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DISTRIBUTION-FREE TESTS AGAINST SERIAL DEPENDENCE : SIGNED OR UNSIGNED RANKS ?

by

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SIGNED OR UNSIGNED RANKS ?

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Optimal rank-based procedures were derived in Hallin et al (1985, 1987) and Hallin and Puri (1988a) for a variety of testing problems arising in time-series analysis, when the underlying innovation densities remain unspecified. Signed-rank analogues have been proposed in Hallin and Puri (1988b) for the case of symmetric unspecified innovation densities. The ARE of unsigned rank-based tests with respect to their signed counterparts however has been shown to be one (same reference). The objective of the present paper is an investigation of the finite-sample behavior of some asymptotically optimal signed-rank tests against first-order serial dependence, as well as a finite-sample comparative study of the actual performances, under non local alternatives, of signed and unsigned ranks. Exact and approximate critical values are provided for various scores (van der Waerden, Wilcoxon, Laplace, Spearman) and various series lengths. The comparative study indicates that, notwithstanding the ARE values of one, signed-rank tests may yield substantially better performances than unsigned ones.

Key words and phrases. Times series, Autoregressive-moving average models, Rank tests, Signed-rank tests.

AMS 1980 subject classifications : 62M10, 62G10.

1. INTRODUCTION

1.1. Rank and signed-rank tests for serial dependence problems

Though rank-based tests against serial dependence can be traced back to the very beginning of nonparametric statistical inference (see Dufour et al. 1982, Bhattacharyya 1984 or Hallin et al. 1985 for a bibliography), no systematic and coherent theoretical investigation of the subject had been undertaken until recently. Asymptotically, locally optimal testing procedures based on (unsigned) ranks have been developed for a variety of problems arising in time-series analysis, when the underlying innovation density remains unspecified in Hallin et al. (1985,1987) and Hallin and Puri(1988a). The small-sample performances of some of these rank-based procedures is discussed in Hallin and Mélard (1988). The asymptotic distribution of serial rank statistics under mixing assumptions has been obtained in Harel and Puri (1987, 1988) and Tran (1988), using two distinct approaches. The multivariate version of the problem is investigated in Hallin et al. (1988).

Whenever the innovation density underlying the process under study can be assumed symmetric, which is often the case in practical situations, the vector of (ordinary) ranks loses its *maximal* invariance property for the benefit of the vector of signed ranks. Signed-rank serial statistics accordingly have been considered in Hallin and Puri (1988b), where it is shown also that optimal (unsigned) rank procedures yield an asymptotic relative efficiency of one with respect to their signed counterparts.

This local asymptotic equivalence of optimal signed and unsigned procedures however does not imply that the advantage of signed ranks over unsigned ones is nill, neither asymptotically (the ARE of a Fisher-Yates test for symmetry with respect to the classical t-test under normality also is one; yet the corresponding deficiency is unbounded as the sample size increases), nor in small-sample situations. One of the objectives of the present paper is to provide numerical evidence for this small-sample superiority of signed ranks over unsigned ones.

1.2. *Outline of the paper*

The paper is divided into two main sections. We start (section 2.1) with the definition of the so-called *signed-rank autocorrelation coefficients*, whose role in rank-based inference can be compared with that of classical autocorrelation coefficients in the classical, normal-theory, context. The convergence to normality of the distributions of these coefficients under the null hypothesis of randomness is investigated, and tables of critical values needed for performing tests of randomness against first-order serial dependence are provided (section 2.2) for various series lengths and various score functions.

Section 3 concentrates on the small-sample behavior of first-order signed-rank autocorrelation coefficients under (non local) alternatives of first-order serial dependence. The performances of these coefficients are compared with those of their parametric and unsigned analogues. This comparison turns out to be quite favourable to signed-rank procedures, indicating that they often are substantially more powerful than their competitors.

Rank-based methods—and more particularly, in the case of symmetrically distributed innovation processes, those which are based on signed-ranks—thus appear, due to their distribution-freeness properties, much more reliable under the null hypothesis than their parametric counterparts. They are at least as powerful, as easy to apply, and considerably more robust. Their role in time-series analysis accordingly should be as important as in other areas such as linear models or experimental design.

2. SIGNED-RANK AUTOCORRELATION COEFFICIENTS

2.1. *Definitions and basic properties*

Let $\underline{X}^{(n)} = (X_1^{(n)}, \ldots, X_n^{(n)})$ be an observed series of length n. Let $H_+^{(n)}$ denote the null hypothesis of *symmetric randomness* under which the observations $X_t^{(n)}$ are independent and identically distributed, according to some unspecified, absolutely continuous distribution function F(x), whith density f(x) satisfying f(-x) = f(x). Denote by $R_{+,t}^{(n)}$ the rank of $|X_t^{(n)}|$ among the absolute values $|X_1^{(n)}|, \ldots, |X_n^{(n)}|$, and by $\text{sgn}(X_t^{(n)}) = I[X_t^{(n)} \geq 0] - I[X_t^{(n)} < 0]$ the sign of $X_t^{(n)}$. Under $H_+^{(n)}$, the vector $\underline{R}_+^{(n)} = (R_{+,1}^{(n)}, \ldots, R_{+,n}^{(n)})$ constitutes (with probability one a random permutation of {1, ..., n}, the variables $\mathrm{sgn}(X_1^{(n)})$, ..., $\mathrm{sgn}(X_n^{(n)})$ are independent and uniformly distributed on {-1;1} and the ranks and signs are mutually independent.

