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RESEARCH ARTICLE

On some validity-robust tests for the homogeneity of concentrations

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In this paper we tackle the problem of testing the homogeneity of concentrations for directional data. All the existing procedures for this problem are parametric procedures based on the assumption of a Fisher-von Mises-Langevin (FvML) distribution. We construct here a so-called pseudo-FvML test and a rank-based Kruskal-Wallis type test for this problem. The pseudo-FvML test improves on the traditional FvML parametric procedures by being asymptotically valid under the whole semiparametric class of rotationally symmetric distributions. Furthermore, it is asymptotically equivalent to the locally and asymptotically most stringent parametric FvML procedure in the FvML case. The Kruskal-Wallis rank-based test is also asymptotically valid under rotationally symmetric distributions and performs nicely under various important distributions. The finite-sample behavior of the proposed tests is investigated by means of a Monte Carlo simulation.

Keywords: directional statistics, homogeneity of concentrations, rank-based tests, robust tests.

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1. Introduction.

In this paper, we tackle a testing problem which is related to so-called directional data. Directional data naturally arise in many different fields such as geology, meteorology and studies of animal behavior to only cite a few. They are commonly viewed as realizations of a random vector $\mathbf{X}$ taking values on the surface of the unit hypersphere $\mathcal{S}^{p-1} := \{\mathbf{v} \in \mathbb{R}^p \,|\, \mathbf{v}'\mathbf{v} = 1\}$, $p \geq 2$. We refer to Mardia and Jupp (2000) for a detailed overview of the traditional techniques and models used to analyze such data.

The probability distribution which is the most often used to model directional data is the Fisher-von Mises-Langevin (FvML) distribution. The latter is characterized by a density function of the form (with respect to the usual surface area measure on hyperspheres)

$$\mathbf{x} \to c_{p,\kappa} \, \exp(\kappa \mathbf{x}'\boldsymbol{\theta}), \tag{1.1}$$

where $\mathbf{x} \in \mathcal{S}^{p-1}$, $\boldsymbol{\theta} \in \mathcal{S}^{p-1}$ is a location parameter, $\kappa > 0$ is a concentration parameter and $c_{p,\kappa}$ is a normalizing constant. The distribution of a FvML random vector $\mathbf{X}$ depends on its "distance" from the fixed point $\boldsymbol{\theta}$ which belongs to the unit sphere. The parameter $\boldsymbol{\theta}$, which can therefore be regarded as a "north pole" (or modal direction) is naturally considered as a location parameter. Now, the concentration parameter κ is closely related to the expectation $\mathrm{E}[\mathbf{X}'\boldsymbol{\theta}]$ (we refer to Section 2 below for a detailed description of the concentration); there exists a one-to-one monotone increasing mapping $A_p : \mathbb{R}^+ \to]0, 1[$ such that $A_p^{-1}(\mathrm{E}[\mathbf{X}'\boldsymbol{\theta}]) = \kappa$. As a direct consequence the concentration parameter κ drives the probability mass in the vicinity of the location $\boldsymbol{\theta}$. A large κ yields to a probability mass which is concentrated around $\boldsymbol{\theta}$ while on the contrary, a small κ yields to less concentration around $\boldsymbol{\theta}$.

Due to their extremely important role, inference problems related to the concentration of a FvML distribution have received a lot of attention in the literature. Maximum likelihood estimation of the concentration has been tackled in Watson (1986) while robust estimation of the concentration has been considered in Fisher (1982), Ko and Guttorp (1988) and Ko (1992). Multi-sample problems related to the concentration are also important in the directional framework. In particular, assuming that m samples of spherical observations with concentrations $\kappa_1, \ldots, \kappa_m$ respectively are available, the problem of testing the homogeneity of the concentrations $\mathcal{H}_0 : \kappa_1 = \ldots = \kappa_m$ is particularly important. It can be regarded as the directional equivalent to the problem of testing the homogeneity of variances in the general univariate or multivariate ANOVA framework. FvML likelihood ratio or FvML score tests for this problem have been studied in Stephens (1969), Larsen *et al.* (2002) and Watamori and Jupp (2005). Recently, Ley and Verdebout (2014) computed the local asymptotic power of the Watamori and Jupp (2005) test and showed that it enjoys the property of being locally and asymptotically most stringent.

To the best of our knowledge, all the testing procedures related to the homogeneity of concentrations are FvML parametric procedures. As a direct consequence, they are only asymptotically valid (in the sense that they reach the asymptotic nominal level constraint) in the same FvML case only. In this paper, we provide *validity-robust* tests for the homogeneity of concentrations. More precisely, we obtain tests for this problem which are asymptotically valid (in the sense that they reach the asymptotic nominal level constraint) whitin the semiparametric class of *rotationally symmetric* distributions which contains the FvML one. We provide two different types of testing procedures. First, we use the fact that the testing problem can be seen as a univariate problem on location parameters (see Section 2) to obtain a so-called pseudo-FvML test. Then, we also use the ranks studied in Ley *et al.* (2013) and Paindaveine and Verdebout (2014) to provide a Kruskal-Wallis type test. Both tests are shown to be valid under any m-tuple of rotationally symmetric distributions. The pseudo-FvML test is asymptotically equivalent to the Watamori and Jupp (2005) score test in the FvML case. It therefore keeps the desirable property of being locally and asymptotically most stringent if the true densities are FvML ones. Out of this FvML case, simulations show that the proposed Kruskal-Wallis test outperforms the pseudo-FvML procedure under some important distributions.

The paper is organized as follows. In Section 2, we collect the main assumptions and notations to be used in the sequel and we describe the Watamori and Jupp (2005) score test together with some of its properties. In Section 3, we introduce our validty-robust test statistics and we study their asymptotic distributions under the null. In Section 4, we compare the empirical levels and powers of our tests with respect to the FvML test of Watamori and Jupp (2005). Finally, an appendix collects the technical details.

2. Assumptions, notations and description of the classical FvML procedure

Let $\mathbf{X}$ stand for a random p-vector having a FvML distribution with location $\boldsymbol{\theta}$ and concentration κ (it has density (1.1)). Letting $\omega_p = 2\pi^{p/2}/\Gamma(p/2)$ be the surface area of $\mathcal{S}^{p-1}$ and writing $B(\cdot,\cdot)$ for the beta function, it is well-known (see Watson (1983)) that the cosine $Y(\boldsymbol{\theta}) := \mathbf{X}'\boldsymbol{\theta}$ has density

$$t \mapsto \frac{\omega_p \, c_{p,\kappa}}{B(\frac{1}{2}, \frac{1}{2}(p-1))} \exp(\kappa t)(1-t^2)^{(p-3)/2}, \quad -1 \leq t \leq 1.$$

As a direct consequence, the parameter κ of a FvML distribution is clearly identified using the identity

$$\mathrm{E}[\mathbf{X}] = \mathrm{E}[Y(\boldsymbol{\theta})]\boldsymbol{\theta} =: A_p(\kappa)\boldsymbol{\theta} = \left(\frac{\int_{-1}^{1} t e^{\kappa t}(1-t^2)^{\frac{p-3}{2}}}{\int_{-1}^{1} e^{\kappa t}(1-t^2)^{\frac{p-3}{2}}} \right) \boldsymbol{\theta},$$

where, letting $I_q(v)$ stand for the modified Bessel function of first kind and of order q, $A_p(.) := I_{p/2}(.)/I_{p/2-1}(.)$ (see Watson (1983) for details). Then, one readily obtains that $\kappa := A_p^{-1}(\mathrm{E}[Y(\boldsymbol{\theta})])$. Since A_p is a one-to-one mapping, inference for the concentration of a FvML distribution can be done equivalently on $\kappa = A_p^{-1}(\mathrm{E}[Y(\boldsymbol{\theta})])$ or on $\mu := \mathrm{E}[Y(\boldsymbol{\theta})]$.

