

# Rationality Properties for Morita Contexts associated to Corings

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ABSTRACT. Given an  $A$ -coring  $C$  with a fixed grouplike element, we can construct a Morita context connecting the dual of the coring with the ring of coinvariants of  $A$ . In this paper, we discuss the image of one of the two connecting maps, and show that it is contained in the rational part of the dual of the coring, at least if the coring is locally projective. We apply our result to entwined modules, and this leads to the introduction of factorizable entwined modules.

## 1. INTRODUCTION

During the past decades, several variations of the notion of Hopf module have been proposed, with applications in various directions, for example relative Hopf modules (in connection with Hopf Galois theory), Yetter-Drinfeld modules (in connection with quantum groups), Long dimodules (in connection with the Brauer group). Doi [11] and Koppinen [16] gave a unification of all these types of modules, nowadays usually called Doi-Hopf modules, and their construction was generalized later by Brzeziński and Majid [6], who introduced entwined modules. In a mathematical review [19], Takeuchi observed that entwined modules and all their special cases can be considered as comodules over a coring, a notion that goes back to Sweedler [18]. This idea has been worked out by Brzeziński in [5], see also [15], [21], [22], and it turned out that many interesting properties of Hopf modules and their generalizations can be at the same time generalized and reformulated more elegantly using the language of corings. For example, Galois corings generalize Hopf Galois extensions, and they can be introduced in such a way that the connection with descent theory is clarified (see [22] or [7, Sec. 4.8]).

Hopf Galois extensions were first considered by Chase and Sweedler [10], and their

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work reveals that there is a close connection with Morita theory: to a comodule algebra, they associate a Morita context that is strict in case the comodule algebra in question is a Hopf Galois extension. This Morita context has been generalized by Doi [12]. In [8] and, independently in [1], it was discussed how Doi's Morita context can be generalized to corings. Of course one then wants to investigate when this context is strict. For one of the two connecting maps, there is no problem to find necessary and sufficient conditions for its surjectivity (see [8]), but for the other one, having values in the dual coring, a satisfactory answer can be given only in the situation where the coring is finitely generated and projective over the groundring  $A$ . In fact we will prove in this paper that surjectivity of this second map  $\mu$  implies that the coring  $\mathcal{C}$  is finitely generated projective (Corollary 5.2). The aim of this paper is to show that - under the condition that  $\mathcal{C}$  is locally projective, the image of the map  $\mu$  is contained in the rational part of the dual of  $\mathcal{C}$  (Proposition 5.1). If  $\mathcal{C}$  satisfies the Weak or Strong Structure Theorem, then  $\mu$  is surjective onto the rational part.

This paper is organized as follows. In Section 2, we recall some generalities about corings, entwined modules and Doi-Hopf modules. In Section 3, we introduce a relative version of local projectivity, and relate it to the so-called  $\alpha$ -condition. In Section 4, we discuss rationality properties of modules over a suitable subring  $\mathcal{R}$  of the dual of a coring; in case the coring is  $\mathcal{R}$ -locally projective, we can introduce the  $\mathcal{R}$ -rational part of such a module. Sections 3 and 4 provide the necessary machinery to state and prove our main results in Section 5. We apply our results to entwined modules in Section 6, and this leads us to the introduction of what we called factorizable entwining structures. In Section 7, we prove that there is an injective map between the two connecting modules in the Morita context, under the condition that the coring  $\mathcal{C}$  is co-Frobenius, which means that there exists a  ${}^*\mathcal{C}$ -linear map from  $\mathcal{C}$  to its left dual.

## 2. PRELIMINARY RESULTS

### Corings

Let  $A$  be a ring. Recall that an  $A$ -coring  $\mathcal{C}$  is an  $A$ -bimodule together with two  $A$ -bilinear maps  $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$ ,  $\Delta_{\mathcal{C}}(c) = c_{(1)} \otimes_A c_{(2)}$  and  $\varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A$ , such that

$$\Delta_{\mathcal{C}}(c_{(1)}) \otimes_A c_{(2)} = c_{(1)} \otimes_A \Delta_{\mathcal{C}}(c_{(2)}) \text{ and } c_{(1)}\varepsilon_{\mathcal{C}}(c_{(2)}) = \varepsilon_{\mathcal{C}}(c_{(1)})c_{(2)} = c$$

for all  $c \in \mathcal{C}$ . A right  $\mathcal{C}$ -comodule  $M$  is a right  $A$ -module together with a map  $\rho^r : M \rightarrow M \otimes_A \mathcal{C}$ ,  $\rho^r(m) = m_{[0]} \otimes m_{[1]}$  such that

$$\rho^r(m_{[0]}) \otimes_A m_{[1]} = m_{[0]} \otimes_A \Delta_{\mathcal{C}}(m_{[1]})$$

and

$$m_{[0]}\varepsilon_{\mathcal{C}}(m_{[1]}) = m$$

for all  $m \in M$ .  $\rho^r$  is called a right  $\mathcal{C}$ -coaction on  $M$ . Left  $\mathcal{C}$ -comodules are introduced in a similar way. A right  $A$ -linear map  $f : M \rightarrow N$  between two right  $\mathcal{C}$ -comodules is called right  $\mathcal{C}$ -colinear if  $f$  preserves the coaction, that is

$$f(m_{[0]}) \otimes_A m_{[1]} = f(m)_{[0]} \otimes_A f(m)_{[1]}$$

for all  $m \in M$ . The category of right  $\mathcal{C}$ -comodules and  $\mathcal{C}$ -colinear maps is denoted by  $\mathcal{M}^{\mathcal{C}}$ .

The left dual  ${}^*\mathcal{C} = {}_A\text{Hom}(\mathcal{C}, A)$  of a coring  $\mathcal{C}$  is a ring with multiplication

$$(f\#g)(c) = g(c_{(1)}f(c_{(2)}))$$

and unit  $\varepsilon_{\mathcal{C}}$ . We have a ring homomorphism  $i : A \rightarrow {}^*\mathcal{C}$ ,  $i(a)(c) = \varepsilon_{\mathcal{C}}(c)a$ . It is easy to verify that

$$(i(a)\#f)(c) = f(ca) \quad \text{and} \quad (f\#i(a))(c) = f(c)a$$

for all  $a \in A$ ,  $f \in {}^*\mathcal{C}$  and  $c \in \mathcal{C}$ .

An element  $x \in \mathcal{C}$  is called grouplike if

$$\Delta_{\mathcal{C}}(x) = x \otimes_A x \text{ and } \varepsilon_{\mathcal{C}}(x) = 1. \quad (1)$$

The set of grouplike elements of  $\mathcal{C}$  is denoted by  $G(\mathcal{C})$ . Grouplike elements correspond bijectively to right (or left)  $\mathcal{C}$ -coactions on  $A$ :

$$\begin{aligned} G(\mathcal{C}) &\cong \{ \rho^r : A \rightarrow A \otimes_A \mathcal{C} \cong \mathcal{C} \mid \rho^r \text{ makes } A \text{ into a right } \mathcal{C}\text{-comodule} \} \\ &\cong \{ \rho^\ell : A \rightarrow \mathcal{C} \otimes_A A \cong \mathcal{C} \mid \rho^\ell \text{ makes } A \text{ into a left } \mathcal{C}\text{-comodule} \} \end{aligned}$$

The right and left coactions  $\rho^r$  and  $\rho^\ell$  corresponding to  $x \in G(\mathcal{C})$  are given by the formulas

$$\rho^r(a) = xa \text{ and } \rho^\ell(a) = ax$$

(where we identify  $\mathcal{C} \otimes_A A$  and  $\mathcal{C}$ ).

If  $G(\mathcal{C}) \neq \emptyset$ , then  $i : A \rightarrow {}^*\mathcal{C}$  is injective, since for every grouplike element  $x$ , the map  $\chi : {}^*\mathcal{C} \rightarrow A$ ,  $\chi(f) = f(x)$  is a left inverse of  $i$ . In this case,  $\varepsilon_{\mathcal{C}}$  is surjective, with right inverse  $\rho^r$  (or  $\rho^\ell$ ).

For a right  $\mathcal{C}$ -comodule  $M$ , we define the set of coinvariants as follows:

$$M^{\text{co}\mathcal{C}} = \{ m \in M \mid \rho(m) = m \otimes_A x \} = \text{Hom}^{\mathcal{C}}(A, M).$$

$B = A^{\text{co}\mathcal{C}} = \{ b \in A \mid bx = xb \}$  is a subring of  $A$  and it is easy to see that  $M^{\text{co}\mathcal{C}}$  is a  $B$ -submodule of  $M$ .  $\mathcal{C}$  is a right  $\mathcal{C}$ -comodule (the coaction is the comultiplication), and  $\mathcal{C}^{\text{co}\mathcal{C}} \cong A$  as a right  $B$ -module:  $\rho^\ell$  is a right  $B$ -linear map from  $A$  to  $\mathcal{C}^{\text{co}\mathcal{C}}$ , with inverse  $\varepsilon_{\mathcal{C}}$ .

Recall that we have a functor  $F : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_{*\mathcal{C}}$ ,  $F(M) = M$  with  $m \cdot f = m_{[0]}f(m_{[1]})$ . For every  $M \in \mathcal{M}_{*\mathcal{C}}$ , we define

$$M^{*\mathcal{C}} = \{ m \in M \mid m \cdot f = mf(x), \text{ for all } f \in {}^*\mathcal{C} \}.$$

In particular,  $A^{*\mathcal{C}} = B'$  is a subring of  $A$ . Obviously  $M^{\text{co}\mathcal{C}} \subset M^{*\mathcal{C}}$ .

### A Morita context associated to a coring

Let  $(\mathcal{C}, x)$  be a coring with a fixed grouplike element. Following [8], we define

$$Q = \{ q \in {}^*\mathcal{C} \mid c_{(1)}q(c_{(2)}) = q(c)x, \text{ for all } c \in \mathcal{C} \} = {}^{\mathcal{C}}\text{Hom}(\mathcal{C}, A)$$

and

$$Q \subset Q' = ({}^*\mathcal{C})^{*\mathcal{C}}.$$

$Q$  is a  $({}^*\mathcal{C}, B)$ -bimodule, and  $A$  is a  $(B, {}^*\mathcal{C})$ -bimodule, with right  ${}^*\mathcal{C}$ -action  $a \leftarrow f = f(xa)$ . Consider the bimodule maps

$$\mu : Q \otimes_B A \rightarrow {}^*\mathcal{C}, \quad \mu(q \otimes_B a) = q\#a$$

$$\tau : A \otimes_{*C} Q \rightarrow B, \quad \tau(a \otimes_{*C} q) = a \lrcorner q = q(xa)$$

Then  $(B, *C, A, Q, \tau, \mu)$  is a Morita context. In a similar way,  $Q'$  is a  $(*C, B')$ -bimodule,  $A$  is a  $(B', *C)$ -bimodule, and we can consider the maps

$$\mu' : Q' \otimes_{B'} A \rightarrow *C \text{ and } \tau' : A \otimes_{*C} Q' \rightarrow B'$$

defined in the same way as  $\mu$  and  $\tau$ . Then  $(B', *C, A, Q', \tau', \mu')$  is also a Morita context, and it was shown in [8] that the two contexts coincide if  $C$  is finitely generated projective as a left  $A$ -module. From [8] we also recall the following result.

**THEOREM 2.1.** *With notation as above, the following statements are equivalent:*

1.  $\tau$  is surjective (and, a fortiori, bijective);
2. there exists  $\Lambda \in Q$  such that  $\Lambda(x) = 1$ ;
3. for every right  $C$ -comodule  $M$ , the map

$$\omega_M : M \otimes_{*C} Q \rightarrow M^{\text{co}C}, \quad \omega_M(m \otimes_{*C} q) = m \cdot q$$

is bijective.

Consider a right  $A$ -submodule  $\mathcal{R}$  of  $*C$  that is closed under multiplication, and let  $\tilde{Q} = Q \cap \mathcal{R}$ . Then we have a Morita context  $(B, \mathcal{R}, A, \tilde{Q}, \tilde{\mu}, \tilde{\tau})$  with  $\tilde{\mu}$  and  $\tilde{\tau}$  the obvious restrictions of the original maps  $\mu$  and  $\tau$ . It is possible that  $\mathcal{R}$  has no unit. For Morita contexts between rings without unit, see [3].

### Galois corings

Let  $(C, x)$  be an  $A$ -coring with a fixed grouplike element. Let

$$B = A^{\text{co}C} = \{b \in A \mid bx = xb\}.$$

We have a pair of adjoint functors  $(F, G)$  between the categories  $\mathcal{M}_B$  and  $\mathcal{M}^C$ , namely, for  $N \in \mathcal{M}_B$  and  $M \in \mathcal{M}^C$ ,

$$F(N) = N \otimes_B A \text{ and } G(M) = M^{\text{co}C}.$$

The unit and counit of the adjunction are

$$\eta_N : N \rightarrow (N \otimes_B A)^{\text{co}C}, \quad \eta_N(n) = n \otimes_B 1$$

$$\varepsilon_M : M^{\text{co}C} \otimes_B A \rightarrow M, \quad \varepsilon_M(m \otimes_B a) = ma$$

We say that  $(C, x)$  satisfies the *Weak Structure Theorem* if  $\varepsilon_M$  is an isomorphism for all  $M \in \mathcal{M}^C$ , that is,  $G = \bullet^{\text{co}C}$  is a fully faithful functor.  $(C, x)$  satisfies the *Strong Structure Theorem* if, in addition, all  $\eta_N$  are isomorphisms, or  $F$  is fully faithful, and therefore  $(F, G)$  is an equivalence of categories.

