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ON EXISTENCE OF LOCATION EQUILIBRIA IN THE 3-FIRM HOTELLING PROBLEM*

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If consumers have a positive probability to purchase from each firm, then centrally agglomerated and/or symmetric dispersed location equilibria may exist in the 3-firm Hotelling problem.

I. INTRODUCTION

IT IS well known that the Hotelling model of spatial competition with three firms admits no equilibrium solution; see Chamberlin [1933] and Lerner and Singer [1937]. Apparently, this non-existence result is associated with the assumption that customers patronize the nearest firm. Now, empirical evidence supports the idea that consumers do not necessarily choose to buy from the closest firm, since they also take variables other than distance into account. As these variables are often unobservable, firms cease to be fully informed about consumers' motivations. Consequently, they can at best determine the shopping behavior of a particular customer *up to a probability distribution.*

The purpose of this note is to reconsider the 3-firm Hotelling problem within a probabilistic framework. It is shown that two different types of equilibria emerge: *centrally agglomerated* equilibria, in which the three firms are clustered, and *symmetric dispersed* equilibria, in which the three firms have distinct locations. These equilibria are compared to the socially optimal locations.

II. THE MODEL

The assumptions of the standard 3-firm Hotelling location model are as follows:

(i) Three firms $i = 1, 2, 3$ locate on a segment of unit length, at locations x_i ($i = 1, 2, 3$) and sell a homogeneous commodity.

(ii) The distribution of customers is uniform on the segment (with unit density), and each of them buys a single unit of the commodity per unit of time.

(iii) Firms produce at zero marginal cost and sell at the same given mill price p; transportation costs increase linearly with distance, so that a

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customer located at x will pay $p+c|x-x_i|$ if he buys from firm i, where $c|x-x_i|$ stands for the transport cost between x and x_i.

(iv) As the mill price is the same, customers buy from the nearest firm. Accordingly, the demand addressed to firm i, denoted by $D_i(x_1, x_2, x_3)$, is given by the measure of the set of consumers closer to x_i than to x_j or x_k.

(v) Firm i chooses its location x_i in order to maximize its profits $\Pi_i(x_1, x_2, x_3) = pD_i(x_1, x_2, x_3)$.

(vi) The equilibrium corresponds to a non-cooperative Nash equilibrium in location, i.e., a triple x_1^*, x_2^*, and x_3^* such that $\Pi_i(x_i^*, x_j^*, x_k^*) \geqslant \Pi_i(x_i, x_j^*, x_k^*)$ for any $x_i \in [0, 1]$, $i = 1, 2, 3$ and $i \neq j \neq k$.

Discontinuities in the profit functions are fundamental for explaining the non-existence of equilibria in the above model. A natural idea is therefore to reformulate the model so as to generate continuous profit functions, like in de Palma *et al.* [1985], where it is assumed that *firms are not able to predict with certainty the decisions of customers.*[1] The reason is that, when they make their buying decisions, customers do take into account variables other than just the full delivered price. Under these circumstances, firms cannot do better but endowing customers with purchasing probabilities, which depend on the delivered prices. To determine these probabilities we assume that, before selecting their locations, firms are able to collect observations on the shopping behavior of customers who can choose between the three alternative firms. Denote by y_h a random variable which takes the value $i = 1$, 2 or 3 when customer h chooses to buy from firm 1, 2, or 3. The firms then estimate the vector of parameters β_i of the following econometric model (see, e.g., Amemiya [1981])

$$\Pr(y_h = i) = \beta_{0i} - \beta_{1i}|x_h - x_i|$$

where the first coefficient can be considered as an index of the "attractiveness" of firm i; $|x_h - x_i|$ is the distance between customer h and firm i, and $\beta_{1i}|x_h - x_i|$ measures the cost of the corresponding trip. Then, if the observations are distributed according to the logistic function we obtain

$$(1) \qquad \Pr(y_h = i) = \frac{\exp[(\beta_{0i} - \beta_{1i}|x_h - x_i|)/\delta]}{\sum_{j=1}^{3} \exp[(\beta_{0j} - \beta_{1j}|x_h - x_j|)/\delta]}$$

where $\delta > 0$ is a normalization constant.

[1] Kohlberg [1982] and Shaked [1982] have tackled the problem in other directions. Kohlberg suggests to add waiting time for service (which in turn depends on the number of customers visiting a firm) as argument of the utility function of each customer. As shown by Kohlberg, this restores continuity of the profit functions, but not existence of a Nash equilibrium. Shaked [1982] suggests to use mixed strategies instead of pure strategies. In spite of the discontinuities, Shaked was able to show that a Nash equilibrium in mixed strategies always exists in the 3-firm Hotelling problem. Though mathematically very elegant, it is hard to imagine that the model with mixed strategies still has sufficient predictive power to account for most locational decisions.