As described in the Introduction, our objective in this paper is to obtain asymptotically valid tests under *rotationally symmetric* distributions. A random vector $\mathbf{X}$, with values on the unit sphere $\mathcal{S}^{p-1}$ of $\mathbb{R}^p$, is said to be *rotationally symmetric* about $\boldsymbol{\theta}(\in \mathcal{S}^{p-1})$ if and only if, for all orthogonal $p \times p$ matrices $\mathbf{O}$ satisfying $\mathbf{O}\boldsymbol{\theta} = \boldsymbol{\theta}$, the random vectors $\mathbf{OX}$ and $\mathbf{X}$ are equal in distribution. If $\mathbf{X}$ is further absolutely continuous (with respect to the usual surface area measure on $\mathcal{S}^{p-1}$), then its density is of the form

$$\begin{aligned} f_{\boldsymbol{\theta}} \, : \, \mathcal{S}^{p-1} &\rightarrow \mathbb{R}^p \\ \mathbf{x} &\rightarrow c_{p,f}\, f(\mathbf{x}'\boldsymbol{\theta}), \end{aligned} \tag{2.2}$$

where $c_{p,f}(> 0)$ is a normalization constant and $f : [-1, 1] \rightarrow \mathbb{R}$ is some nonnegative function—called an *angular function* in the sequel. Obviously, the FvML density is a particular instance of (2.2) obtained by taking $f(u) = \exp(\kappa u)$. Throughout the paper, the following assumption will be required on the data generating process.

Assumption A1. The sample $(\mathbf{X}_{i1}, \ldots, \mathbf{X}_{in_i})$, $i = 1, \ldots, m$ is a collection of mutually independent samples of i.i.d. random vectors such that the n_i observations $\mathbf{X}_{ij}$, $j = 1, \ldots, n_i$ in sample i are independent and admit a common density $f_{\boldsymbol{\theta}_i}$ $(i = 1, \ldots, m)$ of the form (2.2), for some $\boldsymbol{\theta}_i \in \mathcal{S}^{p-1}$ and some angular function f_i $(i = 1, \ldots, m)$ in the collection $\mathcal{F}$ of functions from $[-1, 1]$ to $\mathbb{R}^+$ that are positive and monotone nondecreasing.

Note that if $\mathbf{X}_{i1}, \ldots, \mathbf{X}_{in_i}$ are i.i.d with location $\boldsymbol{\theta}_i$ and angular function f_i, then the projections $Y_{i1}(\boldsymbol{\theta}_i) := \mathbf{X}'_{i1}\boldsymbol{\theta}_i, \ldots, Y_{in_i}(\boldsymbol{\theta}_i) := \mathbf{X}'_{in_i}\boldsymbol{\theta}_i$ are i.i.d with density

$$t \mapsto \tilde{f}_i(t) := C_{p,f_i} f_i(t)(1-t^2)^{(p-3)/2}, \quad -1 \leq t \leq 1,\ i = 1, \ldots, m,$$

where C_{p,f_i} stands for a normalizing constant.

Letting $\mu_i := \mathrm{E}[Y_{i1}(\boldsymbol{\theta}_i)]$, $i = 1, \ldots, m$, it directly follows from the relationship between the concentration κ_i and the location μ_i $(i = 1, \ldots, m)$ in the FvML case that the null hypotheses of homogeneity of concentrations $(\mathcal{H}_0 : \kappa_1 = \ldots = \kappa_m)$ and of homogeneity of the locations $(\mathcal{H}_0 : \mu_1 = \ldots = \mu_m)$ are clearly equivalent (still in the FvML case). Now, letting $\boldsymbol{\vartheta} := (\boldsymbol{\theta}'_1, \ldots, \boldsymbol{\theta}'_m)'$, $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_m)'$ and $\underline{f} := (f_1, \ldots, f_m)$, we will use in the sequel the notation $\mathrm{P}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$ for the joint distribution of the $\mathbf{X}_{ij}$'s under Assumption A1 with location parameters $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ and angular functions $f_1, \ldots, f_m$ such that the cdfs of the projections $\tilde{F}_1(t) := \int_{-1}^{t} \tilde{f}_1(u)\,du, \ldots, \tilde{F}_m(t) := \int_{-1}^{t} \tilde{f}_m(u)\,du$ have expectations $\mu_1, \ldots, \mu_m$ respectively. In this paper, we consider the null hypothesis

$$\mathcal{H}_0^{\mathrm{Hom}} : \cup_{\boldsymbol{\vartheta} \in (\mathcal{S}^{p-1})^m} \cup_{\underline{f} \in (\mathcal{F})^m} \mathrm{P}_{\boldsymbol{\vartheta},(\mu,\ldots,\mu)';\underline{f}},$$

where μ stand for the common value of the $\mu_1, \ldots, \mu_m$. The following homogeneity assumption on the angular densities $f_1, \ldots, f_m$ will be necessary to perform the proposed rank-based test.

Assumption A2. The angular densities $f_1, \ldots, f_m$ of the projections are such that $f_1 = \ldots = f_m$ under $\mathcal{H}_0^{\mathrm{Hom}}$.

Assumption A2 corresponds to a usual homogeneity assumption required to perform a univariate ANOVA based on ranks. Nevertheless, we draw here the reader attention to the fact that no homogeneity assumption on the location parameters $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ is required. Under Assumption A2 and under $\mathcal{H}_0^{\mathrm{Hom}}$, the notation $\underline{f}$ in $\mathrm{P}_{\boldsymbol{\vartheta},(\mu,\ldots,\mu)';\underline{f}}$ could therefore be seen as superfluous since $\underline{f}$ is therefore of the form $\underline{f} = (f_1, \ldots, f_1)$ with $f_1 \in \mathcal{F}$ under $\mathcal{H}_0^{\mathrm{Hom}}$. We will nevertheless use this notation throughout for the sake of clarity and simplicity. Clearly, any m-tuple of FvML densities with the same concentrations belongs to $\mathcal{H}_0^{\mathrm{Hom}}$ under Assumption A2. However, $\mathcal{H}_0^{\mathrm{Hom}}$ contains many more distributions since it is not restricted to the FvML parametric model; both the location parameters $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ and the common angular function under the null play the role of nuisance parameters here. The pseudo-FvML test obtained in Section 3 only requires the following weaker version of Assumption 2.

Assumption A2$'$. Under $\mathcal{H}_0^{\mathrm{Hom}}$, the angular functions $f_1, \ldots, f_m$ are such that $\mathrm{Var}[Y_{11}(\boldsymbol{\theta}_1)] = \ldots = \mathrm{Var}[Y_{m1}(\boldsymbol{\theta}_m)]$.

Now, for the derivation of our asymptotic results, we will need to control the specific sizes n_i, $i = 1, \ldots, m$ via the following assumption.

Assumption A3. Letting $n = \sum_{i=1}^m n_i$, for all $i = 1, \ldots, m$ the ratio $r_i^{(n)} := n_i/n$ converges to a finite constant r_i as $n \to \infty$.

When we work under Assumption A3 in what follows, we simply use the superscript $^{(n)}$ for the different quantities at play and do not specify whether they are associated with a given n_i. Letting $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$ stand for the $m \times m$ block-diagonal matrix with blocks $\mathbf{A}_1, \ldots, \mathbf{A}_m$, we put $\mathbf{r}^{(n)} := \mathrm{diag}(r_1^{(n)}, \ldots, r_m^{(n)})$ in the sequel.

The main competitor for the testing problem considered in this paper is the Watamori and Jupp (2005) score test $\phi_{\mathrm{FvML}}^{(n)}$ which is locally and asymptotically most stringent in the FvML case (see Ley and Verdebout (2014)). Letting $\bar{\mathbf{X}}_i := n_i^{-1} \sum_{j=1}^{n_i} \mathbf{X}_{ij}$, $i = 1, \ldots, m$ and $\hat{\kappa} := A_p^{-1}\left(\sum_{i=1}^m r_i^{(n)} \|\bar{\mathbf{X}}_i\|\right)$, the test ϕ_{FvML} rejects the null hypothesis $\mathcal{H}_0^{\mathrm{Hom}}$ at asymptotic level α when

$$
\begin{aligned}
T_{\mathrm{WJ}}^{(n)} &:= \frac{\sum_{i=1}^m n_i \left(\|\bar{\mathbf{X}}_i\| - A_p(\hat{\kappa})\right)^2}{1 - \frac{p-1}{\hat{\kappa}} A_p(\hat{\kappa}) - (A_p(\hat{\kappa}))^2} \\
&= \frac{\left(\sum_{i=1}^m n_i \|\bar{\mathbf{X}}_i\|^2 - \frac{1}{n}\left(\sum_{i=1}^m n_i \|\bar{\mathbf{X}}_i\|\right)^2\right)}{1 - \frac{p-1}{\hat{\kappa}} A_p(\hat{\kappa}) - (A_p(\hat{\kappa}))^2}
\end{aligned}
$$

exceeds the α-upper quantile of the chi-square distribution with $m-1$ degrees of freedom. The test $\phi_{\mathrm{FvML}}^{(n)}$ is a pure FvML parametric test in the sense that is is valid under FvML distributions only. In the next Section, we provide testing procedures which are valid in the rotationally symmetric case.

