INTRODUCING REALISTIC SAVINGS PATTERNS IN INTERTEMPORAL MODELS *

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Discrete-time intertemporal welfare optimization models often involve the choice of discount factors whose value is determinantal, but usually unknown. In this note, we provide a general equilibrium formulation which involves intertemporal wealth transfers instead. We illustrate our approach with a simple numerical example, and show that in a convex analysis framework, it encompasses a version of the Golden Path Rule.

intertemporal optimization * discount factor * general equilibrium * savings

1. Introduction

A wide variety of economic models attempts to study issues that cannot be confined to particular points in time, but have effects that range over a whole (possibly infinite) interval. The most popular approach is to formulate them as discrete-time, finite-horizon, welfare optimization problems. Often, this involves the choice of time discount factors, or, equivalently, time preference rates tying the single period utility flows together. While this approach enjoys a number of advantages and most notably that of (relative) computational efficiency, it also poses the following problem:

The solution is, in general, quite sensitive to the choice of discount factors, yet, this choice is usually indefensible in an empirically or analytically acceptable manner.

The credibility of the solution is even more questionable when implied key macroeconomic flows exhibit a haphazard evolution pattern, inconsistent with empirical data or intuition (when empirical data are not available). Such a key flow in intertemporal models is the implied savings/consumption ratio. It has the advantage of being amenable to statistical estimation and is thus a valid basis for result evaluation.

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To illustrate the point consider the following simple macroeconomic welfare optimization model (it is a simpler version of ETA, Manne [8]):

$$NP^T = \begin{cases} \max & \sum_{t=1}^{T} \beta_t \log x_t + \beta_{T+1}\Phi_{T+1}(k_{T+1}, I_{T+1}), \\ \text{s.t.} & x_t + I_t \leqslant y_t, & t = 1, \ldots, T, \\ & y_t \leqslant F_t(k_t, l_t, e_t, g_t) - \epsilon_t e_t - \gamma_t g_t, & t = 1, \ldots, T, \\ & k_{t+1} \leqslant \theta k_t + \xi I_{t+1} + \eta I_t, & t = 0, \ldots, T, \\ & k_0,\ I_0 \text{ given}, \end{cases}$$

where

x_t	represents consumption for period t.
k_t	represents the capital at the start of period t; θ is its survival rate.
I_t	represents investment for period t; ξ (resp. η) is the fraction realized immediately (resp. in the next period); $\xi + \eta = 1$.
$\beta_t > 0$	is the time discount factor for period t.
$l_t \in R_+$	is an exogenously specified labor supply for period t (or a technological progress parameter).
$e_t \in R_+$	is electric energy for period t.
$g_t \in R_+$	is non-electric energy for period t.
F_t: $R_+^4 \to R \cup (-\infty\}$	is a four-factor production function for period t. It is assumed to be proper closed and concave.
Φ_{T+1}: $R_+^2 \to R \cup \{-\infty\}$	is a proper, closed and concave terminal stock and investment valuation function. We have used the dual equilibrium method for truncating the underlying infinite-horizon problem. This applies when

$$\begin{aligned} &\beta_{T+t} = \delta^{t-1}\beta_{T+1}, && t \geqslant 1, \quad \delta \text{ is a one-period discount factor}, \\ &l_{T+t} = (1+g_r)^{t-2}\hat{l}, && t \geqslant 2, \quad g_r \text{ is a post-terminal growth rate}, \\ &\epsilon_T = \epsilon, \quad \gamma_T = \gamma, \quad F_t = F, && t \geqslant T+2. \end{aligned}$$

It yields

$$\Phi_{T+1}(k_{T+1}, I_{T+1}) = \begin{cases} \log x_{T+1} + \dfrac{\delta}{1-\delta}\log \hat{x} + K_{T+1}(g_r, \delta), \\ \quad \text{if} \begin{bmatrix} x_{T+1} + \hat{I} \leqslant F_{T+1}(k_{T+1}, l_{T+1}, e_{T+1}, g_{T+1}) - \epsilon_{T+1}e_{T+1} \\ \qquad - \gamma_{T+1}g_{T+1}, \\ \hat{x} + \hat{I} \leqslant F(\hat{k}, \hat{l}, \hat{e}, \hat{g}) - \epsilon\hat{e} - \gamma\hat{g}, \\ \hat{k}(1+g_r-\theta\delta) \leqslant (1+g_r)(1-\delta)\theta k_{T+1} + \delta\eta\hat{I} + (1+g_r)\xi\hat{I} \\ \qquad + \eta(1-\delta)(1+g_r)I_{T+1}, \\ \quad \text{for some} \quad (x_{T+1}, e_{T+1}, g_{T+1}) \geqslant 0, \quad (\hat{x}, \hat{k}, \hat{I}, \hat{e}, \hat{g}) \geqslant 0, \end{bmatrix} \\ -\infty \text{ otherwise}. \end{cases}$$

