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Sufficient Conditions for the Existence of Bound States in a Potential without Spherical Symmetry

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We give in this paper several sufficient conditions for the existence of negative energy bound states in a purely attractive potential without spherical symmetry. These conditions generalize the condition obtained recently by K. Chadan and A. Martin (*C. R. Acad. Sci. Paris* **290** (1980), 151), and can ensure the existence of n bound states. For the spherically symmetric case, one gets simple formulae which are also new.

I. INTRODUCTION

In a recent paper [1], a sufficient condition was given which ensures the existence of at least one bound state for a nonrelativistic particle in a purely attractive potential without spherical symmetry. It is the purpose of the present work to generalize the above condition in order to ensure the existence of n bound states.

As usual, we choose the units in such a way that $\hbar = 2M = 1$, where M is the mass of the particle. The Schrödinger equation then reads

$$H\Psi = [-\Delta + V(\mathbf{r})]\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{1}$$

where Δ is the Laplacian, and $V(\mathbf{r})$ the potential. We shall not enter into the details of various sufficient couditions under which the Hamiltonian is self-adjoint and what are its spectral properties in each case, and we refer the reader to the literature for more details about the brief mathematical discussion which follows [2, 3]. We assume again that the potential is purely attractive: $V \leqslant 0$.

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To be on the safe side concerning the Hamiltonian as a self-adjoint operator with usual spectral properties: given a continuum from 0 to infinity, plus a point spectrum whose negative elements (negative energy bound states) are bounded from below and are finite in number, we assume the following conditions:

$$V(\mathbf{r}) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3). \tag{2}$$

These conditions guarantee that H has a unique self-adjoint extension, and that its spectrum is as expected. Moreover, according to [2, Theorem I.22], the operator

$$\begin{aligned} K(E) &= |V|^{1/2}(E - H_0)^{-1} |V|^{1/2} \\ &= |V(\mathbf{r})|^{1/2} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} |V(\mathbf{r}')|^{1/2}, \qquad \operatorname{Im} k > 0, \end{aligned} \tag{3}$$

is a Hilbert–Schmidt operator for all E off the positive real axis. It is analytic in E in the cut plane, and has a continuous extension to the real axis from above and below, i.e., the limit is also Hilbert–Schmidt, including at $E = 0$. The negative energy bound states Ψ satisfy the homogeneous Fredholm integral equation ($E = -\gamma^2$, $\gamma > 0$)

$$\Phi(\mathbf{r}) = (4\pi)^{-1} \int |V(\mathbf{r})|^{1/2} \frac{e^{-\gamma|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} |V(\mathbf{r}')|^{1/2} \Phi(\mathbf{r}')\, d^3r', \tag{4a}$$

where $\Phi = |V|^{1/2}\Psi$ [2, Theorems III.2 and III.4]. As is known, the possible zero energy (L^2) solution of this equation, which usually corresponds to a resonance rather than a true bound state, is not a solution of $H\Psi = 0$ with Ψ in L^2. This means that, in principle, we should keep the energy strictly negative in the analysis which follows. However, as has been done by Schwinger (see [2, Chapter III] for details), for counting the negative energy bound states (including the possible resonance at zero energy), we can use the zero energy limit of (4a) essentially because the kernel stays Hilbert–Schmidt at $\gamma = 0$. According to Schwinger, the existence or nonexistence of such states is related to whether or not the kernel $K(E = 0)$ has characteristic values (inverse of eigenvalues) less than 1 or not, that is, whether or not one can solve the Fredholm integral equation

$$\Phi = |V|^{1/2} + \int (4\pi)^{-1} |V|^{1/2} \frac{1}{|\mathbf{r} - \mathbf{r}'|} |V|^{1/2} \Phi\, d^3r' \tag{4b}$$

by iteration, and obtain a convergent Born series. Iterating the above equation, we obtain, essentially, the usual Born series of the physicist's scattering equation at zero energy

$$\Psi = 1 + (4\pi)^{-1} \int \frac{|V(\mathbf{r}')|}{|\mathbf{r} - \mathbf{r}'|} \Psi(\mathbf{r}')\, d^3r' \tag{5}$$

whose terms are finite to all orders [2]. Therefore, what we have to study is this last equation, as was done in [1].

As we shall see, the conditions for having bound states are of the form $I_j \geqslant 1$, where I_j are appropriate integrals of the potential. With equality sign, we may have either bound states, or resonances at zero energy. To be sure of having true bound states, it would therefore be sufficient to take the inequality sign. This means that we strengthen the potential, and therefore the possible resonances become true bound states.

