On a risk measure inspired from the ruin probability and the expected deficit at ruin

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Abstract

In this paper, we study a risk measure derived from ruin theory defined as the amount of capital needed to cope in expectation with the first occurrence of a ruin event. Specifically, within the compound Poisson model, we investigate some properties of this risk measure with respect to the stochastic ordering of claim severities. Particular situations where combining risks yield diversification benefits are identified. Closed form expressions and upper bounds are also provided for certain claim severities.

Key words and phrases: Ruin probability, risk measure, classical risk model, maximal aggregate loss, ladder heights, stochastic ordering.

1 Introduction and motivation

Over the last decade, the concept of risk measures has become very popular in the insurance industry, especially with the introduction of the European Solvency II regulation. European insurance companies are now required to hold a level of available capital such that the probability for technical insolvency after 1 year is less than 1 in 200. The Value-at-Risk (VaR) has then emerged as the key instrument to compute the solvency capital requirement.

However, the appropriateness of risk measures defined over fixed time horizons usually advocated in both actuarial theory and practice can be questioned. It is not obvious that a time horizon of 1 year for assessing the safety of a business activity is appropriate when uniformly applied to all companies, as under the current regulation practice. Roughly speaking, the insurance risk over a horizon of 1 year of an insurance company is comparable with the one of a company with n times as many policies (of the same kind) but a time horizon of $1/n$ years. This becomes obvious when comparing the corresponding surplus processes.

Whereas VaR is usually defined in terms of a given time horizon, as explained above, in certain applications it may be more natural to look for measures that give a more robust reflection of the risk inherent in a business activity in a random environment. In particular, the VaR does not consider the possible adverse situations in between or beyond the specified time horizon. Ruin theory precisely accounts for the insured risk during the whole life-time of the business. It is this type of robustness that makes the ruin probability still nowadays an interesting quantity in this context. Moreover, the type of thinking coming from ruin theory is often considered important by practitioners.

The concept of risk measures derived from ruin theory has drawn the attention of researchers in recent years. Dhaene et al. (2003) give ample motivation for an exponential risk measure inherited from the Cramèr-Lundberg upper bound for the ruin probability in a discrete-time ruin model. Cheridito et al. (2006) investigate (in an application of their study on coherent risk measures for unbounded stochastic processes) the Tail-VaR (TVaR) of the maximal aggregate loss encountered by the insurance company, also called the dynamic TVaR. Specifically, they show that it is coherent and hence satisfies many desirable properties. In Trufin et al. (2011), the authors study the properties of a risk measure defined as the amount of initial capital needed to guarantee a certain probability of solvency throughout the lifetime of the process. As it can also be seen as the VaR of the maximal aggregate loss encountered by the insurer, the name of dynamic VaR is sometimes used in the literature.

The dynamic VaR does not distinguish a large deficit at ruin from a small one. However, in practice, once the ruin event occurs, the insurance company needs a capital injection to cover the deficit. Hence, the larger the deficit at ruin is, the higher is the cost required for business continuity. Therefore, in order to assess the riskiness of an insurance business, it is practically relevant to consider a risk measure that combines both the capital needed to ensure a certain probability of solvency and the expected deficit at ruin. This is why, in this paper, we define a new risk measure that is the sum of the dynamic VaR and the expected deficit at ruin and we study its properties. The proposed risk measure enables then

to be prepared for the first of the possible unfavorable events. Thereafter, another strategy can be applied, which is in line with insurance practices since once the insurer faces ruin, it is reasonable to assume that its business strategy will be adapted accordingly. In that sense, the risk measure we consider in this paper appears to be less static than the dynamic TVaR which considers both the premium strategy and the aggregate claim process as being stationary over time.

Overall, over the last few years, the relative position and relation between risk measures that fulfill a list of axioms on the one hand, and classical ruin theory on the other hand, has often been a matter of debate. In the same vein as Trufin et al. (2011), this paper brings additional insights to this debate by considering certain properties of a new risk measure motivated by ruin theory.

Our paper is organized as follows. In section 2, we present some properties and tools of the compound Poisson risk model and we recall the definitions of some stochastic orders used in the following. Next, in section 3, we define the studied risk measure and we also discuss its relevancy and positioning with respect to the dynamic VaR and TVaR. Then, in section 4, we investigate the axiomatic properties of our risk measure while section 5 extends previous results for small required solvency levels. Finally, in section 6, we derive closed form expressions and upper bounds for our risk measure in some particular cases.

2 Risk model and stochastic orders

2.1 Surplus process

In this section, we set up the scene by recalling some properties and tools of the compound Poisson risk model. For more details, see for instance Asmussen and Albrecher (2010). The surplus process (or risk process) is defined as

$$U_t = u + c\,t - S_t, \quad t \geq 0, \tag{2.1}$$

where U_t is the insurer's capital at time t starting from some initial capital $U_0 = u$, c is the (constant) premium income per unit of time and $S_t = \sum_{k=1}^{N_t} X_k$ is the total claim amount up to time t, with N_t the corresponding number of claims, and X_k the size of the kth claim, assumed to be nonnegative. The claim number process $\{N_t,\ t \geq 0\}$ is assumed to be Poisson with constant rate λ. The X_ks are independent and distributed as X, with distribution function F_X. They are furthermore supposed to be independent of $\{N_t,\ t \geq 0\}$. Also, we assume that the premium rate is of the form $c = (1+\eta)\lambda\mu_X$ where $\eta > 0$ is called the loading factor and $\mu_X = \mathbb{E}[X]$. Henceforth, η is assumed to be the same for all the risk processes used in this paper.

The first time that the surplus process becomes negative is denoted T. So

$$T = \begin{cases} \min\{t \geq 0 \,|\, U_t < 0\}; \\ +\infty \quad \text{if } U_t \geq 0 \text{ for all } t, \end{cases}$$

and is called the time of ruin. The ruin probability, namely the probability that ruin ever occurs, is defined as $\psi_X(u) = \Pr[T < \infty]$.

Let $L_X = \max\{S_t - ct \,|\, t \geq 0\}$ be the maximal aggregate loss of the surplus process with distribution function F_{L_X}, i.e. the maximal difference over t between the claim amount and the earned premium up to time t. Obviously, the events $T < \infty$ and $L_X > u$ are equivalent so that

$$\psi_X(u) = \Pr[L_X > u].$$

It is well-known that L_X can be decomposed as

$$L_X = \sum_{j=1}^{M} D_{X,j},$$

where M follows the geometric distribution with success probability $1 - \psi_X(0) = \frac{\eta}{1+\eta}$ and $D_{X,1}, D_{X,2}, \ldots$ are the ladder heights of the loss process which are independent and identically distributed (i.i.d.) as D_X, with distribution function F_{D_X}. In the compound Poisson model, the common distribution function of the $D_{X,j}$'s is precisely given by the integrated tail distribution

$$F_{D_X}(y) = \int_0^y \frac{1 - F_X(x)}{\mu_X} dx, \quad y > 0. \tag{2.2}$$

Note that, in the literature, F_{D_X} is also often called the stationary forward recurrence time, the equilibrium distribution or the residual lifetime. The expectations of D_X and L_X can be expressed in terms of the first two moments of X. In fact, we have

$$\mathbb{E}[D_X] = \frac{\mu_{2,X}}{2\,\mu_X} \tag{2.3}$$

where $\mu_{2,X} = \mathbb{E}[X^2]$, and consequently

$$\mathbb{E}[L_X] = \frac{\mu_{2,X}}{2\,\mu_X \eta} \tag{2.4}$$

as $\mathbb{E}[L_X] = \mathbb{E}[D_X]/\eta$.