Note that this is but one of various competing methods of truncation. Alternatively, we could have used a primal or dual target value approach, or a primal equilibrium method. For a theoretical justification of these methods, as well as some convergence properties, the reader is referred to Grinold [5,6,7] and Evers [4].

The model was calibrated for the whole of the EC countries and was run for a model horizon of 11 five-year (and one end-horizon) periods. For the precise value of the parameters, see Noël, de Groote and Smeers [11]. Figure 1 shows the consumption/income pattern obtained from a constant annual discount

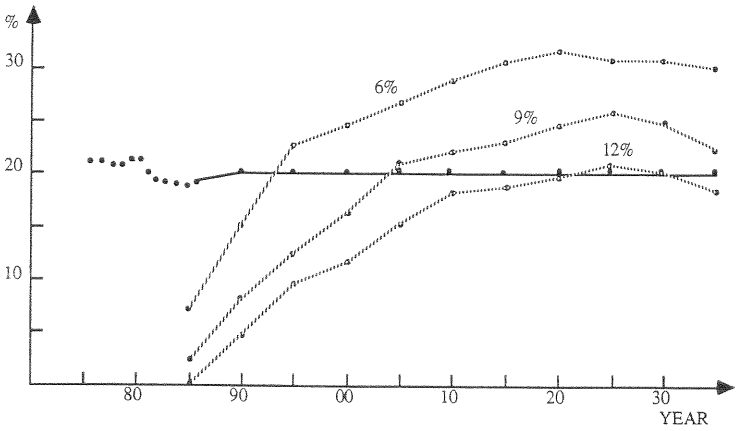

Fig. 1. Consumption/income ratios.

rate of 6%, 9% and 12%, respectively. In all cases it is in sharp contrast with available empirical data (see C.C.E. [3]), which show a limited fluctuation between 21% and 18.5% for the last ten years. We should note that the last periods' behavior may be somewhat perturbed due to end-effects.

Our aim is to reformulate the model so as to ensure that consumption/savings ratios match a predetermined pattern, and avoid the specification of an exogenous time discount factor.

In Section 2 we discuss the general model and present the main ideas. We introduce a general equilibrium formulation, in which the optimizing agents' behavior is independent of the discount factors. Instead, it depends on patterns of intertemporal wealth transfers. These patterns may be empirically measured, or, in certain simple cases, their optimal behavior may be analytically derived. We end by applying our formulation to the numerical example presented earlier, and contrasting our results to those obtained from the optimization approach.

Finally, in Section 3 we deal with a convex activity analysis model in which the capital accumulation mechanism is furthermore restricted. For this model, the results of Section 2 have a particularly meaningful economic interpretation. In particular, we specify a pattern of intertemporal wealth transfers that is consistent with a version of the Golden Rule of Capital Accumulation.

2. The model and main results

In what follows let

$x_t \in R^m_+$	represent consumption for period t.
$I_t \in R^m$	represent investment for period t.
$S^2_t \subseteq R^m$	be a closed convex non-empty set containing 0. It is usually taken to be the whole of R^m or R^m_+.
$k_t \in R^m_+$	represent the capital that is available at the beginning of period t.
$y_t \in R^m$	represent the production vector for period t. Its positive (negative) elements denote outputs (resp. inputs).
$S^1_t \subseteq R^m$	specify the capital/production pairs that are technologically feasible for period t. It is assumed non-empty closed and convex.
$u_t: R^m_+ \to R \cup \{-\infty\}$	is the utility function for period t. It is assumed to be proper closed and concave.
w_t	represent the exogenously specified wealth for period t.

Θ_t, Ξ_{t+1}, H_t specify the capital accumulation process. The capital available at period $t+1$ is assumed to be a (usually increasing) linear function of the previous period's capital and investment outlays and the current period's investment vector. Typically,

Θ_t is a diagonal matrix with non-negative entries at most equal to one; the ith element of Θ_t denotes the capital survival rate of capital of type i in period t.