The canonical coring associated to  $i : B \rightarrow A$  is  $\mathcal{D} = A \otimes_B A$ , with structure maps  $\Delta_{\mathcal{D}} : A \otimes_B A \rightarrow (A \otimes_B A) \otimes_A (A \otimes_B A) \cong A \otimes_B A \otimes_B A$  and  $\varepsilon_{\mathcal{D}} : A \otimes_B A \rightarrow A$  given by

$$\begin{aligned} \Delta_{\mathcal{D}}(a \otimes_B b) &= (a \otimes_B 1) \otimes_A (1 \otimes_B b) = a \otimes_B 1 \otimes_B b \\ \varepsilon_{\mathcal{D}}(a \otimes_B b) &= ab \end{aligned}$$

$1 \otimes_B 1$  is a grouplike element. If  $A$  is faithfully flat as a  $B$ -module, then  $(\mathcal{D}, 1 \otimes_B 1)$  satisfies the Strong Structure Theorem. We have a canonical coring morphism

$$\text{can} : \mathcal{D} \rightarrow C; \quad \text{can}(a \otimes_B b) = axb$$

We say that  $(\mathcal{C}, x)$  is a *Galois coring* if it can be an isomorphism of corings. In this situation, we obviously have an isomorphism between the categories  $\mathcal{M}^{\mathcal{C}}$  and  $\mathcal{M}^{\mathcal{D}}$ , and if  $(\mathcal{D}, 1 \otimes_B 1)$  satisfies the Strong, resp. Weak Structure Theorem (for example if  $A/B$  is faithfully flat, resp.  $A/B$  is flat), then  $(\mathcal{C}, x)$  also satisfies the Strong resp. Weak Structure Theorem. If  $(\mathcal{C}, x)$  satisfies the Weak Structure Theorem, then  $(\mathcal{C}, x)$  is Galois (see [8, Proposition 1.1]).

### Entwined modules

Let  $k$  be a commutative ring,  $A$  a  $k$ -algebra,  $C$  a  $k$ -coalgebra, and  $\psi : C \otimes A \rightarrow A \otimes C$  a  $k$ -linear map satisfying the following four conditions:

$$(ab)_{\psi} \otimes c^{\psi} = a_{\psi} b_{\Psi} \otimes c^{\psi\Psi} \quad (2)$$

$$(1_A)_{\psi} \otimes c^{\psi} = 1_A \otimes c \quad (3)$$

$$a_{\psi} \otimes \Delta_C(c^{\psi}) = a_{\psi\Psi} \otimes c_{(1)}^{\Psi} \otimes c_{(2)}^{\psi} \quad (4)$$

$$\varepsilon_C(c^{\psi})a_{\psi} = \varepsilon_C(c)a \quad (5)$$

Here we used the sigma notation

$$\psi(c \otimes a) = a_{\psi} \otimes c^{\psi} = a_{\Psi} \otimes c^{\Psi}$$

We then call  $(A, C, \psi)$  a (right-right) entwining structure. To an entwining structure  $(A, C, \psi)$ , we can associate an  $A$ -coring  $\mathcal{C} = A \otimes C$ . The structure maps are given by the formulas

$$\begin{aligned} a'(b \otimes c)a &= a'ba_{\psi} \otimes c^{\psi} \\ \Delta_{\mathcal{C}}(a \otimes c) &= (a \otimes c_{(1)}) \otimes_A (1 \otimes c_{(2)}) \\ \varepsilon_{\mathcal{C}}(a \otimes c) &= a\varepsilon_C(c) \end{aligned}$$

An entwined module  $M$  is a  $k$ -module together with a right  $A$ -action and a right  $C$ -coaction, in such a way that

$$\rho^r(ma) = m_{[0]}a_{\psi} \otimes m_{[1]}^{\psi}$$

for all  $m \in M$  and  $a \in A$ . The category  $\mathcal{M}(\psi)_A^C$  of entwined modules and  $A$ -linear  $C$ -colinear maps is isomorphic to the category of right  $\mathcal{C}$ -comodules.

### Doi-Hopf modules

Let  $H$  be a Hopf algebra,  $A$  a right  $H$ -comodule algebra, and  $C$  a right  $H$ -module coalgebra. We call  $(H, A, C)$  a right-right Doi-Hopf structure. We associate an entwining structure  $(A, C, \psi)$  to  $(H, A, C)$  as follows, with  $\psi$  defined by  $\psi(c \otimes a) = a_{[0]} \otimes ca_{[1]}$ . The corresponding entwined modules are called Doi-Hopf modules. They have to satisfy the compatibility relation

$$\rho(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]}$$

for all  $m \in M$  and  $a \in A$ .

### Factorization structures and the smash product

Let  $A$  and  $S$  be  $k$ -algebras, and  $R : S \otimes A \rightarrow A \otimes S$  a  $k$ -linear map. We will write

$$R(s \otimes a) = a_R \otimes s_R = a_r \otimes s_r$$

(summation understood).  $A \#_R S$  will be the  $k$ -module  $A \otimes S$ , with multiplication

$$(a \# s)(b \# t) = ab_R \# s_R t \quad (6)$$

It is straightforward to verify that this multiplication is associative with unit  $1_A \# 1_S$  if and only if

$$R(s \otimes 1_A) = 1_A \otimes s \quad (7)$$

$$R(1_S \otimes a) = a \otimes 1_S \quad (8)$$

$$R(st \otimes a) = a_{Rr} \otimes s_r t_R \quad (9)$$

$$R(s \otimes ab) = a_R b_r \otimes s_{Rr} \quad (10)$$

for all  $a, b \in A$  and  $s, t \in S$ . We then call  $(A, S, R)$  a factorization structure, and  $A \#_R S$  the smash product of  $A$  and  $S$ .

### 3. $\mathcal{R}$ -LOCALLY PROJECTIVE MODULES

Let  $A$  be a ring and  $P$  a left  $A$ -module. Then  ${}^*P = {}_A\text{Hom}(P, A)$  is a right  $A$ -module, with action given by  $(f \cdot a)(p) = f(p)a$ , for  $f \in {}^*P$ ,  $p \in P$  and  $a \in A$ . Let  $\mathcal{R}$  be an abelian group, and  $\varphi : \mathcal{R} \rightarrow {}^*P$  a morphism of abelian groups. For  $M \in \mathcal{M}_A$ , we consider the map

$$\xi_{M, \mathcal{R}} : M \otimes_A P \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{R}, M), \quad \xi_{M, \mathcal{R}}(m \otimes_A p)(r) = m \cdot \varphi(r)(p)$$

Since we will only be interested in  $\mathcal{R}$  as acting on  $P$ , it will be no restriction to assume that  $\varphi$  is injective, and consider  $\mathcal{R}$  as a subgroup of  ${}^*P$ .

If  $\mathcal{R}$  is a right  $A$ -submodule of  ${}^*P$ , then  $\text{Im}(\xi_{M, \mathcal{R}}) \subset \text{Hom}_A(\mathcal{R}, M)$  and  $\xi_{M, \mathcal{R}}$  factors as the composition of a map

$$\xi'_{M, \mathcal{R}} : M \otimes_A P \rightarrow \text{Hom}_A(\mathcal{R}, M)$$

followed by the natural inclusion  $\text{Hom}_A(\mathcal{R}, M) \subset \text{Hom}_{\mathbb{Z}}(\mathcal{R}, M)$ .

DEFINITION 3.1. Let  $\mathcal{R}$  be an additive subgroup of  ${}^*P$ . We say that  $P$  satisfies the  $\alpha$ -condition for  $\mathcal{R}$  iff  $\xi_{M, \mathcal{R}}$  is injective for all  $M \in \mathcal{M}_A$ . If  $P$  satisfies the  $\alpha$ -condition for  ${}^*P$  we just say that  $P$  satisfies the  $\alpha$ -condition.

More specifically, the  $\alpha$ -condition for  $\mathcal{R}$  is equivalent to the following statement: if  $\sum_i m_i f(p_i) = 0$  for all  $f \in \mathcal{R}$ , then  $\sum_i m_i \otimes_A p_i = 0$ .

If  $\mathcal{S}$  is an abelian subgroup of  $\mathcal{R}$ , then obviously

$$\ker \xi_{M, \mathcal{R}} \subset \ker \xi_{M, \mathcal{S}}, \quad \text{Im} \xi_{M, \mathcal{S}} \subset \text{Im} \xi_{M, \mathcal{R}}$$

and the  $\alpha$ -condition for  $\mathcal{S}$  implies the  $\alpha$ -condition for  $\mathcal{R}$ .

EXAMPLE 3.2 (extension of scalars). Let  $B \rightarrow A$  be a ringmorphism, and  $Q \in {}_B\mathcal{M}$ . Then  $P = A \otimes_B Q \in {}_A\mathcal{M}$ , and we have a morphism of right  $B$ -modules

$$\gamma : {}^*Q = {}_B\text{Hom}(Q, B) \rightarrow {}^*P = {}_A\text{Hom}(P, A), \quad \gamma(f)(a \otimes_B q) = af(q)$$

If  $Q$  satisfies the  $\alpha$ -condition for  ${}^*Q$ , then  $P$  also satisfies the  $\alpha$ -condition for  ${}^*Q$  (since  $M \otimes_A P \cong M \otimes_B Q$  for every right  $A$ -module  $M$ ), and therefore  $P$  satisfies the  $\alpha$ -condition for  ${}^*P$  as well.

Recall that the *finite topology* on  ${}^*P$  is the topology generated by the basis of open sets

$$\mathcal{O}(f, p_1, \dots, p_n) = \{g \in {}^*P \mid g(p_i) = f(p_i), 1 \leq i \leq n\}$$

where  $f \in {}^*P$  and  $p_1, \dots, p_n \in P$ .

A subset  $\mathcal{R} \subset {}^*P$  is dense with respect to this topology if and only if for every  $f \in {}^*P$  and  $p_1, \dots, p_n \in P$ , we can find a  $g \in \mathcal{R}$  such that  $g(p_i) = f(p_i)$ , for  $1 \leq i \leq n$ .

Let  $N$  be a subset of  $P$ .  $\{(e_i, f_i) \mid i \in I\} \subset P \times {}^*P$  is called a dual basis of  $N$  if

$$\#\{i \mid e_i^*(n) \neq 0\} < \infty \text{ and } n = \sum_i f_i(n)e_i$$

for every  $n \in N$ .  $P$  is called locally projective if every finite subset (or every finitely generated submodule) of  $P$  has a dual basis, and it is well-known that this is equivalent to  $P$  having the  $\alpha$ -condition for  ${}^*P$ . If  $P$  is finitely generated as an  $A$ -module, or if  $A$  is left perfect, then local projectivity of  $P$  is equivalent to projectivity. We refer to [21] for a survey of known results, and to [14] and [23] for the proofs. In the sequel we will need the following generalization of some results of [14] and [23]; we recover the results of [14] and [23] if we take  $\mathcal{R} = {}^*P$ .

**PROPOSITION 3.3.** *Let  $P$  be a left  $A$ -module,  $\mathcal{R} \subset {}^*P$  an additive subgroup and  $\mathcal{S} = \mathcal{R}A$  the right  $A$ -submodule of  ${}^*P$  generated by  $\mathcal{R}$ . Then the following statements are equivalent :*

- (i)  $P$  satisfies the  $\alpha$ -condition for  $\mathcal{R}$ ;
- (i)'  $P$  satisfies the  $\alpha$ -condition for  $\mathcal{S}$ ;
- (ii)  $\xi_{M, \mathcal{R}}$  is injective for every cyclic right  $A$ -module  $M$ ;
- (ii)'  $\xi_{M, \mathcal{S}}$  is injective for every cyclic right  $A$ -module  $M$ ;
- (iii) For every  $p \in P$ , we have  $p \in \mathcal{R}(p)P$ ;
- (iii)' For every  $p \in P$ , we have  $p \in \mathcal{S}(p)P$ ;
- (iv) Every finitely generated submodule  $N$  of  $P$  has a dual basis contained in  $P \times \mathcal{R}$ ;
- (iv)' Every finitely generated submodule  $N$  of  $P$  has a dual basis contained in  $P \times \mathcal{S}$ ;
- (v)  $P$  satisfies the  $\alpha$ -condition and  $\mathcal{S}$  is dense with respect to the finite topology.

If  $P$  satisfies any of these equivalent conditions, then we say that  $P$  is  $\mathcal{R}$ -locally projective as a left  $A$ -module.