Considering a standardized absolutely continuous distribution function G, (i.e. such that $\int_{-\infty}^{\infty} x\, dG = 0$, $\int_{-\infty}^{\infty} x^2\, dG = 1$), with strongly unimodal symmetric density g of class C^2, Hallin and Puri (1988b) define the signed-rank autocorrelation coefficient of order p associated with g as

$$(2.1) \qquad r_{p;g}^{(n)+} = S_{p;g}^{(n)+}/\sigma_g^{(n)+}$$

where

$$(2.2) \qquad S_{p;g}^{(n)+} = (n-p)^{-1} \sum_{t=p+1}^{n} \mathrm{sgn}(X_t^{(n)} X_{t-p}^{(n)})\phi_g\left(G_+^{-1}\left(\frac{R_{+,t}^{(n)}}{n+1}\right)\right) G_+^{-1}\left(\frac{R_{+,t-p}^{(n)}}{n+1}\right)$$

with

$$G_+ = 2G-1,\ G_+^{-1}(u) = \inf\{x \mid G_+(x) > u\},\ u \in [0;1],\ \phi_g = -d \log g(x)/dx$$

and

$$(2.3) \qquad (\sigma_g^{(n)+})^2 = \left[n(n-1)\right]^{-1} \sum_{i_1 \neq i_2}\sum \phi_g\left(G_+^{-1}\left(\frac{i_1}{n+1}\right)\right) G_+^{-1}\left(\frac{i_2}{n+1}\right)$$

$$= \left[n(n-1)\right]^{-1} \left\{ \sum_{i=1}^{n} \phi_g\left(G_+^{-1}\left(\frac{i}{n+1}\right)\right) \sum_{i=1}^{n} G_+^{-1}\left(\frac{i}{n+1}\right) - \sum_{i=1}^{n} \phi_g\left(G_+^{-1}\left(\frac{i}{n+1}\right)\right) G_+^{-1}\left(\frac{i}{n+1}\right)\right\}$$

Clearly, $(\sigma_g^{(n)+})^2$ is the variance of $(n-p)\, S_{p;g}^{(n)+}$ (under $H_+^{(n)}$) — note that it does not depend on p: $(n-p)^{1/2}\, r_{p;g}^{(n)}$ is thus exactly standardized under $H_+^{(n)}$. Particularizing G, we obtain

- the van der Waerden signed rank autocorrelation coefficients (normal densities)

$$r_{vdw;p}^{(n)+} = \left[\sigma_{vdw}^{(n)+}\,(n-p)\right]^{-1} \sum_{t=p+1}^{n} \mathrm{sgn}(X_t^{(n)} X_{t-p}^{(n)})\ \Phi^{-1}\left[\frac{1}{2} + \frac{R_{+,t}^{(n)}}{2(n+1)}\right] \Phi^{-1}\left[\frac{1}{2} + \frac{R_{+,t-p}^{(n)}}{2(n+1)}\right]$$

where $\Phi(x)$ stands for the standard normal distribution function,

- the Wilcoxon signed-rank autocorrelation coefficients (logistic densities)

$$r_{W;p}^{(n)+} = \left[\sigma_W^{(n)+}\ (n-p)\right]^{-1} \sum_{t=p+1}^{n} \operatorname{sgn}(X_t^{(n)}\ X_{t-p}^{(n)})\ \frac{R_{+,t}^{(n)}}{n+1}\ \log\left[\frac{n+1+R_{+,t-p}^{(n)}}{n+1-R_{+,t-p}^{(n)}}\right],$$

and

- the Laplace signed-rank autocorrelation coefficients (double exponential densities)

$$r_{L;p}^{(n)+} = \left[\sigma_L^{(n)+}\ (n-p)\right]^{-1} \sum_{t=p+1}^{n} \operatorname{sgn}(X_t^{(n)}\ X_{t-p}^{(n)})\ \log\left[1-\frac{R_{+,t-p}^{(n)}}{n+1}\right].$$

Because of its intuitive appeal (and excellent overall performances), we also include in this study a signed version of the Spearman-Wald-Wolfowitz autocorrelation coefficient, the unsigned counterpart of which has been considered by Wald and Wolfowitz (1943) and, under a somewhat different form, by Bartels (1982) :

$$r_{S;p}^{(n)+} = \left[\sigma_S^{(n)+}\ (n-p)\right]^{-1} \sum_{t=p+1}^{n} \operatorname{sgn}(X_t^{(n)}\ X_{t-p}^{(n)})\ R_{+,t}^{(n)}\ R_{+,t-p}^{(n)}$$

It follows from Hallin and Puri (1988b ,Proposition 2.1) that the variance $(\sigma_g^{(n)+})^2$ converges, as $n \to \infty$, to the Fisher information $I(g) = \int_{-\infty}^{x} [\phi_g(x)]^2\ dG$ —provided that the latter is finite. Table 2.1 below provides the values of $\sigma_g^{(n)+}$ for various series lengths and various score functions; the convergence of $(\sigma_g^{(n)+})^2$ to $I(g)$ is monotone from below, not very fast—twice as fast, however, as that of the corresponding variance $(\sigma_g^{(n)})^2$ in the unsigned case (see Hallin and Mélard 1988, Table 1 for a comparison).

