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Singular value analysis for band-pass filtering systems

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Abstract

We consider the problem of restoring a signal degraded by a band-pass filter, in the case where this signal is modulated by a given profile function. We study the properties of the associated singular system and derive analytic expressions for the singular functions in some particular cases.

Introduction

Many inverse problems encountered in practice consist in restoring an object function or input signal from its image or output produced by a degrading linear system. In many instances, the effect of this linear system can be viewed as a filtering process, since the system attenuates or suppresses some frequency components in the Fourier transform of the input signal. In general the systems cuts off the highest frequency components and behaves thus as a low-pass filter. The problem of restoring an object function from its low-pass filtered image has been extensively studied in the literature. In several situations of practical relevance, however, the system also filters off the low-frequency components and it is to be considered as a band-pass filter. We think in particular to optical systems with centrally obstructed pupils and to inverse scattering problems when low-frequency information is missing either because of experimental constraints or because of the limited validity of the linear approximation. In spite of its interest,[1] the restoration problem from band-pass filtered outputs has not received much attention so far. In the present paper, we consider this question in the case where the input signal is weighted by some known profile function.

The band-pass filter with weighted input

Let us assume that the effect of the linear system on the input signal $f(x)$ is described by the following integral operator

$$(Af)(x) = \int_{-\infty}^{+\infty} K(x-y)\, P(y)\, f(y)\, dy \qquad (-\infty < x < +\infty) \tag{1}$$

where the impulse response $K(x)$ is given by

$$K(x) = 2\nu_2 \operatorname{sinc}(2\nu_2 x) - 2\nu_1 \operatorname{sinc}(2\nu_1 x) \qquad 0 \leq \nu_1 < \nu_2 \tag{2}$$

the symbol "sinc" being the usual notation for the following function

$$\operatorname{sinc}(x) = \sin(\pi x)/(\pi x) \tag{3}$$

The corresponding transfer function $\hat{K}(\nu)$, which is the Fourier transform of $K(x)$, is just the characteristic function of the frequency band

$$\begin{aligned} \hat{K}(\nu) &= 1 \quad \text{for } -\nu_2 \leq \nu \leq -\nu_1 \text{ and } \nu_1 \leq \nu \leq \nu_2 \\ &= 0 \quad \text{elsewhere} \end{aligned} \tag{4}$$

The restoration problem associated to this band-pass filter consists in recovering $f(x)$ from continuous or sampled values of the image $g(x) = (Af)(x)$. We will essentially consider here the case of continuous data on the whole real axis.

The "profile" or weighting function $P(x)$ is a given function which can be viewed as an entrance window affecting the object before the filtering. The physical interpretation of $P(x)$ depends of course on the application under study. For example, in Fourier optics, where $f(x)$ represents the transmission function of a coherently illuminated transparency, the profile function allows to describe the field distribution in the illuminating beam, e. g.

a gaussian profile for a laser beam or a sinc - function for a beam focused by diffraction-limited optics. Alternatively, $P(x)$ could be used for expressing some a priori knowledge about the localization of the unknown object $f(x)$. For example, $P(x)$ can be the characteristic function of the support of the object. Such weighted inputs have already been used in the case of a low-pass filter (i. e. $\nu_1 = 0$).[2,3] We assume that the profile function is square integrable

$$\int_{-\infty}^{+\infty} |P(x)|^2 dx < +\infty \tag{5}$$

and that $|P(x)|$ is bounded by 1.

Properties of the singular system

With the assumption (5), it is easily seen that the operator A given by (1) belongs to the class of Hilbert-Schmidt and hence defines a compact operator in $L^2(-\infty,+\infty)$, the space of square integrable functions on the line. A useful tool for analysing the restoration problem for a linear system modelled by a compact operator A is the so-called singular system of A, which is the set of non-trivial solutions $\{\alpha_k; u_k, v_k\}$ of the coupled equations

$$Au_k = \alpha_k v_k \qquad A^* v_k = \alpha_k u_k \tag{6}$$

The quantities α_k, which are real positive numbers by definition, are called the singular values of A and the symbol A^* denotes the adjoint operator

$$(A^* g)(y) = P^*(y) \int_{-\infty}^{+\infty} K(x-y)\, g(x)\, dx \tag{7}$$

where P^* is the complex conjugate of P. The functions u_k and v_k, called the singular functions of A, are also the eigenfunctions (with eigenvalues α_k^2) of the self-adjoint operators A^*A and AA^*, respectively, as seen from (6). The operator AA^* has a simple expression in Fourier space and the corresponding eigenvalue equation for the Fourier transform $\hat{v}_k(\nu)$ is

$$\int_{-\nu_2}^{-\nu_1} \hat{v}_k(\xi)\, \hat{Q}(\nu-\xi)\, d\xi + \int_{\nu_1}^{\nu_2} \hat{v}_k(\xi)\, \hat{Q}(\nu-\xi)\, d\xi = \alpha_k^2\, \hat{v}_k(\nu) \qquad (\nu_1 \le |\nu| \le \nu_2) \tag{8}$$

where $\hat{Q}(\nu)$ is the Fourier transform of $|P(x)|^2$.