*Proof.*  $(i) \Rightarrow (i)'$ ,  $(i)' \Rightarrow (ii)'$ ,  $(i) \Rightarrow (ii)$  and  $(ii) \Rightarrow (ii)'$  are trivial.

$(ii)' \Rightarrow (iii)'$ .

$$\mathcal{S}(p) = \{f(p) \mid f \in \mathcal{S}\} = \left\{ \sum_i f_i(p)a_i \mid f_i \in \mathcal{R}, a_i \in A \right\}$$

is a right ideal of  $A$ , and  $A/\mathcal{S}(p)$  is a cyclic right  $A$ -module. For all  $f \in \mathcal{S}$ ,  $1f(p) = 0$  in  $A/\mathcal{S}(p)$ , hence  $1 \otimes_A p = p = 0$  in  $A/\mathcal{S}(p) \otimes_A P \cong P/\mathcal{S}(p)P$ , and  $p \in \mathcal{S}(p)P$ .

$(iii)' \Rightarrow (iii)$  We know that  $p = \sum_i s_i(p)e_i$ , with  $s_i \in \mathcal{S}$  and  $e_i \in P$ . Since  $\mathcal{S}$  is generated by  $\mathcal{R}$ , we can write  $s_i = \sum_{j_i} r_{j_i} a_{j_i}$  and we find  $p = \sum_i \sum_{j_i} (r_{j_i} a_{j_i})(p)e_i = \sum_i \sum_{j_i} r_{j_i}(p)a_{j_i}e_i$  and so  $p \in \mathcal{R}(p)P$ .

$(iii) \Rightarrow (iv)$  We prove by induction on  $k$  that every finite set  $\{n_1, \dots, n_k\}$  has a finite dual basis. If  $k = 1$ , this follows immediately from  $(iii)$ . Now suppose we have a dual basis  $\{(e_i, r_i) \mid i \in I\}$  for  $n_1, \dots, n_{k-1}$ , and consider then  $n_k - \sum_i r_i(n_k)e_i \in P$  and take a dual basis  $\{(e'_j, r'_j) \mid j \in J\}$  for this element. An easy calculation shows that  $\{(e'_j, r'_j) \mid j \in J\} \cup \{(e_i - \sum_j r'_j(e_i)e'_j, r_i) \mid i \in I\}$  is a dual basis for  $\{n_1, \dots, n_k\}$ .  $(iv) \Rightarrow (iv)'$  is trivial.

$(iv)' \Rightarrow (iv)$  is similar to  $(iii)' \Rightarrow (iii)$ .

$(iv) \Rightarrow (i)$  Let  $M$  be a right  $A$ -module, take  $\sum_i m_i \otimes_A p_i \in M \otimes_A P$  and suppose that  $\sum_i m_i f(p_i) = 0$  for all  $f \in \mathcal{R}$ . From  $(iv)$  we know we can find a dual basis  $\{e_j, r_j\}$  for the elements  $p_i$ .

$$\sum_i m_i \otimes_A p_i = \sum_{i,j} m_i \otimes_A r_j(p_i)e_j = \sum_{i,j} m_i r_j(p_i) \otimes_A e_j = 0$$

$(i) \Rightarrow (v)$ . It follows from the comments following Definition 3.1 that  $P$  satisfies the  $\alpha$ -condition. For every  $f \in {}^*P$  and  $p_1, \dots, p_n \in P$ , we have to find  $g \in \mathcal{S}$  such that  $f(p_i) = g(p_i)$ . From  $(iv)'$ , we know that  $\{p_1, \dots, p_n\}$  has a dual basis  $\{(e_j, s_j) \mid j \in J\} \subset P \times \mathcal{S}$ . We find

$$f(p_i) = f\left(\sum_j s_j(p_i)e_j\right) = \sum_j s_j(p_i)f(e_j) = \sum_j (s_j f(e_j))(p_i)$$

and our statement follows since  $\sum_j s_j f(e_j) \in \mathcal{S}$ .

$(v) \Rightarrow (iv)'$  The  $\alpha$ -condition for  $P$ , gives us a dual basis  $\{(e_i, f_i) \mid i \in I\} \subset P \times {}^*P$  for  $N$ . Since  $\mathcal{S}$  is dense in  ${}^*P$ , we can find elements  $s_i \in \mathcal{S}$ , such that  $f_i$  and  $s_i$  have the same action on  $N$ . Then  $\{(e_i, s_i) \mid i \in I\} \subset P \times \mathcal{S}$  is the dual basis that we are looking for.  $\square$

*Remark 3.4.* Note that the equivalent conditions of Proposition 3.3 do not imply that  $\mathcal{R}$  is dense in the finite topology. If one takes  $P = A \otimes_B Q$  as in Example 3.2, then  ${}^*Q$  is never dense in the finite topology on  ${}^*P$  if  $B \rightarrow A$  is a proper ring extension.

Now let  $P = \mathcal{C}$  be an  $A$ -coring, and fix  $x \in G(\mathcal{C})$ .

DEFINITION 3.5. For  $M \in \mathcal{M}_* \mathcal{C}$  and  $\mathcal{R} \subset {}^* \mathcal{C}$ , we define

$$M^{\mathcal{R}} = \{m \in M \mid m \cdot f = mf(x), \text{ for all } f \in \mathcal{R}\}$$

PROPOSITION 3.6. If  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective over  $A$ , then  $M^{\mathcal{R}} = M^{\text{co}\mathcal{C}}$  for every  $M \in \mathcal{M}^{\mathcal{C}}$ , and  $({}^* \mathcal{C})^{\mathcal{R}} = Q$ .

*Proof.* We have already seen that  $M^{\text{co}\mathcal{C}} \subset M^* \mathcal{C}$ . The same argument shows that  $M^{\text{co}\mathcal{C}} \subset M^{\mathcal{R}}$ . Conversely, take  $m \in M^{\mathcal{R}}$ , and write  $\rho(m) = \sum_{j=1}^n m_j \otimes_A c_j$ . Take a dual basis  $\{(e_i, f_i) \mid i \in I\} \subset \mathcal{C} \times \mathcal{R}$  of  $\{c_1, \dots, c_n, x\}$ . Then

$$\begin{aligned} \rho(m) &= \sum_{i,j} m_j \otimes_A f_i(c_j)e_i = \sum_{i,j} m_j f_i(c_j) \otimes_A e_i \\ &= \sum_i m f_i(x) \otimes_A e_i = \sum_i m \otimes_A f_i(x)e_i = m \otimes_A x \end{aligned}$$



and  $m \in M^{\text{co}\mathcal{C}}$ .

We still have to show that  $(*\mathcal{C})^{\mathcal{R}} \subset Q$ . If  $q \in (*\mathcal{C})^{\mathcal{R}}$ , then  $q\#f = qf(x)$ , for all  $f \in \mathcal{R}$ . Take  $c \in \mathcal{C}$  and a dual basis  $\{(e_i, f_i) \mid i \in I\} \subset \mathcal{C} \times \mathcal{R}$  for  $c_{(1)}q(c_{(2)})$  and  $x$ . Then we find

$$\begin{aligned} c_{(1)}q(c_{(2)}) &= f_i(c_{(1)}q(c_{(2)}))e_i \\ &= q(c)f_i(x)e_i = q(c)x \end{aligned}$$

and it follows that  $q \in Q$ .  $\square$

**COROLLARY 3.7.** *Let  $\mathcal{C}$  be an  $A$ -coring. If  $\mathcal{C}$  is locally projective as a left  $A$ -module, then  $M^{*\mathcal{C}} = M^{\text{co}\mathcal{C}}$  for every  $M \in \mathcal{M}^{\mathcal{C}}$ . In particular,  $B = B'$ . We also have  $Q = Q'$ , and the two Morita contexts of Section 2 coincide.*

### Local units and local projectivity

**DEFINITION 3.8.** Let  $A$  be a ring without unit,  $M \in \mathcal{M}_A$ , and consider subsets  $R \subset A$  and  $N \subset M$ . We say that  $a \in A$  can be multiplicatively approximated from the right by  $R$  on  $N$ , if for every finite subset  $\{n_1, \dots, n_k\} \subset N$ , there exists  $r \in R$  such that  $n_i \cdot a = n_i \cdot r$ , for all  $i \in \{1, \dots, k\}$ .

If every  $a \in A$  can be multiplicatively approximated by  $R$  on  $N$ , then we say  $R$  is a right multiplicative approximation of  $A$  on  $N$ .

If  $A$  has a unit, then  $R$  has right local units if and only if  $1_A$  can be multiplicatively approximated from the right by  $R$  on  $R$ . Furthermore, we have the following result.

**PROPOSITION 3.9.** *Let  $A$  be a ring with unit, and consider an additive subset  $R \subset A$ . Then the following assertions are equivalent:*

- (i)  $R$  is a multiplicative approximation from the right of  $A$  on  $R$ ;
- (ii)  $R$  is a right ideal of  $A$  and has right local units.

*If these conditions hold, then  $R$  is a multiplicative approximation of  $A$  on every right  $A$ -module  $M$  satisfying  $MR = M$ .*

*Proof.*  $(i) \Rightarrow (ii)$  For every finite set of elements  $\{r_1, \dots, r_n\} \subset R \subset A$  and  $a \in A$ , we have an element  $r \in R$  such that  $r_i a = r_i r \in R$ . This means  $R$  is a right ideal, and taking  $a = 1$  we find that  $R$  has right local units.

$(ii) \Rightarrow (i)$   $R$  has right local units, so for all  $r_1, \dots, r_n \in R$ , we can find  $e \in R$  such that  $r_i e = r_i$ . For every  $a \in A$ , we then have  $r_i e a = r_i a$ , and  $ea \in R$ , since  $R$  is a right ideal. The proof of the final statement is similar.  $\square$

These approximation properties have the same flavour as the density properties that we encountered when we discussed  $\mathcal{R}$ -relative local projectivity. So the question arises whether they are related. To be able to define a finite topology, first remark that the rings we used before now need to be duals of modules. So it is very natural to look at duals of corings, since they have both a finite topology and a ring structure.

**PROPOSITION 3.10.** *Let  $\mathcal{C}$  be an  $A$ -coring, and  $\mathcal{R} \subset *\mathcal{C}$  a subring.*

1. *If  $\mathcal{R}$  is dense in the finite topology on  $*\mathcal{C}$ , then  $\mathcal{R}$  is a multiplicative approximation of  $*\mathcal{C}$  from the right on every  $\mathcal{C}$ -comodule (regarded as a  $*\mathcal{C}$ -module). In particular,  $M\mathcal{R} = M$  for every  $M \in \mathcal{M}^{\mathcal{C}}$ .*

2. If  $\mathcal{R}$  is a right ideal in  ${}^*\mathcal{C}$  and has right local units, and  $\mathcal{C}\mathcal{R} = \mathcal{C}$ , then  $\mathcal{R}$  is dense in the finite topology.

*Proof.* 1) Take  $M \in \mathcal{M}^{\mathcal{C}}$ . For every  $m \in M$  and  $f \in {}^*\mathcal{C}$ , we have  $m \cdot f = m_{[0]}f(m_{[1]})$ . Now, by the denseness of  $\mathcal{R}$ , there exists a  $g \in \mathcal{R}$  such that  $f(m_{[1]}) = g(m_{[1]})$ , and so  $m \cdot f = m \cdot g$ .

2) By Proposition 3.9,  $\mathcal{R}$  is a multiplicative approximation of  ${}^*\mathcal{C}$  on  $\mathcal{C}$ . Hence for every finite  $\{c_1, \dots, c_n\} \subset \mathcal{C}$ , and  $f \in {}^*\mathcal{C}$ , there exists  $g \in \mathcal{R}$  such that  $c_{i(1)}f(c_{i(2)}) = c_i \cdot f = c_i \cdot g = c_{i(1)}g(c_{i(2)})$ , for all  $i$ . Applying  $\varepsilon_{\mathcal{C}}$  to both sides, we find that  $f(c_i) = g(c_i)$  for all  $i$ , which means exactly that  $\mathcal{R}$  is dense in the finite topology on  ${}^*\mathcal{C}$ .  $\square$

**COROLLARY 3.11.** *If  $\mathcal{R} \subset {}^*\mathcal{C}$  is a  $\mathcal{C}$ -comodule and an ideal, then  $\mathcal{R}$  is dense in the finite topology if and only if  $\mathcal{R}$  has local units and  $\mathcal{C}$  is unitary as an  $R$ -module.*

For later use, we give the following generalization of a well-known property of Morita contexts, illustrating the connection between local units and local projectivity.