3. Validity robust tests for the homogeneity of concentrations

In this Section, we provide validity-robust tests for $\mathcal{H}_0^{\text{Hom}}$. First, we construct a so-called pseudo-FvML test; that is a test which is valid under any m-tuple of rotationally symmetric densities while keeping the optimality features of the Watamori and Jupp (2005) test under FvML densities. Then, based on the invariance principle, we use the ranks studied in Ley *et al.* (2013) and Paindaveine and Verdebout (2014) to provide a Kruskal-Wallis rank-based test.

3.1. *Pseudo-FvML test*

As explained in the previous Section, the problem of testing the equality of concentrations in the FvML case can be seen as a traditional univariate ANOVA problem on the projections $Y_{11}(\boldsymbol{\theta}_1), \ldots, Y_{mn_m}(\boldsymbol{\theta}_m)$. Therefore, we consider here a natural test statistic for this problem which is simply based on empirical means and variances.

In this sequel, we use the notations $\bar{Y}_i(\boldsymbol{\theta}_i) := n_i^{-1} \sum_{j=1}^{n_i} Y_{ij}(\boldsymbol{\theta}_i)$, $\bar{\mathbf{Y}}(\boldsymbol{\vartheta}) := (\bar{Y}_1(\boldsymbol{\theta}_1), \ldots, \bar{Y}_m(\boldsymbol{\theta}_m))'$ and $\mathbf{S}^{(n)}(\boldsymbol{\vartheta}) := n^{1/2}(\mathbf{r}^{(n)})^{1/2}\bar{\mathbf{Y}}(\boldsymbol{\vartheta})$. Under the null and Assumption A2$'$, let $\sigma^2 := \text{Var}[Y_{11}(\boldsymbol{\theta}_1)] = \ldots = \text{Var}[Y_{m1}(\boldsymbol{\theta}_m)]$ stand for the common value of the variances of the $Y_{ij}(\boldsymbol{\theta}_i)$'s and put $\boldsymbol{\Sigma} := \sigma^2 \mathbf{I}_m$. A typical univariate ANOVA statistic is then defined by

$$T(\boldsymbol{\vartheta})^{(n)} := (\mathbf{S}^{(n)}(\boldsymbol{\vartheta}))' \left(\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Upsilon}^{(n)} \left((\boldsymbol{\Upsilon}^{(n)})' \boldsymbol{\Sigma} \boldsymbol{\Upsilon}^{(n)} \right)^{-1} (\boldsymbol{\Upsilon}^{(n)})' \right) \mathbf{S}^{(n)}(\boldsymbol{\vartheta}), \qquad (3.3)$$

where $\boldsymbol{\Upsilon}^{(n)} := (\mathbf{r}^{(n)})^{1/2}\mathbf{1}_m$ with $\mathbf{1}_\ell := (1, \ldots, 1)' \in \mathbb{R}^\ell$.

Obviously, $T(\boldsymbol{\vartheta})^{(n)}$ is not a genuine statistic since both $\boldsymbol{\vartheta}$ and σ are typically unknown. We show that (i) σ can be estimated consistently and (ii) the substitution of the location parameters $\boldsymbol{\vartheta}$ in $\mathbf{S}^{(n)}(\boldsymbol{\vartheta})$ by root-n consistent estimators have no asymptotic impact under any m-tuple of rotationally symmetric densities.

Point (i) above can be easily obtained: letting $\hat{\boldsymbol{\vartheta}} := (\hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_m)'$ stand for an estimator such that $\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta} = O_{\text{P}}(n^{-1/2})$ (many estimators can be chosen such as spherical means or spherical medians) under $\text{P}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$, a consistent estimator (still under $\text{P}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$) of σ is given by

$$\hat{\sigma}^2 = n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n_i} Y_{ij}^2(\hat{\boldsymbol{\theta}}_i) - \left(n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n_i} Y_{ij}(\hat{\boldsymbol{\theta}}_i) \right)^2 .$$

Point (ii) requires a more careful study. In the following Lemma, we show that $\hat{\boldsymbol{\vartheta}}$ can be used instead of $\boldsymbol{\vartheta}$ in $\mathbf{S}^{(n)}(\boldsymbol{\vartheta})$ under any m-tuple of rotationally symmetric densities without any asymptotic cost.

Lemma 3.1. *Let assumptions A1, A2′ and A3 hold. Then,* $\mathbf{S}^{(n)}(\hat{\boldsymbol{\vartheta}}) - \mathbf{S}^{(n)}(\boldsymbol{\vartheta})$ *is* $o_{\mathrm{P}}(1)$ *as* $n \to \infty$ *under* $\mathrm{P}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$.

Based on (3.3) and Lemma 3.1, the pseudo-FvML test $\varphi_{\dagger}^{(n)}$ we propose rejects the null $\mathcal{H}_0^{\mathrm{Hom}}$ at asymptotic level α when ($\hat{\boldsymbol{\Sigma}} := \hat{\sigma}^2 \mathbf{I}_m$)

$$\begin{aligned} T^{(n)}(\hat{\boldsymbol{\vartheta}}) &:= (\mathbf{S}^{(n)}(\hat{\boldsymbol{\vartheta}}))' \left(\hat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{\Upsilon}^{(n)} \left((\boldsymbol{\Upsilon}^{(n)})' \hat{\boldsymbol{\Sigma}} \boldsymbol{\Upsilon}^{(n)} \right)^{-1} (\boldsymbol{\Upsilon}^{(n)})' \right) \mathbf{S}^{(n)}(\hat{\boldsymbol{\vartheta}}) \\ &= \hat{\sigma}^{-2} \left(\sum_{i=1}^{m} n_i \bar{Y}_i^2(\hat{\boldsymbol{\theta}}_i) - \frac{1}{n} \left(\sum_{i=1}^{m} n_i \bar{Y}_i(\hat{\boldsymbol{\theta}}_i) \right)^2 \right) \end{aligned}$$

exceeds the α-upper quantile of the chi-square distribution with $m-1$ degrees of freedom. The following result summarizes the asymptotic properties of $T^{(n)}(\hat{\boldsymbol{\vartheta}})$.

Proposition 3.1. *Let Assumptions A1, A2′ and A3 hold. Then* $T^{(n)}(\hat{\boldsymbol{\vartheta}})$ *is*

(i) asymptotically chi-square with $m-1$ *degrees of freedom under* $\cup_{\boldsymbol{\vartheta} \in (\mathcal{S}^{p-1})^m} \cup_{\underline{f} \in (\mathcal{F})^m} \mathrm{P}_{\boldsymbol{\vartheta},(\mu,\ldots,\mu)';\underline{f}}$;

(ii) such that $T_{\mathrm{WJ}}^{(n)} - T^{(n)}(\hat{\boldsymbol{\vartheta}}) = o_{\mathrm{P}}(1)$ *as* $n \to \infty$ *under the null and FvML densities (see Section 2 for a definition of* $T_{\mathrm{WJ}}^{(n)}$*).*

Proposition 3.1 entails that the test $\varphi_{\dagger}^{(n)}$ is asymptotically valid under any m-tuple of rotationally symmetric densities and is still locally and asymptotically most stringent (as the Watamori and Jupp (2005) test) under FvML densities.

