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A NOTE ON THE SIMULTANEOUS STABILITY OF TÂTONNEMENT PROCESSES FOR COMPUTING EQUILIBRIA*

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1. INTRODUCTION

This paper is motivated by recent experience which suggests that tâtonnement processes may prove quite efficient in computing equilibria.[2] These processes are in fact today the only effective method of solving equilibrium models of realistic size. Their weakness relative to Scarf type algorithms is that convergence is not guaranteed.

General equilibrium problems can be solved by several tâtonnement processes. Price tâtonnement is of course well known.[3] Mantel [1971] has proposed a welfare weight adjustment process, and compared its stability properties with those of price tâtonnement. Dixon [1975a] introduced a utility adjustment process, and applied it to a simple model. Ginsburgh and Waelbroeck [1975] suggested a fourth process and applied this one and Dixon's to an equilibrium model of the world economy.

The four processes are generated from one optimizing model by means of minor changes, and it is easy to switch to another process when one approach has failed to converge. The model builder has, so to speak, several strings to fit his bow: but is this worthwhile? The answer would be negative if alternative processes were simultaneously stable, so that failure of one implied failure of the others.

It turns out that tâtonnement processes are not simultaneously stable locally — and a fortiori globally. It appears worthwhile therefore to make the minor investment in additional programming which permits switching among the four processes instead of being restricted to only one. Our analysis also generalizes Mantel's [1971] and Dixon's [1975] in considering a wider set of processes and in giving slightly more general results on simultaneous stability.

The paper is organized as follows. In Section 2, we build up the various computational procedures; in Section 3 we show how the processes can be locally approximated, and in Section 4, we turn to the local stability properties.

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[1] We are grateful to M. Todd and to an anonymous referee for very careful reading and for pointing out some serious errors in the previous version.

[2] See, e.g., P. Dixon [1975a, 1975b], I. Adelman and S. Robinson [1978], V. Ginsburgh and J. Waelbroeck [1975, 1976].

[3] A good overview can be found in K. J. Arrow [1974].

2. DEFINITION OF THE TÂTONNEMENT PROCESSES

Let x_i $(i=1, 2,..., r)$ denote the n-dimensional vector of net trades of consumer i and assume his preferences to be represented by the real valued, monotone and strictly concave twice continuously differentiable utility function $U_i(x_i)$; p is an n-vector prices. We also assume that the set of net trades which are at least as good as the no trade position have a lower bound, not attained at the no trade position. Then, we define a competitive equilibrium as

DEFINITION. A Competitive Equilibrium (C. E.) is a bundle of net trades $x_1^*, x_2^*,..., x_r^*$ and a vector of prices p^* such that
(a) x_i^* maximizes $U_i(x_i)$ subject to $p^*x_i \leq 0,\ x_i \in \mathbf{R}^n$, $\quad i = 1, 2,..., r$;
(b) $\sum x_i^* \leq 0$.

THEOREM 1. (Arrow-Debreu and Negishi) *There exist vectors $x_1^*,..., x_r^*$ and a vector $p^* \geq 0$ such that the conditions* (a)–(b) *of the definition are satisfied. There also exist scalars $\alpha_1^* > 0,..., \alpha_r^* > 0$, vectors $x_1^*,..., x_r^*$ and a vector $p^* \geq 0$ such that the solution of the problem* max $\sum \alpha_i^* U_i(x_i)$ *s. t.* $\sum x_i \leq 0$ *satisfies the definition of a C. E.*

Let $G(x)=[U_1(x_1)\ U_2(x_2) \cdots U_r(x_r)]$ be a vector valued function whose elements are the utility functions of the r consumers. Consider the following mathematical program:

$$\text{vecmax } G(x) \tag{2.1}$$

subject to both or either of the following constraints (2.2)

$$\sum x_i \leq 0;\ px_i \leqq 0 \qquad i = 1, 2,..., r; \tag{2.2}$$

$$x_i \in \mathbf{R}^n \qquad i = 1, 2,..., r. \tag{2.3}$$

We show that the four tâtonnement processes, the properties of which will be examined, can formally be derived from the mathematical program (2.1)–(2.3).

Indeed, the classical price tâtonnement process (p-tâtonnement) can be obtained from max $\sum_i U_i(x_i)$ s. t. $px_i \leq 0$ $(i=1, 2,..., r)$. The first order conditions for a maximum are

$$\frac{\partial U_i(x_i)}{\partial x_i} - \lambda_i p = 0 \qquad i = 1, 2,..., r \tag{2.4}$$

$$px_i \leq 0;\ \lambda_i px_i = 0;\ \lambda_i \geq 0 \qquad i = 1, 2,..., r \tag{2.5.1)–(2.5.3}$$

λ_i is a Lagrange multiplier associated to the constraint $px_i \leq 0$. Assuming $\lambda_i > 0$ for all i, (2.4) and (2.5.1) can be solved, leading to individual excess demand functions $x_i(p)$; aggregation of these functions gives the market excess demand functions $\sum_i x_i(p) = f(p)$ and the Walrasian tâtonnement is defined by the system of differential equations $\dot{p} = f(p)$ where $\dot{p} = dp/dt$. It is easy to check that a solution p^* of the system of differential equations with $f(p^*)=0$ satisfies condition (b) of the C. E. Condition (a) is satisfied by the individual utility maximization problems.