In paper [1], considering the totality of the potential, it was shown that a sufficient condition for the existence of at least one bound state would be

$$\operatorname*{Inf}_{|\mathbf{r}|=R} \int_{|\mathbf{r}'|\leqslant R} \frac{d\mathbf{r}'}{4\pi} \frac{|V(r')|}{|\mathbf{r}-\mathbf{r}'|} + R \operatorname*{Inf}_{|\mathbf{r}|=R} \int_{|\mathbf{r}'|>R} \frac{d\mathbf{r}'}{4\pi} \frac{|V(\mathbf{r}')|}{|\mathbf{r}'-\mathbf{r}|\,|\mathbf{r}'|} \geqslant 1. \tag{6}$$

Here, R is arbitrary, and can be chosen at will. Our purpose in the present paper is to find the generalization of (6) which would guarantee the existence of at least n bound states. The method of proof is quite similar to that of [1], to which we refer the reader for details.

Divide the space into spherical shells $\Omega_1, \Omega_2, \ldots, \Omega_n$, $\Omega_j = \{R_{j-1} \leqslant |\mathbf{r}| \leqslant R_j\}$, $j = 1, 2, \ldots, n$, $R_0 = 0$, $R_n = \infty$, and consider the potential (henceforth, we shall write $|\mathbf{r}| = r$, etc.)

$$V_j(\mathbf{r}) = V(\mathbf{r})\theta(R_j - r)\theta(r - R_{j-1}) \tag{7a}$$

so that $V(\mathbf{r}) = \sum V_j(\mathbf{r})$.

Suppose now that each

$$H_j = -\Delta + V_j(\mathbf{r}), \qquad \mathbf{r} \in \Omega_j, \tag{7b}$$

defined as a self-adjoint operator in $L^2(\Omega_j)$ by imposing some boundary condition on $\partial\,\Omega_j$ has one eigenstate with $E < 0$. Would it be then possible for $H = -\Delta + V(\mathbf{r})$, defined as a unique self-adjoint operator in $L^2(\mathbb{R}^3)$ to have n bound states with $E < 0$? The answer is yes provided we impose on each Ω_j the Dirichlet boundary condition

$$\Psi(\mathbf{r}) = 0, \qquad \mathbf{r} \in \partial\Omega_j. \tag{7c}$$

It amounts to putting infinite walls between each shell. For the proof of this lemma and complete references, see [4]. It is now obvious that what we need in order to secure the existence of at least n bound states is the analogue of (6) for each Ω_j with Dirichlet boundary condition. Remember now that the (simple) proof of (6) was based on the zero energy solution of the Schrödinger equation, i.e., Eq. (5), where 1 represents the boundary value, i.e., the value of $\Psi(\infty)$. We must now write the corresponding equation for each Ω_j. We have therefore to get first the Green's function $(\Delta)^{-1}$ for Ω_j with Dirichlet boundary condition. Such a Green's function can easily be found by the method of images known in electrostatics [5], and is given by ($r = |\mathbf{r}|$, etc., $R_{j-1} < r, r' < R_j$):

$$G_j(\mathbf{x}, \mathbf{y}) = (-4\pi)^{-1} \left[\frac{1}{|\mathbf{x} - \mathbf{y}|} + \sum_{n=1}^{\infty} (-)^n \left(\frac{q_n}{|\mathbf{x} - l_-^{(n)} \mathbf{y}/y|} + \frac{q_n'}{|\mathbf{x} - l_+^{(n)} \mathbf{y}/y|} \right) \right], \tag{8}$$

where $l_\pm^{(n)}$ are obtained by recursion from

$$q_0 = q_0' = 1, \qquad l_+^{(0)} = l_-^{(0)} = |\mathbf{y}|, \tag{9a}$$

$$q_{n+1} = q_n' R_{j-1}/l_+^{(n)}, \qquad q_{n+1}' = q_n R_j/l_-^{(n)}, \tag{9b}$$

$$l_-^{(n+1)} = R_{j-1}^2/l_+^{(n)}, \qquad l_+^{(n+1)} = R_j^2/l_-^{(n)}. \tag{9c}$$