3.2. *A Kruskal-Wallis rank-based test*

Another typical procedure for a classical univariate one-way ANOVA problem is the Kruskal-Wallis rank-based procedure (see Kruskal and Wallis (1952)). In this Section, we provide a Kruskal-Wallis type procedure to test for the equality of concentrations. Letting $R_{ij}(\boldsymbol{\vartheta})$ stand for the (univariate) rank of $Y_{ij}(\boldsymbol{\theta}_i)$ among the projections $Y_{11}(\boldsymbol{\theta}_1), \ldots, Y_{1n_1}(\boldsymbol{\theta}_1), Y_{21}(\boldsymbol{\theta}_2), \ldots, Y_{mn_m}(\boldsymbol{\theta}_m)$, put

$$\bar{R}_i(\boldsymbol{\vartheta}) := n_i^{-1} \sum_{j=1}^{n_i} \left(\frac{R_{ij}(\boldsymbol{\vartheta})}{n+1} - \frac{1}{2} \right)$$

and $\bar{\mathbf{R}}(\boldsymbol{\vartheta}) := (\bar{R}_1(\boldsymbol{\vartheta}), \ldots, \bar{R}_m(\boldsymbol{\vartheta}))'$. In this Section, we study a testing procedure which is based on the rank-based quantity $\mathbf{R}^{(n)}(\boldsymbol{\vartheta}) := n^{1/2}(\mathbf{r}^{(n)})^{1/2}\bar{\mathbf{R}}(\boldsymbol{\vartheta})$. Note that such ranks of projections have already been used for problems related with spherical location problems in Ley *et al.* (2013) and Paindaveine and Verdebout (2014). They are invariant with respect to specific monotone transformations on the unit sphere (see Ley *et al.* (2013) for details). Letting $\mathbf{\Gamma} := (1/12)\mathbf{I}_m$, the test we study in this Section rejects the null hypothesis $\mathcal{H}_0^{\mathrm{Hom}}$ for large values of

$$\begin{aligned}
T_{\mathrm{KW}}^{(n)}(\boldsymbol{\vartheta}) &:= \mathbf{R}^{(n)\prime}(\boldsymbol{\vartheta}) \left(\mathbf{\Gamma} - \mathbf{\Upsilon}^{(n)} \left((\mathbf{\Upsilon}^{(n)})'\mathbf{\Gamma}\mathbf{\Upsilon}^{(n)} \right)^{-1} (\mathbf{\Upsilon}^{(n)})' \right) \mathbf{R}^{(n)}(\boldsymbol{\vartheta}) \\
&= 12 \left(\sum_{i=1}^{m} n_i^{-1} \left(\sum_{j=1}^{n_i} \frac{R_{ij}(\boldsymbol{\vartheta})}{n+1} \right)^2 - \frac{1}{n} \left(\sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{R_{ij}(\boldsymbol{\vartheta})}{n+1} \right)^2 \right) \\
&= \frac{12}{(n+1)^2} \sum_{i=1}^{m} n_i \left(\bar{R}_i(\boldsymbol{\vartheta}) - \frac{n+1}{2} \right)^2 .
\end{aligned}$$

Note that $T_{\mathrm{KW}}^{(n)}(\boldsymbol{\vartheta})$ is nothing but the traditional Kruskal and Wallis (1952) statistic computed using the ranks of the projections $Y_{11}(\boldsymbol{\theta}_1), \ldots, Y_{1n_1}(\boldsymbol{\theta}_1), Y_{21}(\boldsymbol{\theta}_2), \ldots, Y_{mn_m}(\boldsymbol{\theta}_m)$. As for the pseudo-FvML procedure of the previous Section, we will need to substitute the location parameters $\boldsymbol{\vartheta}$ with a root-n consistent estimator $\hat{\boldsymbol{\vartheta}} = (\hat{\boldsymbol{\theta}}_1', \ldots, \hat{\boldsymbol{\theta}}_m')'$. To this end we will assume here the existence of an estimator $\hat{\boldsymbol{\vartheta}} = (\hat{\boldsymbol{\theta}}_1', \ldots, \hat{\boldsymbol{\theta}}_m')' \in (\mathcal{S}^{p-1})^m$ which (i) is still root-n consistent (such that $\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta} = O_{\mathrm{P}}(n^{-1/2})$) and (ii) is further *locally and asymptotically discrete*, meaning that it only takes a bounded number of distinct values in $\boldsymbol{\vartheta}$-centered balls of the form $\{\mathbf{t} \in \mathbb{R}^{mk} : n^{1/2}\|\left(\mathbf{r}^{(n)}\right)(\mathbf{t} - \boldsymbol{\vartheta})\| \leq c\}$. Estimators satisfying the above assumption are easy to construct. Indeed the consistency is not a problem and the discretization condition is a purely technical requirement (see Ley *et al.* (2013)) with little practical implications (in fixed-n practice, such discretizations are irrelevant as the radius can be taken arbitrarily large). We will therefore tacitly assume that $\hat{\boldsymbol{\vartheta}} \in (\mathcal{S}^{p-1})^m$ is locally and asymptotically discrete throughout this section.

The derivation of the asymptotic results in this section require the following technical reinforcement of Assumption A1.

Assumption (A4). Under $\mathrm{P}_{\boldsymbol{\vartheta},\boldsymbol{\mu},\underline{f}}$, the cdfs $\tilde{F}_1,\ldots,\tilde{F}_m$ (associated with $\underline{f}$) of the projections $Y_{ij}(\boldsymbol{\theta}_i)$ are such that $u \to \tilde{F}_i(u)$ is almost everywhere differentiable for $i = 1,\ldots,m$.

We have the following result.

Proposition 3.2. *Let Assumptions A1, A2, A3 and A4 hold. Then, under* $\mathrm{P}_{\boldsymbol{\vartheta},\boldsymbol{\mu},\underline{f}}$ *as* $n \to \infty$, *we have that for all* $i = 1,\ldots,m$,

(i) $n_i^{1/2}(\bar{R}_i(\boldsymbol{\vartheta}) - \bar{R}_i(\boldsymbol{\vartheta}, f_i))$ *is* $o_{L^2}(1)$ *where* $\bar{R}_i(\boldsymbol{\vartheta}, f_i) := n_i^{-1}\sum_{j=1}^{n_i}(\tilde{F}_i(Y_{ij}(\boldsymbol{\theta}_i)) - \frac{1}{2})$ *and;*
(ii) $n_i^{1/2}(\bar{R}_i(\hat{\boldsymbol{\vartheta}}) - \bar{R}_i(\boldsymbol{\vartheta}))$ *is* $o_{\mathrm{P}}(1)$.

It directly follows from the combination of points (i) and (ii) in Proposition 3.2 above and from the mutual independence of the $\mathbf{X}_{ij}$'s that $\mathbf{R}^{(n)}(\hat{\boldsymbol{\vartheta}})$ is asymptotically multinormal with mean zero and covariance matrix $\boldsymbol{\Gamma} := (1/12)\mathbf{I}_m$. As a direct consequence,

$$\begin{aligned} T_{\mathrm{KW}}^{(n)}(\hat{\boldsymbol{\vartheta}}) &:= \mathbf{R}^{(n)\prime}(\hat{\boldsymbol{\vartheta}})\left(\boldsymbol{\Gamma} - \boldsymbol{\Upsilon}^{(n)}\left((\boldsymbol{\Upsilon}^{(n)})'\boldsymbol{\Gamma}\boldsymbol{\Upsilon}^{(n)}\right)^{-1}(\boldsymbol{\Upsilon}^{(n)})'\right)\mathbf{R}^{(n)}(\hat{\boldsymbol{\vartheta}}) \\ &= \frac{12}{(n+1)^2}\sum_{i=1}^{m} n_i\left(\bar{R}_i(\hat{\boldsymbol{\vartheta}}) - \frac{n+1}{2}\right)^2 \end{aligned}$$

is aymptotically chi-square with $m-1$ degrees of freedom. The resulting test $\varphi_{\mathrm{KW}}^{(n)}$ rejects the null when $T_{\mathrm{KW}}^{(n)}(\hat{\boldsymbol{\vartheta}})$ exceeds the α-upper quantile of the chi-square distribution with $m-1$ degrees of freedom. This is stated in the following result.

Proposition 3.3. *Let Assumptions A1, A2, A3 and A4 hold. Then* $T_{\mathrm{KW}}^{(n)}(\hat{\boldsymbol{\vartheta}})$ *is asymptotically chi-square with* $m-1$ *degrees of freedom under* $\cup_{\boldsymbol{\vartheta}\in(\mathcal{S}^{p-1})^m}\cup_{\underline{f}\in(\mathcal{F})^m}\mathrm{P}_{\boldsymbol{\vartheta},(\mu,\ldots,\mu)';\underline{f}}$.

The proof readily comes from Proposition 3.2 and the multivariate CLT and is therefore omitted.