As is well known, a similar expression can also be derived from random utility maximization; see, e.g. McFadden [1974]. Assume indeed that firms choose to model the utility of a customer at x purchasing from firm i, as $-p-c\,|x-x_i|+\mu\varepsilon_i(x)$, instead of $-p-c\,|x-x_i|$ as in the standard model. Here, $\mu\varepsilon_i(x)$ is a random variable with zero mean and unit variance, which represents the utility associated with non-observable and/or non-measurable characteristics attributed by a consumer at x to the commodity sold by firm i. It is important to notice that $\varepsilon_i(x)$ is treated as a random variable to reflect the firms' lack of information about this consumer's tastes, and not (necessarily) to reflect inconsistencies in his/her behavior.

Given that consumers are utility-maximizers, and if the $\varepsilon_i(x)$ are identically, independently Gumbel distributed, the probability that the customer at x will purchase from firm i is given by

$$(2) \qquad P_i(x) = \frac{\exp[(-p-c\,|x-x_i|)/\mu]}{\sum_{j=1}^{3} \exp[(-p-c\,|x-x_j|)/\mu]}$$

In this model, μ is positive and can be interpreted as a measure of the degree of heterogeneity in customers' tastes.[2]

At this stage, it is clear that the two expressions (1) and (2) are identical provided that $p = -\beta_{0i}$, $c = \beta_{1i}$ and $\mu = \delta$.

In both cases, we are led to replace the above definition of the demand to firm i by

$$D_i(x_1, x_2, x_3) = \int_0^1 P_i(x; x_1, x_2, x_3)dx$$

i.e., the expected demand. Given the expressions of the probabilities (1) and (2), $D_i(\cdot)$ is now continuous in x_1, x_2 and x_3.

III. RESULTS

III(i). Let us first investigate the possible existence of an *agglomerated equilibrium*, i.e. a Nash equilibrium (x_1^*, x_2^*, x_3^*) which satisfies the additional property $x_1^* = x_2^* = x_3^* = x^*$.

The following results hold (see de Palma *et al.* [1985]):

Proposition 1

(a) If $0 < \mu/c < \infty$, then an agglomerated equilibrium can occur only at the center of the market segment, that is $x^* = 1/2$.
(b) If $\mu/c < 1/6$, then no agglomerated equilibrium exists.
(c) If $\mu/c \geqslant 1/3$, then an agglomerated equilibrium exists.

[2] Another interpretation of the model is obtained when consumers have a variety-seeking behavior which is modelled by means of an entropy-like utility function (see Anderson *et al.* [1986]). In this context, (2) represents the frequency of purchases from firm i over a certain period of time. A larger μ means that the customer at x chooses from a more dispersed shopping pattern.

As can be seen, the existence of an agglomerated equilibrium crucially depends on the value of μ/c, the ratio of the degree of variation in consumers' tastes and the transportation rate. This is so because the choices of consumers are more and more determined by their individual tastes, while the influence of the objective characteristic of firms (here the full price) gets weaker. The profit maximizing choice for firms is then central agglomeration.

The above results deal with the (non-) existence of an agglomerated equilibrium. Nothing is said, however, about the existence of a *dispersed equilibrium*, i.e., an equilibrium such that $x_1^* \neq x_2^* \neq x_3^*$. The issue is discussed now. More precisely, we want to answer the following questions.

(1) Does a dispersed equilibrium exist for some value of μ/c?
(2) Can an agglomerated equilibrium and a dispersed equilibrium simultaneously exist?

The complexity of the problem has made it impossible for us to find solutions analytically. We have therefore decided to resort to numerical computations. For simplicity, we focus on *symmetric equilibria*. The profits of the three firms have been computed for the following locations: $0 \leqslant x_1 < 1/2$, $x_2 = 1/2$ and $x_3 = 1 - x_1$. More specifically, the values of $\pi_1(\pi_3)$ and π_2 have been computed with a grid size of 10^{-3} for integer values of $100\mu/c$ with $0 \leqslant \mu/c \leqslant 0.40$.