A second "adjustment of accounting to shadow prices" (π-tâtonnement) is obtained by consideration of the program $\max \sum_i U_i(x_i)$ s.t. $\sum x_i \leq 0$ and $\pi \cdot x_i \leqq 0$ $(i=1, 2,..., r)$ where π is a vector of accounting prices used in evaluating the budget constraints.

The first order conditions for a maximum are

$$\frac{\partial U_i(x_i)}{\partial x_i} - p - \lambda_i \pi = 0 \qquad i = 1, 2,..., r \tag{2.6}$$

$$\sum x_i \leq 0; \; p\sum x_i = 0; \; p \geq 0 \tag{2.7.1)-(2.7.3}$$

$$\pi x_i \leq 0; \; \lambda_i \pi x_i = 0; \; \lambda_i \geq 0 \qquad i = 1, 2,..., r. \tag{2.8.1)-(2.8.3}$$

Assuming $p>0$, $\lambda_i>0$ $(i=1, 2,..., r)$ the system (2.6)–(2.7.1)–(2.8.1) can be solved for the shadow prices $p(\pi)$ and the π-tâtonnement will be defined by the system of differential equations $\dot{\pi}=p(\pi)-\pi$. To show that for a vector π^* such that $p(\pi^*)-\pi^*=0$ we obtain a C. E., we need

THEOREM 2. (Ginsburgh-Waelbroeck [1975]) *The vectors p^*, π^*, x_i^* $(i=1, 2,..., r)$ such that*

(i) *$x_1^* \cdots x_r^*$ are a solution of* $\max \sum_i U_i(x_i)$ s.t. $\sum x_i \leq 0$ *and* $\pi^* x_i \leq 0$ $(i=1, 2,..., r)$

(ii) *$p^*=\pi^*$ (p^* is a vector of multipliers associated to the constraints $\sum x_i \leq 0$) are a solution of a C. E.*

A third "welfare weight" process (α-tâtonnement) can be obtained by considering Negishi's mathematical program $\max \sum \alpha_i U_i(x_i)$ s.t. $\sum x_i \leq 0$. This problem leads to the following first order conditions

$$\alpha_i \frac{\partial U_i(x_i)}{\partial x_i} - p = 0 \qquad i = 1, 2,..., r \tag{2.9}$$

$$\sum x_i \leq 0; \; p\sum x_i = 0; \; p \geq 0. \tag{2.10.1)-(2.10.3}$$

If we assume that $\alpha_i>0$ all i and $p>0$, we can solve the system (2.9)–(2.10.1), find r net trade vectors $x_i(\alpha)$ $(i=1, 2,..., r)$ and a vector $p(\alpha)$, and compute the "savings" of consumer i as $s_i(\alpha)=-p(\alpha)x_i(\alpha)$; the α-tâtonnement can then very naturally be defined by the differential equations $\dot{\alpha}=s(\alpha)$ where $\dot{\alpha}=d\alpha/dt$.

It has been shown by Negishi [1960] that a solution $\alpha^*>0$ such that $s(\alpha^*)=0$, satisfies both conditions (a) and (b) of a C. E.

The fourth "Pareto Optimum" process (U-tâtonnement) is obtained from the following mathematical program $\max U_r(x_r)$ s.t. $U_i(x_i) \geq \bar{U}_i$ $(i=1, 2,..., r-1)$ and $\sum x_i \leq 0$, where $\bar{U}_i$ are fixed utility levels. Again, the first order conditions for a maximum are

(2.11) $$\alpha_i \frac{\partial U_i(x_i)}{\partial x_i} - p = 0 \qquad i = 1, 2, \ldots, r-1$$

(2.12) $$\frac{\partial U_r(x_r)}{\partial x_r} - p = 0$$

(2.13.1)–(2.13.3) $$-U_i(x_i) + U_i \leq 0; \; \alpha_i[-U_i(x_i) + U_i] = 0; \; \alpha_i \geq 0 \qquad i = 1, 2, \ldots, r-1.$$

(2.14.1)–(2.14.3) $$\sum x_i \leq 0; \; p\sum x_i = 0; \; p \geq 0.$$

The α_i's $(i=1, 2, \ldots, r-1)$ are Lagrange multipliers associated to the constraints $U_i(x_i) \geq U_i$ $(i=1, 2, \ldots, r-1)$.

Assuming $p>0$, $\alpha_i>0$ $(i=1, 2, \ldots, r-1)$, the system (2.11) to (2.14.1) can be solved to give a vector $p(\overline{U})$ and r vectors $x_i(\overline{U})$ $(i=1, 2, \ldots, r)$, where $\overline{U}$ is the vector $(U_1\ U_2 \cdots U_{r-1})$; again we can compute the savings of consumer i, $s_i(\overline{U}) = -p(\overline{U}) \cdot x_i(\overline{U})$; the U-tâtonnement can now be defined by the system of differential equations $\dot{\overline{U}} = s(\overline{U})$, where $\dot{\overline{U}} = d\overline{U}/dt$. Using the same arguments as Negishi, it can be shown that a solution $\overline{U}^*$ such that $s(\overline{U}^*)=0$ satisfies conditions (a) and (b) of a C. E.