4. Monte-Carlo simulations

In this Section, the objective is to compare the levels and the powers of the Watamori and Jupp (2005) FvML score test, the pseudo-FvML test $\varphi_{\dagger}^{(n)}$ and the Kruskal-Wallis type test $\varphi_{\mathrm{KW}}^{(n)}$. Both $\varphi_{\dagger}^{(n)}$ and $\varphi_{\mathrm{KW}}^{(n)}$ are performed using spherical means to estimate location parameters in $T^{(n)}(\hat{\boldsymbol{\vartheta}})$ and $T^{(n)}_{\mathrm{KW}}(\hat{\boldsymbol{\vartheta}})$; that is $\hat{\boldsymbol{\vartheta}} := \left((\hat{\boldsymbol{\theta}}_1^{\mathrm{Mean}})', \ldots, (\hat{\boldsymbol{\theta}}_m^{\mathrm{Mean}})'\right)'$, where $\hat{\boldsymbol{\theta}}_i^{\mathrm{Mean}} := \bar{\mathbf{X}}_i/\|\bar{\mathbf{X}}_i\|$, $i = 1, \ldots, m$.

We generated $M = 10,000$ replications of four pairs of mutually independent samples (with respective sizes $n_1 = 200$ and $n_2 = 250$) of circular ($p = 2$) rotationally symmetric random vectors

$$\boldsymbol{\epsilon}_{\ell;1j}, \quad \ell = 1, 2, 3, 4, \quad j = 1, \ldots, n_1,$$

and

$$\boldsymbol{\epsilon}_{\ell;2j;c}, \quad \ell = 1, 2, 3, 4, \quad j = 1, \ldots, n_2, \quad c = 0, \ldots, 10$$

with FvML densities, wrapped-Cauchy densities, Kato and Jones (2010) densities and cardioid densities : the $\boldsymbol{\epsilon}_{1;1j}$'s have a FvML distribution with concentration $\kappa = 1$ location $\boldsymbol{\theta}_1 = (1, 0)'$ and the $\boldsymbol{\epsilon}_{1;2j;c}$'s have a FvML distribution with concentration $\kappa = 1 + c/15$ and with location $\boldsymbol{\theta}_2 = (-1, 0)'$; the $\boldsymbol{\epsilon}_{2;1j}$'s have a wrapped-Cauchy distribution with concentration $\kappa = .5$ and with location $\boldsymbol{\theta}_1 = (1, 0)'$ and the $\boldsymbol{\epsilon}_{2;2j;c}$'s have a wrapped-Cauchy distribution with concentration $\kappa = .5 + c/50$ and with location $\boldsymbol{\theta}_2 = (-1, 0)'$; the $\boldsymbol{\epsilon}_{3;1j}$'s have a Kato-Jones distribution with parameters $\mu_1 = \pi/3$, $\nu_1 = \pi/4$ and $r_1 = .5$ (using the notations of Kato and Jones (2010)) and concentration $\kappa = 1$ and the $\boldsymbol{\epsilon}_{3;2j;c}$'s have a Kato-Jones distribution with parameters $\mu_2 = \pi/3$, $\nu_2 = \pi/4$ and $r_2 = .5$ and concentration $\kappa = 1 + c/50$; the $\boldsymbol{\epsilon}_{4;1j}$'s have a cardioid distribution with location $\boldsymbol{\theta}_1 = (1, 0)'$ and concentration $\kappa = .4$ and the $\boldsymbol{\epsilon}_{4;2j;c}$'s have a cardioid distribution with location $\boldsymbol{\theta}_2 = (-1, 0)'$ and concentration $\kappa = .4 + c/50$.

The rotationally symmetric vectors $\boldsymbol{\epsilon}_{\ell;1j}$'s and $\boldsymbol{\epsilon}_{\ell;2j;0}$'s have all been generated with a common concentration κ while the concentrations of the $\boldsymbol{\epsilon}_{\ell;1j}$'s and the $\boldsymbol{\epsilon}_{\ell;2j;c}$'s for $c = 1, \ldots, 10$ are increasingly (with c) under the alternative to the null hypothesis of a common concentration. Power curves based on the asymptotic chi-square critical values at nominal level 5% are plotted in Figures 1, 2, 3 and 4 below.

The inspection of Figures 1, 2, 3 and 4 reveals nice results:

(i) The pseudo-FvML test and the Kruskal-Wallis rank-based test are valid under any couple of densities. They reach the 5% nominal level constraint under any considered pair of densities. The Watamori and Jupp (2005) is valid in the FvML case only;

(ii) The pseudo-FvML test and the Watamori and Jupp (2005) test are clearly equivalent in the FvML case;

(iii) The proposed Kruskal-Wallis type test outperforms the pseudo-FvML test under wrapped-Cauchy and Kato-Jones distributions.

5. Appendix: proofs

Proof of Lemma 3.1. It is well-known that any rotationally symmetric random vector $\mathbf{X}$ with location $\boldsymbol{\theta}$ is such that

$$\mathrm{E}[\mathbf{X}] = \mathrm{E}[\mathbf{X}'\boldsymbol{\theta}]\boldsymbol{\theta} = \mathrm{E}[Y(\boldsymbol{\theta})]\boldsymbol{\theta}, \tag{5.4}$$

see Watson (1983). Now, using the delta method applied to the mapping $\mathbf{x} \to \mathbf{x}/\|\mathbf{x}\|$, it is easy to show that for $i = 1, \ldots, m$, as $n \to \infty$,

$$n_i^{1/2}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i) = n_i^{1/2}\left(\frac{\hat{\boldsymbol{\theta}}_i}{\|\hat{\boldsymbol{\theta}}_i\|} - \frac{\boldsymbol{\theta}_i}{\|\boldsymbol{\theta}_i\|}\right) = (\mathbf{I}_p - \mathrm{proj}(\boldsymbol{\theta}_i))\, n^{1/2}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i) + o_{\mathrm{P}}(1), \tag{5.5}$$

where $\mathrm{proj}(\mathbf{A}) := \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}$ is the projection (onto $\mathrm{span}(\mathbf{A})$) operator. Using (5.4), (5.5) and the law of large numbers, we obtain that for $i = 1, \ldots, m$,

$$\begin{aligned}
n_i^{-1/2}\sum_{j=1}^{n_i} Y_{ij}(\hat{\boldsymbol{\theta}}_i) &= n_i^{-1/2}\sum_{j=1}^{n_i} \mathbf{X}'_{ij}\hat{\boldsymbol{\theta}}_i \\
&= n_i^{-1/2}\sum_{j=1}^{n_i} Y_{ij}(\boldsymbol{\theta}_i) + n_i^{-1/2}\sum_{j=1}^{n_i} \mathbf{X}'_{ij}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i) \\
&= n_i^{-1/2}\sum_{j=1}^{n_i} Y_{ij}(\boldsymbol{\theta}_i) + \left(n_i^{-1}\sum_{j=1}^{n_i} \mathbf{X}'_{ij}\right) n_i^{1/2}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i) \\
&= n_i^{-1/2}\sum_{j=1}^{n_i} Y_{ij}(\boldsymbol{\theta}_i) + \mathrm{E}[Y_{i1}(\boldsymbol{\theta}_i)]\boldsymbol{\theta}'_i\,(\mathbf{I}_p - \mathrm{proj}(\boldsymbol{\theta}_i))\, n^{1/2}(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i) + o_{\mathrm{P}}(1) \\
&= n_i^{-1/2}\sum_{j=1}^{n_i} Y_{ij}(\boldsymbol{\theta}_i) + o_{\mathrm{P}}(1)
\end{aligned}$$

as $n \to \infty$; which which directly yields the desired result. $\square$

Proof of Proposition 3.1. The spherical mean $\hat{\boldsymbol{\theta}}_i^{\mathrm{Mean}} = \bar{\mathbf{X}}_i/\|\bar{\mathbf{X}}_i\|$ is a root-n_i consistent estimator of $\boldsymbol{\theta}_i$ (i=1, ..., m). As a direct consequence, using the same arguments as proof