The following results have been obtained:

Proposition 2

(a) If $\mu/c < 0.157$, there exists no symmetric equilibrium.
(b) If $0.157 \leqslant \mu/c < 1/6$, then there exists no agglomerated equilibrium, but symmetric dispersed equilibria do exist.
(c) If $1/6 \leqslant \mu/c < 0.27$, then there exist both agglomerated and symmetric dispersed equilibria.
(d) If $\mu/c \geqslant 0.27$, then there exists an agglomerated equilibrium, but symmetric dispersed equilibria vanish.

Figure 1 depicts the four possible regions. In region 1, there is no equilibrium; in region 2, there exist only dispersed equilibria; in region 4, there exist only agglomerated equilibria while in region 3, both types of equilibria coexist. For μ positive but very small relative to c, the profit functions are continuous but no equilibrium occurs. This is because the shape of the profit functions is still very close to what it is for $\mu = 0$. A minimal amount of heterogeneity is therefore required to restore existence. When c is not too small relative to μ, that is transport costs still matter, each firm may enjoy a local monopoly in equilibrium. Otherwise, the agglomeration is the only possible outcome. It is interesting to point out that, in the dispersed equilibrium, the distance between the two peripheral firms decreases monotonically as μ/c increases from 1/6 to 0.27. It should be apparent, therefore, that *μ/c acts as an agglomerative factor* in the present revision of the 3-firm Hotelling problem.

Our results can be given a nice interpretation in terms of product specification by competitive firms. As is standard in the theory of product differentiation, c can be viewed as the loss parameter in the consumers' utility derived from the observable characteristic. For small values of c, consumers