Ξ_t, H_t are non-negative diagonal matrices summing up to the identity matrix. The ith element of Ξ_t (resp. H_t) denotes the fraction of investment of type i of period t that is effective in the same (resp. next) period.

Φ_{T+1}: $R_+^m \times S_{T+1}^2 \to R \cup \{-\infty\}$ is a closed and increasing proper concave function that evaluates the terminal levels of capital and investment.

The feasible set, X^T, is defined as follows:

$$X^T = \left\{(x_t, k_t, y_t, I_t)_1^T, I_{T+1}, k_{T+1} / (x_t, k_t, y_t, I_t)_1^T, I_{T+1}, k_{T+1} \text{ satisfies (1)}\right\},$$

$$\begin{cases} x_t + I_t \leqslant w_t + y_t, & t = 1, \ldots, T, \\ k_t \leqslant \Theta_{t-1} k_{t-1} + \Xi_t I_t + H_{t-1} I_{t-1}, & t = 1, \ldots, T+1, \\ (k_t, y_t) \in S_t^1, & t = 1, \ldots, T, \\ I_t \in S_t^2, & t = 1, \ldots, T+1, \\ I_0, k_0 \text{ given}, & \\ k_t, x_t \geqslant 0, & t = 1, \ldots, T, \\ k_{T+1} \geqslant 0. & \end{cases} \tag{1}$$

Let the present value prices be π_t for capital goods, and p_t for all other consumption goods – hereafter termed simply goods.

If k_t units of capital and y_t^- units of goods are inputs in the production process at period t, they yield y_t^+ units of goods in the same period, and $\Theta_t k_t$ units of capital in the next (capital that survived). Therefore, the net present value of profits due to productive activities is

$$\pi_{t+1} \Theta_t k_t + p_t y_t - \pi_t k_t.$$

If I_t units of consumable goods are invested in period t, they yield $\Xi_t I_t$ units of capital in the same period, and $H_t I_t$ units of capital in the next. Therefore, the net present value of profits due to investment activities is

$$\pi_{t+1} H_t I_t + \pi_t \Xi_t I_t - p_t I_t.$$

Let the wealth shares ν_{it} (resp. μ_{it}, λ_{it}), $t = 1, \ldots, T+1$, $i = 0, \ldots, T$ denote the share of the ith period's endowment (resp. production profit, investment profit) allocated to period t. Obviously, we require

$$(\lambda_{ij}, \mu_{ij}, \nu_{ij})_{i=0,\ldots,T}^{j=1,\ldots,T+1} \geqslant 0, \qquad \sum_{j=1}^{T+1} (\lambda_{ij}, \mu_{ij}, \nu_{ij}) = (1, 1, 1), \quad \nu_{0j} = 0.$$

Then the budget of period t is given by

$$M_t = \sum_{i=1}^{T} \nu_{it}(p_i w_i) + \sum_{i=0}^{T} \mu_{it}(\pi_{i+1} \Theta_i k_i + p_i y_i - \pi_i k_i) + \sum_{i=0}^{T} \lambda_{it}(\pi_{i+1} H_i I_i + \pi_i \Xi_i I_i - p_i I_i).$$

For any wealth shares (λ, μ, ν), and discount factors $(\beta_t)_1^{T+1}$, we define a respective general equilibrium problem, EQ^T, and a welfare optimization problem, P^T, as follows:

EQ^T: *Find a set of quantities* $(x_t, k_t, y_t, I_t)_0^T$, k_{T+1}, I_{T+1}, *and prices* $(p_t, \pi_t)_0^T \geqslant 0$ *that satisfy conditions* (i)–(vi).

(i) $(x_t)_1^T$ solve the *consumers'* subproblems; to maximize their utility subject to their budget constraint,

$$\max_{x_t} u_t(x_t) \quad \text{s.t.} \quad p_t x_t \leqslant M_t,$$

and it is understood that the terms in which π_0, p_0 appear are to be dropped in the above formulation.