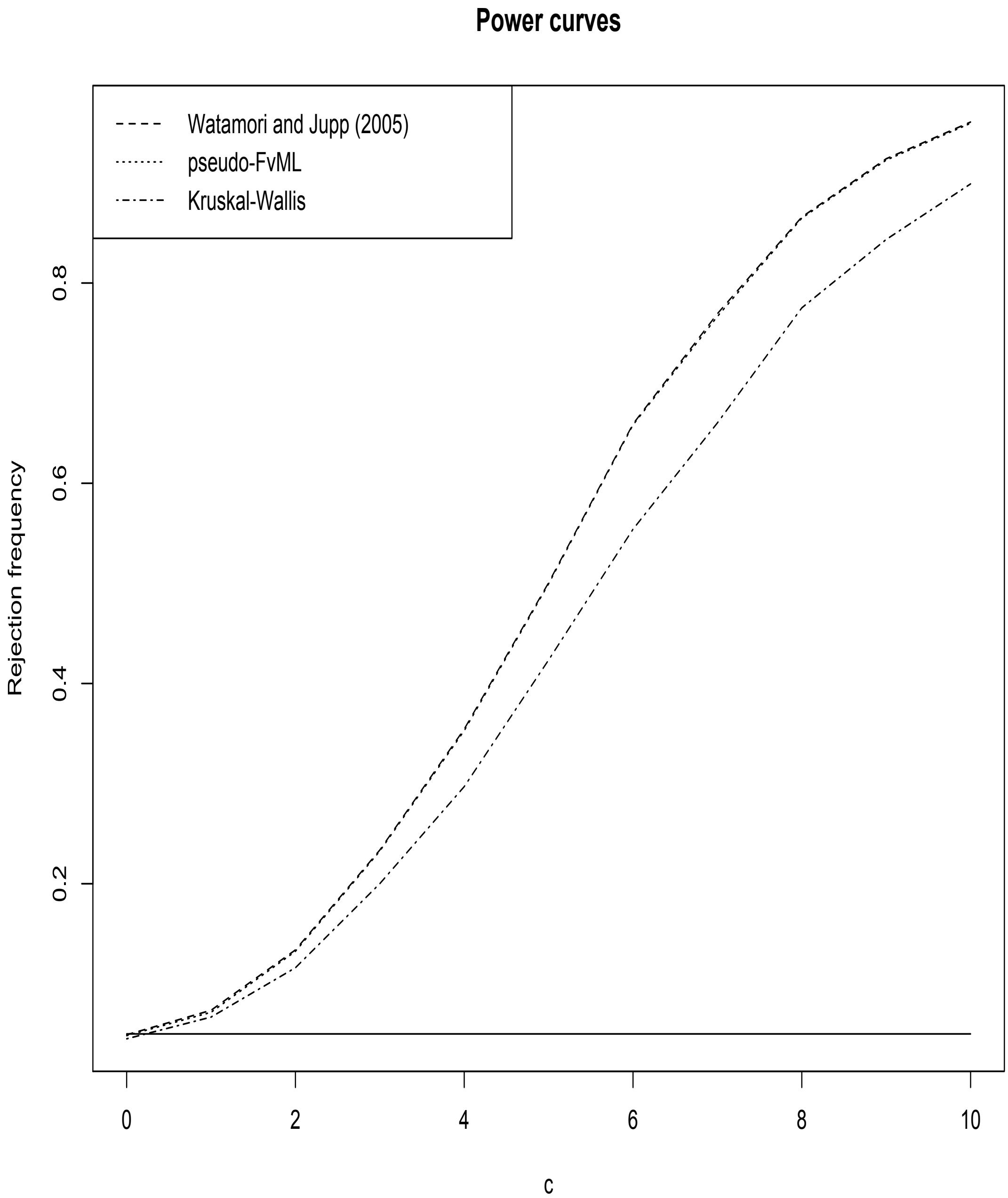

Figure 1. Power curves of the Watamori and Jupp (2005) test, the pseudo-FvML test and the Kruskal-Wallis test under FvML distributions with $\boldsymbol{\theta}_1 = (1,0)'$ and $\boldsymbol{\theta}_2 = (-1,0)'$. The concentration under the null is $\kappa = 1$. Sample sizes are $n_1 = 200$ and $n_2 = 250$. The number of replications is 10000.

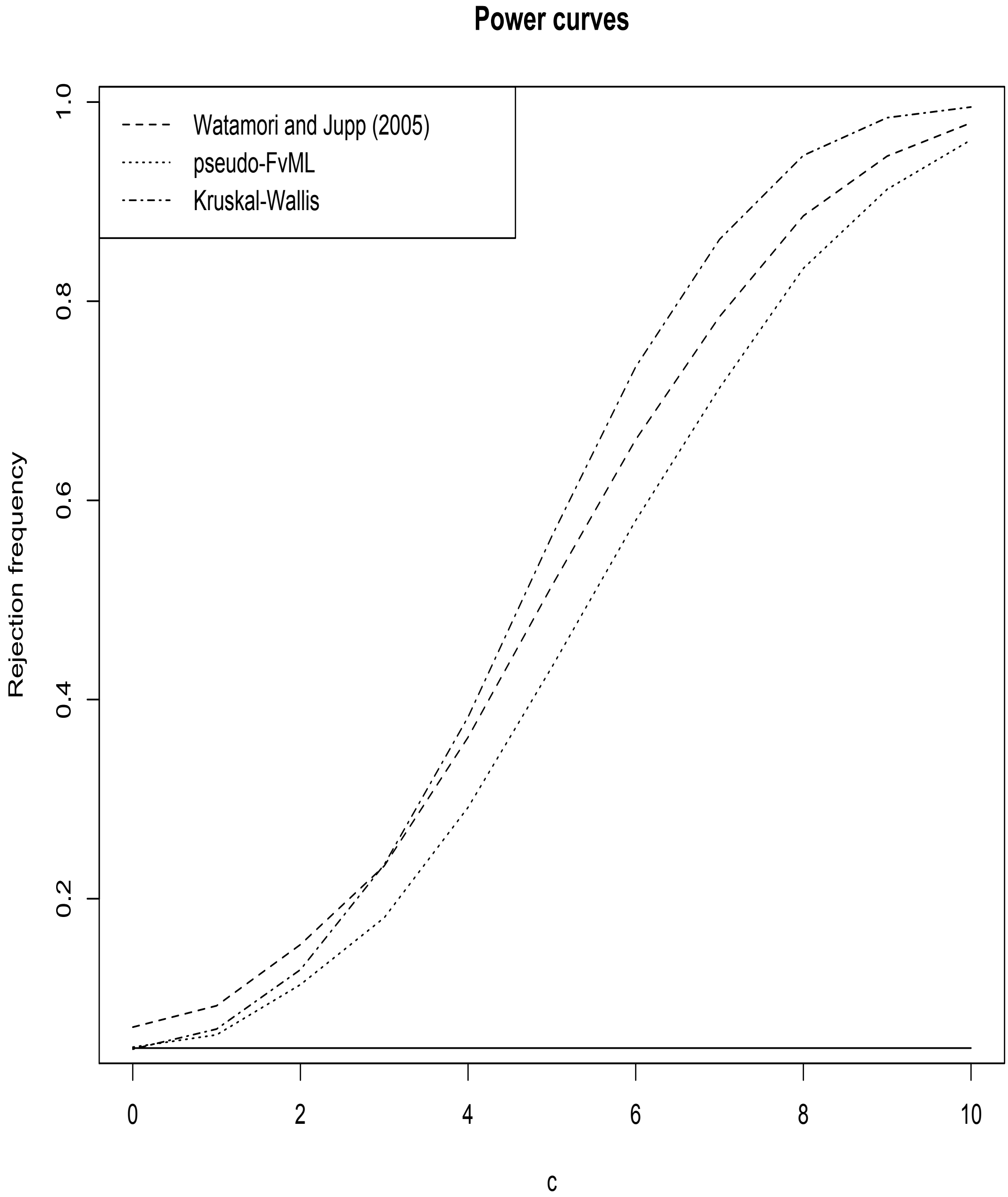

Figure 2. Power curves of the Watamori and Jupp (2005) test, the pseudo-FvML test and the Kruskal-Wallis test under wrapped-Cauchy distributions with $\boldsymbol{\theta}_1 = (1,0)'$ and $\boldsymbol{\theta}_2 = (-1,0)'$. The concentration under the null is $\kappa = .5$. Sample sizes are $n_1 = 200$ and $n_2 = 250$. The number of replications is 10000.