n	5	10	20	50	100	200	400	∞
$\sigma_{vdw}^{(n)+}$	0.6208	0.7460	0.8388	0.9171	0.9516	0.9724	0.9845	1
$\sigma_{W}^{(n)+}$	0.7068	0.8228	0.9048	0.9723	1.0022	1.0206	1.0316	1.0472
$\sigma_{L}^{(n)+}$	1.0095	1.1275	1.2199	1.3041	1.3448	1.3715	1.3884	1.4142
$\sigma_{S}^{(n)+}$	10.11	36.95	140.62	851.62	3370.00	13407.00	53480.00	∞

Table 2.1. Exact standard errors $\sigma_{vdw}^{(n)+}$, $\sigma_{W}^{(n)+}$, $\sigma_{L}^{(n)+}$ and $\sigma_{S}^{(n)+}$ for various series lengths.

2.2. *Distribution under* $H_+^{(n)}$

Hallin and Puri (1988b) have shown that asymptotically most powerful tests for the null hypothesis $H_+^{(n)}$ of randomness against local alternatives of positive first-order serial dependence can be based on signed-rank autocorrelation coefficients of order 1 [dropping useless subscripts, denote them by $r_f^{(n)+}$]. Such tests reject $H_+^{(n)}$ whenever $r_f^{(n)+} > (r_f^{(n)+})_{1-\alpha}$ [whenever $r_f^{(n)+} < (r_f^{(n)+})_\alpha$ if alternatives of negative serial dependence are considered]. The critical values $(r_f^{(n)+})_\alpha$ and $(r_f^{(n)+})_{1-\alpha}$ to be used are the exact α-and $(1-\alpha)$-quantiles of $r_f^{(n)+}$ under $H_+^{(n)}$; since the distribution of $r_f^{(n)+}$ clearly is symmetric with respect to zero, $(r_f^{(n)+})_\alpha = -(r_f^{(n)+})_{1-\alpha}$.

The exact distribution of $r_f^{(n)+}$ under $H_+^{(n)}$ can be obtained by enumerating the $2^n\, n!$ possible combinations of signs and permutations for $X_1^{(n)}, \ldots, X_n^{(n)}$. This method has been used here for n = 4 to 8. The complete distributions of $r_{vdw}^{(4)+}$, $r_W^{(4)+}$, $r_L^{(4)+}$ and $r_S^{(4)+}$ (at least, the positive part of it) has been displayed in Table 2.2. Note that the number of possible values, for n as small as 4, is pretty large (especially in the Wilcoxon case), and allows for, e.g. non trivial one-sided 5 % tests. For n larger than 4, this number is so high that displaying the whole distribution was not possible : Table 2.3 accordingly provides (1-α)-quantiles of $r_{vdW}^{(n)+}$, $r_{W}^{(n)+}$, $r_{L}^{(n)+}$ and $r_{S}^{(n)+}$, for usual probability levels α, n = 5 to 8. The latter were obtained as follows. The range of each statistic has been divided into intervals of the form $[10^{-3}(i \pm 0.5)]$, $i \in \mathbb{Z}$, and the frequency of each of such intervals among the $2^n\, n!$ possible values has been recorded ; the critical values in Table 2.3 then were taken as the middle of the interval containing the appropriate quantile.

The number $2^n\, n!$ of possible values however increases at such a rate that approximate methods had to be considered for $n > 8$. Three types of approximating methods have been adopted : (a) Monte-Carlo approximations, (b) beta approximations and (c) normal approximations. The normal approximation appears satisfactory for $n \geqslant 100$, the beta one for $n \geqslant 25$. The Monte-Carlo method was used when none of the preceding was found acceptable.

(a) Monte-Carlo approximations. Table 2.4 provides Monte-Carlo approximations for n = 9 to 25. As in Hallin and Mélard (1988), a random sample of N = 100 000 permutations of {1, ..., n} was generated through the NAG routine GØ5EHF. Pseudo-random signs were generated separately, using the NAG routine GØ5CAF to obtain a pseudo-random number uniformly distributed over [± 0.5], then considering its sign only. In order to take into account the symmetry of the null distribution of $r_f^{(n)+}$, the empirical quantile of order 1-2α of $|r_f^{(n)+}|$ has been taken as an approximation for $(r_f^{(n)+})_{1-\alpha}$. An approximate standard error (s.e.), hence approximate confidence limits, can be computed for the latter, using the asymptotic normal distribution of $(n-1)^{1/2}\, r_f^{(n)+}$ (Cramér 1946, p. 369-370) :

$$\text{s.e.} = \left[\frac{\alpha}{2N(n-1)}\left(\frac{1-2\alpha}{\varphi^2(r_\alpha)}\right)\right]^{1/2} .$$

(with $\varphi(r_\alpha) = (2\pi)^{-1/2}\, e^{-r_\alpha^2/2}$; r_α stands for the estimated value of the empirical quantile under study).