It can be easily verified that indeed G_j vanishes whenever $\mathbf{x}$ is on the spheres R_{j-1} or R_j, and that it is strictly negative inside. To show this last point, we use the fact that $G_j(\mathbf{X}, \mathbf{X}')$ is a harmonic function except at $\mathbf{X} = \mathbf{X}'$. Therefore, it cannot have local maxima and minima inside Ω_j punctured at $\mathbf{X}'$. Therefore, since G_j vanishes on $\partial \Omega_j$, it cannot vanish elsewhere. At $\mathbf{X} = \mathbf{X}'$, it reaches its absolute minimum, $-\infty$. All these facts are well known [6, 7]. Notice also that, when $R_{j-1} \to 0$, or $R_j \to +\infty$, or both, we recover the Green's function for $0 < r < R$, $r > R$, or the entire space, respectively [6]:

$$G_{[0,R]} = -\frac{1}{4\pi} \left[\frac{1}{|\mathbf{x} - \mathbf{y}|} - \frac{R}{y\,|\mathbf{x} - R^2\mathbf{y}/y^2|} \right], \qquad x, y < R, \tag{10a}$$

$$G_{[R,\infty)} = -\frac{1}{4\pi} \left[\frac{1}{|\mathbf{x} - \mathbf{y}|} - \frac{R}{y\,|\mathbf{x} - R^2\mathbf{y}/y^2|} \right], \qquad x, y > R \tag{10b}$$

II. Bound States with Dirichlet Conditions

We consider now the Schrödinger equation (1), with Dirichlet boundary conditions, in each domain Ω_j. Since the potential is well behaved (locally L^2), and the domain Ω_j is finite, we have only a discrete infinity of eigenvalues $E_1 < E_2 < \cdots$ tending to $+\infty$, and bounded from below. They are given by the L^2 solutions of the Fredholm equation

$$\Psi_n = \int_{\Omega_j} G_j[V - E_n]\, \Psi_n\, d^3r'. \tag{11a}$$

When the potential is weak, we have $E_1 > 0$. Increasing the strength of the potential and making it more attractive, there appears a situation where the ground state E_1

crosses the value zero and becomes negative. Just at that point, the inhomogeneous integral equation at zero energy

$$\Psi = 1 + \int_{\Omega_j} G_j V \Psi \, d^3r' \tag{11b}$$

fails to have a solution because of the fact that the ground-state wave function, i.e., the solution of the homogeneous equation, is always strictly positive (Ref. [3, Vol. IV, p. 201 ff]), and therefore is not orthogonal to 1. Before such a situation occurs, the inhomogeneous equation has a unique solution, which is given by iteration. Since all the terms of this series are positive (remember that G_j and V are both negative) Ψ itself is positive, and we are in a situation very similar to that treated in [1].

Indeed, in [1], the starting point was the well-kbown fact that when the potential is weak (no bound states present), the Eq. (5) could be solved by iteration, and that the Born series thus obtained is convergent. Each term of this series being positive, we obtain that the solution Ψ itself is everywhere bounded and positive. We then choose an arbitrary sphere of radius R, which divides the space into two regions, take the infimum of Ψ inside the sphere, and the infimum of $|\mathbf{r}|\Psi$ outside, replace them into the integral equation (5), and obtain that a necessary condition for having no bound states is $I < 1$, where I is the left-hand side of (6). It follows that (6) is a sufficient condition for the existence of at least one bound state.

The reason why we take the infimum of $|\mathbf{r}|\Psi$ outside the sphere R instead of the infimum of Ψ itself, as for the interior region, is that the asymptotic behaviour of the right-hand side of (5) for large $\mathbf{r}$ is given by $1 + c|\mathbf{r}|^{-1}$. It is then obvious that the infimum of Ψ for $|\mathbf{r}| > R$ would be 1, reached for $|\mathbf{r}| = \infty$, and this leads to no significant result.

In order to see how the method works in the present case, let us consider in detail the problem for the sphere Ω_1

$$\Psi(\mathbf{r}) = 1 + \int_{|\mathbf{r}'| \leqslant R_1} d^3r' \, G(\mathbf{r}, \mathbf{r}') \, V(\mathbf{r}') \, \Psi(\mathbf{r}'), \tag{12}$$

where G is given by (10a). Remember that both G and V are negative, and for X, $Y < R_1$, $\Delta_X G(\mathbf{X}, \mathbf{Y}) = \delta(\mathbf{X} - \mathbf{Y})$.