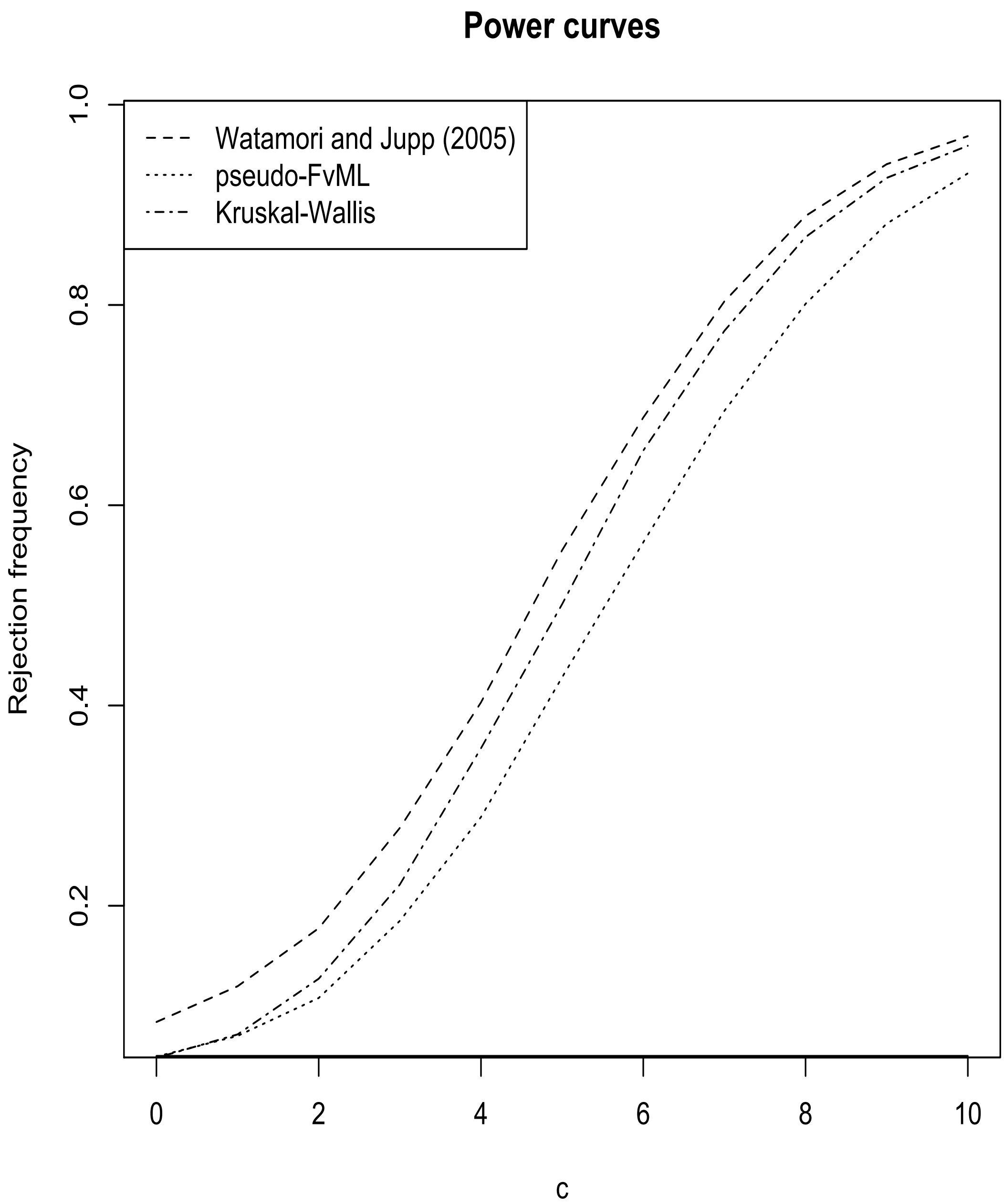

Figure 3. Power curves of the Watamori and Jupp (2005) test, the pseudo-FvML test and the Kruskal-Wallis test under Kato-Jones distributions with parameters $\mu_1 = \mu_2 = \pi/3$, $\nu_1 = \nu_2 = \pi/4$ and $r_1 = r_2 = .5$ (using the same notations as in the Kato-Jones paper). The common concentration under the null is $\kappa = 1$. Sample sizes are $n_1 = 200$ and $n_2 = 250$. The number of replications is 10000.

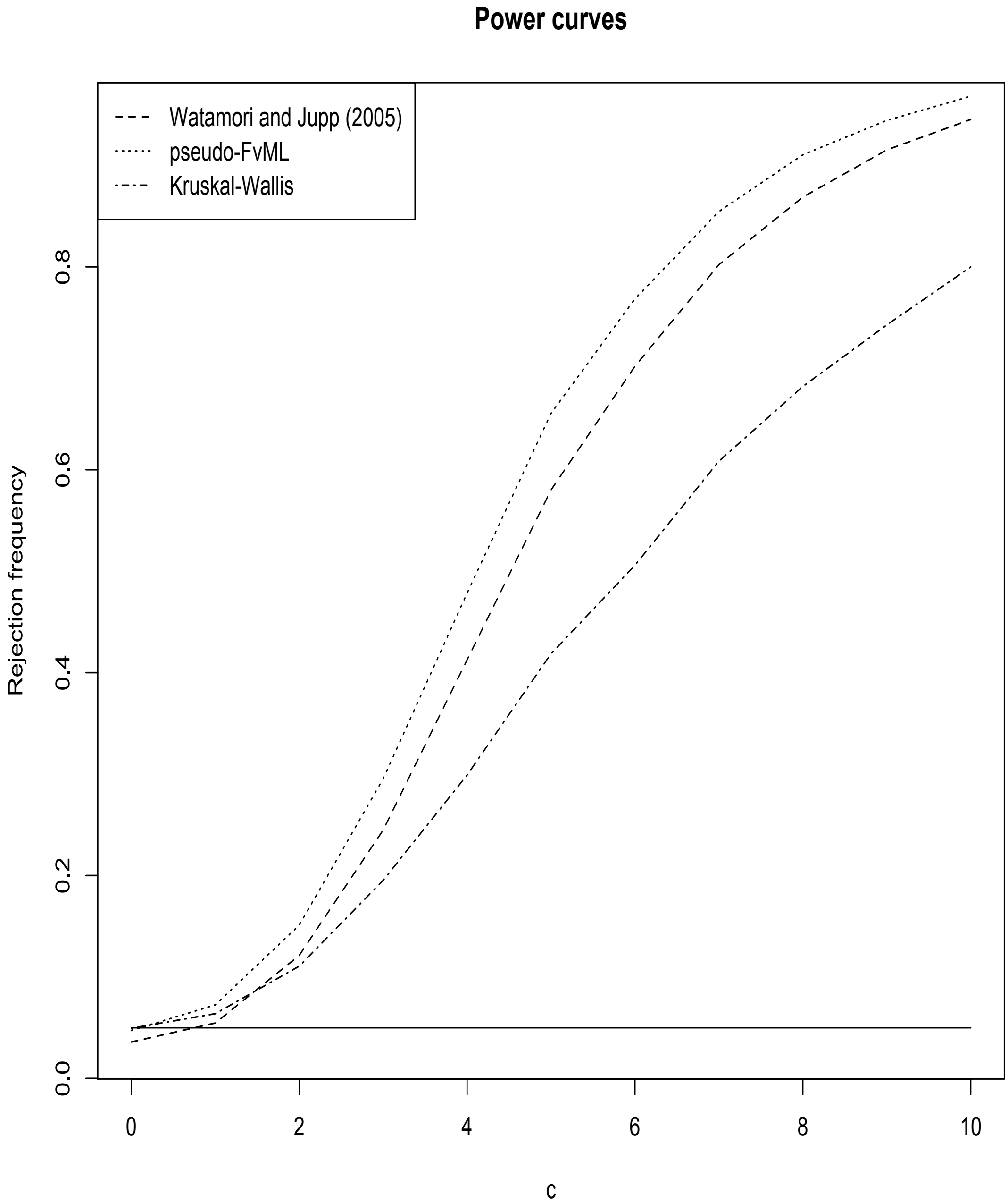

Figure 4. Power curves of the Watamori and Jupp (2005) test, the pseudo-FvML test and the Kruskal-Wallis test under cardioid distributions with $\boldsymbol{\theta}_1 = (1, 0)'$ and $\boldsymbol{\theta}_2 = (-1, 0)'$. The concentration under the null is $\kappa = .4$. Sample sizes are $n_1 = 200$ and $n_2 = 250$. The number of replications is 10000.