**PROPOSITION 3.12.** *Let  $A$  be a ring with left local units, and  $(A, B, P, Q, f, g)$  a Morita context. If  $f : P \otimes_B Q \rightarrow A$  is surjective, then  $P$  is locally projective as a left  $B$ -module. If  $A$  has right local units, then  $Q \in \mathcal{M}_A$  is locally projective.*

*Proof.* Take  $e \in A$  such that  $p = ep$ , and  $\sum_i p_i \otimes_B q_i \in f^{-1}\{e\}$ . Then

$$p = ep = \sum_i (p_i \otimes_B q_i)p = \sum_i p_i g(q_i \otimes_A p)$$

and we have a dual basis  $\{(p_i, g(q_i \otimes_A -)) \mid i \in I\}$ .  $\square$

#### 4. $\mathcal{R}$ -RATIONAL MODULES

Let  $\mathcal{C}$  be an  $A$ -coring, and  $\mathcal{R}$  a subring (without unit) of  ${}^*\mathcal{C}$ .  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective if and only if

$$\xi_{M, \mathcal{R}} : M \otimes_A \mathcal{C} \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{R}, M), \quad \xi_{M, \mathcal{R}}(m \otimes_A c)(r) = m \cdot r(c)$$

is injective for all  $M \in \mathcal{M}_A$ .

Let  $T$  be the subring of  ${}^*\mathcal{C}$  generated by  $A$  and  $\mathcal{R}$ . Then  $T$  is the  $\mathbb{Z}$ -module generated by elements of the form  $a_1 \# r_1 \# a_2 \# r_2 \# \dots \# r_n \# a_{n+1}$  with  $a_i \in A$  and  $r_i \in \mathcal{R}$ . Note that  $T$  has a unit, and that  $\mathcal{S} = \mathcal{R}A \subset T$ . For every  $M \in \mathcal{M}_T$ , we define

$$\delta_{M, \mathcal{R}} : M \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{R}, M), \quad \delta_{M, \mathcal{R}}(m)(f) = m \cdot f$$

If  $\mathcal{R}$  is also a right  $A$ -module, then  $\text{Im}(\delta_{M, \mathcal{R}}) \subset \text{Hom}_A(\mathcal{R}, M)$ .

**DEFINITION 4.1.**  $M \in \mathcal{M}_T$  is called  $\mathcal{R}$ -rational if  $\delta_{M, \mathcal{R}}(M) \subset \xi_{M, \mathcal{R}}(M \otimes_A \mathcal{C})$ , or, equivalently, if for every  $m \in M$ , there exist finitely many  $m_i \in M$  and  $c_i \in \mathcal{C}$  such that  $m \cdot f = \sum_i m_i f(c_i)$ , for all  $f \in \mathcal{R}$ .  $\mathcal{RM}_{\mathcal{R}}$  will be the full subcategory of  $\mathcal{M}_T$  consisting of  $\mathcal{R}$ -faithful  $\mathcal{R}$ -rational  $T$ -modules.

**PROPOSITION 4.2.** *Assume that  $\mathcal{R}$  is a subring of  ${}^*\mathcal{C}$ , such that  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective. Then we have the following properties:*

1. every cyclic submodule of an  $\mathcal{R}$ -rational module  $M$  is finitely generated as an  $A$ -module;

2. the direct sum of a family of  $\mathcal{R}$ -rational modules is again rational;
3. any quotient of a rational  $\mathcal{R}$ -module is rational;
4. any submodule of a rational  $\mathcal{R}$ -module is rational.

*Proof.* The proof of the first three statements is similar to the proof of [13, Theorem 2.2.6]. For the proof of part 4, take  $n \in N$  and  $f \in \mathcal{R}$ . Then we know that  $n \cdot f = n_i f(c_i)$ , with  $n_i \in M$  and  $c_i \in \mathcal{C}$ . We have to show that we can find  $n'_j \in N$  and  $c'_j \in \mathcal{C}$  with the same property. By Proposition 3.3, there exists a dual basis  $\{(e_j, r_j) \mid j \in J\} \subset \mathcal{C} \times \mathcal{R}$  for the  $c_i$ . We easily compute that

$$n \cdot f = n_i f(c_i) = n_i f(r_j(c_i)e_j) = n_i r_j(c_i) f(e_j) = n \cdot r_j f(e_j)$$

using the fact that  $M$  is  $\mathcal{R}$ -rational. Since  $N$  is an  $\mathcal{R}$ -module, we know that  $n \cdot r_j \in N$ . Now just take  $n'_j = n \cdot r_j$  and  $c'_j = e_j$ .  $\square$

Let  $R$  be a ring and  $S$  and  $T$  two subrings of  $R$ . Observe that  $ST \subset TS$  if and only if for all  $s \in S$  and  $t \in T$ , we can find  $s_i \in S$  and  $t_i \in T$  such that  $st = \sum_i t_i s_i$  in  $R$ . We present several examples below.

EXAMPLES 4.3. 1. Let  $(H, A, C)$  be a right-right Doi-Hopf structure over a commutative ring  $k$ . As we have seen,  $\mathcal{C} = A \otimes C$  is an  $A$ -coring.  $C^*$  is a subring of  ${}^* \mathcal{C} \cong \text{Hom}(C, A)$ , and  $AC^* \subset C^*A$ , since

$$\begin{aligned} (i(a)\#f)(b \otimes c) &= f((b \otimes c)a) = ba_{[0]}f(ca_{[1]}) \\ &= bf(ca_{[1]})a_{[0]} = ((a_{[1]} \rightarrow f)\#i(a_{[0]}))(b \otimes c) \end{aligned}$$

Recall that the left  $H$ -action on  $C^*$  is given by  $(h \rightarrow f)(b \otimes c) = bf(ch)$ . It follows that  $i(a)\#f = (a_{[1]} \rightarrow f)\#i(a_{[0]})$ , and  $A\#C^* \subset C^*\#A$ , as needed. We will see in Section 5 that the same property does not hold for entwining structures, leading to the introduction of factorizable entwining structures.

2. Let  $S$  and  $T$  be two  $k$ -algebras,  $k$  is a commutative ring, and  $(T, S, R)$  a factorization structure.

$$i_T : T \rightarrow T\#S, i_T(t) = t\#1_S \text{ and } i_S : S \rightarrow T\#S, i_S(s) = 1_T\#s$$

are algebra maps, and  $ST \subset TS$  since

$$s \cdot t = (1\#s)(t\#1) = t_R\#s_R = (t_R\#1)(1\#s_R) = t_R \cdot s_R \in TS$$

for all  $s \in S$  and  $t \in T$ .

Remark 4.4. Example 1 has been our motivation to work with subrings  $\mathcal{R} \subset {}^* \mathcal{C}$  that are not necessarily right  $A$ -modules, as is usually done in the literature. In the setting of Example 1, we can consider  $\#(C, A)$ -rational modules, as well as  $C^*$ -modules (see [12]), and  $C^*$  is not a right  $A$ -module.

Let  $M \in \mathcal{M}_A$ , and recall that an  $A$ -module is subgenerated by  $M$  if it is isomorphic to a subobject of a quotient of a direct sum of copies of  $M$ .  $\sigma[M]$ , the full subcategory of  $\mathcal{M}_A$  consisting of modules subgenerated by  $M$  is the smallest full Grothendieck subcategory of  $\mathcal{M}_A$  containing  $M$  (see [20]).

LEMMA 4.5. *Let  $\mathcal{R}$  be a subring (without unit) of  ${}^* \mathcal{C}$ ,  $T$  the subring of  ${}^* \mathcal{C}$  generated by  $\mathcal{R}$  and  $A$ , and  $S = \mathcal{R}A$ . Then the following assertions are equivalent:*

1.  $A\mathcal{R} \subset \mathcal{R}A$ ;

2.  $\mathcal{S}$  is a ring and a left  $A$ -module, and  $T = A + \mathcal{S}$ ;
3.  $\mathcal{S}$  is  $T$ -bimodule;
4.  $\mathcal{S}$  is a left  $A$ -module.

In this case:

1.  $\mathcal{R}\mathcal{M}_{\mathcal{S}} = \mathcal{R}\mathcal{M}_{\mathcal{R}}$ ;
2.  $\text{Hom}_{\mathbb{Z}}(\mathcal{S}, M) \in \mathcal{M}_T$ , for all  $M \in \mathcal{M}_T$ ;
3.  $\xi_{M, \mathcal{S}} \in \mathcal{M}_T$  for all  $M \in \mathcal{M}_T$ .

*Proof.* 1)  $\Rightarrow$  2). Take a generator  $g = a_1 \# r_2 \# \cdots \# r_n \# a_{n+1}$  of  $T$  as a  $\mathbb{Z}$ -module. If  $n = 0$ , then  $g = a \in A$ . If  $n > 1$ , then it follows from  $A\mathcal{R} \subset \mathcal{R}A$  that  $g \in \mathcal{S}$ , hence  $T = A + \mathcal{S}$ .  $\mathcal{S}$  is closed under multiplication since  $\mathcal{S}^2 = \mathcal{R}A\mathcal{R}A \subset \mathcal{R}^2A^2 \subset \mathcal{R}A = \mathcal{S}$ .  $\mathcal{S}$  is a left  $A$ -module since  $A\mathcal{S} = A\mathcal{R}A \subset \mathcal{R}A^2 = \mathcal{S}$ .

2)  $\Rightarrow$  3).  $\mathcal{S}$  is a ring, so it is an  $\mathcal{S}$ -bimodule. It is a right  $A$ -module, and, by 2), a left  $A$ -module. Since  $T = A + \mathcal{S}$ ,  $\mathcal{S}$  is a  $T$ -bimodule.

3)  $\Rightarrow$  4) is trivial.

4)  $\Rightarrow$  1). If  $\mathcal{S}$  is a left  $A$ -module, then  $A\mathcal{R} \subset A\mathcal{S} \subset \mathcal{S} = \mathcal{R}A$ .

1)  $\mathcal{R} \subset \mathcal{S}$ , so  $\mathcal{S}$ -rationality implies  $\mathcal{R}$ -rationality. Conversely, if  $M$  is  $\mathcal{R}$ -rational, then for all  $m \in M$  and  $\sum_i f_i \# a_i \in \mathcal{S}$ , we have

$$m \cdot \sum_i f_i \# a_i = \sum_i (m \cdot f_i) a_i = \sum_i (m_{[0]} f_i(m_{[1]})) a_i = m_{[0]} \left( \sum_i f_i \# a_i \right) (m_{[1]})$$

and it follows that  $M$  is  $\mathcal{S}$ -rational. It is easy to see that  $M$  is  $\mathcal{R}$ -faithful if and  $M$  is  $\mathcal{S}$ -faithful.

2) Take  $\varphi \in \text{Hom}_{\mathbb{Z}}(\mathcal{S}, M)$ ,  $f \in \mathcal{S}$  and  $g \in T$ . Then  $g \# f \in \mathcal{S}$ , and we define  $\varphi \cdot g$  by  $(\varphi \cdot g)(f) = \varphi(g \# f)$ .

3)  $\xi_{M, \mathcal{S}} : M \otimes_A \mathcal{C} \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{S}, M)$  is right  $T$ -linear since

$$\begin{aligned} \xi_{M, \mathcal{S}}((m \otimes_A c) \cdot g)(s) &= \xi_{M, \mathcal{S}}(m \otimes_A c_{(1)} g(c_{(2)}))(s) = m \cdot s(c_{(1)} g(c_{(2)})) \\ &= m \cdot (g \# s)(c) = \xi_{M, \mathcal{S}}(m \otimes_A c)(g \# s) = \left( (\xi_{M, \mathcal{S}}(m \otimes_A c)) \cdot g \right)(s) \end{aligned}$$

□

**COROLLARY 4.6.** *Let  $\mathcal{C}$  be an  $A$ -coring, and  $\mathcal{R}$  a subring of  ${}^* \mathcal{C}$ . If  $A\mathcal{R} \subset \mathcal{R}A$  and  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective, then  $\mathcal{M}^{\mathcal{C}}$  is a subcategory of  $\sigma[\mathcal{C}_{\mathcal{S}}]$  and  $\sigma[\mathcal{C}_{\mathcal{S}}]$  is full subcategory of  $\mathcal{R}\mathcal{M}_{\mathcal{R}}$ .*

*Proof.* Take  $M \in \mathcal{M}^{\mathcal{C}}$ . Then there exists a nonempty set  $I$  such that  $M$  is isomorphic as a  $\mathcal{C}$ -comodule (and a fortiori as an  $\mathcal{S}$ -module) to a subobject of a direct sum  $\mathcal{C}^{(I)}$  of copies of  $\mathcal{C}$ , and therefore  $M \in \sigma[\mathcal{C}_{\mathcal{S}}]$ .