We assume now, as was done in [1], that there are no bound states, and proceed along similar lines. We introduce a sphere of radius R, $R < R_1$, and consider (12) in $r \leqslant R$ and $R \leqslant r \leqslant R_1$ separately. Since Ψ is positive and bounded (convergent Born series), and the Green's function vanishes on the sphere R_1, it is obvious, according to (12), that the infimum of Ψ for $R \leqslant |\mathbf{r}| \leqslant R_1$ is 1, and is reached on the sphere R_1. We have therefore to study first the exact behaviour of Ψ near the boundary $|\mathbf{r}| = R_1$. We are looking for a behaviour independent of angles, similar to $|\mathbf{r}|^{-1}$ of the previous case (see lemma below), and so it is sufficient to assume here that the potential is spherically symmetric. We then easily obtain (see the Appendix)

$$\Psi(\mathbf{r}) = 1 + C \frac{R_1 - r}{rR_1} + o(R_1 - r), \tag{13}$$

where $r = |\mathbf{r}|$ and C is a positive constant. Now we go back to the general case (no spherical symmetry). Let

$$M_1 = \operatorname*{Inf}_{|\mathbf{r}|\leqslant R} \Psi(\mathbf{r}), \tag{14a}$$

$$M_2 = \operatorname*{Inf}_{R_1 \geqslant |\mathbf{r}| \geqslant R} \left(\frac{r}{R_1 - r}\right) \Psi(\mathbf{r}), \tag{14b}$$

Using these definitions in (12), we easily obtain

$$\begin{aligned} M_1 \geqslant 1 &+ M_1 \operatorname*{Inf}_{|\mathbf{r}|\leqslant R} \int_{|\mathbf{r}'|\leqslant R} |GV|\, d^3r' \\ &+ M_2 \operatorname*{Inf}_{|\mathbf{r}|\leqslant R} \int_{R\leqslant|\mathbf{r}'|\leqslant R_1} d^3r'\, |GV|\, \frac{R_1 - r'}{r'} \end{aligned} \tag{15a}$$

and

$$\begin{aligned} M_2 \geqslant \left(\frac{R}{R_1 - R}\right) &+ M_1 \operatorname*{Inf}_{R\leqslant|\mathbf{r}|\leqslant R_1} \left(\frac{r}{R_1 - r}\right) \int_{|\mathbf{r}'|\leqslant R} d^3r'\, |GV| \\ &+ M_2 \operatorname*{Inf}_{R\leqslant|\mathbf{r}|\leqslant R_1} \left(\frac{r}{R_1 - r}\right) \int_{R\leqslant|\mathbf{r}'|\leqslant R_1} d^3r' \left(\frac{R_1 - r'}{r'}\right) |GV|. \end{aligned} \tag{15b}$$

In order to go further, we need the folloling:

LEMMA. *Let $\mathbf{X} \in \Omega_j$, where Ω_j is the domain (finite or infinite) $R_{j-1} \leqslant |\mathbf{X}| \leqslant R_j$ introduced before, and $G_j(\mathbf{X}, \mathbf{Y})$ the corresponding Green's function, given by* (8), (10a), *or* (10b). *Let r be arbitrary, $R_{j-1} \leqslant r \leqslant R_j$, and $W \geqslant 0$, and define*

$$F(\mathbf{X}) = -\int_{\Omega_j} G_j(\mathbf{X}, \mathbf{Y})\, W(\mathbf{Y})\, d^3Y,$$

$$f(r) = \operatorname*{Inf}_{|\mathbf{X}|=r} F(\mathbf{X}),$$

where W, besides being nonnegative, satisfies the same integrability conditions as those given at the beginning for the potential (locally L^2,..., etc.). Then $r(r - R_{j-1})^{-1} f(r)$ is a decreasing function of r, and $R_j r(R_j - r)^{-1} f(r)$ an increasing function, for r in the interval (R_{j-1}, R_j). Notice that both products are positive. Also, because of the vanishing of G_j on the boundary, the first one vanishes at R_j, and the second one at R_{j-1}.

For the proof of this lemma, see the Appendix. We only notice here the following facts. Making $R_{j-1} \to 0$, we find that $f(r)$ is a decreasing function of r inside the sphere R_j. This is a simple consequence of the fact that $f(r)$, defined as the infimum of superharmonic functions $F(\mathbf{X})$, is itself a superharmonic function [7]. Indeed, we have $\Delta F(\mathbf{X}) = -W(\mathbf{X}) < 0$, which is just one of the definitions of superharmonic functions (remember that a superharmonic function is just minus a subharmonic function). We can also make $R_j \to \infty$. We get then that $r f(r)$ is an increasing function of r outside the sphere R_{j-1}. When both $R_{j-1} = 0$ and $R_j = \infty$, we find that $f(r)$ is decreasing, and $r f(r)$ increasing, for all $r > 0$. These two properties were shown in [1], and were used to prove (6).