Finally, for the purpose of comparing the various approximation methods, the three approximate values (viz., the Monte-Carlo, beta and normal ones) are provided for $r_{vdw}^{(8)+}$, $r_{w}^{(8)+}$, $r_{L}^{(8)+}$ and $r_{S}^{(8)+}$ along with the exact ones in Table 2.3. The concordance between Monte-Carlo and exact values appears to be quite good, but the normal and beta approximations cannot be considered satisfactory. Similarly, the bottom of Table 2.4 allows for a comparison of Monte-Carlo and beta approximations for n = 25, indicating that the latter method can be used safely for $n \geq 25$.

(b) *Beta approximations*. The beta distributions considered here are over the range [-1,1], with mean zero and variance (n-1). For each value of n, the same approximation is thus provided for all autocorrelation coefficients. Beta approximations have been used for n = 26 to 100 (Table 2.5); still for comparison purposes, the normal approximation for n = 100 is also provided.

(c) *Normal approximations*. The normal approximation appears to be sufficient, for most purposes, for $n \geq 100$, and relies on the asymptotic standard normal distribution of $(n-1)^{1/2}\, r_f^{(n)+}$.

3. THE POWER OF SIGNED-RANK TESTS AGAINST FIRST-ORDER SERIAL DEPENDENCE

3.1. *Asymptotic Relative Efficiencies*

The classical parametric, normal—theory test statistic used for testing randomness against first-order serial dependence (see e.g. Anderson 1971, chapter 6) is the sample autocorrelation coefficient of order one

$$(3.1) \qquad r_1^{(n)} = \frac{\sum_{t=2}^{n} (X_t^{(n)} - \bar{X}^{(n)})(X_{t-1}^{(n)} - \bar{X}^{(n)})}{\sum_{t=1}^{n} (X_t^{(n)} - \bar{X}^{(n)})^2}$$

($\bar{X}^{(n)}$ here denotes the sample mean). The asymptotic relative efficiencies (AREs) of tests based on unsigned versions of rank autocorrelation coefficients with respect to each other and with respect to the parametric test based on (3.1)—or on the asymptotically equivalent Durbin-Watson or von Neumann ratio tests—were derived in Hallin et al. (1985). Since signed-rank autocorrelations are asymptotically equivalent, under $H_+^{(n)}$ and hence under contiguous alternatives, to their unsigned counterparts (Hallin and Puri 1988b, Proposition 3.1), ARE values remain the same for signed ranks as for unsigned ones. The mutual ARE's of signed van der Waerden, Wilcoxon, Laplace and Spearman tests, as well as their ARE's with respect to the parametric tests based on (3.1), are provided in Table 3.1 below, under various densities.

As expected, tests based on $r_{vdW}^{(n)+}$, $r_W^{(n)+}$ and $r_L^{(n)+}$ are asymptotically most efficient under normal, logistic and double-exponential alternatives, respectively—actually, they are (one-sided) uniformly most powerful and (two-sided) maximin against local alternatives of first-order (AR(1), MA(1), ARMA(1)) dependence.

3.2. *Signed or unsigned ranks ?*

If asymptotic relative efficiences were to be considered as the only relevant measure of performance, there would be little motivation for using signed-rank tests instead of unsigned ones. Asymptotic relative efficiencies however should not be overemphasized in the present context, neither from an asymptotic point of view (unsigned tests are likely to present unbounded deficiencies with respect to their signed counterparts), nor from a finite-sample, nonlocal-alternative point of view.

Our objective here is to provide numerical evidence of the advantage of signed-ranks over unsigned ones for finite series lengths.

Monte-Carlo experiments have been conducted by Hallin and Mélard (1988) to investigate the power of several parametric tests and the unsigned versions of the four rank autocorrelation tests defined in Section 2. Two sequences of pseudo-random deviates, distributed uniformly over [0,1] were generated and divided into 5000 series of length n=20 and n=100, respectively. These series were turned into normal, logistic and double-exponential white noise series $\{\varepsilon_t;\ t = 1, \ldots, n\}$ by means of the appropriate inverse distribution transformations. Finally, stationary AR(1) series were obtained from these pseudo-white noise series, and for each of the five autoregressive parameter values θ = 1/2, 1/4, 1/8, 1/16 and 1/32 [the method used does not require "warming up" the series—see Hallin and Mélard (1988) for details]. As in this latter paper, the following one-sided tests against positive first-order serial dependence were considered :

(a) the unsigned van der Waerden (vdW), Wilcoxon (W), Laplace (L) and Spearman (S) rank autocorrelation tests, based on the critical values provided in Hallin and Mélard (1988)'s Tables 4 (n=20) and 5 (n=100), respectively.

(b) the Ljung-Box (1978) (LB), Moran (1948) (M) and Dufour-Roy (1985) (DR) versions of the first-order sample autocorrelation test, based on the test statistics

$$(n^2 + 2n)^{1/2}\ r_1^{(n)}/(n-1)^{1/2} \qquad \text{(LB)},$$

$$(n-1)^{1/2}\ (n\ r_1^{(n)} + 1)/(n-2) \qquad \text{(M)},$$

and

$$(n-1)^{1/2}\ (n\ r_1^{(n)} + 1)/(n^2 - 2n)^{1/2} \qquad \text{(DR)},$$

respectively, whose distributions under $H_+^{(n)}$ are approximately standard normal [for a comparative discussion of these three parametric procedures, we refer to Dufour and Roy (1985) or Hallin and Mélard (1988)].