We are now faced with four tâtonnement processes

$$\begin{array}{ll} p\text{-tâtonnement} & \dot{p} = f(p) \\ \pi\text{-tâtonnement} & \dot{\pi} = p(\pi) - \pi \\ \alpha\text{-tâtonnement} & \dot{\alpha} = s(\alpha) \\ U\text{-tâtonnement} & \dot{\overline{U}} = s(\overline{U}) \end{array}$$

and we show that, locally, they are not necessarily stable or unstable simultaneously.

3. LOCAL APPROXIMATION OF THE PROCESSES

To simplify the analysis, we make the following assumptions:

ASSUMPTION 1. (Normalization of equilibrium welfare weights.) The utility functions are chosen in such a way that, at equilibrium, $\alpha_i=1$ $(i=1, 2, \ldots, r)$. Since $\lambda_i = \frac{1}{\alpha_i}$ at equilibrium,[4] we also have $\lambda_i=1$ $(i=1, 2, \ldots, r)$.

ASSUMPTION 2. (Positiveness of prices at equilibrium.) At equilibrium, $p>0$. This will be ensured if the utility functions are strictly monotone.

From these two assumptions, it follows that, at equilibrium, $\sum x_i=0$, $U_i(x_i)=U_i$ and $px_i=0$ all i.

[4] For the proof of this, see, e.g., Negishi [1960].

The mathematical derivations which follow are lengthy but not difficult. In each case, a local approximation of the tâtonnement process is defined; this approximation is obtained in three steps:

(a) the conditions holding at the optimum of each problem defined in Section 2 are totally differentiated, in the neighborhood of equilibrium;

(b) the resulting linear system is solved for the unknowns;

(c) the tâtonnement is defined in terms of the solution obtained in Step (b).

We will need well-known results on the inversion of partitioned negative definite matrices, which we briefly recall in Lemmas 1 and 2.

LEMMA 1. *Let* $A=\begin{bmatrix} T & R \\ R' & W \end{bmatrix}$ *be a full rank symmetric matrix; if* T *is a full rank matrix,* A^{-1} *can be obtained as follows:*

$$(3.1)\quad A^{-1}=\begin{bmatrix} T^{-1}+T^{-1}R(W-R'T^{-1}R)^{-1}R'T^{-1} & -T^{-1}R(W-R'T^{-1}R)^{-1} \\ -(W-R'T^{-1}R)^{-1}R'T^{-1} & (W-R'T^{-1}R)^{-1} \end{bmatrix}.$$

If T *and* W *are full rank matrices, then we also have*

$$(3.2)\quad A^{-1}=\begin{bmatrix} (T-RW^{-1}R')^{-1} & -(T-RW^{-1}R')^{-1}RW^{-1} \\ -(W-R'T^{-1}R)^{-1}R'T^{-1} & (W-R'T^{-1}R)^{-1} \end{bmatrix}.$$

LEMMA 2. *Let* A *be as in Lemma* 1 *and negative definite. Then,* $(W-R'T^{-1}R)^{-1}$ *and* $(T-RW^{-1}R')^{-1}$ *are negative definite.*

It will also be useful to define the following matrices:

$V_i=\dfrac{\partial^2 U_i(x_i)}{\partial x_i^2}$ is the $n\times n$ matrix of second derivatives of consumer i's utility function, evaluated at equilibrium.

$Q=\begin{bmatrix} V_1 & 0\cdots\cdots 0 \\ 0 & V_2\cdots\cdots 0 \\ \vdots & \vdots \qquad \vdots \\ 0 & 0\cdots\cdots V_r \end{bmatrix}$ is a block diagonal $rn\times rn$ matrix.

$J'=[I\cdots I\cdots\cdots I]$ is an $n\times rn$ matrix with I the $n\times n$ identity matrix.
$\bar{J}'$ is the same matrix as J', with the last row deleted.

$$P = \begin{bmatrix} p & 0\cdots\cdots 0 \\ 0 & p\cdots\cdots 0 \\ \vdots & \vdots \quad\quad \vdots \\ 0 & 0\cdots\cdots p \end{bmatrix}$$ is an $rn \times r$ matrix; p and 0 are column vectors of n elements; p is the equilibrium price.

$\bar{P}$ is the same matrix as P, with the last column deleted.
Note that for the π process, we will need a matrix Π, identical to P, but with prices π instead of p. However, since both matrices are evaluated at equilibrium, by Theorem 2, $P \equiv \Pi$. For notational convenience, we will express all processes in terms of the P matrix.

$X = [x_1 \, x_2 \cdots x_r]$ is an $n \times r$ matrix; x_i is a column vector of n elements (the net trades of consumer i at equilibrium)

$\bar{X} = [x_1 \, x_2 \cdots x_{r-1}]$

$\bar{\bar{X}}$ is the same matrix as X, with the last row deleted.

