t)e^{\frac{a}{d}s} \in L^p_k\}$ for some constant b_0

- there is $R^- > 0$ such that $f((-\infty, R^-]) \times [0,1]) \subset \mathbb{R} \times N^-$ and writing $f(r,t) = (\theta^-, x^-, y^-)(r,t)$ for $(r,t) \in (-\infty, R^-]) \times [0,1]$ we have $(s(r,t) + s - b_1, \theta^-(r,t) - T^- t, x^-(r,t), y^-(r,t)) \in L^{a,p}_k = \{g(r,t); g(r,t)e^{\frac{a}{d}-s} \in L^p_k\}$ for some constant b_0.

It follows from the formulas of appendix A, that if we take $a < \delta_0$, $p \geq 2$ and $k \geq 0$ then for any cylindrical almost complex structure $\overline{J}$ as defined in section 2.1.2, all the elements of $\mathcal{M}(c^+, c^-, L, \widehat{L}; \overline{J})$ are in the space $\mathfrak{B}^{a,p}_k(c^+, c^-)$.

We now construct a Banach bundle $\mathcal{Z}$ over $\mathfrak{B}^{a,p}_k(c^+, c^-)$. The fiber $\mathcal{Z}_f$ over $f \in \mathfrak{B}^{a,p-1}_k(c^+, c^-)$ will be $L^{a,p-1}_k(\wedge^{0,1}(f))$, which consists of $L^{a,p-1}_k$ $0,1$-forms of $\mathbb{R} \times [0,1]$ with values in $f^*(T(\mathbb{R} \times Y))$.

It is clear from the asymptotic formulas of appendix A and elliptic regularity (see [3]), that the operator $\overline{\partial}_{\overline{J}}$ takes elements of $\mathfrak{B}^{a,p}_k(c^+, c^-)$ to a section of $\mathcal{Z}$; in this way $\overline{\partial}_{\overline{J}}$ can be seen as a section of $\mathcal{Z}$. Moreover, the moduli space $\mathcal{M}(c^+, c^-, L, \widehat{L}; \overline{J})$ consists exactly of the intersection of the section zero section of $\mathcal{Z}$ and the section $\overline{\partial}_{\overline{J}}$.

As $\overline{\partial}_{\overline{J}}$ is a differentiable section one can consider its differential $D\overline{\partial}_{\overline{J}}(\widetilde{v})$ for elements $\widetilde{v} \in \mathcal{M}(c^+, c^-, L, \widehat{L}; \overline{J})$. The tangent space to $\mathcal{M}(c^+, c^-, L, \widehat{L}; \overline{J})$ at $\widetilde{v}$ can then be identified with the kernel of $D\overline{\partial}_{\overline{J}}(\widetilde{v})$. With the setup, it is shown in [2] (see also [6]) that $D\overline{\partial}_{\overline{J}}(\widetilde{v})$ is a Fredholm operator.

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