2.2 Stochastic orders

In what follows, we recall the definitions of some stochastic orders that will be useful in the present study. We refer the interested readers to, e.g., Müller and Stoyan (2002), Denuit et al. (2005) or Shaked and Shanthikumar (2007).

Given two random variables X and Y, X precedes Y in the usual stochastic order, denoted as $X \preceq_{\text{st}} Y$, if

$$\overline{F}_X(u) \leq \overline{F}_Y(u) \quad \text{for all } u,$$

where $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$. The latter is also equivalent to the inequality $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ for any non-decreasing function g such that the expectations exist.

The usual stochastic order compares the sizes of the risks. The convex order focuses on their variabilities and enables to compare two risks with identical means. When $\mu_X = \mu_Y$, one says that X precedes Y in the convex order, denoted as $X \preceq_{\text{cx}} Y$, when

$$\int_t^\infty \overline{F}_X(u)\,du \leq \int_t^\infty \overline{F}_Y(u)\,du \quad \text{for all } t. \tag{2.5}$$

The inequality in (2.5) can be equivalently written as

$$\mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+] \quad \text{for all } t, \tag{2.6}$$

where, for any real number r, we let r_+ denote the positive part of r; that is, $r_+ = r$ if $r \geq 0$ and $r_+ = 0$ if $r < 0$. From (2.6) it follows that $X \preceq_{\text{cx}} Y$ if and only if $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ for all convex functions $g : \mathbb{R}^+ \to \mathbb{R}$, provided the expectations exist.

Another important variability order in our setting is the dilation order. Unlike the convex order, it does not depend on the location of the underlying distributions. One says that X is smaller than Y in the dilation order if

$$X - \mu_X \preceq_{\text{cx}} Y - \mu_Y,$$

denoted $X \preceq_{\text{dil}} Y$.

The increasing convex, or stop loss, order combines the aspects of size (as $\preceq_{\text{st}}$) and variability (as $\preceq_{\text{cx}}$ or $\preceq_{\text{dil}}$). By definition, X is said to be smaller than Y in the stop-loss order, denoted as $X \preceq_{\text{icx}} Y$, when inequality (2.5) (or, equivalently (2.6)) holds. Similarly, $X \preceq_{\text{icx}} Y$ if and only if $\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)]$ for all non-decreasing convex functions $g : \mathbb{R}^+ \to \mathbb{R}$, provided the expectations exist. In case $\mu_X = \mu_Y$, the orders $\preceq_{\text{cx}}$, $\preceq_{\text{dil}}$ and $\preceq_{\text{icx}}$ are equivalent.

For the rest of this section, X and Y are assumed to be nonnegative random variables with finite means.

In the sequel, we also make use of the mean residual life order and the harmonic mean residual life order, extensively used in reliability theory. We define the mean residual life function of X, denoted m_X, as $m_X(t) = \mathbb{E}[X - t | X > t]$, for all t such that $\Pr[X > t] > 0$, provided the expectations exist. If X is the lifetime of a device, $m_X(t)$ then expresses the conditional expected residual life of the device at time t, given that the device is still alive. Let X and Y such that $m_X(t) \leq m_Y(t)$ for all $t \geq 0$. Then X is said to be smaller than Y in the mean residual life order, denoted $X \preceq_{\text{mrl}} Y$. It follows that $X \preceq_{\text{mrl}} Y$ if and only if

$$\frac{\int_t^\infty \overline{F}_X(u)du}{\overline{F}_X(t)} \leq \frac{\int_t^\infty \overline{F}_Y(u)du}{\overline{F}_Y(t)} \quad \text{for all } t \geq 0.$$

It is said that X is smaller than Y in the harmonic mean residual life order, denoted $X \preceq_{\text{hmrl}} Y$ if the harmonic mean of m_X is smaller than the harmonic mean of m_Y, i.e.

$$\left[\frac{1}{t}\int_0^t \frac{1}{m_X(u)}du\right]^{-1} \leq \left[\frac{1}{t}\int_0^t \frac{1}{m_Y(u)}du\right]^{-1} \quad \text{for all} \quad t \geq 0.$$

It can be proved that $X \preceq_{\text{hmrl}} Y$ if and only if $\frac{\int_t^\infty \overline{F}_X(u)du}{\mu_X} \leq \frac{\int_t^\infty \overline{F}_Y(u)du}{\mu_Y}$ for all $t \geq 0$, which is equivalent to $D_X \preceq_{\text{st}} D_Y$.

Finally, we define the new better than used in expectation order. The random variable X is said to be smaller than Y in the new better than used in expectation order, denoted $X \preceq_{\text{nbue}} Y$, when

$$\frac{m_X(F_X^{-1}(u))}{m_Y(F_Y^{-1}(u))} \leq \frac{\mu_X}{\mu_Y} \quad \text{for all} \quad u \in [0,1].$$

Occasionally, we also refer to the following stochastic orders, namely:
(1) the dispersive order (denoted $\preceq_{\text{disp}}$):

$$X \preceq_{\text{disp}} Y \Leftrightarrow F_X^{-1}(\beta) - F_X^{-1}(\alpha) \leq F_Y^{-1}(\beta) - F_Y^{-1}(\alpha) \quad \text{whenever} \quad 0 < \alpha \leq \beta < 1;$$

(2) the star order (denoted $\preceq_*$), the support of X is assumed to be an interval (finite or infinite):

$$X \preceq_* Y \Leftrightarrow \frac{F_Y^{-1}(u)}{F_X^{-1}(u)} \text{ is increasing in } u \in (0,1);$$

(3) the decreasing mean residual life order (denoted $\preceq_{\text{dmrl}}$), the supports of X and Y are assumed to be of the form $[0,a)$, $a > 0$ (a can be infinity):

$$X \preceq_{\text{dmrl}} Y \Leftrightarrow \frac{\frac{1}{\mu_Y}\int_{F_Y^{-1}(u)}^\infty \overline{F}_Y(x)dx}{\frac{1}{\mu_X}\int_{F_X^{-1}(u)}^\infty \overline{F}_X(x)dx} \text{ is increasing in } u \in [0,1];$$

(4) the excess wealth order (denoted $\preceq_{\text{ew}}$):

$$X \preceq_{\text{ew}} Y \Leftrightarrow \frac{1}{1-u}\int_u^1 \left(F_Y^{-1}(x) - F_X^{-1}(x)\right) dx \text{ is increasing in } u \in (0,1);$$

and (5) the total time on test transform order (denoted $\preceq_{\text{ttt}}$):

$$X \preceq_{\text{ttt}} Y \Leftrightarrow \int_0^{F_X^{-1}(u)} \overline{F}_X(x)dx \leq \int_0^{F_Y^{-1}(u)} \overline{F}_Y(x)dx \quad \text{for all} \quad u \in (0,1).$$

3 Risk measure ξ_ϵ

In Trufin et al. (2011), the authors consider the amount of initial capital needed in order to bound the ruin probability by ϵ and study its properties. That is, they assume a fixed safety loading η and study the ruin-consistent VaR risk measure defined as

$$\rho_\epsilon[X] = \inf\{v \geq 0 | \psi_X(v) \leq \epsilon\},$$

which can also be seen as a VaR applied to the transformed risk L_X (the maximal aggregate loss), namely

$$\rho_\epsilon[X] = F_{L_X}^{-1}(1-\epsilon) = \inf\{v \geq 0 | F_{L_X}(v) \geq 1-\epsilon\}. \tag{3.1}$$

In words, $\rho_\epsilon[X]$, also called dynamic VaR, is the smallest amount of capital needed such that the ultimate ruin probability ψ_X for a risk process with individual claim sizes distributed as X is at most equal to some specified probability level ϵ.