(3.3) $$F = J' \, Q^{-1} \, J = \sum_{i=1}^{r} V_i^{-1}$$

(3.4) $$\bar{F} = \bar{J}' \, Q^{-1} \, \bar{J}$$

(3.5) $$S = J' \, Q^{-1} \, P$$

(3.6) $$\bar{S} = J' \, Q^{-1} \, \bar{P}$$

(3.7) $$\bar{\bar{S}} = \bar{J}' \, Q^{-1} \, P$$

(3.8) $$D = P' \, Q^{-1} \, P$$

(3.9) $$\bar{D} = \bar{P}' \, Q^{-1} \, \bar{P}.$$

Two more lemmas will be needed.

LEMMA 3. *The matrices* $(\bar{F} - \bar{\bar{S}} \, D^{-1} \, \bar{\bar{S}}')$ *and* $(\bar{D} - \bar{S}' \, F^{-1} \, \bar{S})$, *are negative definite.*

PROOF.[5] We give the proof for the first matrix. The same type of argument holds for the second matrix.

(a) We first show that $(\bar{F} - \bar{\bar{S}} \, D^{-1} \, \bar{\bar{S}}')$ is negative semi-definite. Using (3.4), (3.7) and (3.8), we have

$$\bar{F} - \bar{\bar{S}} \, D^{-1} \, \bar{\bar{S}}' = \bar{J}'[Q^{-1} - Q^{-1} \, P(P' \, Q^{-1} \, P)^{-1} \, P' \, Q^{-1}]\bar{J} = \bar{J}' \, M \, \bar{J}$$

with $M = Q^{-1} - Q^{-1} \, P(P' \, Q^{-1} \, P)^{-1} \, P' \, Q^{-1}$. It can easily be checked that $M = M' \, Q \, M$, so that $\bar{J}' \, M \, \bar{J} = \bar{J}' \, M' \, Q \, M \, \bar{J} = (M \, \bar{J})' \, Q \, (M \, \bar{J})$. Since Q is a block

[5] We are grateful to the referee for simplification of the proof.

diagonal matrix, the blocks of which are negative definite matrices (as matrices of second derivatives of strictly concave utility functions), Q is negative definite and $(M\bar{J})'\,Q(M\bar{J})$ is negative semi-definite. So is $\bar{F}-\bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}'$.

(b) We now show that $\bar{F}-\bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}'$ is non singular. Clearly, the matrix $[\bar{J}\;P]$ has full column rank. Since Q^{-1} is non singular, so is $\begin{bmatrix}\bar{J}'\\P'\end{bmatrix}Q^{-1}[\bar{J}\;P]$. But using (3.4), (3.7) and (3.8), we have

$$\begin{bmatrix}\bar{J}'\\ \\P'\end{bmatrix}Q^{-1}[\bar{J}\quad P]=\begin{bmatrix}\bar{F} & \bar{\bar{S}}\\ \\ \bar{\bar{S}}' & D\end{bmatrix}.$$

Postmultiplying this result by $\begin{bmatrix}I\\-D^{-1}\bar{\bar{S}}'\end{bmatrix}$ gives $\begin{bmatrix}\bar{F}-\bar{\bar{S}}D^{-1}\bar{\bar{S}}'\\0\end{bmatrix}$ and $\bar{F}-\bar{\bar{S}}D^{-1}\bar{\bar{S}}'$ must be non singular.

(c) It is a well-known result that a negative semi-definite matrix is negative definite if it is non singular.[6] Hence $\bar{F}-\bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}'$ is negative definite.

LEMMA 4. *Let Q be negative definite and M of full column rank. Then* $G=\begin{bmatrix}Q & M\\M' & 0\end{bmatrix}$ *is non singular.*

PROOF. Consider the system $\begin{bmatrix}Q & M\\M' & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$. We show that the only solution is $\begin{bmatrix}x\\y\end{bmatrix}=0$. Consider the system $M'x=0$. But $M'x=-M'\,Q^{-1}\,My=0$. Since M is of full column rank, $M'\,Q^{-1}\,M$ is non singular and the only solution is $y=0$. Then, $Qx+My=Qx=0$ has $x=0$ as the only solution. Hence G cannot be singular.

We are now ready to handle the local approximations.

3.1. *p-tâtonnement.* Total differentiation of (2.4)–(2.5.1) with respect to x_i, λ_i and p gives

$$\left.\begin{aligned} V_i\,dx_i - p d\lambda_i - \lambda_i dp &= 0\\ x_i' dp + p' dx_i \qquad &= 0\end{aligned}\right\}\qquad i=1,2,\ldots,r.$$

Writing these relations for all consumers, and using Assumption 1 ($\lambda_i=1$), we have

$$\begin{bmatrix}Q & P\\ \\P' & 0\end{bmatrix}\begin{bmatrix}dx\\ \\-d\lambda\end{bmatrix}=\begin{bmatrix}J\\ \\-X'\end{bmatrix}dp$$

where $dx'=(dx_1,\ldots,dx_r)$, $d\lambda'=(d\lambda_1,\ldots,d\lambda_r)$ and $dp'=(dp_1,\ldots,dp_n)$. As is well-known, this system defines demand as a zero homogeneous function of prices. As a result, the p-tâtonnement defined in this way converges possibly to a price

[6] See, e.g., Gantmacher [1960, p. 305].

ray and not to a unique price vector. Usually the difficulty is solved by normalizing the prices. Setting $p_n = 1$, $dp_n = 0$, the equations can be written:

$$\begin{bmatrix} Q & P \\ P' & 0 \end{bmatrix} \begin{bmatrix} dx \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \bar{J} \\ -\bar{X}' \end{bmatrix} d\bar{p}$$

where $d\bar{p}' = (dp_1 \; dp_2 \cdots dp_{n-1})$.