Finally, for the purpose of comparing signed and unsigned rank procedures, the same series have been subjected to

(c) the signed van der Waerden (vdW+), Wilcoxon (W+), Laplace (L+) and Spearman (S+) rank tests, based on Table 2.4 (n=20) and Table 2.5 (n=100), respectively.

The resulting rejection percentages, providing estimations of the powers of the corresponding tests, are listed in Table 3.2 (n=20) and Table 3.3(n=100). Confidence intervals at an approximate 95 % confidence level for the actual powers can be obtained by adding and subtracting $\delta(p_0)$ to the printed rejection percentage values p_0. If two approximately equal rejection percentages ($p_1 \approx p_2 \approx p_0$) are to be compared, equality of the corresponding probabilities can be rejected at a 5 % probability level if the observed differences are larger than $\Delta(p_0)$. Table 3.4 contains the values of δ and Δ for several rejection percentage values p_0. These limits are used in Table 3.2 and 3.3 to determine the most powerful tests in each column.

Even a rough inspection of Tables 3.2 and 3.3 reveals how spectacular the advantage of signed-ranks over unsigned ones : for n=20, the power increase can be as high as 0.20 (for the signed Spearman testing procedure with respect to the unsigned, traditional one, at $\theta = 1/2$ under double-exponential density) and 0.10 (for the signed van der Waerden test, with respect to the unsigned one, still at $\theta = 1/2$, under normal density). The profits of course are less impressive for n=100, and for small values of θ—in accordance with the local asymptotic equivalence of signed and unsigned procedures—but they still are quite substantial.

If compared to their parametric competitors, signed-rank tests confirm and strongly reinforce the conclusions of Hallin and Mélard (1988) about the unsigned ones. Whereas, under Gaussian assumptions, the unsigned van der Waerden test seems slightly less powerful than the Moran and Dufour-Roy procedures, the signed van der Waerden test, still under Gaussian conditions, appears significantly more powerful than Moran's . Under non-Gaussian densities , the advantage of the signed van der Waerden test over all parametric tests based on (3.1) is plain and clear. Remark that the signed Spearman test behaves almost as well—sometimes better than— the two other signed-rank procedures, though the performances of the unsigned Spearman procedure are somewhat poor. Whether signed or unsigned, the behaviour of the Laplace test remains somewhat unsatisfactory. As mentioned in Hallin and Mélard (1988), this probably indicates that the optimality of the Laplace test under double-exponential densities is of a purely local nature.

Whenever the underlying innovation densities can be assumed symmetric, and whatever the sample size, signed-rank tests are thus both substantially more powerful and more reliable than parametric tests based on sample auto-correlation coefficients—not to mention their robustness properties. They are also more powerful and easier to compute than their unsigned analogues—compare e.g. the expression for $\sigma_g^{(n)+}$ given in (2.3) with equation (2.11) in Hallin and Mélard (1988). For all these reasons, and though several theoretical points (such as alignment problems) require further investigation, the developement of appropriate rank-based methods probably constitutes one of the most promising topics for the future of time-series analysis.

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Van der Waerden		Wilcoxon				Laplace		Spearman	
r	$P(r^{(n)} \le r)$	r	$P(r^{(n)} \le r)$			r	$P(r^{(n)} \le r)$	r	$P(r^{(n)} \le r)$
0.009	0.5000	0.002	0.5000			0.063	0.5000	0.000	0.5104
0.010	0.4896	0.007	0.4948			0.162	0.4375	0.049	0.4896
0.055	0.4792	0.023	0.4896			0.217	0.4063	0.099	0.4479
0.056	0.4687	0.031	0.4844			0.302	0.3750	0.148	0.4167
0.078	0.4583	0.047	0.4740			0.316	0.3438	0.198	0.4063
0.112	0.4479	0.060	0.4583			0.415	0.3125	0.247	0.3854
0.145	0.4375	0.065	0.4531			0.456	0.2500	0.296	0.3646
0.159	0.4271	0.081	0.4479			0.569	0.2188	0.346	0.3333
0.181	0.4167	0.116	0.4427			0.654	0.1875	0.395	0.2813
0.189	0.4062	0.120	0.4375			0.695	0.1563	0.445	0.2604
0.192	0.3958	0.140	0.4323	0.528	0.2135	0.794	0.1250	0.494	0.2396
0.258	0.3854	0.159	0.4271	0.535	0.2083	0.808	0.0938	0.544	0.2083
0.293	0.3750	0.181	0.4219	0.548	0.2031	0.948	0.0625	0.593	0.1875
0.313	0.3646	0.183	0.4167	0.559	0.1979	1.047	0.0313	0.642	0.1791
0.321	0.3542	0.185	0.4115	0.563	0.1823			0.692	0.1458
0.322	0.3437	0.201	0.4062	0.586	0.1771			0.741	0.1250
0.325	0.3333	0.221	0.3958	0.593	0.1719			0.791	0.1146
0.343	0.3229	0.232	0.3854	0.596	0.1667			0.840	0.1042
0.360	0.3125	0.235	0.3802	0.606	0.1615			0.889	0.0729
0.361	0.3021	0.238	0.3750	0.623	0.1563			0.988	0.0417
0.368	0.2917	0.241	0.3698	0.651	0.1510			1.087	0.0313
0.425	0.2813	0.243	0.3646	0.685	0.1458			1.137	0.0104
0.437	0.2708	0.256	0.3594	0.698	0.1406				
0.447	0.2604	0.280	0.3542	0.722	0.1354				
0.455	0.2500	0.283	0.3490	0.733	0.1302				
0.458	0.2396	0.299	0.3385	0.742	0.1250				
0.515	0.2292	0.326	0.3333	0.743	0.1198				
0.522	0.2188	0.350	0.3281	0.746	0.1146				
0.523	0.2083	0.351	0.3229	0.767	0.1094				
0.558	0.1979	0.357	0.3177	0.780	0.1042				
0.568	0.1875	0.378	0.3125	0.838	0.0990				
0.579	0.1771	0.384	0.3073	0.842	0.0937				
0.591	0.1667	0.396	0.3021	0.858	0.0885				
0.673	0.1563	0.399	0.2969	0.862	0.0833				
0.694	0.1458	0.404	0.2865	0.866	0.0729				
0.702	0.1354	0.406	0.2812	0.879	0.0677				
0.737	0.1250	0.415	0.2760	0.885	0.0625				
0.770	0.1146	0.419	0.2708	0.896	0.0573				
0.805	0.1042	0.440	0.2604	0.913	0.0521				
0.827	0.0937	0.443	0.2552	0.916	0.0469				
0.835	0.0833	0.462	0.2500	0.924	0.0417				
0.892	0.0729	0.466	0.2448	1.029	0.0365				
0.939	0.0521	0.470	0.2396	1.036	0.0313				
0.959	0.0417	0.480	0.2344	1.087	0.0260				
1.071	0.0313	0.498	0.2292	1.107	0.0156				
1.093	0.0208	0.521	0.2240	1.124	0.0104				
1.140	0.0104	0.524	0.2188	1.145	0.0052				