(ii) $(k_t, y_t)_1^T$ solve the *producers'*, subproblems; to maximize the net present value of production profits,

$$\max_{k_t, y_t} \pi_{t+1}\Theta_t k_t + p_t y_t - \pi_t k_t \quad \text{s.t.} \quad (k_t, y_t) \in S_t^1, \quad k_t \geqslant 0 \quad \text{and } k_0 \text{ is given.}$$

(iii) $(I_t)_1^T$ solve the *investors'* subproblems; to maximize the net present value of investment profits,

$$\max_{I_t} \pi_{t+1} H_t I_t + \pi_t \Xi_t I_t - p_t I_t \quad \text{s.t.} \quad I_t \in S_t^2 \quad \text{and } I_0 \text{ is given.}$$

(iv) k_{T+1}, I_{T+1} solve the *terminal condition* subproblem,

$$\max_{k_{T+1}, I_{T+1}} \Phi_{T+1}(k_{T+1}, I_{T+1}) \quad \text{s.t.} \quad \pi_{T+1} k_{T+1} - \pi_{T+1}\Xi_{T+1} I_{T+1} \leqslant M_{T+1}, \quad I_{T+1} \in S_{T+1}^2.$$

(v) The *no-excess-demand* conditions are satisfied,

$$x_t + I_t \leqslant w_t + y_t, \qquad t = 1, \ldots, T,$$
$$k_{t+1} \leqslant \Theta_t k_t + \Xi_{t+1} I_{t+1} + H_t I_t, \qquad t = 0, \ldots, T.$$

(vi) The *Walras law* is satisfied,

$$\sum_{t=1}^{T} p_t(x_t + I_t - w_t - y_t) + \sum_{t=1}^{T+1} \pi_t(k_t - \Theta_{t-1} k_{t-1} - \Xi_t I_t - H_{t-1} I_{t-1}) = 0,$$

and

$$P^T\colon \max \sum_{t=1}^{T} \beta_t u_t(x_t) + \beta_{T+1}\Phi_{T+1}(k_{T+1}, I_{T+1}) \quad \text{subject to (1).}$$

We now turn to discuss the relationships between these two problems. First, note that for any set of discount factors for which P^T is stable and finite, there exists a primal (quantity) solution $(x_t, k_t, y_t, I_t)_0^T$, k_{T+1}, I_{T+1}, and supporting dual prices $(p_t, \pi_t)_0^T$, $\pi_{T+1} \geqslant 0$. Such a pair will be denoted as an optimal primal–dual solution. Then we can show:

Theorem 1. *Let $(\beta_t)_1^{T+1}$ be discount factors for which P^T is stable and finite, and $(x_t, k_t, y_t, I_t)_0^T$, k_{T+1}, I_{T+1}, $(p_t, \pi_t)_0^T$, π_{T+1} a primal–dual solution. Then, there exist wealth shares (λ, μ, ν) for which it also solves EQ^T.*

Proof. If $(x_t, k_t, y_t, I_t)_1^T$, I_{T+1}, k_{T+1} solves P^T with optimal dual prices $(p_t, \pi_t)_1^T$, π_{T+1}, it is clear that conditions (ii), (iii), (v), (vi) are satisfied for any wealth shares. It is also clear that we may find wealth shares such that the budget constraints of optimization problems (i) and (iv) are all satisfied with equality (e.g., let $\nu_{it} = \mu_{it} = \lambda_{it} = m_t / \sum_{t=1}^T m_t$, where $m_t = p_t w_t + \pi_{t+1}\Theta_t k_t - \pi_t k_t + p_t y_t + \pi_t \Xi_t I_t + \pi_{t+1} H_t I_t - p_t I_t =$ revenue generated in period t). Note that conditions (ii) and (iii) guarantee the non-negativity of these wealth shares. From the Kuhn–Tucker conditions for P^T, it follows that the Kuhn–Tucker conditions for (i) and (iv) are satisfied with dual prices: $1/\beta_t$. Under our assumptions of convexity and stability they are both necessary and sufficient for optimality. □

The above result relates the solutions of the welfare optimization problem P^T to those of the equilibrium problem EQ^T with an appropriately chosen set of shares. This is only part of the link between the two formulations. It can also be shown that each solution to the equilibrium problem EQ^T solves the welfare optimization problem P^T with an appropriately chosen set of discount factors. For this we shall need two additional assumptions:

Assumption 1 (constraint qualification). For every choice of positive discount factors, P^T is finite and stable in terms of RHS perturbations.