From the above lemma, it follows immediately that the infima in (15a) are reached on the intermediate sphere R. Calling these infima J_1 and J_2 respectively, we obtain

$$M_1 \geqslant 1 + M_1 J_1 + M_2 J_2$$

and

$$M_2 \geqslant \frac{R}{R_1 - R}(1 + M_1 J_1 + M_2 J_2)$$

since the infima in (15b) are also reached on the sphere R. Combining these two inequalities we get

$$M_1 J_1 + M_2 J_2 \geqslant (1 + M_1 J_1 + M_2 J_2)\left(J_1 + \frac{R J_2}{R_1 - R}\right).$$

It follows from this last inequality that a necessary condition for the validity of our assumptions:

No bound states $\to$ convergence of the Born series $\to$ positivity of Ψ,

is $J_1 + R J_2/(R_1 - R) < 1$. Therefore, a sufficient condition for having at least one bound state with the potential $V(\mathbf{r})\theta(R_1 - r)$ is

$$\begin{aligned} &\operatorname*{Inf}_{|\mathbf{r}|=R} \int_{|\mathbf{r}'|\leqslant R} d^3r'\, G(\mathbf{r}, \mathbf{r}')\, V(\mathbf{r}') \\ &\qquad + \left(\frac{R}{R_1 - R}\right) \operatorname*{Inf}_{|\mathbf{r}|=R} \int_{R_1 > |\mathbf{r}'| > R} d^3r'\, G(\mathbf{r}, \mathbf{r}')\, V(\mathbf{r}') \frac{R_1 - |\mathbf{r}'|}{|\mathbf{r}'|} \geqslant 1, \qquad (16)\end{aligned}$$

where G is the Green's function (10a). This condition is, of course, stronger than condition (6) with $V(\mathbf{r})\theta(R_1 - r)$ because we impose now the vanishing of Ψ on R_1, whereas (6) was obtained without imposing any condition at finite distances. However, as we saw in the introduction, Dirichlet boundary conditions are necessary in order to be sure of the additivity of the number of bound states when we add up several regions, each with one bound state.

In the case of spherical symmetry, we find, by using the expansion [8]

$$\frac{1}{|\mathbf{r} - \boldsymbol{\rho}|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{\rho}{r}\right)^l P_l(\cos\theta) \qquad (17)$$

valid for $\rho < r$, where θ is the angle between $\mathbf{r}$ and $\boldsymbol{\rho}$, and integrating over the angles

$$\left(\frac{1}{R} - \frac{1}{R_1}\right) \int_0^R r^2 \,|\, V(r)|\, dr + \frac{R}{R_1 - R} \int_R^{R_1} \frac{(R_1 - r)^2}{R_1} \,|\, V(r)|\, dr \geqslant 1. \tag{18}$$

Making $R_1 \to \infty$ in (16) and (18), we get, respectively, formula (6), and an old result of Calogero for the radial case [9]

$$\frac{1}{R} \int_0^R r^2 \,|\, V(r)|\, dr + R \int_R^\infty |\, V(r)|\, dr \geqslant 1. \tag{19}$$

Condition (16), although not easily amenable to explicit and simple calculations for potentials with complicated shapes, is nevertheless not too complicated and computer calculations seem feasible. Notice also that R is arbitrary, and can be chosen at will.

Consider now the problem for $|\, \mathbf{r} \,| \geqslant R_2$, with the Green's function given by (10b), which vanishes on the sphere R_2, and assume again that there are no bound states. It is shown in the Appendix that the behaviour of the solution of

$$\Psi(\mathbf{r}) = 1 + \int_{|\mathbf{r}'| > R_2} d^3r'\, G(\mathbf{r}')\, V(\mathbf{r}')\, \Psi(\mathbf{r}') \tag{20}$$

with a spherically symmetric potential, near the sphere R_2, is given by

$$\Psi = 1 + C\,\frac{r - R_2}{r} + o(r - R_2). \tag{21}$$

We introduce now again an arbitrary sphere of radius $R (> R_2)$, and consider separately the two regions $R_2 \leqslant |\, \mathbf{r} \,| \leqslant R$ and $|\, \mathbf{r} \,| \geqslant R$. Let

$$M_1 = \operatorname*{Inf}_{R_2 \leqslant |\mathbf{r}| \leqslant R} \Psi(\mathbf{r}) \left(\frac{r}{r - R_2}\right),$$
$$M_2 = \operatorname*{Inf}_{r \geqslant R} |\, \mathbf{r} \,|\, \Psi(\mathbf{r}).$$