Table 2.2 Exact distributions of $r_{vdW}^{(4)+}$, $r_{W}^{(4)+}$, $r_{L}^{(4)+}$ and $r_{S}^{(4)+}$ under $H_{+}^{(n)}$.

n	α	.100	.050	.025	.010	.005
5	VdW	.687	.803	.920	1.025	1.078
	W	.675	.812	.923	1.015	1.076
	L	.666	.815	.897	.988	.988
	S	.667	.816	.915	1.013	1.088
6	VdW	.600	.734	.830	.930	.989
	W	.602	.736	.827	.928	.982
	L	.604	.737	.819	.913	.948
	S	.608	.749	.834	.933	.989
7	VdW	.545	.671	.766	.864	.921
	W	.546	.672	.769	.863	.917
	L	.554	.682	.748	.832	.866
	S	.548	.681	.778	.875	.920
8	VdW	.502	.623	.716	.810	.868
	W	.503	.623	.716	.810	.865
	L	.508	.621	.710	.791	.831
	S	.501	.625	.720	.820	.873
Monte-Carlo approximation (n=8)						
	VdW	.502	.623	.716	.811	.868
	W	.503	.623	.717	.810	.865
	L	.508	.620	.708	.792	.833
	S	.501	.625	.720	.817	.876
standard error		.0014	.0017	.0022	.0032	.0040
beta approximation (n=8)		.507	.621	.707	.789	.834
normal approximation (n=8)		.484	.622	.741	.879	.974

Table 2.3. Exact upper critical values for one-sided tests of $H^{(n)}$ based on the signed-rank autocorrelation coefficients $r^{(n)+}_{vdW}$, $r^{(n)+}_{W}$, $r^{(n)+}_{L}$ and $r^{(n)+}_{S}$ respectively, for n = 5 , 6 , 7 , 8 , with Monte-Carlo , beta approximate values for n = 8 . The numbers in parentheses are approximate standard errors associated with the Monte-Carlo critical values (for n = 8).