Note that it need only be verified for $(\beta_1, \ldots, \beta_{T+1}) = (1, 0, \ldots, 0) \cdots (0, \ldots, 0, 1)$, and is obviously satisfied whenever the Slater constraint qualification holds.

Assumption 2 (minimum wealth for a given set of wealth shares). For every choice of positive discount factors, the resulting income M_t, $t = 1, \ldots, T+1$ is positive when evaluated at the optimal solution of the resulting P^T and its supporting price vector.

It is more usual to assume conditions which ensure that Assumption 2 holds. Here we assume it directly, because there is a wide variety of circumstances for which it holds, but none of them encompasses all cases of interest. For example, it would be sufficient to guarantee that

$$\forall t = 1, \ldots, T+1, \quad \exists i, \tau: \sum_{j=1}^{T} \nu_{jt} w_j^i > 0 \quad \text{and} \quad (u_\tau^+)^i > 0,$$

where $(u_\tau^+)^i$ denotes the ith right derivative of u_τ.

The condition above assures us that each consumer holds some proportion of a desired good. It is a weaker form of the minimum wealth assumption introduced by Negishi [10].

Now we can state the converse to Theorem 1:

Theorem 2. *If Assumption* 1 *holds, for any wealth shares* (λ, μ, ν) *satisfying Assumption* 2, *there exists a set of discount factors* $(\beta_t)_1^{T+1}$ *and a primal–dual solution of* P^T *that also solves* EQ^t.

Proof. Similar to [10]; the only difference is that in our case the application of duality theory is guaranteed by Assumption 1, rather than the Slater constraint qualification originally used by Negishi. □

The general equilibrium model developed in this section has the characteristic that its formulation does not involve discount factors but wealth shares. Each period of the welfare optimization problem corresponds to a 'consumer' and one need only specify the profit share of each economic activity that this consumer will enjoy. A reasonable approach would be to require that the economic activities of period t add only to the income of that same period and the next $t+1$ (as savings). That would correspond to requiring

$$(\lambda_{ij}, \mu_{ij}, \nu_{ij}) = (0, 0, 0) \quad \text{whenever} \quad j > i+1 \quad \text{or} \quad j < i,$$

and would simplify our notation significantly. Such restricted transfers would not, in general, suffice to generate all the possible choices of discount factors, and consequently Theorem 1 would no longer be valid; they are, nonetheless, the most plausible candidates.

To illustrate our approach we have applied our equilibrium formulation to the example presented in the introduction, NP^T. The allocation of revenues between current consumption and savings was fixed to follow the path shown in Figure 1, i.e., to agree with the empirical data for 1985 and follow the last ten years' average (about 20%); from then on wealth shares were independent of the origin of the revenue (labor, capital). The computations were performed using the Manne–Chao–Wilson [9] algorithm as implemented in Boucher and Smeers [1]. The individual agents retained in the model are endowed with the utility functions (i) and (iv). The terminal condition subproblem requires the specification of a discount rate which is what the method is in principle designed to avoid. This inconsistency is part of the assumptions introduced in order to define the (terminal) agent (investment pattern in the primal method, dual variables evaluation, ...). The method should thus be viewed as a means to obtain sensible savings patterns within the horizon of interest and not necessarily over an infinite one. The method finds the weights, which, when applied to the utility functions of the individual agents, result in an optimization

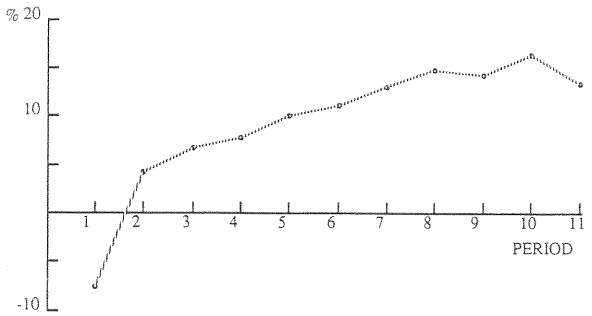

Fig. 2. Social discount rates.

problem whose solution is the desired equilibrium. In the multitemporal context of this problem, these weights transform the single period utility functions into an additively separable multitemporal utility index; the ratio of the weights of two successive periods is then equal to one plus the discount rate that prevails between them. The implied discount rates are readily available without extra computational effort: they are shown in Figure 2.