It is clear that  $\mathcal{C}$  is an  $\mathcal{R}$ -rational  $\mathcal{S}$ -module, and it follows from Proposition 4.2 that every  $M \in \sigma[\mathcal{C}_{\mathcal{S}}]$  is an  $\mathcal{R}$ -rational  $\mathcal{S}$ -module. □

**PROPOSITION 4.7.** *Let  $\mathcal{C}$  be an  $A$ -coring,  $\mathcal{R}$  a subring of  ${}^* \mathcal{C}$ , and  $M \in \mathcal{R}\mathcal{M}_{\mathcal{R}}$ . If  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective and  $A\mathcal{R} \subset \mathcal{R}A$ , then  $\delta_{M, \mathcal{R}}$  defines a right  $\mathcal{C}$ -comodule structure on  $M$ .*

*Proof.* From the  $\mathcal{R}$ -rationality of  $M$  and the fact that  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective, it follows that, for any  $m \in M$ , there exists a unique  $\sum_i m_i \otimes_A c_i \in M \otimes_A \mathcal{C}$  such that  $m \cdot f = \sum_i m_i f(c_i)$ , for every  $f \in \mathcal{R}$ . So we have a well-defined map

$\delta_M : M \rightarrow M \otimes_A \mathcal{C}$ ,  $\delta_M(m) = \sum_i m_i \otimes_A c_i$ , which is equal to  $\delta_{M, \mathcal{R}}$  if we regard the injection  $\xi_{M, \mathcal{R}}$  as an inclusion. Let us use the notation

$$\delta_M(m) = m_{[0]} \otimes_A m_{[1]}$$

We will show that  $\delta_M$  defines a  $\mathcal{C}$ -comodule structure on  $M$ . First,  $\delta_M$  is right  $A$ -linear. Since  $A\mathcal{R} \subset \mathcal{R}A$ , there exist  $a_k$  and  $f_k$  such that  $i(a)\#f = \sum_k f_k\#i(a_k)$ , hence

$$\begin{aligned} (ma) \cdot f &= m \cdot (i(a)\#f) = m \cdot \left( \sum_k f_k\#i(a_k) \right) \\ &= \sum_k (m \cdot f_k)a_k = \sum_k (m_{[0]} \cdot f_k(m_{[1]}))a_k \\ &= \sum_k m_{[0]} \cdot (f_k(m_{[1]})a_k) = \sum_k m_{[0]} \cdot (f_k\#i(a_k))(m_{[1]}) \\ &= m_{[0]} \cdot (i(a)\#f)(m_{[1]}) = m_{[0]} \cdot f(m_{[1]}a) \end{aligned}$$

for all  $f \in \mathcal{R}$ ,  $m \in M$  and  $a \in A$ , so  $\delta_M(ma) = m_{[0]} \otimes_A m_{[1]}a = \delta_M(m)a$ .

Let us next show the mixed coassociativity. For all  $m \in M$ , we have to show that

$$y_m = (I \otimes_A \Delta_{\mathcal{C}})\delta_M(m) - (\delta_M \otimes_A I)\delta_M(m) = 0$$

For all  $f, g \in \mathcal{R}$ , we have

$$\begin{aligned} m \cdot (f\#g) &= m_{[0]}(f\#g)(m_{[1]}) \\ &= m_{[0]}g((m_{[1]})_{(1)})f((m_{[1]})_{(2)}) \\ &= (I \otimes_A g) \circ (I \otimes_A I \otimes_A f)(m_{[0]} \otimes_A (m_{[1]})_{(1)} \otimes_A (m_{[1]})_{(2)}) \\ &= (I \otimes_A g) \circ (I \otimes_A I \otimes_A f)(I \otimes_A \Delta_{\mathcal{C}})\delta_M(m) \\ &= (m \cdot f) \cdot g = (m_{[0]} \cdot f(m_{[1]})) \cdot g \\ &= (m_{[0]})_{[0]}g((m_{[0]})_{[1]}f(m_{[1]})) \\ &= (I \otimes_A g) \circ (I \otimes_A I \otimes_A f)((m_{[0]})_{[0]} \otimes_A (m_{[0]})_{[1]} \otimes_A m_{[1]}) \\ &= (I \otimes_A g) \circ (I \otimes_A I \otimes_A f)(\delta_M \otimes_A I)\delta_M(m) \end{aligned}$$

where we used the right  $A$ -linearity of  $\delta_{M, \mathcal{R}}$ . It follows that

$$(I \otimes_A g)((I \otimes_A I \otimes_A f)(y_m)) = 0$$

for all  $f, g \in \mathcal{R}$ . Using the  $\alpha$ -property for  $M$ , we find

$$(I \otimes_A I \otimes_A f)(y_m) = 0$$

and, using the  $\alpha$ -property for  $M \otimes_A \mathcal{C}$ , we find that  $y_m = 0$ , as needed.

Finally, for every  $f \in \mathcal{R}$ , we have, using the mixed coassociativity,

$$\begin{aligned} (m - m_{[0]}\varepsilon_{\mathcal{C}}(m_{[1]})) \cdot f &= m \cdot f - m_{[0]}f(m_{[1]}\varepsilon_{\mathcal{C}}(m_{[2]})) \\ &= m \cdot f - m_{[0]}f(m_{[1]}) = m \cdot f - m \cdot f = 0 \end{aligned}$$

From the fact that  $M$  is faithful as a right  $\mathcal{R}$ -module, we then deduce that  $m = m_{[0]}\varepsilon_{\mathcal{C}}(m_{[1]})$ .  $\square$

It was proved in [21] that  $\mathcal{M}^{\mathcal{C}} = \sigma[\mathcal{C}^*_{\mathcal{C}}]$  if and only if  ${}^*\mathcal{C}$  is locally projective as an  $A$ -module. We will now generalize this result.

PROPOSITION 4.8. *Let  $\mathcal{C}$  be an  $A$ -coring and  $\mathcal{R}$  a subring of  ${}^* \mathcal{C}$ . If  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective and  $A\mathcal{R} \subset \mathcal{R}A$ , then the categories  $\mathcal{R}\mathcal{M}_{\mathcal{R}}$ ,  $\mathcal{R}\mathcal{M}_{\mathcal{S}}$ ,  $\sigma[\mathcal{C}_{\mathcal{S}}]$  and  $\mathcal{M}^{\mathcal{C}}$  are isomorphic full subcategories of  $\mathcal{M}_{\mathcal{S}}$ .*

*Proof.* Suppose first  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective. We define a functor  $F : \mathcal{R}\mathcal{M}_{\mathcal{R}} \rightarrow \mathcal{M}^{\mathcal{C}}$  as follows:  $F(M) = M$  as a right  $A$ -module, with  $\mathcal{C}$ -comodule structure as in Proposition 4.7; for  $f : M \rightarrow N$  in  $\mathcal{R}\mathcal{M}_{\mathcal{R}}$ , we put  $F(f) = f$ .

Let us first prove that  $f$  is right  $\mathcal{C}$ -colinear, as needed. Take  $m \in M$ , then  $\delta_M(f(m)) = f(m)_{[0]} \otimes_A f(m)_{[1]}$ , if and only if  $f(m) \cdot g = f(m)_{[0]} g(f(m)_{[1]})$  for every  $g \in \mathcal{R}$ . But  $f(m) \cdot g = f(m \cdot g) = f(m_{[0]} g(m_{[1]})) = f(m_{[0]}) g(m_{[1]})$ , since  $f$  is right  $\mathcal{R}$ -linear and right  $A$ -linear. Using the  $\alpha$ -condition, we find  $f(m)_{[0]} \otimes_A f(m)_{[1]} = f(m_{[0]}) \otimes_A m_{[1]}$ .

Finally, it is easy to see that  $F(M) = M$  if  $M \in \mathcal{M}^{\mathcal{C}}$ .  $\square$

DEFINITION 4.9. Assume that  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective, and let  $M$  be a right  $\mathcal{R}$ -module. We define the  $\mathcal{R}$ -rational part of  $M$  as

$$M^{\mathcal{R}\text{-rat}} = \delta_{M, \mathcal{R}}^{-1}(\xi_{M, \mathcal{R}}(M \otimes_A \mathcal{C}))$$

The rational part of  $M$  is by definition the  ${}^* \mathcal{C}$ -rational part:  $M^{\text{rat}} = M^{* \mathcal{C}\text{-rat}}$ .

Observe that  $m \in M^{\mathcal{R}\text{-rat}}$  if and only if there exist  $m_i \in M$  and  $c_i \in \mathcal{C}$  such that  $m \cdot f = \sum_i m_i \cdot f(c_i)$  for all  $f \in \mathcal{R}$ .  $M$  is  $\mathcal{R}$ -rational if and only if  $M^{\mathcal{R}\text{-rat}} = M$ . Obviously  $M^{\mathcal{R}} \subset M^{\mathcal{R}\text{-rat}}$ . If  $\mathcal{R} \subset \mathcal{R}' \subset {}^* \mathcal{C}$ , then  $M^{\mathcal{R}'} \subset M^{\mathcal{R}}$ . If  $M \in \mathcal{M}^{\mathcal{C}}$  and  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective, then  $M^{\mathcal{R}'} = M^{\mathcal{R}} = M^{\text{co}\mathcal{C}}$ .

PROPOSITION 4.10. *Let  $\mathcal{R} \subset {}^* \mathcal{C}$  be a subring, assume that  $\mathcal{C}$  is  $\mathcal{R}$ -locally projective, and take  $M \in \mathcal{M}_{* \mathcal{C}}$ .*

1. *Let  $\mathcal{R}'$  be another subring of  ${}^* \mathcal{C}$ . If  $\mathcal{R} \subset \mathcal{R}'$ , then  $\mathcal{C}$  is also  $\mathcal{R}'$ -locally projective, and*

$$M^{\mathcal{R}'\text{-rat}} \subset M^{\mathcal{R}\text{-rat}} \text{ and } ({}^* \mathcal{C})^{\mathcal{R}'\text{-rat}} = ({}^* \mathcal{C})^{\mathcal{R}\text{-rat}}$$

*If  $M$  is  $\mathcal{R}'$ -rational, then  $M$  is also  $\mathcal{R}$ -rational.  ${}^* \mathcal{C}$  is  $\mathcal{R}'$ -rational if and only if  ${}^* \mathcal{C}$  is  $\mathcal{R}$ -rational.*

2. *Fix a grouplike element  $x \in \mathcal{C}$ , and assume that*

$$\varepsilon_{M, \mathcal{R}} : M^{\mathcal{R}} \otimes_B A \rightarrow M, \varepsilon_{M, \mathcal{R}}(m \otimes a) = ma$$

*is surjective. If  $A\mathcal{R} \subset \mathcal{R}A$ , then  $M$  is  $\mathcal{R}$ -rational.*

3.  *$M^{\mathcal{R}\text{-rat}}$  is a right  $\mathcal{R}$ -submodule of  $M$ . If  $A\mathcal{R} \subset \mathcal{R}A$ , then it is a right  $T$ -module, and consequently it is the biggest  $T$ -submodule of  $M$  that is  $\mathcal{R}$ -rational.  $M^{\mathcal{R}\text{-rat}}$  is then also a  $\mathcal{C}$ -comodule.*

*Proof.* 1) The first statement is obvious. Take  $g \in ({}^* \mathcal{C})^{\mathcal{R}\text{-rat}}$ . Then there exist  $g_j \in {}^* \mathcal{C}$  and  $c_j \in \mathcal{C}$  such that  $g \# h = \sum_j g_j \# i(h(c_j))$ , for all  $h \in \mathcal{R}$ . Now  $(g \# h)(c) = h(c_{(1)} g(c_{(2)}))$  and  $\sum_j (g_j \# i(h(c_j)))(c) = \sum_j g_j(c) h(c_j) = \sum_j h(g_j(c) c_j)$ , so it follows that

$$h(c_{(1)} g(c_{(2)}) - \sum_j g_j(c) c_j) = 0$$

for all  $h \in \mathcal{R}$ .  $\mathcal{C}$  satisfies the  $\alpha$ -property for  $\mathcal{R}$ , so we have a dual basis  $\{(e_i, r_i) \mid i \in I\} \subset \mathcal{C} \times \mathcal{R}$ , and we find that

$$c_{(1)}g(c_{(2)}) - \sum_j g_j(c)c_j = \sum_{i \in I} r_i \left( c_{(1)}g(c_{(2)}) - \sum_j h(g_j(c)c_j) \right) e_i = 0$$

Applying  $f \in \mathcal{R}'$ , it follows that  $g\#f = \sum_j g_j\#f(c_j)$  hence  $g \in (*\mathcal{C})^{\mathcal{R}'\text{-rat}}$ .

2) Take  $m \in M$  and  $f \in \mathcal{R}$ . There exist  $m_k \in M^{\mathcal{R}}$  and  $a_k \in A$  such that  $m = \sum_k m_k a_k$ . For any  $k$ , we can find  $a_{kl} \in A$  and  $f_{kl} \in \mathcal{R}$  such that  $a_k\#f = \sum_l f_{kl}\#a_{kl}$ . Now

$$\begin{aligned} m \cdot f &= \sum_k (m_k a_k) \cdot f = \sum_k m_k \cdot (a_k\#f) = \sum_{k,l} m_k \cdot (f_{kl}\#a_{kl}) \\ &= \sum_{k,l} (m_k \cdot f_{kl}) a_{kl} = \sum_{k,l} (m_k f_{kl}(x)) a_{kl} = \sum_{k,l} m_k (f_{kl}(x) a_{kl}) \\ &= \sum_{k,l} m_k ((f_{kl}\#a_{kl})(x)) = \sum_k m_k ((a_k\#f)(x)) \end{aligned}$$

and it follows that  $m \in M^{\mathcal{R}\text{-rat}}$ .