The matrix on the left hand side of this last expression has full rank, by Lemma 4. Using (3.1) to invert this matrix, we end up with

$$dx = [Q^{-1} \bar{J} - Q^{-1} P(P' Q^{-1} P)^{-1} P' Q^{-1} \bar{J} - Q^{-1} P(P' Q^{-1} P)^{-1} \bar{X}'] d\bar{p}.$$

Total excess demand $f(p)$ is equal to $J'x$ so that $\dot{p} = df = J'dx$; but we need only $\dot{\bar{p}} = \bar{J}'dx$. Hence, using (3.4), (3.7) and (3.8), we find

$$\dot{\bar{p}} = \bar{J}'dx = [\bar{F} - \bar{S}\, D^{-1}(\bar{\bar{S}}' + \bar{X}')](\bar{p} - \bar{p}^*) \tag{3.10}$$

where $d\bar{p} = \bar{p} - \bar{p}^*$ is the deviation from the equilibrium value $\bar{p}^*$.

3.2. *π-tâtonnement.* From Theorem 2, it is clear that, at the solution point for which $p = \pi$, one of the constraints $\pi x_i = 0$ or one of the n balances $\sum x_i = 0$ will be redundant, and can be omitted. If the n-th of these balances is omitted, we have as first order conditions for a maximum

$$\frac{\partial U_i}{\partial x_i} - p - \lambda_i \pi = 0 \qquad i = 1, 2, \ldots, r$$

$$\sum \bar{x}_i = 0$$

$$\pi x_i = 0 \qquad i = 1, 2, \ldots, r$$

where $\bar{x}_i$ is the vector of net trades of the $n-1$ first commodites for trader i.

Totally differentiating these expressions with respect to x_i, λ_i and p gives

$$V_i\, dx_i - dp - \lambda_i d\pi - \pi d\lambda_i = 0 \qquad i = 1, 2, \ldots, r$$

$$\sum d\bar{x}_i = 0$$

$$x_i' d\pi + \pi' dx_i = 0 \qquad i = 1, 2, \ldots, r.$$

Taking into account that the n-th element of both vectors dp and $d\pi$ is identically zero (the normalization implies p_n and π_n constant), we can write these equations under matrix form, as:

$$\begin{bmatrix} Q & \bar{J} & \Pi \\ \bar{J}' & 0 & 0 \\ \Pi' & 0 & 0 \end{bmatrix} \begin{bmatrix} dx \\ -d\bar{p} \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \bar{J} \\ 0 \\ -\bar{X}' \end{bmatrix} d\bar{\pi} \tag{3.11}$$

with $d\bar{p}' = (dp_1, dp_2, \ldots, dp_{n-1})$ and $d\bar{\pi}' = (d\pi_1, d\pi_2, \ldots, d\pi_{n-1})$. The matrix on the left hand side of (3.11) is non singular (Lemma 4).

Let $N=[\bar{J}\ \Pi]$ and invert first the matrix $\begin{bmatrix} Q & N \\ N' & 0 \end{bmatrix}$ using (3.1)

$$(3.12)\quad \begin{bmatrix} Q & N \\ N' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} Q^{-1} - Q^{-1}N(N'Q^{-1}N)^{-1}N'Q^{-1} & Q^{-1}N(N'Q^{-1}N)^{-1} \\ (N'Q^{-1}N)^{-1}N'Q^{-1} & -(N'Q^{-1}N)^{-1} \end{bmatrix}.$$

Note that

$$(N'Q^{-1}N) = \begin{bmatrix} \bar{J}' \\ \Pi' \end{bmatrix} Q^{-1}[\bar{J}\ \Pi] = \begin{bmatrix} \bar{F} & \bar{\bar{S}} \\ \bar{\bar{S}}' & D \end{bmatrix}.$$

We now use (3.2) to invert $(N'\,Q^{-1}\,N)$. This gives

$$(N'Q^{-1}N)^{-1} = \begin{bmatrix} (\bar{F} - \bar{\bar{S}}D^{-1}\bar{\bar{S}}')^{-1} & -(\bar{F} - \bar{\bar{S}}D^{-1}\bar{\bar{S}}')^{-1}\bar{\bar{S}}D^{-1} \\ -(D - \bar{\bar{S}}'\bar{F}^{-1}\bar{\bar{S}})^{-1}\bar{\bar{S}}'\bar{F}^{-1} & (D - \bar{\bar{S}}'\bar{F}^{-1}\bar{\bar{S}})^{-1} \end{bmatrix}.$$

Inserting this result into (3.12), we derive, from (3.11) the following expression for $d\bar{p}$:

$$(3.13)\qquad d\bar{p} = -(\bar{F} - \bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}')^{-1}[\bar{F} - \bar{\bar{S}}\,D^{-1}(\bar{\bar{S}}' + \bar{\bar{X}}')]d\bar{\pi}.$$