A natural extension of ρ_ϵ is the Tail-VaR (or average VaR) defined as

$$\overline{\rho}_\epsilon[X] = \frac{1}{\epsilon}\int_0^\epsilon \rho_w[X]\,dw.$$

Alternatively one can rewrite (see Trufin et al. (2011))

$$\begin{aligned}\overline{\rho}_\epsilon[X] &= \rho_\epsilon[X] + \frac{1}{\epsilon}\,\mathbb{E}\left[(L_X - \rho_\epsilon[X])_+\right] \\ &= \mathbb{E}[L_X \big| L_X > F_{L_X}^{-1}(1-\epsilon)].\end{aligned}$$

From the above it is clear that the risk measure $\overline{\rho}_\epsilon[X]$ represents the amount of capital needed to be able to cope "in expectation" also with the insurance loss in those problematic cases that occur with probability less than ϵ. It is often called dynamic TVaR in the literature Cheridito et al. (2006) study the properties of the dynamic TVaR as a functional on the space of all bounded càdlàg processes. In particular, they show that $\overline{\rho}_\epsilon[X]$ is a coherent risk measure when viewed as a function of L_X.

In ruin theory it has been common to study related but different quantities. Instead of looking at the maximal deficit that occurs over the lifetime of the process once the "ruin" boundary is crossed (an event with probability less than ϵ), the first crossing of the ruin boundary and its magnitude is often considered. It is then natural to consider the amount of capital needed to cope in expectation with the first occurrence of an "ϵ-event", defined as

$$\xi_\epsilon[X] = \rho_\epsilon[X] + \mathbb{E}[|U_T| \,\big|\, T < \infty],$$

where T is in this context the time of ruin of the risk process $(\rho_\epsilon[X] + ct - S_t)_{t\geq 0}$, i.e. an insurance risk process for which the initial capital has been chosen in such a way that ruin occurs with probability less than ϵ. One could rephrase the rationale behind the measure ξ_ϵ as to consider the expected deficit at the first occurrence of an ϵ-event. It is practically intuitive to study ξ_ϵ as it is less static than its $\overline{\rho}_\epsilon$ counterpart and realistically there will be an intervention of the management into the dynamics of the process once an ϵ-event has occurred. So it is consistent to allocate enough capital to cope (in expectation) with the first ϵ-event and thereafter another strategy or allocation principle can be applied. Although the basic intention behind considering ruin-related risk measures is to look for robust long-term strategies, the idea of ξ_ϵ is to be prepared for the first of the possible unfavorable events and perhaps reallocate capital afterwards. In that way it can be considered to substantially improve upon $\overline{\rho}_\epsilon$.

The quantity $\mathbb{E}[|U_T| \,\big|\, T < \infty]$ is well understood in risk theory and in several situations (semi)explicit expressions for it are available. Specifically, a well-known result in the classical risk model (see for instance Lin and Willmot (2000)) is that the expected deficit at ruin of

a compound Poisson risk process with initial capital u can be written as

$$\mathbb{E}[|U_T| \,|T < \infty] = \int_u^\infty \frac{\psi_X(x)}{\psi_X(u)} dx - \frac{\mu_{2,X}}{2\mu_X \eta}.$$

As a result, ξ_ϵ can be expressed as

$$\xi_\epsilon[X] = \rho_\epsilon[X] + \frac{1}{\epsilon} \int_{\rho_\epsilon[X]}^\infty \psi_X(x) dx - \frac{\mu_{2,X}}{2\mu_X \eta}$$

or equivalently, by (2.4) and (3.1),

$$\begin{aligned} \xi_\epsilon[X] &= F_{L_X}^{-1}(1-\epsilon) + \frac{1}{\epsilon} \int_{F_{L_X}^{-1}(1-\epsilon)}^\infty \overline{F}_{L_X}(x)\, dx - \mathbb{E}[L_X] \\ &= \mathrm{TVaR}[L_X; 1-\epsilon] - \mathbb{E}[L_X], \end{aligned} \tag{3.2}$$

which will be very useful in our analysis. Since $\overline{\rho}_\epsilon[X] = \mathrm{TVaR}[L_X; 1-\epsilon]$, we note that there also exists a direct link between ξ_ϵ and $\overline{\rho}_\epsilon$, namely

$$\xi_\epsilon[X] = \overline{\rho}_\epsilon[X] - \mathbb{E}[L_X]. \tag{3.3}$$

In the present paper, we study the risk measure ξ_ϵ within the compound Poisson risk model (2.1). Also, we only consider the relevant cases where $\epsilon \in (0, 1/(1+\eta))$, since otherwise, the required capital ρ_ϵ is simply zero. As for ρ_ϵ and $\overline{\rho}_\epsilon$, ξ_ϵ does not explicitly depend on the claim frequency λ. This is because neither the ruin probability nor the expected deficit at ruin is influenced by the volume of the portfolio reflected in λ as we are allowed to switch to any operational time scale. In other words, the classical model values the time diversification. In fact, as mentioned in Trufin et al. (2011), the fixed value of η ensures that the corresponding value of the premium rate c compensates for the change of volume in the portfolio. As a result, only the distribution of the claim amount matters, i.e. ξ_ϵ only depends on X (and on η) as is the case for ρ_ϵ and $\overline{\rho}_\epsilon$. This may be considered as an advantage or disadvantage depending on the field of application. We note that the vast majority of the results in ruin theory consider η as fixed when the effect of switching from one severity distribution to another is studied. We also call attention to the fact that throughout this paper, we implicitly work with nonnegative claim severities of finite means.

4 Properties of ξ_ϵ

In this section, we explore stochastic order inequalities that capture the idea that a claim severity Y is more dangerous than a claim severity X, in the sense of $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$. Moreover, we also investigate under which conditions ξ_ϵ recognizes diversification effect.

Beforehand, we briefly discuss some classes of random variables. The random variable X (or its distribution) is said to be increasing (decreasing) mean residual lifetime or IMRL (DMRL) if $m_X(x)$ is nondecreasing (nonincreasing) for $x \geq 0$. The class IMRL (DMRL) is included in the class of the new worse (better) than used in convex order class or NWUC (NBUC) class, where $\overline{F}_{D_X}(x+y) \geq (\leq) \overline{F}_{D_X}(x)\overline{F}_X(y)$, for all $x \geq 0$ and $y \geq 0$. A larger class of random variables is the new worse (better) than used in expectation or NWUE (NBUE) class, with $m_X(x) \geq (\leq) m_X(0)$, or equivalently $\overline{F}_{D_X}(x) \geq (\leq) \overline{F}_X(x)$, for $x \geq 0$.

4.1 Risk ordering

We begin by considering the maximal aggregate losses associated to X and Y. As shown in the next proposition, the dilation order naturally arises when comparing $\xi_\epsilon[X]$ with $\xi_\epsilon[Y]$.