From (6) and (7), we see that $v_k(x)$ is a bandlimited function to the band $\nu_1 \le |\nu| \le \nu_2$. The functions v_k form an orthogonal basis in the space of bandlimited functions. Moreover, it follows from (7) and (6) that

$$u_k(x) = (1/\alpha_k)\, P^*(x)\, v_k(x) \tag{9}$$

The restoration problem has a unique solution or, in other words, the operator A is invertible when $\hat{P}(\nu)$, the Fourier transform of P, is analytic and, in particular, when $P(x)$ vanishes outside of some bounded support. In such a case the singular functions u_k form a basis in $L^2(-\infty,+\infty)$. On the other hand, when $P(\nu)$ has a bounded support, A cannot be invertible and the singular functions u_k span only the orthogonal complement of the null-space of A, i. e. of the set of input functions $f(x)$ which produce a zero output.

From (9) and from the orthogonality properties of the u_k and v_k, it is seen that the singular functions $v_k(x)$ satisfy a double-orthogonality relation[4] which generalizes the well-known double orthogonality of the prolate spheroidal wave functions.[5] The v_k reduce indeed to the prolate functions in the particular case of a low-pass filter and of a profile function which is the characteristic function of a finite interval, say (-1,+1).

The separability condition

In some circumstances, the solutions of equations (6) or (8) can be reduced to a problem relative to a low-pass filter. This happens when $\hat{Q}(\nu)$ vanishes for $|\nu|$ greater than a given value, say ν_0. In such a case, if

$$\nu_0 \le 2\nu_1 \tag{10}$$

equation (8) splits into two separate equations, one for the positive-frequency part and one for the negative-frequency part of $\hat{v}_k(\nu)$. Each part can be determined from the singular functions of the corresponding low-pass filter and a two-fold degeneracy occurs.

The sinc - profile

Let us now consider the case where the profile function is given by

$$P(x) = \sin(\pi x)/(\pi x) \tag{11}$$

The function $\hat{Q}(\nu)$ is then the following triangular function

$$\Lambda(\nu) = 1 - |\nu| \quad \text{for } |\nu| \leq 1$$
$$0 \quad \text{elsewhere} \tag{12}$$

We have put here $\nu_0 = 1$, for simplicity, but this condition is not restrictive. When $\nu_1 \geq 1/2$, the separability condition holds and equation (8) can be solved explicitly, at least when $\nu_2 - \nu_1 \leq 1$. The expression of $v_k(x)$ and the conditions which determine the singular values can then be easily derived from the results relative to the low-pass filter with a sinc - profile.[3]

When the separability condition does not hold, we can still solve explicitly equation (8), whatever be ν_1, provided that $\nu_2 \leq 1/2$. Indeed, when putting (12) into (8) and differentiating two times, we see that the functions $\hat{v}_k(\nu)$ satisfy the following differential equation

$$\frac{d^2\hat{v}_k(\nu)}{d^2\nu} + (2/\alpha_k^2)\,\hat{v}_k(\nu) = 0 \qquad \nu_1 \leq |\nu| \leq \nu_2 \tag{13}$$

Hence, they are harmonic functions and by direct insertion of a harmonic function in (8), we find that they are even or odd and have the following explicit expression for $\nu_1 \leq |\nu| \leq \nu_2$

$$\hat{v}_k(\nu) = \cos[\beta_k(|\nu| - \nu_1)] \qquad \text{for even } k$$
$$\hat{v}_k(\nu) = \mathrm{sign}(\nu)\cos[\beta_k(|\nu| - \nu_2)] \qquad \text{for odd } k \tag{14}$$

where $\mathrm{sign}(\nu) = 1$ for $\nu \geq 0$ and $\mathrm{sign}(\nu) = -1$ for $\nu < 0$. The allowed values for β_k are the solutions of the transcendental equations

$$\mathrm{tg}\,[\beta_k(\nu_2 - \nu_1)] = 1/[\beta_k(1 - \nu_2)] \qquad \text{for even } k$$
$$\mathrm{tg}\,[\beta_k(\nu_2 - \nu_1)] = 1/(\beta_k \nu_1) \qquad \text{for odd } k \tag{15}$$

The singular values are then given by

$$\alpha_k = \sqrt{2}/\beta_k \tag{16}$$

Thanks to equation (9), the singular system in now completely determined.

Other explicitly solvable cases

We have also been able to determine explicitly the singular systems corresponding to other choices of the profile function. These results have been reported elsewhere [3] and are relative to the cases where $P(x)$ is proportional to the impulse response (2) and where $P^2(x)$ is a Lorentzian function.

Regularized solutions of the restoration problem

In order to overcome the problems of existence and uniqueness, the restoration problem is to be solved by means of the standard concept of generalized solutions.[6] The so-called generalized solution $f^\dagger$ of the equation $(Af)(x) = g(x)$ is orthogonal to the null-space of A and admits the following expansion onto the singular system of the compact operator A

$$f^\dagger = \sum_k (1/\alpha_k)\,(g, v_k)\,u_k \tag{17}$$

where (g, v_k) denotes the usual scalar product in L^2. However, restoration by means of formula (17) is unstable in the presence of noise, since the singular values accumulate to zero. Regularized (i. e. stable approximate) solutions are obtained by suppressing in the sum (17) all terms for which $\alpha_k \leq \epsilon/E$, where $(E/\epsilon)^2$ is the signal-to-noise ratio.

When analytic expressions for the singular functions are not available, the singular system of A can be computed numerically.[2] Such a computation is easier for sampled data, since it reduces to a matrix diagonalization. When the number of data points is sufficiently high, we can obtain very good approximations of the singular system corresponding to continuous data. Numerical examples of singular systems and of restored signals are to be presented during the oral discussion of the paper.

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