Since we defined the π-tâtonnement by $\dot{\pi}=p(\pi)-\pi$, locally we have,

$$dp - d\pi = (p - p^*) - (\pi - \pi^*) = p - \pi = \dot{\pi}.$$

Hence, we have, after substracting $d\bar{\pi}$ from both sides of (3.13)

$$(3.14)\qquad \dot{\bar{\pi}} = -(\bar{F} - \bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}')^{-1}[2(\bar{F} - \bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}') - \bar{\bar{S}}\,D^{-1}\,\bar{\bar{X}}'](\bar{\pi} - \bar{\pi}^*).$$

3.3. *α-tâtonnement.* Total differentiation of (2.6)–(2.7.1) with respect to x_i, α_i and p gives

$$\frac{\partial U_i(x_i)}{\partial x_i}d\alpha_i + \alpha_i\,V_i\,dx_i - dp = 0 \qquad i = 1, 2,\ldots, r$$

$$\sum_i dx_i = 0.$$

Replacing $\frac{\partial U_i(x_i)}{\partial x_i}$ by $\frac{p}{\alpha_i}$ and taking into account Assumption 1 ($\alpha_i=1$) these equations can also be written for all i as:

$$\begin{bmatrix} Q & J \\ J' & 0 \end{bmatrix}\begin{bmatrix} dx \\ -dp \end{bmatrix} = -\begin{bmatrix} P \\ 0 \end{bmatrix}d\alpha$$

dx and dp have already been defined, $d\alpha'=(d\alpha_1\cdots d\alpha_r)$. The same problem of normalization arises as with the p-tâtonnement.

Setting $\alpha_r=1$ reduces the system to

$$\begin{bmatrix} Q & J \\ J' & 0 \end{bmatrix}\begin{bmatrix} dx \\ -dp \end{bmatrix} = -\begin{bmatrix} \bar{P} \\ 0 \end{bmatrix}d\bar{\alpha}$$

with $\bar{\alpha}' = (\alpha_1 \cdots \alpha_{r-1})$.

By Lemma 4, $\begin{bmatrix} Q & J \\ J' & 0 \end{bmatrix}$ is non singular. Using (3.1), the solution of the system can be written:

$$dx = [-Q^{-1} - Q^{-1} J(J' Q^{-1} J)^{-1} J' Q^{-1}]\bar{P} d\bar{\alpha} \tag{3.15}$$

$$dp = (J' Q^{-1} J)^{-1} J' Q^{-1} \bar{P} d\bar{\alpha}. \tag{3.16}$$

Savings of consumer i are equal to $-p\, x_i$. Differentiating this result leads to $ds_i = -x_i'\, dp - p'\, dx_i$ or, in matrix notation, for all consumers, except consumer r:

$$d\bar{s} = -\bar{X}'\, dp - \bar{P}'\, dx.$$

Then, using (3.3), (3.6), (3.9), (3.15) and (3.16), we have

$$d\bar{s} = [\bar{D} - (\bar{S}' + \bar{X}')F^{-1}\, \bar{S}]d\bar{\alpha}$$

or, since $\dot{\bar{\alpha}} = d\bar{s}$

$$\dot{\bar{\alpha}} = [\bar{D} - (\bar{S}' + \bar{X}')F^{-1}\, \bar{S}](\bar{\alpha} - \bar{\alpha}^*) \tag{3.17}$$

where $\bar{\alpha} - \bar{\alpha}^*$ is the deviation from the equilibrium value $\bar{\alpha}^*$.

3.4. *U-tâtonnement.* Total differentiation of (2.11), (2.12), (2.13.1) and (2.14.1) with respect to x_i, α_i, p and U_i gives

$$V_r\, dx_r - dp = 0$$

$$\frac{\partial U_i(x_i)}{\partial x_i} d\alpha_i + \alpha_i\, V_i\, dx_i - dp = 0 \qquad i = 1, 2, \ldots, (r-1)$$

$$\frac{\partial U_i(x_i)}{\partial x_i} dx_i - d\, U_i = 0 \qquad i = 1, 2, \ldots, (r-1)$$

$$\sum_i dx_i = 0.$$

Since $\dfrac{\partial U_i(x_i)}{\partial x_i} = \dfrac{p}{\alpha_i}$ and $\alpha_i = 1$ at equilibrium, we can write these relations in matrix form as:

$$\begin{bmatrix} Q & J & \bar{P} \\ J' & 0 & 0 \\ \bar{P}' & 0 & 0 \end{bmatrix} \begin{bmatrix} dx \\ -dp \\ d\bar{\alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d\bar{U} \end{bmatrix}$$

with $d\bar{\alpha} = [d\alpha_1 \cdots d\alpha_{r-1}]$ and $d\bar{U} = [dU_1 \cdots dU_{r-1}]$. By Lemma 4, the matrix on the left is non-singular.