Reasoning exactly as before, and using again our lemma, we obtain that a sufficient condition for having at least one bound state with the potential $V(\mathbf{r})\theta(r - R_2)$ in $|\, \mathbf{r} \,| \geqslant R_2$ with Dirichlet condition on R_2 is

$$\left(\frac{R}{R - R_2}\right) \operatorname*{Inf}_{|\mathbf{r}| = R} \int_{R_2 \leqslant |\mathbf{r}'| \leqslant R} G(\mathbf{r}, \mathbf{r}')\, V(\mathbf{r}') \left(\frac{|\, \mathbf{r}' \,| - R_2}{|\, \mathbf{r}' \,|}\right) d^3r'$$
$$+ R \operatorname*{Inf}_{|\mathbf{r}| = R} \int_{|\mathbf{r}'| > R} G(\mathbf{r}, \mathbf{r}')\, \frac{V(\mathbf{r}')}{|\, \mathbf{r}' \,|}\, d^3r' \geqslant 1, \tag{22}$$

where G is given by (10b). Making here $R_2 \to 0$, we obtain again, as expected, formula (6). When the potential is spherically symmetric, we obtain, after integration over the angles, the sufficient condition

$$\left(\frac{1}{R - R_2}\right) \int_{R_2}^R (r - R_2)^2 \,|\, V(r)|\, dr + (R - R_2) \int_R^\infty |\, V(r)|\, dr \geqslant 1. \tag{23}$$

Again, in (22) and (23), R is arbitrary, and can be varied in order to obtain the optimum inequality.

For the general case of a finite spherical shell $0 < R_1 \leqslant |\mathbf{r}| \leqslant R_2 < \infty$, it is shown in the Appendix that we have behaviour similar to (13) for $r \to R_2$ and (21) for $r \to R_1$. Starting now from the integral equation (12), introducing R, and

$$M_1 = \operatorname*{Inf}_{R_1 \leqslant |\mathbf{r}| \leqslant R} \left(\frac{r}{r - R_1}\right) \Psi(\mathbf{r}),$$

$$M_2 = \operatorname*{Inf}_{R \leqslant |\mathbf{r}| \leqslant R_2} \left(\frac{r}{R_2 - r}\right) \Psi(\mathbf{r})$$

and following the same procedure as before, we obtain, thanks to our lemma, that the condition

$$\left(\frac{R}{R - R_1}\right) \operatorname*{Inf}_{|\mathbf{r}| = R} \int_{R_1 \leqslant |\mathbf{r}'| \leqslant R} G(\mathbf{r}, \mathbf{r}')\, V(\mathbf{r}') \left(\frac{|\mathbf{r}'| - R_1}{|\mathbf{r}'|}\right) d^3r'$$
$$+ \left(\frac{R}{R_2 - R}\right) \operatorname*{Inf}_{|\mathbf{r}| = R} \int_{R \leqslant |\mathbf{r}'| \leqslant R_2} G(\mathbf{r}, \mathbf{r}')\, V(\mathbf{r}') \left(\frac{R_2 - |\mathbf{r}'|}{|\mathbf{r}'|}\right) d^3r' \geqslant 1, \qquad (24)$$

where G is given by the general formula (8) with $j = 2$, is sufficient to secure the existence of at least one bound state for the potential $V(\mathbf{r})\theta(r - R_1)\theta(R_2 - r)$ with Dirichlet condition on the spheres R_1 and R_2. Again, R is arbitrary here, and can be varied in order to make the inequality optimal. Making $R_1 \to 0$, $R_2 \to \infty$, or both, we recover our previous conditions.

In the case of spherical symmetry, integrating over the angles (tedious but straightforward), and using formulae (30) and (31) of the Appendix, leads to the condition

$$\Big\{\left(\frac{1}{R - R_1}\right) \int_{R_1}^{R} (r - R_1)\, r\, |V(r)|\, dr + \left(\frac{R}{R_2 - R}\right) \int_{R}^{R_2} |V(r)|\, (R_2 - r)\, dr$$
$$+ \left(\frac{R_1}{(R_2 - R_1)(R - R_1)}\right) \int_{R_1}^{R} (r - R_2)(r - R_1)\, |V(r)|\, dr$$
$$+ \left(\frac{R}{(R_2 - R)(R_2 - R_1)}\right) \int_{R}^{R_2} |V(r)|\, (r - R_1)(r - R_2)\, dr$$
$$- \left(\frac{R}{(R_2 - R_1)(R - R_1)}\right) \int_{R_1}^{R} (r - R_1)^2\, |V(r)|\, dr - \left(\frac{R_1}{(R_2 - R)(R_2 - R_1)}\right)$$
$$\times \int_{R}^{R_2} |V(r)|\, (R_2 - r)^2\, dr\Big\} \geqslant 1 \qquad (25)$$

from which all our previous conditions can be obtained by taking the appropriate limits.