n	α	.100	.050	.025	.010	.005
9	VdW	.469	.583	.673	.768	.828
	W	.469	.582	.675	.768	.828
	L	.473	.582	.672	.752	.789
	S	.467	.587	.682	.777	.835
10	VdW	.442	.552	.637	.728	.783
	W	.442	.551	.637	.726	.781
	L	.443	.551	.631	.712	.757
	S	.421	.553	.642	.736	.793
11	VdW	.416	.522	.606	.697	.751
	W	.416	.522	.607	.696	751
	L	.419	.522	.605	.682	.729
	S	.415	.522	.610	.703	.758
12	VdW	.396	.497	.581	.666	.718
	W	.396	.497	.580	.666	.717
	L	.399	.499	.575	.655	.700
	S	.394	.498	.583	.671	.727
13	VdW	.380	.477	.555	.638	.691
	W	.380	.477	.556	.637	.691
	L	.382	.478	.553	.632	.676
	S	.380	.479	.558	.643	.700
14	VdW	.363	.455	.534	.620	.673
	W	.363	.456	.533	.618	.671
	L	.365	.457	.530	.607	.652
	S	.361	.455	.535	.625	.679
15	VdW	.349	.442	.516	.597	.648
	W	.349	.443	.517	.595	.648
	L	.352	.441	.514	.589	.638
	S	.349	.442	.518	.600	.655
16	VdW	.339	.428	.499	.579	.628
	W	.339	.428	.500	.579	.628
	L	.340	.428	.498	.573	.621
	S	.338	.428	.502	.583	.632
17	VdW	.326	.412	.483	.560	.609
	W	.327	.411	.482	.559	.609
	L	.327	.412	.479	.554	.599
	S	.325	.412	.485	.563	.614
18	VdW	.318	.402	.472	.547	.596
	W	.318	.402	.471	.546	.594
	L	.318	.401	.468	.540	.584
	S	.317	.402	.472	.549	.599
19	VdW	.307	.389	.457	.534	.582
	W	.306	.389	.457	.534	.581
	L	.309	.388	.455	.525	.569
	S	.306	.388	.459	.537	.586
20	VdW	.299	.379	.445	.518	.565
	W	.299	.378	.444	.518	.564
	L	.300	.379	.444	.512	.555
	S	.298	.378	.446	.521	.567
21	VdW	.292	.370	.436	.505	.553
	W	.293	.370	.435	.507	.553
	L	.293	.370	.432	.502	.548
	S	.292	.370	.436	.513	.560
22	VdW	.284	.361	.424	.495	.538
	W	.284	.361	.424	.494	.538
	L	.286	.361	.423	.489	.532
	S	.283	.361	.425	.497	.541
23	VdW	.276	.351	.413	.480	.525
	W	.276	.351	.412	.481	.525
	L	.277	.350	.412	.478	.520
	S	.276	.350	.413	.484	.530
24	VdW	.271	.343	.403	.471	.514
	W	.272	.343	.403	.470	.513
	L	.272	.344	.401	.466	.507
	S	.270	.343	.404	.472	.516
25	VdW	.266	.339	.397	.466	.512
	W	.268	.338	.398	.464	.510
	L	.267	.339	.397	.460	.501
	S	.265	.338	.400	.467	.513

standard error (n= 25)

	.100	.050	.025	.010	.005
standard error (n=25)	.0007	.0009	.0012	.0017	.0022
beta approximation (n=25)	.265	.337	.396	.462	.505
normal approximation (n=25)	.262	.336	.400	.475	.526

Table 2.4. Approximate Monte-Carlo upper critical values for one-sided tests of $H_{+}^{(n)}$ based on the signed rank autocorrelation coefficients $r_{vdW}^{(n)+}$, $r_{W}^{(n)+}$, $r_{L}^{(n)+}$ and $r_{S}^{(n)+}$, respectively, for n=9 (1) 25, with beta and normal approximate values for n=25. The numbers in parentheses are approximate standard errors associated with the Monte-Carlo critical values (for n=25).

n	.100	.050	.025	.010	.005
26	.260	.330	.388	.453	.496
27	.255	.323	.381	.445	.487
28	.250	.317	.374	.437	.479
29	.245	.311	.367	.430	.471
30	.241	.306	.361	.423	.463
32	.233	.296	.349	.409	.449
34	.225	.287	.339	.397	.436
36	.219	.279	.329	.386	.424
38	.213	.271	.320	.376	.413
40	.207	.264	.312	.367	.403
42	.202	.257	.304	.358	.393
44	.197	.251	.297	.350	.384
46	.192	.246	.291	.342	.376
48	.188	.240	.285	.335	.368
50	.184	.235	.279	.328	.361
55	.175	.224	.266	.313	.345
60	.168	.214	.254	.300	.330
65	.161	.206	.244	.288	.317
70	.155	.198	.235	.278	.306
75	.150	.191	.227	.268	.296
80	.145	.185	.220	.260	.286
85	.140	.180	.213	.252	.278
90	.136	.174	.207	.245	.270
95	.133	.170	.202	.238	.263
100	.129	.165	.197	.232	.256
standard error	.0004	.0005	.0006	.0008	.0011
normal approximation	.129	.165	.197	.234	.259

Table 2.5. Beta upper critical values for one-sided tests of $H_+^{(n)}$ based on signed-rank autocorrelation coefficients of order one, for n = 26 to 100, with normal critical values for n = 100.

	Sample auto-correlation	van der Waerden	Wilcoxon	Laplace	Spearman	
Sample autocorrelation	1.000	1.000	1.055	1.631	1.096	normal
	1.000	0.955	0.912	1.230	1.000	logistic
	1.000	0.816	0.674	0.500	0.790	double-exponential
van der Waerden	1.000	1.000	1.055	1.631	1.096	normal
	1.047	1.000	0.954	1.288	1.047	logistic
	1.226	1.000	0.827	0.613	0.968	double-exponential
Wilcoxon	0.948	0.948	1.000	1.556	1.039	normal
	1.098	1.048	1.000	1.349	1.097	logistic
	1.483	1.209	1.000	0.742	1.171	double-exponential
Laplace	0.613	0.613	0.643	1.000	0.672	normal
	0.813	0.776	0.741	1.000	0.813	logistic
	2.000	1.631	1.348	1.000	1.580	double-exponential
Spearman	0.912	0.912	0.962	1.488	1.000	normal
	1.000	0.955	0.912	1.230	1.000	logistic
	1.266	1.033	0.854	0.633	1.000	double-exponential

Table 3.1. Asymptotique Relative Efficiencies of (signed or unsigned) rank autocorrelation tests with respect to each other and with respect to the first-order sample autocorrelation tests , under normal ,logistic and double-exponential densities .