We shall not elaborate much on the negativity of the discount rate (-7%) between the first and second periods. This results from the fact that the initial stock of capital is far too large compared to its equilibrium value. The monotonicity of the discount rate, growing from 4% in period 3 to 15% in period 9, is worth noting. It is in sharp contrast with the usual assumption of a constant discount rate frequently found in economic planning models.

3. Convex activity analysis and the Golden Path Rule

We restrict our model by requiring

(i) (Simplified capital accumulation process) $\Xi_t = 0, \quad \Theta_t = H_t =$ Identity matrix,

(ii) (Investment free of sign restriction) $S_t^2 = R^m$,

(iii) (Convex activity production processes) $(k_t, y_t) \in S_t^1 \Leftrightarrow \begin{cases} \exists R^n \in z_t \geqslant 0, \\ y_t \leqslant A_t(z_t), \\ k_t \geqslant K_t(z_t), \end{cases}$

where

$z_t \in R_+^n$ denotes the activity intensity vector for period t.

A_t: $R^n \to R^m$ is a closed proper concave function specifying production for period t. Positive (negative) elements denote outputs (inputs); $A_t(0) = 0$.

K_t: $R^n \to R_+^m$ is a closed proper convex function representing capital utilization for period t; $K_t(0) = 0$.

We note that since $\Xi_t = 0$ the terminal conditions' evaluation will depend only on the terminal capital stock. Therefore, we may write

$$\Phi_{T+1}(k_{T+1}, I_{T+1}) = \phi_{T+1}(k_{T+1}).$$

Then P^T becomes

$$CP^T = \begin{cases} \max & \sum_{t=1}^{T} \beta_t u_t(x_t) + \beta_{T+1}\phi_{T+1}(k_{T+1}), \\ \text{s.t.} & x_t + k_{t+1} - k_t \leqslant w_t + A_t(z_t), & t = 1, \ldots, T, \\ & K_t(z_t) \leqslant k_t, & t = 1, \ldots, T, \\ & x_t,\ k_{t+1},\ z_t \geqslant 0, & t = 1, \ldots, T, \\ & k_1 \text{ given.} \end{cases}$$

Let us denote by r_t the capital utilization prices (rents). Then, for any shares $(\lambda_{ij}, \mu_{ij}, \nu_{ij})_{i=1,\dots,T}^{j=1,\dots,T+1} \geqslant 0$, $\sum_{j=1}^{T+1}(\lambda_{ij}, \mu_{ij}, \nu_{ij}) = (1, 1, 1)$, in a fashion similar to EQ^T, we define the following general equilibrium problem:

CEQ^T: Find a set of quantities $(x_t, k_t, z_t)_1^T$, k_{T+1}, and prices $(p_t, r_t)_1^T \geqslant 0$ that satisfy conditions (i)–(vi).

(i) $(x_t)_1^T$ solve the *consumers'* subproblems; to maximize their utility,

$$\max_{x_t} u_t(x_t) \quad \text{s.t.} \quad p_t x_t \leqslant M_t,$$

where

$$M_t = \sum_{i=1}^{T} \nu_{it}(p_i w_i) + \sum_{i=1}^{T} \mu_{it}(p_i k_i + r_i k_i - p_{i-1} k_i) + \sum_{i=1}^{T} \lambda_{it}(p_i A_i(z_i) - r_i K_i(z_i)).$$

(ii) $(k_t)_2^T$ solve the *capital holders'* subproblems; to maximize the net present profits due to capital formation,

$$\max_{k_t} p_t k_t + r_t k_t - p_{t-1} k_t \quad \text{s.t.} \quad k_t \geqslant 0 \quad \text{and} \quad k_1 \text{ is given.}$$

(iii) $(z_t)_1^T$ solve the *producing firms'* subproblems; to maximize the net present value of production profits,

$$\max_{z_t} p_t A_t(z_t) - r_t K_t(z_t) \quad \text{s.t.} \quad z_t \geqslant 0.$$

(iv) k_{T+1} solves the *terminal condition* subproblem,

$$\max_{k_{T+1}} \phi(k_{T+1}) \quad \text{s.t.} \quad p_T k_{T+1} \leqslant M_{T+1}.$$

(v) The *no-excess-demand* conditions are satisfied,

$$x_t + k_{t+1} - k_t \leqslant w_t + A_t(z_t), \qquad t = 1, \dots, T,$$
$$K_t(z_t) \leqslant k_t, \qquad t = 1, \dots, T.$$