Proposition 1. *$\xi_\epsilon[X] \leq \xi_\epsilon[Y]$ if and only if $L_X \preceq_{dil} L_Y$.*

Proof. By (3.2), it amounts to prove that $\mathrm{TVaR}[L_X; 1-\epsilon] - \mathbb{E}[L_X] \leq \mathrm{TVaR}[L_Y; 1-\epsilon] - \mathbb{E}[L_Y]$ if and only if $L_X \preceq_{\mathrm{dil}} L_Y$. Now, as we have $L_X - \mathbb{E}[L_X] \preceq_{\mathrm{cx}} L_Y - \mathbb{E}[L_Y]$ if and only if $\mathrm{TVaR}[L_X - \mathbb{E}[L_X]; 1-\epsilon] \leq \mathrm{TVaR}[L_Y - \mathbb{E}[L_Y]; 1-\epsilon]$ (see, e.g., Proposition 3.4.8 in Denuit et al. (2005)), the proof is complete. □

From the above proposition, we directly get the following property for ξ_ϵ. We recall that X is said to be smaller than Y in the 3-convex order, denoted $X \preceq_{\text{3-cx}} Y$ if and only if $\mu_X = \mu_Y$, $\mu_{2,X} = \mu_{2,Y}$ and $\mathbb{E}[(X-t)_+^2] \leq \mathbb{E}[(Y-t)_+^2]$ for all t.

Property 1. *The risk measure ξ_ϵ agrees with the 3-convex order, that is $X \preceq_{3\text{-}cx} Y$ implies $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.*

Proof. Using Theorem 3.A.65 in Shaked and Shanthikumar (2007), we have $X \preceq_{\text{3-cx}} Y$ if and only if $D_X \preceq_{\mathrm{cx}} D_Y$. Also, since the convex ordering is preserved under compounding of independent risks (see e.g. Property 3.4.39 in Denuit et al. (2005)), we directly get $L_X \preceq_{\mathrm{cx}} L_Y$. □

Menezes et al. (1980) give an interpretation to the order $\preceq_{\text{3-cx}}$: if $X \preceq_{\text{3-cx}} Y$, then, of course, X and Y have the same mean and variance, but X then has smaller rightside risk than Y.

The stochastic inequality $L_X \preceq_{\mathrm{dil}} L_Y$ is a necessary and sufficient condition for $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$. We now obtain a sufficient condition for $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.

Proposition 2. *If $D_X \preceq_{dil} D_Y$, then $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.*

Proof. First we note that the dilation order is closed under convolution. Indeed, for all $m = 1, 2, \ldots$, it follows from $D_X \preceq_{\mathrm{dil}} D_Y$ that

$$\sum_{i=1}^{m} (D_{X,i} - \mathbb{E}[D_X]) \preceq_{\mathrm{cx}} \sum_{i=1}^{m} (D_{Y,i} - \mathbb{E}[D_Y]),$$

or equivalently

$$\sum_{i=1}^{m} D_{X,i} - m\mathbb{E}[D_X] \preceq_{\mathrm{cx}} \sum_{i=1}^{m} D_{Y,i} - m\mathbb{E}[D_Y],$$

i.e.

$$\sum_{i=1}^{m} D_{X,i} \preceq_{\mathrm{dil}} \sum_{i=1}^{m} D_{Y,i}. \tag{4.1}$$

Next, let us prove that $L_X = \sum_{i=1}^{M} D_{X,i} \preceq_{\mathrm{dil}} L_Y = \sum_{i=1}^{M} D_{Y,i}$. In that goal, it is convenient to write M as a sum of n i.i.d. random variables with $n = \min\{k \in \mathbb{N} : \mathbb{E}[M]/k < 1\}$. So, as M follows a geometric distribution, we can write $M = \sum_{k=1}^{n} M_k^{(n)}$, where the

$M_k^{(n)}$'s are i.i.d. negative binomial random variables with mean equals to $\mathbb{E}[M]/n < 1$. Using the fact that the dilation order is closed under convolutions, it is then sufficient to show that $\sum_{i=1}^{M_1^{(n)}} D_{X,i} \preceq_{\text{dil}} \sum_{i=1}^{M_1^{(n)}} D_{Y,i}$. To this end, for all $m = 1, 2, \ldots$, we notice that

$$\sum_{i=1}^{m} D_{X,i} - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_X] \preceq_{\text{icx}} \sum_{i=1}^{m} D_{Y,i} - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_Y].$$

Indeed, since $D_X \preceq_{\text{dil}} D_Y$ yields $D_X \preceq_{\text{icx}} D_Y$ (see Belzunce et al. (1997)), it follows that $\mathbb{E}[D_X] \leq \mathbb{E}[D_Y]$. Moreover, as $\mathbb{E}[M_1^{(n)}] < 1$, we have

$$0 \leq m\mathbb{E}[D_X] \leq \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_X] - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_Y] + m\mathbb{E}[D_Y],$$

and hence

$$\begin{aligned}\sum_{i=1}^{m} D_{X,i} - (\mathbb{E}[M_1^{(n)}]\mathbb{E}[D_X] - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_Y] + m\mathbb{E}[D_Y]) \quad &\preceq_{\text{icx}} \quad \sum_{i=1}^{m} D_{X,i} - m\mathbb{E}[D_X] \\ &\preceq_{\text{cx}} \quad \sum_{i=1}^{m} D_{Y,i} - m\mathbb{E}[D_Y]\end{aligned}$$

by equation (4.1). Therefore, we get

$$\sum_{i=1}^{m} D_{X,i} - (\mathbb{E}[M_1^{(n)}]\mathbb{E}[D_X] - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_Y] + m\mathbb{E}[D_Y]) \preceq_{\text{icx}} \sum_{i=1}^{m} D_{Y,i} - m\mathbb{E}[D_Y]$$

and thus

$$\sum_{i=1}^{m} D_{X,i} - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_X] \preceq_{\text{icx}} \sum_{i=1}^{m} D_{Y,i} - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_Y].$$

Now, as the stop-loss order is closed under mixtures (see, e.g., Theorem 4.A.8 in Shaked and Shanthikumar (2007)), it follows that $\sum_{i=1}^{M_1^{(n)}} D_{X,i} - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_X] \preceq_{\text{cx}} \sum_{i=1}^{M_1^{(n)}} D_{Y,i} - \mathbb{E}[M_1^{(n)}]\mathbb{E}[D_Y]$, which completes the proof. □

It is worth noting that it is easier to compare the ladder heights D_X and D_Y with the dilation order rather than the corresponding maximal aggregate losses L_X and L_Y.

The following result states a characterization of the condition $D_X \preceq_{\text{dil}} D_Y$ by means of the distributions of X and Y.

Proposition 3. *$D_X \preceq_{dil} D_Y$ if and only if*

$$\frac{1}{\mu_X} \int_{t+\frac{\mu_{2,X}}{2\mu_X}}^{\infty} \left[x - \left(t + \frac{\mu_{2,X}}{2\mu_X}\right)\right] \overline{F}_X(x)\, dx \leq \frac{1}{\mu_Y} \int_{t+\frac{\mu_{2,Y}}{2\mu_Y}}^{\infty} \left[x - \left(t + \frac{\mu_{2,Y}}{2\mu_Y}\right)\right] \overline{F}_Y(x)\, dx \quad \textit{for all} \quad t. \tag{4.2}$$

Proof. From (2.5), $D_X \preceq_{\text{dil}} D_Y$ if and only if

$$\int_{t+\mathbb{E}[D_X]}^{\infty} \overline{F}_{D_X}(x)\,dx \leq \int_{t+\mathbb{E}[D_Y]}^{\infty} \overline{F}_{D_Y}(x)\,dx \quad \text{for all} \quad t.$$

By (2.2) and (2.3), the latter inequality can be rewritten as

$$\frac{1}{\mu_X} \int_{t+\frac{\mu_{2,X}}{2\mu_X}}^{\infty} \int_{x}^{\infty} \overline{F}_X(u)\,du\,dx \leq \frac{1}{\mu_Y} \int_{t+\frac{\mu_{2,Y}}{2\mu_Y}}^{\infty} \int_{x}^{\infty} \overline{F}_Y(u)\,du\,dx \quad \text{for all} \quad t.$$

Then, integrating by parts leads to inequality (4.2). □

We note that in the special case where $X \preceq_{\text{3-cx}} Y$, (4.2) becomes $\mathbb{E}[(X-t)_+^2] \leq \mathbb{E}[(Y-t)_+^2]$ for all t, which is always fulfilled by definition of the 3−convex order.