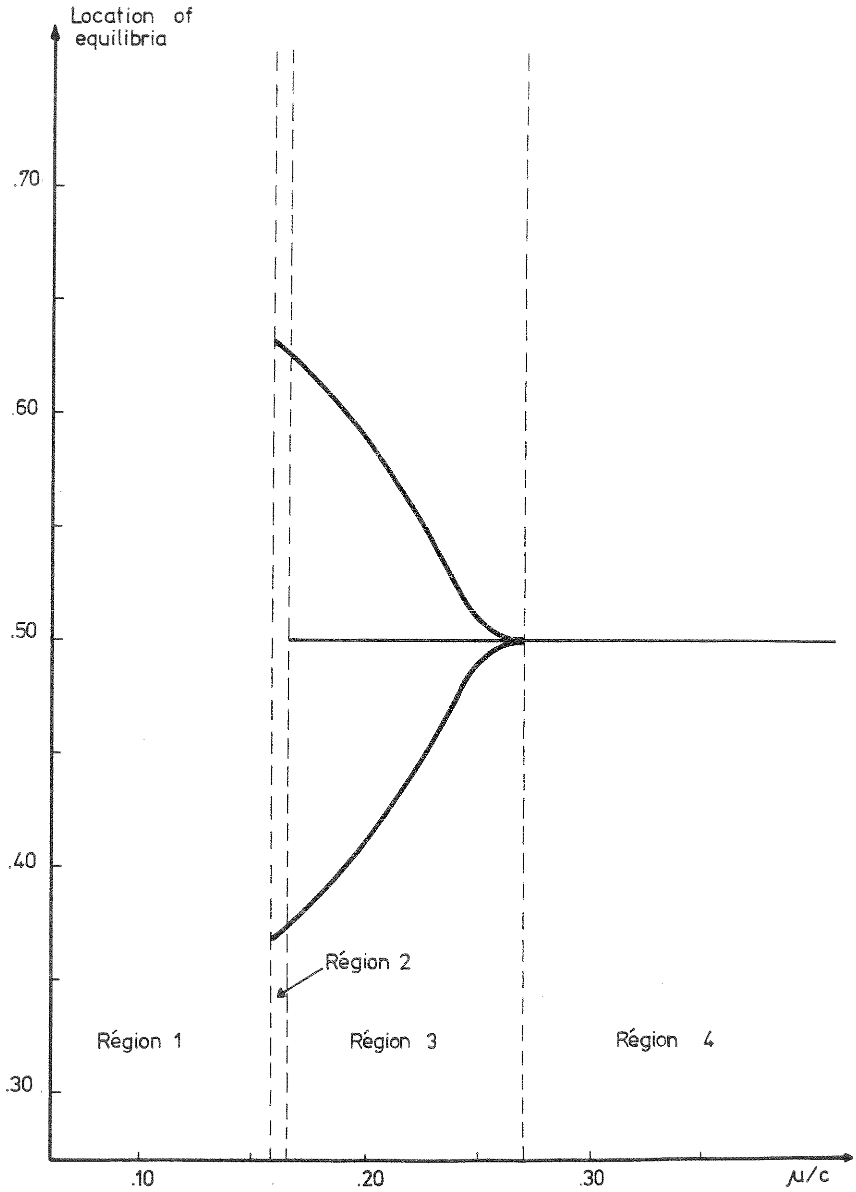

Figure 1
Location of Equilibria as a Function of μ/c

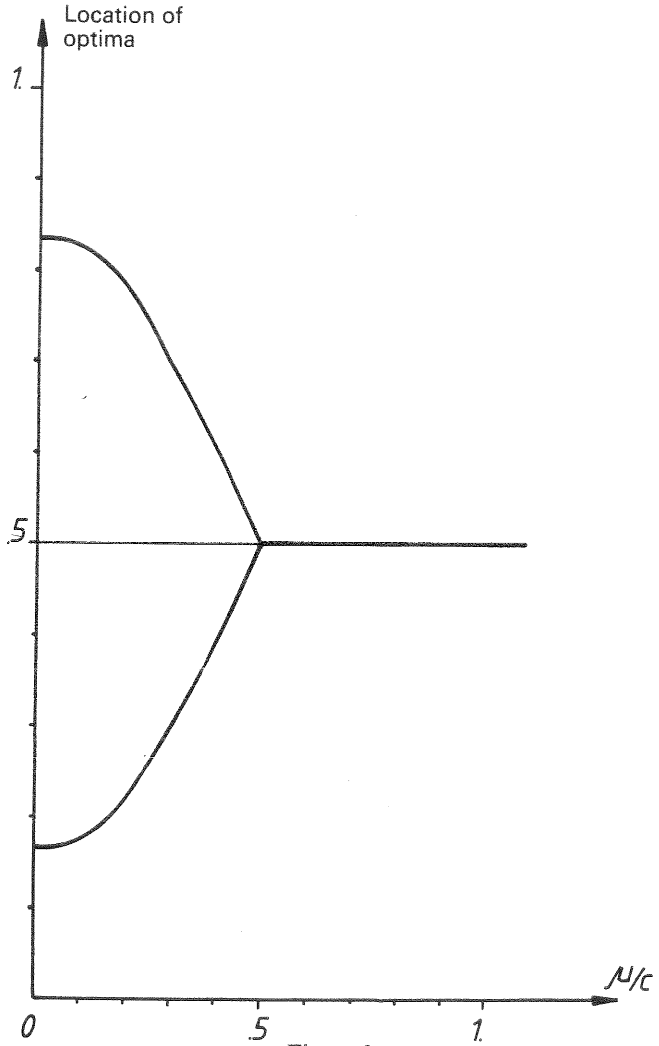

Figure 2
Location of Optima as a Function of μ/c

pay little attention to the variations in this characteristic and one may expect μ to be large relative to c; and vice versa for large values of c. If this holds, the model seems to nicely describe what happens, for instance, on the car market. Presumably, indeed, rich consumers have very specific preferences for some luxury brands, but less wealthy consumers care less about the specific standard brand they buy. Accordingly, in the first case, μ/c is "small" and there is a large variety in the group of luxury cars (contrast Mercedes with Rolls Royce and Ferrari) while, in the second, μ/c is "large" and popular cars look very much the same.

III(ii). We have also computed the socially optimal locations. Small and Rosen [1981] have derived the consumer surplus function associated with the logit. In the present context, this function can be written as $(\mu/c)\int_0^1 \sum_{j=1}^{3} \exp(-c\,|x-x_i|/\mu)dx$. Applying the FOC for symmetric locations gives rise to a transcendental equation that we have solved numerically, using the same grid size and the same values of μ/c as above. The results, depicted in Figure 2, are as follows:

Proposition 3

(a) When $\mu/c = 0$, the optimal locations are $x_1^0 = 1/6$, $x_2^0 = 1/2$ and $x_3^0 = 5/6$.
(b) As μ/c increases, the distance between the two peripheral firms decreases.
(c) For $\mu/c \geqslant 0.5$, the three firms are agglomerated.

A comparison of Figures 1 and 2 reveals that, for $\mu/c < 0.5$, the equilibrium locations (whenever they exist) are always *more concentrated* than the socially optimal locations. This is because the utility function underlying the above consumer surplus function embodies a dispersion effect (see Anderson *et al.* [1986]), whereas, in equilibrium, firms want to be close to the largest number of consumers, thus leading to less dispersion.[3]

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[3] For μ/c sufficiently large, but smaller than 0.5, the optimal configuration can be sustained as an equilibrium by a tax on transport. On the other hand, for μ/c small, this is not possible because the optimal configuration does not correspond to any equilibrium.

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