(vi) The *Walras law* is satisfied,

$$\sum_{t=1}^{T} p_t(x_t + k_{t+1} - k_t - w_t - A_t(z_t)) + \sum_{t=1}^{T} (r_t(K_t(z_t) - k_t) = 0.$$

The interpretation of the subproblems is entirely analogous to that of EQ^T. The only difference is that for the activity analysis model we have succeeded in decoupling the capital utilization part of the problem from the rest of the productive activities. If k_{t+1} units of capital are set aside in period t, in period $t+1$ they yield k_{t+1} units (all the capital survives) at a price $p_{t+1} + r_{t+1}$ (joint price for consumption and rental). Therefore, the net present value of profits due to capital formation is

$$p_{t+1} k_{t+1} + r_{t+1} k_{t+1} - p_t k_{t+1}.$$

Similarly, if productive activities in period t are z_t, then, the actual inputs are $A_t(z_t)^-$ units of goods at price p_t, and $K_t(z_t)$ units of capital rented at price r_t; the outputs are $A_t(z_t)^+$ units of goods in the same period (at price p_t). Therefore, the net present value of profits due to productive activities is

$$p_t A_t(z_t) - r_t K_t(z_t).$$

It is precisely this property that will allow us to implement the Golden Rule. But first, we note that in a fashion similar to that of Section 2, we may prove direct analogues of Theorems 1 and 2. The analogue to Theorem 1 is

Theorem 3. *Let $(\beta_t)_1^{T+1}$ be discount factors for which CP^T is stable and finite, and $(x_t, k_t, z_t)_1^T$, k_{T+1}, $(p_t, r_t)_1^T$ a primal–dual solution. Then, there exist wealth shares (λ, μ, ν) for which it also solves CEQ^T.*

The analogues of Assumptions 1 and 2 for the new setting are given below:

Assumption 3 (constraint qualification). For every choice of positive discount factors, CP^T is finite and stable in terms of RHS perturbations.

Assumption 4 (minimum wealth for a given set of wealth shares). For every choice of positive discount factors, the resulting income M_t, $t = 1, \ldots, T+1$ is positive when evaluated at the optimal solution of the resulting CP^T and its supporting price vector.

We note in passing, that the stability required by Assumption 3 includes perturbations to the capital utilization constraints. The converse to Theorem 3 is:

Theorem 4. *If Assumption 3 holds, for any wealth shares* (λ, μ, ν) *satisfying Assumption 4 there exists a set of discount factors* (β_t) *and a primal–dual solution of* CP^T *that also solves* CEQ^T.

The key issue is to devise a method for specifying the wealth shares. Focusing on transfer patterns in which income generated at a given period may be passed on only to the next, let us set

$$\begin{aligned} &\nu_{tt} = 1, \quad \nu_{jt} = 0 && \text{whenever} \quad j \neq t, \\ &\lambda_{tt} = 1, \quad \lambda_{jt} = 0 && \text{whenever} \quad j \neq t, \\ &\mu_{t,t+1} = 1, \quad \mu_{jt} = 0 && \text{whenever} \quad j \neq t+1. \end{aligned}$$

Then, we observe that

$$M_t = m_t + \sigma_{t-1} - \sigma_t, \quad \text{where}$$

$$m_t = p_t w_t + p_t A_t(z_t) - r_t K_t(z_t) + p_t k_t + r_t k_t - p_{t-1} k_t, \qquad t = 1, \ldots, T \quad (\text{and } m_{T+1} = 0)$$

is income generated at period t.

$$\sigma_t = p_t k_t + r_t k_t - p_{t-1} k_t, \qquad t = 1, \ldots, T \quad \text{and} \quad \sigma_0 = \sigma_{T+1} = 0$$

represents the part of current income m_t that is not consumed immediately, but passed on to the next period as savings. Observe also that from its definition, σ_t is easily seen to be the share of current income generated by capital alone. This rule, 'savings = capital's share of income', is one of the versions of the Golden Rule of Capital Accumulation first introduced by Phelps (see, e.g., Phelps [12]). He has shown that it describes behavior that maximizes per capita consumption in certain one-sectoral models, and Burmeister–Dobell [2] have generalized it to some multi-sectoral models. Their version, however, although equivalent to that of Phelps for the one-sectoral model, differs from the one proposed here.

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