Based on Proposition 2, we are now in position to establish the next property for ξ_ϵ.

Property 2. *Let X or Y be IMRL. Then the risk measure ξ_ϵ agrees with the mrl-order, that is $X \preceq_{mrl} Y$ implies $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.*

Proof. Since $X \preceq_{\text{mrl}} Y$ and X or Y is IMRL, we get $D_X \preceq_{\text{disp}} D_Y$ (see, e.g., Theorem 3.B.21 in Shaked and Shanthikumar (2007)). This, in turn, implies $D_X \preceq_{\text{dil}} D_Y$ (see, e.g., Theorem 3.B.16 in Shaked and Shanthikumar (2007)). □

In the following example, we highlight some conditions on the parameters of two IMRL distributions, namely exponential and Pareto distributions, that are necessary and sufficient for stochastic dominance with mrl-order. These conditions are taken from Heilmann and Schröter (1991).

Example 1. (i) Let X be exponentially distributed with mean $1/a$ and Y be exponentially distributed with mean $1/a'$. Then $X \preceq_{\text{mrl}} Y$ if and only if $a \geq a'$. (ii) Let X be Pareto distributed with parameters a and b, that is, X has distribution function $1 - (\frac{a}{x+a})^b$, $x > 0$, and let Y be Pareto distributed with parameters a' and b' with $\min(b, b') > 1$. Then $X \preceq_{\text{mrl}} Y$ is equivalent to $\frac{b-1}{b'-1} \geq \max(\frac{a}{a'}, 1)$.

In the particular cases where X is DMRL and Y is IMRL, one can relax the mrl-order condition required in Property 2, as stated below.

Property 3. *Let X be DMRL and Y be IMRL. Then the risk measure ξ_ϵ agrees with the hmrl-order, that is $X \preceq_{hmrl} Y$ implies $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.*

Proof. As X is DMRL and Y is IMRL, Theorem 4.2 in Lin and Willmot (2000) leads to

$$\mathbb{E}[|U_T^{(X)}| \big| T < \infty] \leq \mathbb{E}[D_X] \text{ and } \mathbb{E}[|U_T^{(Y)}| \big| T < \infty] \geq \mathbb{E}[D_Y].$$

Now, since $X \preceq_{\text{hmrl}} Y$, we have $\mathbb{E}[D_X] \leq \mathbb{E}[D_Y]$ and $\rho_\epsilon[X] \leq \rho_\epsilon[Y]$. So, we get

$$\begin{aligned} \xi_\epsilon[X] = \rho_\epsilon[X] + \mathbb{E}[|U_T^{(X)}| \big| T < \infty] &\leq \rho_\epsilon[X] + \mathbb{E}[D_X] \\ &\leq \rho_\epsilon[Y] + \mathbb{E}[D_Y] \leq \rho_\epsilon[Y] + \mathbb{E}[|U_T^{(Y)}| \big| T < \infty] = \xi_\epsilon[Y]. \end{aligned}$$

□

Other interesting properties for ξ_ϵ are given in the next result, also derived from Proposition 2.

Property 4. *(i) Let X and Y such that $\mu_X \leq \mu_Y$. Then the risk measure ξ_ϵ agrees with the nbue-order, that is $X \preceq_{nbue} Y$ implies $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.*

(ii) Let X and Y have support of the form $[0, a)$ with $\mu_X \leq \mu_Y$. Then the risk measure ξ_ϵ agrees with the star-order, that is $X \preceq_* Y$ implies $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.

(iii) Let X and Y have support of the form $[0, a)$ with $\mu_X \leq \mu_Y$. Then the risk measure ξ_ϵ agrees with the dmrl-order, that is $X \preceq_{dmrl} Y$ implies $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.

Proof. Using Theorem 2.2 in Kochar (1989), we know that $X \preceq_{\text{nbue}} Y$ with $\mu_X \leq \mu_Y$ implies $D_X \preceq_{\text{disp}} D_Y$, and hence $D_X \preceq_{\text{dil}} D_Y$. For items (ii) and (iii), it suffices to notice that in the cases where X and Y have support of the form $[0, a)$, each of the stochastic inequalities $X \preceq_* Y$ and $X \preceq_{\text{dmrl}} Y$ implies $X \preceq_{\text{nbue}} Y$. □

We note that the condition $X \preceq_{\text{nbue}} Y$ may be expressed in terms of other stochastic orders (for more details see Shaked and Shanthikumar (2007)). Indeed, for nonnegative random variables X and Y, it can be proved that $X \preceq_{\text{nbue}} Y$ is equivalent to $\frac{X}{\mu_X} \preceq_{\text{ew}} \frac{Y}{\mu_Y}$ and to $\frac{X}{\mu_X} \preceq_{\text{ttt}} \frac{Y}{\mu_Y}$.

There exist some links among the stochastic orders used in Properties 2, 3 and 4. Obviously, if $X \preceq_{\text{mrl}} Y$, then we have $X \preceq_{\text{hmrl}} Y$. Also, if $\mu_X \leq \mu_Y$, with the help of Theorems 4.B.23 and 3.A.28 in Shaked and Shanthikumar (2007), the stochastic inequality $X \preceq_{\text{nbue}} Y$ implies $X \preceq_{\text{hmrl}} Y$. As X and Y are nonnegative, if $\mu_X = \mu_Y$, there exist the following relations between mrl and nbue stochastic orders: (i) when X or Y or both are DMRL, then $X \preceq_{\text{nbue}} Y$ implies $X \preceq_{\text{mrl}} Y$; (ii) when X or Y or both are IMRL, then $X \preceq_{\text{mrl}} Y$ implies $X \preceq_{\text{nbue}} Y$; (iii) when X is DMRL and Y is IMRL, then $X \preceq_{\text{nbue}} Y$ is equivalent to $X \preceq_{\text{mrl}} Y$. The proof follows from relation (4.B.12), Theorem 3.C.5 and Theorem 3.C.6 in Shaked and Shanthikumar (2007).

Corollary 1. *Let X be NBUE and Y be exponentially distributed with $\mu_X \leq \mu_Y$. Then $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$.*

Proof. From Theorem 4.B.25 in Shaked and Shanthikumar (2007), it yields $X \preceq_{\text{nbue}} Y$, and the proof follows from Property 4 (i). □

This is very useful, as for exponentially distributed claims we are able to derive a closed form expression for our risk measure. Consequently, for any NBUE claim severity X, one can obtain an explicit upper bound for $\xi_\epsilon[X]$. Further details can be found in section 6.

4.2 Homogeneity and subadditivity

We show that ξ_ϵ also satisfies the following desirable property.