We have therefore completed our programme of finding conditions which would secure the existence of at least n bound states. Indeed, as was explained in the introduction, if we can divide the space into n spherical shells $(R_{j-1}, R_j)\, j = 0, 1, ..., n$, $R_0 = 0$, $R_n = \infty$, in such a way that (24) or (25) are satisfied for each shell with the appropriate Green's function (8), (10a) or (10b), we are sure that the potential $V(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^3$, has at least n bound states.

So far, we have assumed that the potential satisfies (2). However, this condition is not really necessary because it excludes $r^{-2+\epsilon}$ behaviour at the origin and $r^{-2-\epsilon}$ at infinity. As is shown in [2, 3], all that is needed is to assume the Rollnik condition

$$\int \frac{|V(\mathbf{r})|\,|V(\mathbf{r}')|}{|\mathbf{r}-\mathbf{r}'|^2}\, d^3r\, d^3r' < \infty. \tag{26}$$

In fact, it is obvious from our results that they are valid under very general assumptions, i.e., as long as the integrals are meaningful. For the spherically symmetric case, it is obvious, as expected, that what is needed is only

$$\int_0^\infty r\,|V(r)|\, dr < \infty. \tag{27}$$

Like many other bounds on the number of bound states, it is easily seen here also that our bounds are saturated by δ-function potentials

$$V(r) = -g\,\delta(r - r_0). \tag{28}$$

Indeed, considering this potential in the domain $R_1 \leqslant r \leqslant R_2$ with Dirichlet boundary conditions, one finds that the exact condition for having one bound state is (it is necessarily an S-state)

$$g \geqslant \left(\frac{1}{r_0 - R_1} + \frac{1}{R_2 - r_0}\right). \tag{29}$$

The same conclusion is reached by considering our formula (25), and by choosing $R = r_0 - \epsilon$ or $r_0 + \epsilon$.

We mention also another kind of sufficient condition for having at least one bound state in each domain Ω_j, which is [1]

$$\operatorname{Tr} K_j^{(3)} - \operatorname{Tr} K_j^{(2)} > 0, \tag{30}$$

where the positive kernel K_j is given by

$$K_j = -|V|^{1/2} G_j |V|^{1/2}, \tag{31}$$

G_j being the Green's function (8), and the superscripts (2) and (3) meaning the second and the third iterates of K_j.

We end up by reminding the reader that there are various necessary conditions (Bargmann, Schwinger, Calogero, Glaser–Grosse–Martin–Thirring, Martin, Ghirardi–Rimini, Cwickel–Simon, etc.) which must be satisfied in order to have n bound states [2, 3]. Therefore, before trying to see whether a potential admits n bound states, one must verify that the necessary conditions are indeed satisfied. Also, one can choose, instead of spherical shells used in this paper, other shapes (cubes, ellipsoids,...) provided one can calculate the corresponding Green's functions. Finally, there is no difficulty for generalizing our results to spaces of higher dimensions $N > 3$ [1].

APPENDIX

(a) *Behaviour of Ψ near the Boundaries*

Let us consider first the case of the sphere Ω_1 with radius R_1, when Ψ is the solution of Eq. (12). The potential $V = V(r)$ being spherically symmetric by assumption, we separate the integrations for $r' \leqslant r$ and for $r' \geqslant r$, and we use expansion (17) for the two terms of the Green's function (10a). When integrating over the angles, only the term $l = 0$ gives a non-zero contribution of $4\pi/r$, where $r > \rho$. Hence we get

$$\begin{aligned}\Psi &= 1 + \left(\frac{1}{r} - \frac{1}{R}\right) \int_0^r r'^2 \mid V(r') \mid \Psi(r')\, dr' \\ &\quad + \int_r^{R_1} \left(1 - \frac{r'}{R_1}\right) r' \mid V(r') \mid \Psi(r')\, dr' \\ &= 1 + \frac{R_1 - r}{rR_1} \Big[\int_0^{R_1} r'^2 \mid V(r') \mid \Psi(r')\, dr' \\ &\quad + R_1 \frac{R_1 - r}{2} \mid V(r_1) \mid \Psi(R_1) + o(R_1 - r) \Big], \end{aligned} \tag{A1}$$

which proves the behaviour (13) for r tending to R_1.