Defining $M = [J\ \bar{P}]$, we first invert the partitioned matrix $\begin{bmatrix} Q & M \\ M' & 0 \end{bmatrix}$ using (3.1). The result is:

$$\begin{bmatrix} Q & M \\ M' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} Q^{-1} - Q^{-1}M(M'Q^{-1}M)^{-1}M'Q^{-1} & Q^{-1}M(M'Q^{-1}M)^{-1} \\ (M'Q^{-1}M)^{-1}M'Q^{-1} & -(M'Q^{-1}M)^{-1} \end{bmatrix}.$$

We now have to invert $(M'Q^{-1}M)$. But

$$M'Q^{-1}M = \begin{bmatrix} J' \\ \bar{P}' \end{bmatrix} Q^{-1}[J\ \bar{P}] = \begin{bmatrix} F & \bar{S} \\ \bar{S}' & \bar{D} \end{bmatrix}.$$

Using (3.1), we obtain

$$(M'Q^{-1}M)^{-1} = \begin{bmatrix} F^{-1} + F^{-1}\bar{S}(\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}\bar{S}'F^{-1} & -F^{-1}\bar{S}(\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1} \\ -(\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}\bar{S}'F^{-1} & (\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1} \end{bmatrix}.$$

After some further matrix algebra, we end up with

$$dx = -Q^{-1}(JF^{-1}\bar{S} - \bar{P})(\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}d\bar{U}$$

$$dp = -F^{-1}\bar{S}(\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}d\bar{U}.$$

Finally, noting that, as in the α-tâtonnement, $\bar{s} = -\bar{X}'p$ and $d\bar{s} = -\bar{X}'dp - \bar{P}'dx$, we find

$$d\bar{s} = [\bar{X}'F^{-1}\bar{S} + \bar{P}'Q^{-1}(JF^{-1}\bar{S} - \bar{P})](\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}d\bar{U}$$

$$= -[\bar{D} - (\bar{S}' + \bar{X}')F^{-1}\bar{S}](\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}d\bar{U}$$

and

(3.18) $$\dot{\bar{U}} = -[\bar{D} - (\bar{S}' + \bar{X}')F^{-1}\bar{S}](\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}(\bar{U} - \bar{U}^*)$$

where $\bar{U} - \bar{U}^*$ is the deviation from the equilibrium value $\bar{U}^*$.

4. LOCAL STABILITY PROPERTIES

For easy reference, we group the four approximations so far obtained

(4.1) $$\dot{\bar{p}} = [\bar{F} - \bar{\bar{S}}D^{-1}(\bar{\bar{S}}' + \bar{\bar{X}}')](\bar{p} - \bar{p}^*)$$

(4.2) $$\dot{\bar{\pi}} = -(\bar{F} - \bar{\bar{S}}D^{-1}\bar{\bar{S}}')^{-1}[2(\bar{F} - \bar{\bar{S}}D^{-1}\bar{\bar{S}}') - \bar{\bar{S}}D^{-1}\bar{\bar{X}}'](\bar{\pi} - \bar{\pi}^*)$$

(4.3) $$\dot{\bar{\alpha}} = [\bar{D} - (\bar{X}' + \bar{S}')F^{-1}\bar{S}](\bar{\alpha} - \bar{\alpha}^*)$$

(4.4) $$\dot{\bar{U}} = -[\bar{D} - (\bar{X}' + \bar{S}')F^{-1}\bar{S}](\bar{D} - \bar{S}'F^{-1}\bar{S})^{-1}(\bar{U} - \bar{U}^*).$$

We now turn to local stability. We shall see that, in general, the four processes are not necessarily simultaneously stable. To study the question, we first recall a few results on the stability of systems of linear differential equations.

LEMMA 5. *The system of differential equations $\dot{x}=Ax$ is stable*[7]

(a) *if all the characteristic roots of A have negative real parts.*
(b) *(Negative Definiteness) if A is (quasi) negative definite*[8]
(c) *(Diagonal Dominance) if $a_{ii}<0$ all i and there exist numbers $c_i>0$ such that $c_i|a_{ii}|>\sum_{j\neq i} c_j|a_{ij}|$ all i.*

LEMMA 6.[9] *Let G be a quasi negative definite matrix and H be a positive definite symmetric matrix. Then the roots of HG (and GH) have negative real parts.*

4.1. *Simultaneously stable systems.* There are two general cases in which the four systems can be shown to be stable at the same time. These cases are discussed in the theorems and corollaries which follow.

THEOREM 3. *If the income effects vanish, i.e., if $\bar{\bar{S}}\,D^{-1}\,\bar{\bar{X}}'=0$ and $\bar{X}'\,F^{-1}\,\bar{S}=0$, the processes are simultaneously locally stable.*

PROOF. If $\bar{\bar{S}}\,D^{-1}\,\bar{\bar{X}}'$ and $\bar{X}'\,F^{-1}\,\bar{S}=0$, we have

$$\begin{aligned}
\dot{\bar{p}} &= (\bar{F} - \bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}')(\bar{p} - \bar{p}^*)\\
\dot{\bar{\pi}} &= -2(\bar{\pi} - \bar{\pi}^*)\\
\dot{\bar{\alpha}} &= (\bar{D} - \bar{S}'\,F^{-1-1}\,\bar{S})(\bar{\alpha} - \bar{\alpha}^*)\\
\dot{\bar{U}} &= -(\bar{U} - \bar{U}^*)
\end{aligned}$$

In Lemma 3, we proved that $\bar{F}-\bar{\bar{S}}\,D^{-1}\,\bar{\bar{S}}'$ and $\bar{D}-\bar{S}'\,F^{-1}\,\bar{S}$ are negative definite. Processes p and α are thus stable. It is obvious that the π and U processes are stable since $-I$ is negative definite.