Property 5. *The risk measure ξ_ϵ is positively homogeneous, that is, $\xi_\epsilon[aX] = a\xi_\epsilon[X]$ for any constant $a > 0$.*

Proof. Multiplying the annual claim amount by a has the same effect on the annual premium income, so that the initial capital needed to have an ultimate ruin probability of ϵ is also multiplied by a. The surplus process is then also multiplied by a and hence the expected deficit at ruin as well. Formally, replacing X with aX changes L_X into aL_X and so, from (3.2), $\xi_\epsilon[aX] = a\xi_\epsilon[X]$ since $\mathrm{TVaR}[aL_X; 1-\epsilon] = a\mathrm{TVaR}[L_X; 1-\epsilon]$. □

The notion of positive homogeneity can be interpreted as the independence with respect to the monetary unit used.

We now consider claim severities X and Y that are identically distributed (i.d.). From the last property, we are in position to derive some conditions on X and Y such that ξ_ϵ is subadditive, i.e. such that the inequality $\xi_\epsilon[X+Y] \leq \xi_\epsilon[X] + \xi_\epsilon[Y]$ holds true. The subadditivity property reflects the idea that the risk can be reduced by diversification. Since X and Y are i.d., it comes $\xi_\epsilon[X] + \xi_\epsilon[Y] = 2\xi_\epsilon[Y] = \xi_\epsilon[2Y]$. Hence, to highlight some situations where ξ_ϵ is subadditive, we have to compare claim severities $X+Y$ and $2Y$ (or, equivalently, $\frac{X+Y}{2}$ and Y). Using the results obtained in section 4.1, we get the following proposition.

Proposition 4. *Let X and Y be identically distributed.*

(i) The risk measure ξ_ϵ is subadditive if and only if $L_{\frac{X+Y}{2}} \preceq_{dil} L_Y$.

(ii) The risk measure ξ_ϵ is subadditive if one of the followings holds:

(a) $D_{\frac{X+Y}{2}} \preceq_{dil} D_Y$;

(b) X is IMRL (so is Y) so that $\frac{X+Y}{2} \preceq_{mrl} Y$;

(c) $\frac{X+Y}{2} \preceq_{nbue} Y$.

We note that condition (ii-c) is always satisfied when (ii-b) holds, since $\frac{X+Y}{2}$ and Y have the same mean.

Corollary 2. *Let X and Y be identically distributed and IMRL (so they are NWUE) so that $\frac{X+Y}{2}$ is NBUE. Then $\xi_\epsilon[X+Y] \leq \xi_\epsilon[X] + \xi_\epsilon[Y]$.*

Proof. Using Theorem 3.A.58 in Shaked and Shanthikumar (2007), $\frac{X+Y}{2} \preceq_{\mathrm{mrl}} Y$. □

Example 2. Let X and Y be i.i.d. $Gamma(\theta, \sigma)$ random variables, with density function $f(x) = \frac{\sigma}{\Gamma(\theta)}(\sigma x)^{\theta-1}e^{-\sigma x},\ x, \theta, \sigma > 0$. If $\theta \in [0.5, 1]$, then $\xi_\epsilon[X+Y] \leq \xi_\epsilon[X] + \xi_\epsilon[Y]$. Indeed, as $\theta \leq 1$, X and Y are IMRL. Moreover, $X+Y$ follows a $Gamma(2\theta, \sigma)$ distribution with $2\theta \geq 1$ and hence is NBUE.

Let us note that the exponential distribution is a particular case of the gamma distribution, with $\theta = 1$. So, the previous example also applies for exponentially distributed severities.

5 Enhanced results for small values of ϵ

In practice, insurance companies consider small values for the level ϵ. In this section, we show that one can weaken the requirements on X and Y in Properties 2, 3 and 4 when $\epsilon \to 0$. The only condition we impose is $X \preceq_{\text{hmrl}} Y$. In particular, the subadditivity results obtained in Trufin et al. (2011) are also valid for ξ_ϵ.

Afterwards, we restrict our analysis to claim severities for which Lundberg coefficients exist. In that case, we show that for all ϵ smaller than a predefined bound b, one only needs to compare the Lundberg coefficients of X and Y to order the corresponding risk measures.

Proposition 5. *Let X and Y such that $L_X \preceq_{icx} L_Y$. Hence, for $\epsilon \to 0$, the inequality $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$ holds true.*

Proof. We know that $L_X \preceq_{\text{icx}} L_Y$ if and only if $\overline{\rho}_\epsilon[X] \leq \overline{\rho}_\epsilon[Y]$ (see, e.g., Proposition 3.4.8 in Denuit et al. (2005)) or, equivalently,

$$\xi_\epsilon[X] + \mathbb{E}[L_X] \leq \xi_\epsilon[Y] + \mathbb{E}[L_Y],$$

which may be further written as $\frac{\xi_\epsilon[X]}{\xi_\epsilon[Y]} \leq 1 + \frac{\mathbb{E}[L_Y]-\mathbb{E}[L_X]}{\xi_\epsilon[Y]}$. Now, since $\lim_{\epsilon\to 0} \xi_\epsilon[Y] = +\infty$, we get

$$\lim_{\epsilon\to 0} \frac{\xi_\epsilon[X]}{\xi_\epsilon[Y]} \leq 1.$$

□

The condition $L_X \preceq_{\text{icx}} L_Y$ may be expressed in terms of the ladder height distributions of X and Y, as stated in the following proposition.

Proposition 6. *Let X and Y such that $D_X \preceq_{icx} D_Y$. Hence, for $\epsilon \to 0$, the inequality $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$ holds true.*

Proof. It suffices to notice that the stop-loss order is closed under compounding. □

We directly get the following property.

Property 6. *Let X and Y such that $X \preceq_{hmrl} Y$. Hence, for $\epsilon \to 0$, the inequality $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$ holds true.*

Proof. $X \preceq_{\text{hmrl}} Y$ is equivalent to $D_X \preceq_{\text{st}} D_Y$, which yields $D_X \preceq_{\text{icx}} D_Y$ (see Theorems 2.B.2 and 4.A.34 in Shaked and Shanthikumar (2007)). □

The next result is an immediate consequence of Property 6.

Corollary 3. *Let X and Y be i.d. so that $\frac{X+Y}{2} \preceq_{hmrl} Y$. Hence ξ_ϵ is subadditive for $\epsilon \to 0$.*

The coming property highlights some situations where the condition $\frac{X+Y}{2} \preceq_{\text{hmrl}} Y$ is fulfilled. The proof directly results from Property 3.2 (i) and (ii) in Trufin et al. (2011).

Property 7. *(i) The risk measure ξ_ϵ is subadditive for exchangeable risks, that is, $\xi_\epsilon[X+Y] \leq \xi_\epsilon[X] + \xi_\epsilon[Y]$ if X and Y satisfy $\Pr[X \leq t_1, Y \leq t_2] = \Pr[X \leq t_2, Y \leq t_1]$ for all t_1 and t_2.*

(ii) If X and Y are negatively quadrant dependent and identically distributed, that is, the inequality $\Pr[X \le t_1, Y \le t_2] \le \Pr[X \le t_1]\Pr[Y \le t_2]$ holds for all t_1 and t_2, then ξ_ϵ is subadditive.

We note that i.i.d. claim severities X and Y are particular cases of exchangeable risks.