3) For  $m \in M^{\mathcal{R}\text{-rat}}$ , there exists a unique  $m_{[0]} \otimes_A m_{[1]} \in M \otimes_A \mathcal{C}$  such that  $m \cdot f = m_{[0]}f(m_{[1]})$ , for all  $f \in \mathcal{R}$ . For all  $f, g \in \mathcal{R}$ , we then have

$$(m \cdot f) \cdot g = m \cdot (f\#g) = m_{[0]}(f\#g)(m_{[1]}) = m_{[0]}g(m_{[1](1)})f((m_{[1](2)}))$$

and this means that  $m \cdot f \in M^{\mathcal{R}\text{-rat}}$ , as needed. If  $A\mathcal{R} \subset \mathcal{R}A$ , then the same argument as in the first part of the proof of Proposition 4.7 shows that  $(ma) \cdot f = m_{[0]}f(m_{[1]}a)$ , for all  $f \in \mathcal{R}$ , hence  $ma \in M^{\mathcal{R}\text{-rat}}$  if  $m \in M^{\mathcal{R}\text{-rat}}$  and  $a \in A$ . It follows from Proposition 4.8 that  $M^{\mathcal{R}\text{-rat}}$  is a right  $\mathcal{C}$ -comodule.  $\square$

### 5. THE IMAGE OF $\mu$ AND $\mu'$

Let  $(\mathcal{C}, x)$  be a coring with a fixed grouplike element. We have seen in Section 2 how we can associate Morita contexts  $(B, *\mathcal{C}, A, Q, \tau, \mu)$  and  $(B', *\mathcal{C}, A, Q', \tau', \mu')$  to  $(\mathcal{C}, x)$ . We will now apply the rationality results obtained above, to discuss the image of  $\mu$  and  $\mu'$ .

**PROPOSITION 5.1.** *Let  $(\mathcal{C}, x)$  be a coring with a fixed grouplike element, and assume that  $\mathcal{C}$  is locally projective as a left  $A$ -module. Then we have the following properties:*

1.  $(*\mathcal{C})^{\text{rat}}$  is a two-sided ideal of  $*\mathcal{C}$ ;
2.  $(*\mathcal{C})^{\text{rat}}$  has local units if and only if  $(*\mathcal{C})^{\text{rat}}$  is dense in  $*\mathcal{C}$ , with respect to the finite topology;
3.  $\text{Im } \mu' \subseteq (*\mathcal{C})^{\text{rat}}$  and  $((*\mathcal{C})^{\text{rat}})^{\text{co}\mathcal{C}} = Q$ .

*Proof.* 1) We have already seen in Proposition 4.10 that  $(*\mathcal{C})^{\text{rat}}$  is a right ideal. Let us show that it is also a left ideal. For  $f \in (*\mathcal{C})^{\text{rat}}$  and  $g, h \in *\mathcal{C}$ , we have

$$(g\#f)\#h = g\#(f\#h) = g\#(f_{[0]}\#h(f_{[1]})) = (g\#f_{[0]})\#h(f_{[1]})$$

It follows that  $(g\#f)_{[0]} \otimes_A (g\#f)_{[1]} = g\#f_{[0]} \otimes_A f_{[1]}$ , and  $g\#f \in (*\mathcal{C})^{\text{rat}}$ .

2) follows from Corollary 3.11.

3) We have to show that  $q\#a \in (*\mathcal{C})^{\text{rat}}$ , for all  $q \in Q'$  and  $a \in A$ . For every  $f \in *\mathcal{C}$ , we compute

$$(q\#a\#f)(c) = f(c_{(1)}q(c_{(2)})a) = f(q(c)xa) = q(c)f(xa)$$

This proves that  $(q\#a)_{[0]} \otimes_A (q\#a)_{[1]} = q \otimes_A xa$  and  $q\#a \in (*\mathcal{C})^{\text{rat}}$ .  $((*\mathcal{C})^{\text{rat}})^{\text{co}\mathcal{C}} = Q$ , because  $f \in ((*\mathcal{C})^{\text{rat}})^{\text{co}\mathcal{C}}$  if and only if

$$f\#g = f_{[0]}\#g(f_{[1]}) = f\#g(x)$$

for all  $g \in *\mathcal{C}$ , if and only if  $f \in Q$ . □

**COROLLARY 5.2.** *Assume that  $\mathcal{C}$  is locally projective as a left  $A$ -module. If  $\mu' : Q' \otimes_B A \rightarrow *\mathcal{C}$  is surjective, then  $\mathcal{C}$  is finitely generated and projective as a left  $A$ -module.*

*Proof.* If  $\mu'$  is surjective, then it follows from Proposition 5.1 that  $*\mathcal{C} = (*\mathcal{C})^{\text{rat}}$ . Put  $\rho(\varepsilon) = \sum_i f_i \otimes_A c_i \in *\mathcal{C} \otimes_A \mathcal{C}$ . This means that  $\varepsilon\#f = f = \sum_i f_i\#f(c_i)$ , for all  $f \in *\mathcal{C}$ . Every right  $*\mathcal{C}$ -module  $M$  is rational: for all  $m \in M$ , we have that  $m \cdot f = \sum_i (m \cdot f_i)f(c_i)$ , and this means that  $\rho(m) = \sum_i m \cdot f_i \otimes_A c_i$ . In particular, for  $M = \mathcal{C}$ , we find, for all  $c \in \mathcal{C}$ :

$$\rho(c) = c_{(1)} \otimes_A c_{(2)} = \sum_i c \cdot f_i \otimes_A c_i = \sum_i c_{(1)}f_i(c_{(2)}) \otimes_A c_i$$

Applying  $\varepsilon$  to the first factor, we find

$$\begin{aligned} c &= \varepsilon(c_{(1)})c_{(2)} = \sum_i \varepsilon(c_{(1)}f_i(c_{(2)}))c_i \\ &= \sum_i \varepsilon(c_{(1)})f_i(c_{(2)})c_i = \sum_i f_i(\varepsilon(c_{(1)})c_{(2)})c_i = \sum_i f_i(c)c_i \end{aligned}$$

and it follows that  $\{(c_i, f_i) \mid i = 1, \dots, n\}$  is a finite dual basis of  $\mathcal{C}$ . □

We can now prove the main result of this paper.

**THEOREM 5.3.** *Let  $\mathcal{C}$  be locally projective as a left  $A$ -module, and assume that  $\tau' : A \otimes_{*\mathcal{C}} Q' \rightarrow B'$  is surjective. For any  $M \in \mathcal{M}^{\mathcal{C}}$ , we consider the map*

$$\Omega_M : M \otimes_{*\mathcal{C}} (*\mathcal{C})^{\text{rat}} \rightarrow M, \quad \Omega_M(m \otimes_{*\mathcal{C}} f) = m_{[0]}f(m_{[1]})$$

*The following statements are equivalent:*

1.  $(\mathcal{C}, x)$  satisfies the Strong Structure Theorem;
2.  $(\mathcal{C}, x)$  satisfies the Weak Structure Theorem;
3.  $\mu : Q \otimes_B A \rightarrow (*\mathcal{C})^{\text{rat}}$  and  $\Omega_M$  are bijective, for all  $M \in \mathcal{M}^{\mathcal{C}}$ ;
4.  $\mu : Q \otimes_B A \rightarrow (*\mathcal{C})^{\text{rat}}$  and  $\Omega_M$  are surjective, for all  $M \in \mathcal{M}^{\mathcal{C}}$ .

*Proof.* We know from Corollary 3.7 that  $B = B'$ ,  $Q = Q'$ ,  $\mu = \mu'$  and  $\tau = \tau'$ . For every  $M \in \mathcal{M}^{\mathcal{C}}$ , we have  $M^{\text{co}\mathcal{C}} = M^{*\mathcal{C}}$ , and we have the following commutative



diagram

$$\begin{array}{ccc}
 M \otimes_{*\mathcal{C}} Q \otimes_B A & \xrightarrow{\omega_M \otimes_B I_A} & M^{\text{co}\mathcal{C}} \otimes_B A \\
 \downarrow I_M \otimes_{*\mathcal{C}} \mu & & \downarrow \varepsilon_M \\
 M \otimes_{*\mathcal{C}} (*\mathcal{C})^{\text{rat}} & \xrightarrow{\Omega_M} & M
 \end{array} \tag{11}$$

$\omega_M$  is defined as in Theorem 2.1, and is an isomorphism because  $\tau$  is surjective.

1)  $\Rightarrow$  2) is trivial.

2)  $\Rightarrow$  3). Take  $M = (*\mathcal{C})^{\text{rat}}$ . Then  $M^{\text{co}\mathcal{C}} = Q$ , and

$$\varepsilon_M = \mu : M^{\text{co}\mathcal{C}} \otimes_B A \cong Q \otimes_B A \rightarrow M = (*\mathcal{C})^{\text{rat}}$$

is an isomorphism.  $\Omega_M$  is an isomorphism because all the other maps in the commutative diagram (11) are isomorphisms.

3)  $\Rightarrow$  4) is trivial.

4)  $\Rightarrow$  1). It follows immediately from the commutative diagram (11) that every  $\varepsilon_M$  is surjective. For every right  $\mathcal{C}$ -comodule  $P$ , we have a commutative diagram

$$\begin{array}{ccc}
 P^{\text{co}\mathcal{C}} \otimes_B A \otimes_{*\mathcal{C}} Q & \xrightarrow{\varepsilon_P \otimes I_Q} & P \otimes_{*\mathcal{C}} Q \\
 \downarrow I \otimes_B \tau & & \downarrow \omega_P \\
 P^{\text{co}\mathcal{C}} \otimes_B B & \xrightarrow{\cong} & P^{\text{co}\mathcal{C}}
 \end{array}$$

$\tau$  and  $\omega_P$  are isomorphisms, so  $\varepsilon_P \otimes I_Q$  is an isomorphism.

Now take  $P = \text{Ker } \varepsilon_M \in \mathcal{M}^{\mathcal{C}}$ . Then

$$P^{\text{co}\mathcal{C}} \cong P \otimes_{*\mathcal{C}} Q = \text{Ker } (\varepsilon_M) \otimes_{*\mathcal{C}} Q \cong \text{Ker } (\varepsilon_M \otimes_{*\mathcal{C}} I_Q) = 0$$

Here we used the fact that  $Q$  is finitely generated and projective as a left  $*\mathcal{C}$ -module, and this follows from the fact that  $\tau$  is surjective, and using the Morita Theorems. Now  $\varepsilon_P : 0 = P^{\text{co}\mathcal{C}} \otimes_B A \rightarrow P$  is surjective, so  $P = \text{Ker } \varepsilon_M = 0$ , and  $\varepsilon_M$  is injective. Finally it follows from [8, Prop. 2.5] that the unit maps  $\eta_N$  are isomorphisms, for all  $N \in \mathcal{M}_B$ .  $\square$

**PROPOSITION 5.4.** *If  $A$  is flat as left  $B$ -module, then  $(\mathcal{C}, x)$  is Galois if and only if it satisfies the Weak Structure Theorem.*

*Proof.* The Weak Structure Theorem means that  $\varepsilon_M : M^{\text{co}\mathcal{C}} \otimes_B A \rightarrow M$  is an isomorphism for all  $M \in \mathcal{M}^{\mathcal{C}}$ . If  $M = \mathcal{C}$ , then  $M^{\text{co}\mathcal{C}} \cong A$  and  $\varepsilon_M$  is the canonical map. Conversely, if  $(\mathcal{C}, x)$  is Galois, we have to show that  $\varepsilon_M$  is an isomorphism for all  $M \in \mathcal{M}^{\mathcal{C}}$ . We have isomorphisms

$$M \cong M \square_{\mathcal{C}} \mathcal{C} \cong M \square_{\mathcal{C}} (A \otimes_B A) \cong (M \square_{\mathcal{C}} A) \otimes_B A = M^{\text{co}\mathcal{C}} \otimes_B A$$

The flatness of  $A$  implies that the cotensor product is associative. The composition of these isomorphisms is the inverse of  $\varepsilon_M$ .  $\square$

**COROLLARY 5.5.** *With notation and assumptions as in Theorem 5.3, assume that  $(*\mathcal{C})^{\text{rat}}$  has local units (which is the case if  $(*\mathcal{C})^{\text{rat}}$  is dense in the finite topology). Then the four equivalent statements of Theorem 5.3 are equivalent to  $(\mathcal{C}, x)$  being Galois. In this situation,  $A$  and  $Q$  are locally projective as a left, resp. a right  $B$ -module.*

*Proof.* The local projectivity of  $A$  and  $Q$  follows from Proposition 3.12. Since locally projectivity implies flatness (see [14]), the other statement follows from Proposition 5.4.  $\square$

*Remark 5.6.* If  $(*\mathcal{C})^{\text{rat}}$  has local units, then  $\Omega_M$  is surjective for every  $M \in \mathcal{M}^{\mathcal{C}}$ , and this condition can then be dropped in Corollary 5.5. In this case, we can restrict the Morita context to  $(*\mathcal{C})^{\text{rat}}$ . The unital  $(*\mathcal{C})^{\text{rat}}$ -modules are precisely the  $\mathcal{C}$ -comodules: if  $M = M \otimes_{(*\mathcal{C})^{\text{rat}}} (*\mathcal{C})^{\text{rat}}$ , then  $M$  is a  $\mathcal{C}$ -comodule since  $(*\mathcal{C})^{\text{rat}}$  is a  $\mathcal{C}$ -comodule, and since  $(*\mathcal{C})^{\text{rat}}$  has local units, it is dense in the finite topology, so we can approximate  $\varepsilon_{\mathcal{C}}$  and find that all  $\mathcal{C}$ -comodules are unital as  $(*\mathcal{C})^{\text{rat}}$ -modules. Furthermore, the tensor product over  $*\mathcal{C}$  coincides with the tensor product over  $(*\mathcal{C})^{\text{rat}}$ , which is easy to see if one uses Proposition 3.10. We will use this restricted context in Section 7.