In the case of the region $|\,\mathbf{r}\,| = r \geqslant R_2$, the behaviour of the solution Ψ of (20) near the boundary is obtained in a similar way:

$$\begin{aligned}\Psi &= 1 + \left(1 - \frac{R_2}{r}\right) \int_r^{\infty} r' \mid V(r') \mid \Psi(r')\, dr' \\ &\quad + \frac{1}{r} \int_{R_2}^{r} \left(1 - \frac{R_2}{r}\right) r'^2 \mid V(r') \mid \Psi(r')\, dr' \\ &= 1 + \left(\frac{r - R_2}{r}\right) \Big[\int_{R_2}^{\infty} r' \mid V(r') \mid \Psi(r')\, dr' \\ &\quad + R_2 \mid V(R_2) \mid \Psi(R_2) \left(\frac{r - R_2}{2}\right) + o(r - R_2) \Big], \end{aligned} \tag{A2}$$

which yields the behaviour (21) for $r \to R_2$.

In the case of the finite spherical shell $0 < R_{j-1} \leqslant r \leqslant R_j < +\infty$, i.e., of the integral equation (12) with the Green's function (8), we find, after similar but lengthy calculations,

$$\Psi = 1 + C\left(\frac{r - R_{j-1}}{r}\right) + o(r - R_{j-1}), \qquad r \to R_{j-1}\,, \tag{A3}$$

$$\Psi = 1 + C'\frac{R_j - r}{R_j r} + o(R_j - r), \qquad r \to R_j\,, \tag{A4}$$

where C and C' are positive constants. The computations are too long for writing them down explicitly but present no new difficulty with respect to the previous cases, provided one uses the following summation formulas

$$\sum_1^\infty (-)^n\, q_n = \frac{R_{j-1}}{R_j - R_{j-1}}\left(1 - \frac{R_j}{y}\right), \tag{A5a}$$

$$\sum_1^\infty (-)^n\, \frac{q'_n}{l_+^{(n)}} = \frac{1}{R_j - R_{j-1}}\left(\frac{R_{j-1}}{y} - 1\right), \tag{A5b}$$

which can be easily derived from (9a), (9b), (9c).

(b) *Proof of the Lemma*

The function $F(\mathbf{X}) = \int_{\Omega_j} - G_j(\mathbf{X}, \mathbf{Y}) W(\mathbf{Y})\, d^3Y$ is superharmonic in Ω_j since $\Delta F(\mathbf{X}) = -W(\mathbf{X}) < 0$. The function $f(r)$ defined as the infimum of superharmonic functions $F(\mathbf{X})$ is also superharmonic and it follows that (see [10] for the differentiality properties of singular integrals)

$$\Delta f = \frac{1}{r^2}\frac{d}{dr}(r^2 f') = \frac{1}{r^2}(r^2 f'' + 2rf') < 0. \tag{A6}$$

In order to check that $R_j r(R_j - r)^{-1} f(r)$ is an increasing function, let us verify that its derivative is positive, or that $(R_j - r)\, rf' + R_j f > 0$. This results from the fact that the quantity $(d/dr)[(R_j - r)\, rf' + R_j f] = (R_j - r)(rf'' + 2f')$ is negative as follows from (32). Integrating this expression from r to R_j we get

$$[(R_j - r)(rf') + R_j f] > [(R_j - r)\, rf' + R_j f]_{r=R_j}$$

the right-hand side vanishing because of the assumptions on $W(\mathbf{y})$ and the boundary conditions on G_j.

On the other hand, the function $r(r - R_{j-1})^{-1} f(r)$ is a decreasing function. Indeed, its derivative is proportional to $r(r - R_{j-1}) f' - R_{j-1} f$, which is a negative quantity. This follows from (A6) since

$$\frac{d}{dr}[r(r - R_{j-1}) f' - R_{j-1} f] = (r - R_{j-1})(2f' + rf'') < 0.$$

Integrating this inequality between R_{j-1} and r we have

$$[r(r - R_{j-1})f' - R_{j-1}f] < [r(r - R_{j-1})f' - R_{j-1}f]_{r=R_{j-1}},$$

where the right-hand side is again zero.

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