Corollary 4. *If $X = \beta V_1$ and $Y = \gamma V_2$, where V_1 and V_2 are identically distributed as V, then ξ_ϵ is subadditive.*

Proof. Using a similar argument as in Property 3.2 (iii) in Trufin et al. (2011), $\beta V_1 + \gamma V_2 \preceq_{\text{cx}} (\beta+\gamma)V$. Since they have the same means, we have $\beta V_1 + \gamma V_2 \preceq_{\text{hmrl}} (\beta+\gamma)V$. Therefore, $\xi_\epsilon[X+Y] = \xi_\epsilon[\beta V_1 + \gamma V_2] \le \xi_\epsilon[(\beta+\gamma)V] = (\beta+\gamma)\xi_\epsilon[V] = \beta\xi_\epsilon[V] + \gamma\xi_\epsilon[V] = \xi_\epsilon[\beta V] + \xi_\epsilon[\gamma V] = \xi_\epsilon[X] + \xi_\epsilon[Y]$, as needed. □

We now consider in the remaining of this section claim severities for which Lundberg coefficients exist. We recall that for a claim severity X, the Lundberg coefficient κ_X is the unique strictly positive solution of the equation

$$1 + (1+\eta)\,\mu_X\, t = M_X(t), \tag{5.1}$$

where $M_X(t)$ is the moment generating function (m.g.f.) of X. In this setting, given two claim severities X and Y, we derive in the following proposition an upper bound b on ϵ such that, for $\epsilon \in (0, b)$, the inequality $\kappa_X > \kappa_Y$ implies $\xi_\epsilon[X] \le \xi_\epsilon[Y]$.

Beforehand, we recall that the probability of ruin admits exponential bounds (see Taylor (1976)), i.e.

$$C_X^- e^{-\kappa_X u} \le \psi_X(u) \le C_X^+ e^{-\kappa_X u},$$

where $C_X^+(C_X^-) = \sup_{x\ge 0}(\inf_{x\ge 0}) \frac{\overline{F}_X(x)}{\int_x^\infty e^{\kappa_X(y-x)} dF_X(y)}$. Obviously, $C_X^+ \le 1$. Moreover, Lin (1996) shows that if claim severities are NBUC, one can use $C_X^- = \frac{1}{1+\eta}$. Therefore, given two claim severities X and Y, there exists C_X^+ in the interval $[\frac{1}{1+\eta}, 1]$ so that we have

$$\psi_X(u) \le C_X^+ e^{-\kappa_X u} \quad \text{and} \quad \frac{1}{1+\eta} e^{-\kappa_Y u} \le \psi_Y(u), \tag{5.2}$$

which will be useful in the next proposition.

Proposition 7. *Let X and Y be NBUC such that $\kappa_X > \kappa_Y$. Then the inequality $\xi_\epsilon[X] \le \xi_\epsilon[Y]$ holds true at least for all $\epsilon \in (0, b)$, where*

$$b = \min\left\{ \exp\left(-\frac{\kappa_X \kappa_Y}{\kappa_X - \kappa_Y} \left[\frac{1}{\kappa_X} \ln(C_X^+) + \frac{1}{\kappa_Y} \ln(1+\eta) + \frac{1}{2\eta}\left(\frac{\mu_{2,Y}}{\mu_Y} - \frac{\mu_{2,X}}{\mu_X} \right) \right] \right), \frac{1}{1+\eta} \right\}. \tag{5.3}$$

Proof. Let $h(\epsilon) = \epsilon\,(\xi_\epsilon[X] - \xi_\epsilon[Y])$. Then, it yields

$$h(\epsilon) = \epsilon(\rho_\epsilon[X] - \rho_\epsilon[Y]) + \int_{\rho_\epsilon[X]}^{\infty} \psi_X(x)dx - \int_{\rho_\epsilon[Y]}^{\infty} \psi_Y(x)dx - \frac{\epsilon}{2\eta}\left(\frac{\mu_{2,X}}{\mu_X} - \frac{\mu_{2,Y}}{\mu_Y} \right)$$

and

$$h'(\epsilon) = \rho_\epsilon[X] - \rho_\epsilon[Y] - \frac{1}{2\eta}\left(\frac{\mu_{2,X}}{\mu_X} - \frac{\mu_{2,Y}}{\mu_Y}\right).$$

In the latter equation, we use the fact that

$$\frac{d}{d\epsilon}\left\{\epsilon\rho_\epsilon[X] + \int\limits_{\rho_\epsilon[X]}^{\infty} \psi_X(x)dx\right\} = \rho_\epsilon[X].$$

Since $\rho_\epsilon[X] = \overline{F}_{L_X}^{-1}(\epsilon)$, it follows that

$$h'(\epsilon) = \overline{F}_{L_X}^{-1}(\epsilon) - \overline{F}_{L_Y}^{-1}(\epsilon) - \frac{1}{2\eta}\left(\frac{\mu_{2,X}}{\mu_X} - \frac{\mu_{2,Y}}{\mu_Y}\right).$$

Furthermore, as $\psi_X(u)$ is decreasing one-to-one function, the inequalities in (5.2) imply that $\overline{F}_{L_X}^{-1}(\epsilon) \leq -\frac{1}{\kappa_X}\ln\left(\frac{\epsilon}{C_X^+}\right)$ and $\overline{F}_{L_Y}^{-1}(\epsilon) \geq -\frac{1}{\kappa_Y}\ln\left((1+\eta)\epsilon\right)$, for $\epsilon < \frac{1}{1+\eta}$. Then, we get

$$\begin{aligned} h'(\epsilon) &\leq -\frac{1}{\kappa_X}\ln\left(\frac{\epsilon}{C_X^+}\right) + \frac{1}{\kappa_Y}\ln(\epsilon(1+\eta)) - \frac{1}{2\eta}\left(\frac{\mu_{2,X}}{\mu_X} - \frac{\mu_{2,Y}}{\mu_Y}\right) \\ &= \frac{\kappa_X - \kappa_Y}{\kappa_X\kappa_Y}\ln(\epsilon) + \frac{1}{\kappa_X}\ln(C_X^+) + \frac{1}{\kappa_Y}\ln(1+\eta) - \frac{1}{2\eta}\left(\frac{\mu_{2,X}}{\mu_X} - \frac{\mu_{2,Y}}{\mu_Y}\right). \end{aligned}$$

This implies that $h'(\epsilon) \leq 0$ for all $\epsilon \in (0, b)$, since $\kappa_X > \kappa_Y$. Moreover,

$$\begin{aligned} h(0_+) &= \lim_{\epsilon\to 0_+} \epsilon(\rho_\epsilon[X] - \rho_\epsilon[Y]) \\ &\leq \lim_{\epsilon\to 0_+} \epsilon\left[\frac{1}{\kappa_X}\ln(C_X^+) + \frac{1}{\kappa_Y}\ln(1+\eta) + \frac{\kappa_X - \kappa_Y}{\kappa_X\kappa_Y}\ln(\epsilon) - \frac{1}{2\eta}\left(\frac{\mu_{2,X}}{\mu_X} - \frac{\mu_{2,Y}}{\mu_Y}\right)\right] = 0, \end{aligned}$$

which completes the proof. □

Example 3. To illustrate this latter result, let $\eta = 0.4$, Y be a mixture of two exponential distributions with means 1/3 and 1/7 and weights 1/2, and X be exponentially distributed such that $\mu_X = \mu_Y$. Therefore,

$$\overline{F}_Y(x) = \frac{1}{2}e^{-3x} + \frac{1}{2}e^{-7x}, \quad \overline{F}_X(x) = e^{-\frac{21}{5}x}.$$

As shown in Example IX.3.2 in Asmussen and Albrecher (2010), $\psi_Y(u) = \frac{24}{35}e^{-u} + \frac{1}{35}e^{-6u}$, so $\kappa_Y = 1$. For X, the ruin probability has the form (see Willmot and Lin (1998)) $\psi_X(u) = \frac{1}{1+\eta}e^{-\frac{\eta u}{(1+\eta)\mu_X}}$. Thus, $\kappa_X = \frac{\eta}{(1+\eta)\,\mu_X} = \frac{6}{5}$ and $C_X^+ = \frac{1}{1+\eta} = \frac{5}{7}$. Therefore, $\kappa_X > \kappa_Y$ and one can apply the results from the latter proposition. We find $\mu_Y = \mu_X = \frac{5}{21}$, $\mu_{2,Y} = \frac{58}{441}$, $\mu_{2,X} = \frac{50}{441}$ and hence, $\xi_\epsilon[X] < \xi_\epsilon[Y]$ for all $\epsilon \in (0, b)$ with $b = \frac{5}{7}e^{-4/7} = 0.40337$.