## 6. FACTORIZABLE ENTWINING STRUCTURES

Let  $k$  be a commutative ring, and consider a right-right entwining structure  $(A, C, \psi)$ . We call  $(A, C, \psi)$  factorizable if there exists a map  $\alpha : A \rightarrow A \otimes \text{End}(C)$  such that  $\psi$  factorizes as follows:

$$\psi = (I_A \otimes \theta) \circ (\alpha \otimes I_C) \circ \tau : \\ C \otimes A \xrightarrow{\tau} A \otimes C \xrightarrow{\alpha \otimes I_C} A \otimes \text{End}(C) \otimes C \xrightarrow{I_A \otimes \theta} A \otimes C$$

where  $\tau : C \otimes A \rightarrow A \otimes C$  is the switch map, and  $\theta : \text{End}(C) \otimes C \rightarrow C$  is the evaluation map. Using the notation  $\alpha(a) = a_{\alpha} \otimes \lambda^{\alpha}$  (summation implicitly understood), this means that  $\psi(c \otimes a) = a_{\alpha} \otimes \lambda^{\alpha}(c)$ . We will say that  $(A, C, \psi)$  is completely factorizable if  $\psi$  has an inverse  $\varphi$ , and  $(A, C, \psi)$  and  $(A, C, \varphi)$  are both factorizable.

**EXAMPLES 6.1.** 1) Let  $H$  be a bialgebra,  $(H, A, C)$  a left-right Doi-Hopf structure. The corresponding entwining structure  $(A, C, \psi)$  is factorizable: Take  $\alpha(a) = a_{[0]} \otimes m_{a_{[1]}}$ , where  $m_h : C \rightarrow C$ ,  $m_h(c) = ch$ , for all  $c \in C$ .

2) Assume that  $A$  is finitely generated projective. We know from [17] that  $(A, C, \psi)$  is induced by a Doi-Hopf structure over a bialgebra, so  $(A, C, \psi)$  is factorizable. This can also be seen directly: let  $\{(e_i, e_i^*) \mid i = 1, \dots, n\}$  be a finite dual basis of  $A$ , and define

$$\alpha(a) = \sum_i e_i \otimes \langle e_i^*, a_{\psi} \rangle \bullet^{\psi}$$

3) Now assume that  $C$  is finitely generated projective, and let  $\{(c_i, c_i^*) \mid i = 1, \dots, n\}$  be a finite dual basis of  $C$ . Then  $(A, C, \psi)$  is factorizable: if we take  $\alpha(a) = a_{\psi} \otimes c_i^{\psi} c_i^*$ , then we easily compute that

$$a_{\alpha} \otimes \lambda^{\alpha}(c) = a_{\psi} \otimes c_i^{\psi} \langle c_i^*, c \rangle = a_{\psi} \otimes c^{\psi}$$

Factorizable entwining structures are close to Doi-Hopf structures: the philosophy is that the bialgebra  $H$  is replaced by the algebra  $\text{End}(C)$ . Of course this is just philosophy, since there are no bialgebra structures on  $\text{End}(C)$  with the composition as multiplication. We will see that factorizable entwining structures are more general than Doi-Hopf structures, and also that there exist non-factorizable entwining structures. Our examples are inspired by the examples given in [17]. Let  $(A, C, \psi)$  be an entwining structure over a field  $k$ . Recall from [17] that we can construct the following endomorphisms of  $A$ , for every  $c \in C$  and  $c^* \in C^*$ :

$$T_{c,c^*} : A \rightarrow A, \quad T_{c,c^*}(a) = \langle c^*, c^\psi \rangle a_\psi$$

If  $(A, C, \psi)$  originates from a Doi-Koppinen structure  $(A, C, H)$ , then  $T_{c,c^*}(a) = c^*(a_{[1]}c)a_{[0]}$ , and we see that every  $H$ -subcomodule of  $A$  is  $T_{c,c^*}$ -invariant. Since we are working over a field, every  $a \in A$  is contained in a finite dimensional  $H$ -subcomodule of  $A$ , so every  $a \in A$  is contained in a finite dimensional  $T_{c,c^*}$ -invariant subspace of  $A$ . This property will be used in the examples in the sequel.

In a similar way, if  $(A, C, \psi)$  originates from an alternative Doi-Hopf datum, then every  $c \in C$  lies in a finite dimensional  $T_{a,a^*}$ -invariant subspace of  $C$ . Here  $T_{a,a^*}$  is defined by  $T_{a,a^*}(c) = a^*(a_\psi)c^\psi$ , for every  $a \in A$ ,  $a^* \in A^*$  and  $c \in C$ . Remark that  $\alpha(a) = \sum_i e_i \otimes T_{a,e_i^*}$  in Example 6.1 2).

We will now give an example of a factorizable entwining structure that does not originate from a Doi-Hopf datum.

EXAMPLE 6.2. Let  $A = k\langle (X_i)_{i \in I_1} \rangle$  be the free algebra with a family of generators indexed by  $I_1 = \mathbb{N}$  or  $I_1 = \mathbb{Z}$ .

Let  $C$  be the  $k$ -module with free basis  $\{1, t\} \cup \{t_i \mid i \in I_2\}$ , where  $I_2 = \mathbb{N}$  or  $I_2 = \mathbb{Z}$ . We put a coalgebra structure on  $k$  by making 1 grouplike and  $t$  and  $t_i$  primitive.

We now define the entwining map  $\psi$ . For every  $a \in A$  and  $c \in C$  we define  $\psi(a \otimes 1) = a \otimes 1$  and  $\psi(1 \otimes c) = 1 \otimes c$ . Furthermore we define

$$\psi(X_{i_1} \cdots X_{i_n} \otimes t) = X_{i_1+1} \cdots X_{i_n+1} \otimes t$$

$$\psi(X_{i_1} \cdots X_{i_n} \otimes t_j) = X_{i_1+1} \cdots X_{i_n+1} \otimes t_{j+n}$$

and extend  $\psi$  linearly. A straightforward computation shows that  $(A, C, \psi)$  is an entwining structure. Let us show that  $(A, C, \psi)$  is factorizable. By linearity, it suffices to define  $\alpha$  on elements of the form  $a = X_{i_1} \cdots X_{i_n} \otimes c$ . Write  $c = \tilde{c} + \bar{c}t + \sum_i c_i t_i$  with  $\tilde{c}, \bar{c}, c_i \in k$ . Then we have

$$\begin{aligned} \psi(a \otimes c) &= a + \tilde{c} + X_{i_1+1} \cdots X_{i_n+1} \otimes (\bar{c}t + \sum_j c_j t_{j+n}) \\ &= a \otimes \lambda_1^a(c) + X_{i_1+1} \cdots X_{i_n+1} \otimes \lambda_2^a(c) \end{aligned}$$

where  $\lambda_1^a, \lambda_2^a : C \rightarrow C$  are defined by

$$\lambda_1^a(1) = 1, \quad \lambda_1^a(t) = 0, \quad \lambda_1^a(t_i) = 0$$

$$\lambda_2^a(1) = 0, \quad \lambda_2^a(t) = t, \quad \lambda_2^a(t_i) = t_{i+1}$$

so we find that  $\alpha(a) = a \otimes \lambda_1^a + X_{i_1+1} \cdots X_{i_n+1} \otimes \lambda_2^a$  and  $(A, C, \psi)$  is factorizable. Let us show that there is no Doi-Hopf structure inducing  $(A, C, \psi)$ . Take  $c^* \in C^*$  such that  $\langle c^*, t \rangle = 1$  (this is possible since we work over a field). Then  $T_{c^*,t}(X_i) = X_{i+1}$ , so every  $T_{c^*,t}$ -invariant subspace that contains  $X_0$  is infinite dimensional.

In a similar way, we find that  $(A, C, \psi)$  does not originate from an alternative Doi-Hopf datum: take  $a^* \in A^*$  such that  $a^*(X_1) = 1$ ; then we find that  $T_{a^*, X_0}(t_i) = t_{i+1}$ , so every  $T_{a^*, X_0}$ -invariant subspace containing  $t_0$  is infinite dimensional.

If  $I_1 = I_2 = \mathbb{Z}$ , then  $\psi$  is bijective, and  $\varphi = \psi^{-1}$  has the same properties as  $\psi$ , so we find a completely factorizable entwining structure that cannot be derived from an (alternative) Doi-Hopf structure.

Remark that we could also have taken

$$\psi(X_{i_1} \cdots X_{i_n} \otimes t_j) = X_{i_1+1} \cdots X_{i_n+1} \otimes t_{j+\sum_k i_k}$$

Adapting Example 6.2, we can give an example of an entwining structure that is not factorizable.

EXAMPLE 6.3. Example 6.2 Let  $A$  and  $C$  be as in Example 6.2, and let  $\psi : A \otimes C \rightarrow A \otimes C$  be defined by

$$\psi(X_{i_1} \cdots X_{i_n} \otimes t_j) = X_{i_1+j} \cdots X_{i_n+j} \otimes t_{j+n}$$

Consider the  $k$ -linear map  $p : C \rightarrow k$ , given by  $p(1) = p(t) = p(t_i) = 1$ . If  $(A, C, \psi)$  is factorizable, then for all  $a \in A$ , the set

$$A_a = \{(I_A \otimes p)\psi(a \otimes c) \mid c \in C\}$$

is contained in a finite dimensional subspace of  $A$ . This is not the case, since  $A_{X_1}$  contains  $X_2, X_3, \dots$ . Hence  $(A, C, \psi)$  is not factorizable.

Let  $(A, C, \psi)$  be a right-right entwining structure, and  $\mathcal{C} = A \otimes C$  the associated  $A$ -coring. The left dual ring is  ${}^*C = {}_A\text{Hom}(A \otimes C, A) \cong \#(C, A)$ , with multiplication

$$(f\#g)(c) = f(c_{(2)})_\psi g(c_{(1)}^\psi)$$

$A$  and  $(C^*)^{op}$  are then subalgebras of  $\#(C, A)$ , via the algebra monomorphisms  $i : A \rightarrow \#(C, A)$  and  $j : (C^*)^{op} \rightarrow \#(C, A)$  given by

$$i(a)(c) = \varepsilon(c)a \text{ and } j(c^*)(a) = \langle c^*, c \rangle 1_A$$

and we easily compute that

$$(j(c^*)\#i(a))(c) = \langle c^*, c \rangle a$$

for all  $a \in A$ ,  $c \in C$  and  $c^* \in C^*$ .

PROPOSITION 6.4. *If  $(A, C, \psi)$  is a factorizable entwining structure, then*

$$i(A)j((C^*)^{op}) \subset j((C^*)^{op})i(A)$$

*and consequently  $i(A)j((C^*)^{op})$  is a subalgebra of  $\#(C, A)$ .*

*Proof.* For all  $a \in A$ ,  $c \in C$  and  $c^* \in C^*$ , we have

$$\begin{aligned} (i(a)\#j(c^*))(c) &= i(a)(c_{(2)})_\psi j(c^*)(c_{(1)}^\psi) = \varepsilon(c_{(2)})a_\psi \langle c^*, c_{(1)}^\psi \rangle 1_A \\ &= \langle c^*, c^\psi \rangle a_\psi = \langle c^* \circ \lambda^\alpha, c \rangle a_\alpha = \left( j(c^* \circ \lambda^\alpha)\#j(a_\alpha) \right)(c) \end{aligned}$$

hence

$$i(a)\#j(c^*) = j(c^* \circ \lambda^\alpha)\#j(a_\alpha) \in j((C^*)^{op})\#i(A)$$

as needed.  $\square$

If  $C^*$  is  $C$ -locally projective as a  $k$ -module (e.g.  $C^*$  is projective and  $C$  is dense in  $C^{**}$ , this is for example the case when  $k$  is a field), then  $j \otimes i : (C^*)^{op} \otimes A \rightarrow \#(C, A)$  is injective, and we obtain

**COROLLARY 6.5.** *Let  $(A, C, \psi)$  be a factorizable entwining structure, and assume that  $C^*$  is  $C$ -locally projective as a  $k$ -module. Then we have a factorization structure  $((C^*)^{op}, A, R)$ , with  $R : A \otimes (C^*)^{op} \rightarrow (C^*)^{op} \otimes A$  given by  $R(a \otimes c^*) = c^* \circ \lambda^a \otimes a_\alpha$ , and  $j \otimes i : (C^*)^{op} \#_R A \rightarrow \#(C, A)$  is an algebra monomorphism.*

*If  $(A, C, \psi)$  is a completely factorizable entwining structure, then  $R$  is bijective.*

## 7. CO-FROBENIUS PROPERTIES

Let  $\mathcal{C}$  be an  $A$ -coring with a fixed grouplike element, and consider the Morita contexts introduced in Section 2. It was shown in [8] that the connecting modules  $A$  and  $Q$  are isomorphic if the ring morphism  $A \rightarrow {}^*\mathcal{C}$  is Frobenius. In this Section, we will present a weaker property under more general assumptions.