of Lemma 3.1 just above, we have that

$$\begin{aligned} n_i^{-1/2}\|\bar{\mathbf{X}}_i\| &= n_i^{-1/2}\bar{\mathbf{X}}_i'\hat{\boldsymbol{\theta}}_i^{\mathrm{Mean}} \\ &= n_i^{-1/2}\bar{\mathbf{X}}_i'\hat{\boldsymbol{\theta}}_i + o_{\mathrm{P}}(1) \\ &= n_i^{-1/2}\sum_{j=1}^{n_i} Y_{ij}(\boldsymbol{\theta}_i) + o_{\mathrm{P}}(1) \end{aligned}$$

as $n \to \infty$ under FvML densities (and under any other m-tuples of rotationally symmetric densities as well). The result then easily follows from the fact that under the null in the FvML case (with common concentration κ),

$$\mathrm{Var}(Y_{11}(\boldsymbol{\theta}_1)) = \ldots = \mathrm{Var}(Y_{m1}(\boldsymbol{\theta}_m)) = 1 - \frac{p-1}{\kappa}A_p(\kappa) - (A_p(\kappa))^2,$$

see Watson (1983). □

Proof of Proposition 3.2. Part (i) follows easily from Hájek's classical result for linear rank statistics (see Hájek and Šidák (1967)).We therefore only prove part (ii) in details. Following Lemma 4.4 in Kreiss (1987), the local discreteness of $\hat{\boldsymbol{\vartheta}}$ allows to replace non-random perturbations of the form $\boldsymbol{\vartheta}^{(n)} := \boldsymbol{\vartheta} + n^{-1/2}(\mathbf{r}^{(n)})^{-1/2}\mathbf{t}^{(n)}$ with $\mathbf{t}^{(n)} := (\mathbf{t}_1^{(n)}, \ldots, \mathbf{t}_m^{(n)})$ such that $\boldsymbol{\vartheta} + n^{-1/2}(\mathbf{r}^{(n)})^{-1/2}\mathbf{t}^{(n)}$ still belongs to $(\mathcal{S}^{p-1})^m$ by $\hat{\boldsymbol{\vartheta}}$ because of its $n^{1/2}(\mathbf{r}^{(n)})^{1/2}$-consistency. Note that if $\mathbf{t}_i^{(n)}$ $(i = 1, \ldots, m)$ is such that $\boldsymbol{\theta}_i^{(n)} := \boldsymbol{\theta}_i + n_i^{-1/2}\mathbf{t}_i^{(n)}$ still belongs to $\mathcal{S}^{p-1}$, we have that

$$\begin{aligned} 1 &= (\boldsymbol{\theta}_i^{(n)})'\boldsymbol{\theta}_i^{(n)} \\ &= (\boldsymbol{\theta}_i + n_i^{-1/2}\mathbf{t}_i^{(n)})'(\boldsymbol{\theta}_i + n_i^{-1/2}\mathbf{t}_i^{(n)}) \\ &= 1 + 2n_i^{-1/2}\boldsymbol{\theta}_i'\mathbf{t}_i^{(n)} + o(n_i^{-1/2}) \end{aligned} \tag{5.6}$$

It directly follows from (5.6) that $n_i^{-1/2}\boldsymbol{\theta}_i'\mathbf{t}_i^{(n)}$ is $o(n_i^{-1/2})$ or equivalently that $\boldsymbol{\theta}_i'\mathbf{t}_i^{(n)}$ is $o(1)$ as $n \to \infty$.

Now, from part (i) we know that $n_i^{1/2}(\bar{R}_i^{(n)}(\boldsymbol{\vartheta}) - \bar{R}_i^{(n)}(\boldsymbol{\vartheta}, f_i)) = o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$ as $n \to \infty$. Similarly, $n_i^{1/2}(\bar{R}_i^{(n)}(\boldsymbol{\vartheta}^{(n)}) - \bar{R}_i^{(n)}(\boldsymbol{\vartheta}^{(n)}, f_i)) = o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}^{(n)},\boldsymbol{\mu};\underline{f}}$ as $n \to \infty$. Hence, from contiguity, $n_i^{1/2}(\bar{R}_i^{(n)}(\boldsymbol{\vartheta}^{(n)}) - \bar{R}_i^{(n)}(\boldsymbol{\vartheta}^{(n)}, f_i))$ is also $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$ as $n \to \infty$. This entails that the claim holds if $n_i^{1/2}(\bar{R}_i^{(n)}(\boldsymbol{\vartheta}^{(n)}, f_i)) - \bar{R}_i^{(n)}(\boldsymbol{\vartheta}, f_i))$ is $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$ as $n \to \infty$. Under Assumption A4, the law of large numbers and (5.4) entail that

$$\begin{aligned}
n_i^{1/2}(\bar{R}_i^{(n)}(\boldsymbol{\vartheta}^{(n)}, f_i)) - \bar{R}_i^{(n)}(\boldsymbol{\vartheta}, f_i)) &= n_i^{-1/2} \sum_{j=1}^{n_i} \tilde{F}_i(\mathbf{X}'_{ij}\boldsymbol{\theta}_i^{(n)}) - \tilde{F}_i(\mathbf{X}'_{ij}\boldsymbol{\theta}_i) \\
&= n_i^{-1} \sum_{j=1}^{n_i} \tilde{f}_1(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)\mathbf{X}'_{ij}\mathbf{t}_i^{(n)} + o_{\mathrm{P}}(1) \\
&= \mathrm{E}\left[\tilde{f}(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)\mathbf{X}'_{ij}\right]\mathbf{t}_i^{(n)} + o_{\mathrm{P}}(1) \\
&= \mathrm{E}\left[\tilde{f}(\mathbf{X}'_{ij}\boldsymbol{\theta}_i)\mathbf{X}'_{ij}\boldsymbol{\theta}_i\right]\boldsymbol{\theta}'_i\mathbf{t}_i^{(n)} + o_{\mathrm{P}}(1)
\end{aligned}$$

which is $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\boldsymbol{\mu};\underline{f}}$ as $n \to \infty$ (see the comment just below (5.6)).

□

References

Fisher, R. A. (1982). Robust estimation of the concentration parameter of Fisher's distribution on the sphere. *Journal of the Royal Statistical Society Series C*, 31, 152–154.

Hájek, J. and Šidák, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.

Kato, S. and Jones, M.C. (2010). A family of distributions on the circle with links to, and applications arising from, Mbius transformation. *Journal of the American Statistical Association*, 102, 249–262.

Ko, D. (1992). Robust estimation of the concentration parameter of the von Mises-Fisher distribution. *Annals of Statistics*, 20, 917–928.

Ko, D. and Guttorp, P. (1988). Robustness of estimators for directional data. *Annals of Statistics*, 16, 609-618.

Kreiss, J.P. (1987). On adaptive estimation in stationary ARMA processes. *Annals of Statistics*, 15, 112–133.

Kruskal, W. H. and Wallis, W. A. (1952). Use of ranks in one-criterion variance analysis. *Journal of the American statistical Association*, 47(260), 583-621.

Larsen, P.V., Blsild, P., Srensen, M.K. (2002). Improved likelihood ratio tests on the von Mises-Fisher distribution. *Biometrika*, 89, 947–951.

Ley, C., Swan, Y., Thiam, B. and Verdebout, T. (2013). Optimal R-estimation of a spherical location. *Statistica Sinica*, 23, 305–332.

Ley, C. and Verdebout, T. (2014). Local powers of optimal one- and multi-sample tests for the concentration of Fisher-von Mises-Langevin distributions. *International Statistical Review*, to appear.

Mardia, K. V. and Jupp, P.E. (2000). *Directional Statistics.* Wiley, New York.

Paindaveine, D. and Verdebout, T. (2014). Optimal rank-based tests for the location parameter of a rotationally symmetric distribution on the hypersphere. *In M. Hallin, D. Mason, D. Pfeifer, and J. Steinebach Eds, Mathematical Statistics and Limit Theorems: Festschrift in Honor of Paul Deheuvels. Springer,* to appear.

Stephens, M. A. (1969). Multi-sample tests for the Fisher distribution for directions. *Biometrika*, 56, 169–181.

Watson, G.S. (1983). *Statistics on Spheres*, Wiley, New York.

Watson, G.S. (1986). Some estimation theory on the sphere. *Annals of the Institute of Statistical Mathematics*, 38, 263–275.

Watamori, Y. and Jupp, P. E. (2005). Improved likelihood ratio and score tests on concentration parameters of von Mises-Fisher distributions. *Statistics and Probability Letters*, 72, 93–102