Remark 1. From Lundberg's equation (5.1) it follows that for two claim severities X and Y with $\mu_X < \mu_Y$ (resp. $\mu_X \leq \mu_Y$), a sufficient condition for the inequality $\kappa_X > \kappa_Y$ is that $M_X(t) \leq M_Y(t)$ (resp. $M_X(t) < M_Y(t)$) for $t \in (0, \kappa_Y)$.

Corollary 5. *Let X and Y be NBUC such that $\mu_X < \mu_Y$. If $X \preceq_{icx} Y$, then the inequality $\xi_\epsilon[X] \leq \xi_\epsilon[Y]$ holds true at least for all $\epsilon \in (0, b)$, where b is given by* (5.3).

Proof. It suffices to notice that the function $g(x) = e^{t\,x}$ is a non-decreasing convex function and hence $M_X(t) \leq M_Y(t)$ by definition of the increasing convex order. □

We notice that if $\mu_X < \mu_Y$ and $X \preceq_{\text{hmrl}} Y$, we have that $M_X(t) \leq M_Y(t)$ for $t > 0$, since the m.g.f. is a convex function and hence, $\kappa_X > \kappa_Y$. So, when $\mu_X < \mu_Y$, the requirement $\kappa_X > \kappa_Y$ appears to be less restrictive (provided the existence of κ_X and κ_Y) than the condition $X \preceq_{\text{hmrl}} Y$ derived when $\epsilon \to 0$, as expected.

Property 8. *Let X and Y be i.d. and NBUC (we exclude the case $\Pr[X = Y] = 1$). Then, ξ_ϵ is subadditive at least for $\epsilon \in (0, b^*)$, where b^* is given by* (5.3)*, replacing X by $\frac{X+Y}{2}$.*

Proof. By Theorem 2.1 in Li (2004), it comes $M_{\frac{X+Y}{2}}(t) \leq M_Y(t)$. The proof is based on Lemma 2.1 in Pellerey (2000). A closer look at the latter lemma yields the strict inequality $M_{\frac{X+Y}{2}}(t) < M_Y(t)$. The proof then follows from Proposition 7 and Remark 1. □

To end this section, we note that Theorem 2.1 in Li (2004) tells us even more, namely that $M_{\alpha X+(1-\alpha)Y}(t) \leq M_Y(t)$ for all t. In a similar fashion as done in the previous property, one can prove that ξ_ϵ is convex, i.e., if X and Y are i.d. and $\alpha \in (0, 1)$, then

$$\xi_\epsilon[\alpha X + (1-\alpha)Y] \leq \xi_\epsilon[Y] \quad \text{for } \epsilon \in (0, b^+),$$

where b^+ is obtained from (5.3), replacing X by $\alpha X + (1-\alpha)Y$.

6 Closed form expressions

In this section, we show that one may obtain closed form expressions for $\xi_\epsilon[X]$ in some particular cases for X.

Example 4. Suppose that X is exponentially distributed. It is well-known that the ruin probability has the form (see for instance Bowers et al. (1997)):

$$\psi_X(u) = \frac{1}{1+\eta} e^{-\frac{\eta u}{(1+\eta)\mu_X}} \quad \text{for} \quad u \geq 0.$$

In this case, one can easily invert $\psi_X(x)$, which leads to

$$\rho_\epsilon[X] = \psi_X^{-1}(\epsilon) = -\frac{(1+\eta)\mu_X}{\eta} \ln(\epsilon(1+\eta)).$$

As X is exponentially distributed, $\mathbb{E}[|U_T| \,\big|\, T < \infty] = \mu_X$, and hence

$$\xi_\epsilon[X] = \mu_X \left[1 - \left(1 + \frac{1}{\eta}\right) \ln(\epsilon(1+\eta))\right]. \tag{6.1}$$

In particular, for X and Y exponentially distributed such that $\mu_X < \mu_Y$, it comes

$$\xi_\epsilon[X] - \xi_\epsilon[Y] = (\mu_X - \mu_Y)\left[1 - \left(1 + \frac{1}{\eta}\right)\ln(\epsilon(1+\eta))\right] < 0,$$

as already stated in Example 1. We also notice that

$$\lim_{\epsilon \to \frac{1}{1+\eta}} \xi_\epsilon[X] = \mu_X,$$

as expected since $\rho_{\frac{1}{1+\eta}}[X] = 0$.

Remark 2. The closed form expression (6.1) can also be used to obtain upper bounds for our risk measure when the claim severity X follows more complicated distributions. Indeed, suppose that the claim severity X is NBUE. Then,

$$\xi_\epsilon[X] \leq \mu_X\left[1 - \left(1 + \frac{1}{\eta}\right)\ln(\epsilon(1+\eta))\right].$$

The latter inequality results from Corollary 1, by choosing Y exponentially distributed such that $\mu_Y = \mu_X$. In particular, this result holds for all $Gamma(\theta, \sigma)$ distributions with shape parameter $\theta > 1$, since these are NBUE.

Example 5. Suppose that X follows a mixture of two exponentials and its density function is $f_X(x) = q\,\tau_1\,e^{-\tau_1 x} + (1-q)\,\tau_2\,e^{-\tau_2 x},\ x \geq 0$, with $0 < q < 1$. Then, the ruin probability has the form (Dufresne and Gerber (1988))

$$\psi_X(x) = C_1 e^{-R_1 x} + C_2 e^{-R_2 x}, \quad x \geq 0, \tag{6.2}$$

where $R_1 < R_2$ are two positive roots of $(1+\eta)R^2 - [\nu + \eta(\tau_1 + \tau_2)]R + \eta\tau_1\tau_2 = 0$, with $\nu = \frac{(1-q)\tau_1^2 + q\tau_2^2}{(1-q)\tau_1 + q\tau_2}$, $C_1 = \frac{\nu - R_1}{(1+\eta)(R_2 - R_1)}$ and $C_2 = \frac{R_2 - \nu}{(1+\eta)(R_2 - R_1)}$. Obviously, $C_1 + C_2 = \frac{1}{1+\eta}$. Thus, we get

$$\begin{aligned}
\xi_\epsilon[X] &= \rho_\epsilon[X] + \frac{1}{\epsilon}\int_{\rho_\epsilon[X]}^{\infty} \psi_X(x)dx - \frac{\mu_{2,X}}{2\eta\mu_X} \\
&= \rho_\epsilon[X] + \frac{1}{\epsilon}\left(\frac{C_1}{R_1}e^{-R_1\rho_\epsilon[X]} + \frac{C_2}{R_2}e^{-R_2\rho_\epsilon[X]}\right) - \left(\frac{C_1}{R_1} + \frac{C_2}{R_2}\right),
\end{aligned}$$

with $\rho_\epsilon[X]$ the solution of the equation $C_1 e^{-R_1\rho_\epsilon[X]} + C_2 e^{-R_2\rho_\epsilon[X]} = \epsilon$.

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