We call an  $A$ -coring  $\mathcal{C}$  right co-Frobenius if there exists an injective right  ${}^*\mathcal{C}$ -linear map  $j : \mathcal{C} \rightarrow {}^*\mathcal{C}$ .

If  $\mathcal{C}$  is co-Frobenius and locally projective as a left  $A$ -module, then  $I = \text{Im } j \subset ({}^*\mathcal{C})^{\text{rat}}$ .

**PROPOSITION 7.1.** *If  $\mathcal{C}$  is co-Frobenius and locally projective as a left  $A$ -module, then there exists an injective right  $B$ -linear map  $J : A \rightarrow Q$ .*

*Proof.* By Proposition 5.1 we have to prove that the injection  $j : \mathcal{C} \rightarrow ({}^*\mathcal{C})^{\text{rat}}$  induces an injection  $J : A \cong \mathcal{C}^{\text{co}\mathcal{C}} = \text{Hom}^{\mathcal{C}}(A, \mathcal{C}) \rightarrow \text{Hom}^{\mathcal{C}}(A, ({}^*\mathcal{C})^{\text{rat}}) \cong Q$ . For every  $\phi \in \text{Hom}^{\mathcal{C}}(A, \mathcal{C})$ , let  $J(\phi) = j \circ \phi \in \text{Hom}^{\mathcal{C}}(A, ({}^*\mathcal{C})^{\text{rat}})$ .  $J$  is injective: if  $J(\phi) = j \circ \phi = 0$ , then  $\phi = 0$ , since  $j$  is injective. If we view  $J$  as a map  $A \rightarrow Q$ , then  $J$  is given by  $J(a) = j(ax)$ , which is obviously right  $B$ -linear.  $\square$

*Remark 7.2.* If  $A$  is a (commutative) field and  $\mathcal{C}$  is left and right co-Frobenius. It is shown in [13, Prop. 5.5.3], that there exists a surjective (and a fortiori a bijective)  $B$ -linear map  $J$  as above.

Let  $H$  be a co-Frobenius Hopf algebra, and  $A$  an  $H$ -comodule algebra. Then  $A \otimes H$  is an  $A$ -coring, with grouplike element  $1_A \otimes 1_H$ . In this case  ${}^*\mathcal{C} = \#(H, A)$ . In [4] it is shown that  $Q \cong A$ , if we restrict the Morita context from  $\#(H, A)$  to  $A \# H^{*\text{rat}}$ . In Proposition 7.8, we will generalize this to completely factorizable entwining structures.

**LEMMA 7.3.** *Let  $A$  and  $B$  be two algebras over a commutative ring  $k$  and  $R : B \otimes A \rightarrow A \otimes B$  a factorization map. Then  $R$  is a  $(B, A)$ -bimodule morphism from  $B \otimes A$  to  $A \#_R B$ .*

*Proof.* We show that  $R$  is a left  $B$ -linear:

$$R(b'b \otimes a) = a_R \# (b'b)_R = a_{Rr} \# b'_r b_R = (1 \# b')(a_R \# b_R) = b' R(a \otimes b)$$

The right  $A$ -linearity of  $R$  can be handled in a similar way.  $\square$

Let  $(A, C, \psi)$  be a factorizable (right, right) entwining structure over  $k$ . In Section 6, we have seen that we have a smash product  $C^{*\text{op}} \#_R A$  and a ring morphism

$C^{*\text{op}}\#_R A \rightarrow \#(C, A)$ . If  $C$  is locally projective as a  $k$ -module (e.g.  $k$  is a field), then, by extension of scalars,  $A \otimes C$  is a  $C^*$ -locally projective left  $A$ -module.

Consider  $\#(C, A)$  as a right  $\#(C, A)$ -module. By Proposition 4.10, we have that

$$\#(C, A)^{\#(C, A)\text{-rat}} = \#(C, A)^{C^*\text{-rat}}$$

and we will denote this module by  $\#(C, A)^{\text{rat}}$ .  $(C^{*\text{op}}\#_R A)^{\text{rat}}$  will be the  $C^{*\text{op}}$ -rational part of  $C^{*\text{op}}\#_R A$  as a right  $C^{*\text{op}}$ -module.

LEMMA 7.4. *With notation as above,  $(C^{*\text{op}}\#_R A)^{\text{rat}}$  is a two-sided ideal of  $C^{*\text{op}}\#_R A$  and a subring of  $\#(C, A)^{\text{rat}}$ .*

*Proof.* It is obvious that  $(C^{*\text{op}}\#_R A)^{\text{rat}} \subset \#(C, A)^{\text{rat}}$ .

Furthermore, since  $C^{*\text{op}}\#_R A$  is a right  $C^{*\text{op}}$ -module and a right  $A$ -module, and since  $A\#C^{*\text{op}} \subset C^{*\text{op}}\#A$ ,  $(C^{*\text{op}}\#_R A)^{\text{rat}}$  is also a right  $C^{*\text{op}}$  and a right  $A$ -module, by part 3) of Proposition 4.10, and we conclude that  $(C^{*\text{op}}\#_R A)^{\text{rat}}$  is a right ideal in  $C^{*\text{op}}\#_R A$ .

In order to prove that  $(C^{*\text{op}}\#_R A)^{\text{rat}}$  is also a left ideal, take  $b\#g \in C^{*\text{op}}\#_R A$ ,  $a\#f \in (C^{*\text{op}}\#_R A)^{\text{rat}}$  and  $h \in C^*$ . We compute

$$((b\#g)(a\#f))h = (b\#g)((a\#f)h) = (b\#g)((a_i\#f_i)h(c_i)) = ((b\#g)(a_i\#f_i))h(c_i)$$

and we find that  $(b\#g)(a\#f) \in (C^{*\text{op}}\#_R A)^{\text{rat}}$ , as needed.  $\square$

COROLLARY 7.5. *The Morita context associated to the  $A$ -coring  $\mathcal{C} = A \otimes C$  can be restricted to a Morita context*

$$(B, (C^{*\text{op}}\#_R A)^{\text{rat}}, A, \tilde{Q}, \tilde{\tau}, \tilde{\mu})$$

with  $\tilde{Q} = Q \cap (C^{*\text{op}}\#_R A)^{\text{rat}}$  and  $\tilde{\tau}, \tilde{\mu}$  the restricted maps.

LEMMA 7.6. *Let  $(A, C, \psi)$  be a completely factorizable entwining structure, and assume that  $A$  is a free  $k$ -module (e.g.  $k$  is a field). Then*

$$(C^{*\text{op}}\#_R A)^{\text{rat}} \cong (A \otimes C^{*\text{op}})^{\text{rat}} = A \otimes (C^{*\text{op}})^{\text{rat}}$$

and

$$(A \otimes (C^{*\text{op}})^{\text{rat}})^{C^*} = A \otimes ((C^{*\text{op}})^{\text{rat}})^{C^*}$$

as  $(A, C^{*\text{op}})$ -bimodules.

*Proof.* It follows from Lemma 7.3 that  $R$  is a morphism of  $(A, C^{*\text{op}})$ -bimodules and from Corollary 6.5, that  $R$  is bijective, hence  $(C^{*\text{op}}\#_R A)^{\text{rat}} \cong (A \otimes C^{*\text{op}})^{\text{rat}}$ . Obviously  $A \otimes (C^{*\text{op}})^{\text{rat}} \subset (A \otimes C^{*\text{op}})^{\text{rat}}$ . Choose a basis  $\{e_i\}_{i \in I}$  of  $A$ . Then for every  $\sum_i a_i \otimes f_i \in (A \otimes C^{*\text{op}})^{\text{rat}}$ , we can find  $f_{ij} \in C^*$  such that  $\sum_i a_i \otimes f_i = \sum_{i,j \in I} e_j \otimes f_{ij}$ , with only finitely many of the  $f_{ij}$  different from zero.

For every  $g \in C^*$ , we find

$$\left(\sum_i a_i \otimes f_i\right)g = \left(\sum_i a'_i \otimes f'_i\right)g(c_i) = \left(\sum_{i,j} e_j \otimes f'_{ij}\right)g(c_i) = \sum_{i,j} e_j \otimes g(c_i)f'_{ij}$$

We also have that  $(\sum_i a_i \otimes f_i)g = (\sum_{i,j} e_j \otimes f_{ij})g = \sum_{i,j} e_j \otimes gf_{ij}$ , and we find that  $gf_{ij} = \sum_i g(c_i)f'_{ij}$ , since  $\{e_i\}_{i \in I}$  is a basis of  $A$ . It follows that  $f_{ij} \in (C^{*\text{op}})^{\text{rat}}$  and  $(A \otimes C^{*\text{op}})^{\text{rat}} \subset A \otimes (C^{*\text{op}})^{\text{rat}}$ . The final statement can be proved in a similar way.  $\square$

LEMMA 7.7. *If  $(A, C, \psi)$  is a factorizable entwining structure over a field  $k$ , then  $(ab)_\alpha \otimes \lambda^\alpha = a_\alpha b_\beta \otimes \lambda^\beta \circ \lambda^\alpha$  and  $1_\alpha \otimes \lambda^\alpha = 1 \otimes I_C$ .*

Let  $\tilde{B} = \{b \in A \mid \alpha(b) = b \otimes I_C\}$ . Then  $\tilde{B} \subseteq B$  and in case of a factorizable entwining structure arising from a  $H$ -comodule algebra  $A$  (i.e. a Doi-Koppinen structure  $(A, H, H)$ ), we have  $\tilde{B} = B = \{a \in A \mid a_{[0]} \otimes a_{[1]} = a \otimes 1_H\}$ .

PROPOSITION 7.8. *Let  $(A, C, \psi)$  be a completely factorizable entwining structure, where  $C$  is a left and right co-Frobenius coalgebra over a field  $k$ . With notation as above,  $\tilde{Q} \cong A$  as right  $\tilde{B}$ -modules.*

*Proof.* It follows from Propositions 3.6 and 5.1 that  $Q = (\#(C, A)^{\text{rat}})^{\text{co}C} = (\#(C, A)^{\text{rat}})^{C^*}$ . Applying Lemma 7.4, we find that

$$\begin{aligned} \tilde{Q} &= (\#(C, A)^{\text{rat}})^{C^*} \cap (C^{*\text{op}} \#_R A)^{\text{rat}} = ((C^{*\text{op}} \#_R A)^{\text{rat}})^{C^*} \\ &\cong (A \otimes (C^{*\text{op}})^{\text{rat}})^{C^*} \cong A \otimes ((C^{*\text{op}})^{\text{rat}})^{C^*} \end{aligned}$$

$f \in ((C^{*\text{op}})^{\text{rat}})^{C^*}$  if and only if  $gf = g(x)f$  for every  $g \in C^*$ . If  $C$  is left and right co-Frobenius, then by (the dual version of) Remark 7.2, we find an isomorphism  $k \rightarrow ((C^{*\text{op}})^{\text{rat}})^{C^*}$ .

$t = j(x)$  is a generator for  $((C^{*\text{op}})^{\text{rat}})^{C^*}$ , so we find a bijective map  $\varphi : A \rightarrow \tilde{Q}$  given by  $\varphi(a) = t \circ \lambda^\alpha \# a_\alpha$ . We finally have to show that  $\varphi$  is right  $\tilde{B}$ -linear:

$$\begin{aligned} \varphi(ab) &= t \circ \lambda^\alpha \# (ab)_\alpha = t \circ \lambda^\beta \circ \lambda^\alpha \# a_\alpha b_\beta \\ &= t \circ I_C \circ \lambda^\alpha \# a_\alpha b = t \circ \lambda^\alpha \# a_\alpha b \\ &= (t \circ \lambda^\alpha \# a_\alpha) b = \varphi(a) b \end{aligned}$$

□

Remark 7.9. We can now transport the  $C^*$ -module structure from  $\tilde{Q}$  to  $A$ . In the case where  $C = H$ , this is done in [4] using the distinguished grouplike element. Factorizability implies that  $(C^{*\text{op}})^{\text{rat}} \#_R A$  is an  $A$ -bimodule. This is a generalization of the well-known fact that, for a Hopf algebra  $H$ ,  $H^{*\text{rat}}$  is an  $H$ -bimodule.

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