	Normal density					Logistic density					Double exponential density				
n=20	θ = 1/2	θ = 1/4	θ = 1/8	θ =1/16	θ =1/32	θ = 1/2	θ = 1/4	θ = 1/8	θ =1/16	θ =1/32	θ = 1/2	θ = 1/4	θ = 1/8	θ =1/16	θ =1/32
VdW	59.16	24.04	11.92*	7.84*	6.38*	60.50	24.66	12.24*	7.92*	6.40*	62.98	26.54	12.86	8.24*	6.44*
VdW+	68.12**	26.68**	12.46*	8.08*	6.58**	69.14*	28.00*	13.00*	8.04*	6.66**	71.74*	31.00*	14.16*	8.42*	6.80*
W	58.26	23.46	11.66*	8.02*	6.24*	60.62	24.76	12.30*	8.16**	6.30*	65.24	28.24	13.42	8.68*	6.60*
W+	66.88*	25.96*	12.60*	7.76*	6.36*	69.32**	28.26**	13.24**	7.96*	6.44*	73.48**	32.44**	15.18**	8.60*	6.76*
L	44.12	18.90	10.10	6.60	5.74*	48.12	20.90	10.82	6.76	5.84*	54.80	25.72	13.38	8.18*	6.14*
L+	52.88	20.28	10.32	7.20	6.02*	56.72	22.80	11.18	7.52*	6.18*	63.82	29.70	14.60*	8.82**	6.84**
S	47.52	19.22	9.92	6.68	5.40	49.20	20.30	10.18	6.74	5.48	51.58	21.96	12.04	8.30*	6.42*
S+	66.26	25.24*	12.68**	7.86*	6.02*	68.16*	26.64*	12.96*	7.94*	6.10*	71.50	30.40	14.24*	8.24*	6.26*
L B	53.46	17.82	7.76	4.74	3.70	53.72	17.66	7.42	4.68	3.52	54.16	17.18	6.80	4.30	3.36
M O	62.82	25.46*	12.24*	8.22**	6.40*	62.84	25.42	11.94	7.80*	6.12*	63.88	24.96	11.52	7.06	5.78
D R	59.56	22.54	10.54	6.90	5.24	60.24	22.20	10.22	6.38	5.08	60.74	22.08	9.54	6.14	4.72

(a)

Table 3.2. Rejection percentage for series length n=20 of the null hypothesis of randomness H_+ under first-order autoregressive dependence $X_t - \theta X_{t-1} = \varepsilon_t$, with ε_t a white noise series with normal, logistic or double-exponential density, using one-sided unsigned and signed van der Waerden (vdW and vdW_+), Wilcoxon (W and W_+), Laplace (L and L_+) and Spearman(S and S_+) rank-tests and the Ljung-Box (LB), Moran (M) and Dufour-Roy (DR) versions of the sample first-order autocorrelation tests respectively, at a % probability level. Asterisk denote the winner(s) in each column, allowing multiple comparison intervals at a 95 % confidence level. Absolute winner(s) are identified with a double asterisk.

	Normal density				Logistic density				Double exponential densities			
n=100	θ = 1/4	θ = 1/8	θ =1/16	θ =1/32	θ = 1/4	θ = 1/8	θ =1/16	θ =1/32	θ = 1/4	θ = 1/8	θ =1/16	θ =1/32
VdW	78.60	33.72*	14.76*	9.00*	79.84	34.66	15.10*	9.10*	84.42	38.70	16.38	9.60
VdW+	80.20**	34.38*	15.50**	9.00*	81.86*	35.60*	15.70*	9.20*	85.84	39.52	17.08	9.70
W	77.16	32.72	14.42*	8.84*	81.32*	35.64*	15.26*	9.18*	88.36	42.58*	17.94*	9.96
W+	78.60	33.54*	15.06*	8.86*	83.06**	36.66**	16.16**	9.24*	89.90**	43.50**	18.76*	10.20*
L	60.68	24.62	11.24	7.32	69.08	28.38	12.98	7.76	84.72	41.28	18.66*	9.84
L+	62.24	24.04	12.04	8.04*	71.32	29.18	13.60	8.54*	86.04	43.50**	19.16**	11.18**
S	73.64	31.06	13.94	8.54*	76.78	32.60	14.50	8.82*	82.66	37.64	16.88	10.08*
S+	76.90	32.80	14.78*	9.12**	79.76	34.76*	15.38*	9.32**	85.66	40.10	17.26	10.24*
L B	77.00	30.30	12.72	7.58	77.22	30.58	12.56	7.34	77.60	30.46	12.28	7.04
M O	79.70*	34.90**	15.18*	8.90*	80.14	34.62	14.78*	8.94*	80.56	33.92	14.68	8.60
D R	79.24*	34.18*	14.64*	8.70*	79.68	33.92	14.54	8.62*	80.12	33.28	14.22	8.34

Table 3.3. Same rejection percentages as in Table 3.2. for series length n=100 , and with $\theta=(1/2)^i$, i=2 , ... , 5 .

Table 3.4 - Values of δ and Δ used for building approximate 95 % confidence intervals for the actual power and testing equality of actual rejection probabilities

Rejection percentage p_o	95 or 5	90 or 10	80 or 20	70 or 30	60 or 40	50
$\delta(p_o)$	.60	.83	1.10	1.23	1.36	1.38
$\Delta(p_o)$	.85	1.17	1.56	1.74	1.92	1.95