COROLLARY 1. *If there is no trade at equilibrium, the processes are simultaneously locally stable.*

PROOF. The no trade at equilibrium situation implies $X=0$, $\bar{X}=0$ and $\bar{\bar{X}}=0$; hence $\bar{\bar{S}}\,D^{-1}\,\bar{\bar{X}}'=0$ and $\bar{X}'\,F^{-1}\,\bar{S}=0$ and the assumptions of Theorem 3 are satisfied.

COROLLARY 2. *If the matrices V_i $(i=1, 2,\dots, r)$ are all identical, the processes are simultaneously locally stable.*

PROOF. We show that $\bar{\bar{S}}\,D^{-1}\,\bar{\bar{X}}'$ and $\bar{X}'\,F^{-1}\,\bar{S}$ vanish, as in Theorem 3. Indeed, if all V_i's are equal to V, D^{-1} is a diagonal matrix with diagonal terms

[7] There are some other cases which we do not consider here, e.g., Mukerji [1972], Ohyama [1972] and Rader [1977].
[8] The matrix A, not necessarily symmetric, is quasi negative definite if, for $x\neq 0$, $x'Ax<0$.
[9] For a proof, see Karlin [1954, p. 332].

equal to $(p' V^{-1} p)^{-1}$, a scalar. Hence, $\bar{\bar{S}} D^{-1} \bar{\bar{X}}' = \tilde{D}^{-1} \bar{\bar{S}} \bar{\bar{X}}' = \tilde{D}^{-1} J' Q^{-1} P \bar{\bar{X}}' = \tilde{D}^{-1} \tilde{I}[V^{-1} p \, V^{-1} p \cdots V^{-1} p]\bar{\bar{X}}' = \tilde{D}^{-1} \tilde{I} V^{-1} p \sum \bar{\bar{x}}_i' = 0$ ($\tilde{D}$ is a diagonal $(n-1)$ matrix with diagonal terms equal to $(p' V^{-1} p)^{-1}$; $\tilde{I}$ is the identity matrix with the last row deleted).

Likewise, if $V_i = V$, $F = r V^{-1}$ and $\bar{X}' F^{-1} \bar{S} = \frac{1}{r} \bar{X}' V[V^{-1} p \cdots V^{-1} p] = \frac{1}{r} \bar{X}'[p \cdots p] = 0$.

The assumptions of Theorem 3 are thus satisfied and the result follows.

Instability can thus result only from income effects — a well-known conclusion of classical tâtonnement analysis, which carries over to the broader range of processes considered here. The "income effects" are respectively

$$\begin{array}{rl} p \text{ process:} & -\bar{\bar{S}} D^{-1} \bar{X}' \\ \pi \text{ process:} & [\bar{F} - \bar{\bar{S}} D^{-1} \bar{\bar{S}}']^{-1} \bar{\bar{S}} D^{-1} \bar{X}' \\ \alpha \text{ process:} & -\bar{X}' F^{-1} \bar{S} \\ U \text{ process:} & \bar{X}' F^{-1} \bar{S}[\bar{D} - \bar{S}' F^{-1} \bar{S}]^{-1}. \end{array}$$

It does not seem possible to draw up general rules about these matrices, which would ensure the simultaneous stability of the processes. The only result which can apparently be stated is the weak theorem

THEOREM 4. *If the matrix of the p (or α) process is quasi negative definite, then the p and π (or α and U) processes are simultaneously stable.*

PROOF. The conclusion follows from Lemma 6.

COROLLARY 3. *If the matrix $\bar{\bar{S}} D^{-1} \bar{X}'$ is quasi positive semi definite, then the p and the π processes are simultaneously stable. If the matrix $\bar{X}' F^{-1} \bar{S}$ is quasi positive semi definite, the α and the U processes are simultaneously stable.*

PROOF. We know that $\bar{F} - \bar{\bar{S}} D^{-1} \bar{\bar{S}}'$ is negative definite. The assumption of the corollary implies that $[\bar{F} - \bar{\bar{S}} D^{-1}(\bar{X} + \bar{\bar{S}}')]$ is negative quasi definite. Using Theorem 4, we see that the p and π processes are stable. Similar reasoning proves the assertion concerning the α and U processes.

Some other (rather unlikely) cases can be found in which some of the processes would be simultaneously stable. One of them is given in Mantel [1971, p. 429] for the p- and the α-process. Others can be found by imposing severe restrictions on the matrices of the processes, which would be meaningless from the economic viewpoint and impossible to check in practice.

4.2. *Other cases.* It is obvious that the conditions stated for simultaneous stability are far from being general. Indeed, it seems in general impossible to infer the properties of one process from the properties of another. Mantel gives an interesting example of this in showing that if there are two consumers and several goods, the α-tâtonnement will always converge, although the p-tâtonnement is not necessarily convergent.

It seems thus worthwhile to devise alternative adjustment rules, the properties of which are likely to be different.

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