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Abstract

This paper derives simple and plausible conditions under which ambiguity aversion raises the demand for (self-) insurance and self-protection when the effort is furnished one period before the realization of the uncertainty. Unlike the recent contribution made by [Alary D., Gollier C., Treich N., 2013. The effect of ambiguity aversion on insurance and self-protection. The Economic Journal], I show that in the most usual situations in which the level of ambiguity does not increase with the level of effort, a clear and positive answer can be given to the question: Does ambiguity aversion raise the level of effort?

Keywords: Non-expected utility, Self-protection, Self-insurance, Ambiguity, Prudence

JEL Classification: D61, D81, D91, G11.

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1 Introduction

In a recent contribution, Alary, Gollier, and Treich (2013) introduced the notion of ambiguity in self-insurance and self-protection models. By extending a work initiated by Snow (2011), they offered these tools originally designed for the study of risks a broader scope of application. However, if they managed to obtain general results when studying self-insurance, it must be noted that the results they obtained for self-protection concern only a restricted, rather implausible, range of situations. To draw more generally applicable conclusions, this paper proposes a treatment of these more plausible situations by allowing investment to precede the realization of uncertainty. As in the expected utility model of Menegatti (2009), prudence is shown to play a central role and is generally positively associated with self-protection.

Self-insurance and self-protection, in the terminology of Ehrlich and Becker (1972), are two risk management tools used to deal with the risk of facing a monetary loss when market insurance is not available. In both situations, a decision maker (DM) has the opportunity to undertake an *effort* to modify the distribution of a given risk. In particular, the effort in a self-insurance model corresponds to the amount of money invested to reduce the size of the loss occurring in the bad state of the world, while in a self-protection model (also called prevention model), the effort is the amount invested to reduce the probability of being in the bad state.
Though these two models have received a great deal of attention in the literature during the last few years, it is noteworthy to mention that this literature has, until now, generally only focused on simple one-period, two-state models inside the expected utility framework. Although monoperiodic risky models are well adapted to describe a certain number of situations, they seem to be too restrictive in at least two dimensions to describe a large number of other important issues. First, there exist many situations requiring self-insurance or self-protection in real life in which the decision to invest and the realization of uncertainty does not take place at the same point in time. A long period of time may pass between these two events, leading to the necessity of taking intertemporal considerations into account. To illustrate this assertion, think for example of an individual living for two periods. This individual faces the risk of heart attack when he becomes old and has the choice, when he is young, to practice sport or not. Sport is costly but has the advantage of either reducing the probability of a heart attack with which a potentially important fixed loss is associated, or of reducing the severity of an attack that happens with a fixed

probability[1]. In this example, it is clear that many years may separate the moment at which the effort decision is taken and the moment the uncertainty realizes. A single period setting may therefore not be very appropriate to model such a problem. The analysis of self-protection and self-insurance in a two-period environment shows that the monoperiod results could be easily extended in the self-insurance case, while in the most usual situations concerning self-protection (i.e. for events characterized by a low probability of accident), the notion of prudence plays a central role[2].
The second limitation of the self-insurance and self-protection models studied in the literature is that they remain in the (subjective) expected utility framework, and are therefore unable to deal with other kinds of uncertainty besides *risk*[3]. In many real-life problems however, the nature of the uncertainty considered cannot be limited to risk since the probabilities associated with the realization of uncertain events cannot always be objectively known. Think for example to the probability of heart attack discussed above, or to the possibility of catastrophic climate change events. In this kind of situations, ambiguity plays a central role, and the attitude agents generally manifest towards this additional source of uncertainty need to be taken into account. The subjective expected utility theory that assumes ambiguity neutrality is therefore inconsistent in this context. Indeed, as first shown by Ellsberg (1961) and later confirmed by a number of experimental studies[4], the uncertainty on the probabilities of a random event (i.e. ambiguity) often leads the decision maker to violate the axiom of reduction of compound lotteries in the sense that it makes him over-evaluate less desirable outcomes. It is therefore important to take this individual behavior (called ambiguity aversion) into account when considering problems in the presence of ambiguity.

In this paper, I present models of self-insurance and self-protection that do not suffer from these two limitations. Each model takes the form of a simple two-period model incorporating the theory Klibanoff, Marinacci, and Mukerji (2005, 2009) de-

[1] Think for example that doing sport enables to lower recovery costs, thanks to a better physical condition.

[2] In particular prudent and risk-averse expected utility maximizers exert more effort than the risk-neutral agent as shown by Menegatti (2009) and Berger (2010).

[3] The probability distributions are therefore assumed to be known with certainty. In particular, those models implicitly assume the absence of any kind of ambiguity, or equivalently, assume that agents are ambiguity-neutral (and therefore behave as subjective expected utility maximizers in the sense of Savage (1954)). Notably exceptions are the recent papers of Snow (2011) and Alary, Gollier, and Treich (2013).

[4] See Einhorn and Hogarth (1986), Viscusi and Chesson (1999), and Ho et al. (2002) among others.

veloped to deal with ambiguity. The timing of the decision process is very simple: in the first period, a DM chooses the level of effort he wants to exert to either affect the probability of being in a state in which ambiguity is concentrated in second period (self-protection), or to affect the level of wealth in this ambiguous state (self-insurance). Using this setting, I derive the conditions under which ambiguity aversion raises the demand for insurance, self-insurance and self-protection.

This paper hence constitutes the missing piece of the puzzle on the study of self-insurance and self-protection initiated by the work of Ehrlich and Becker (1972). It is both an extension of Snow (2011) and Alary, Gollier, and Treich (2013) in the sense that it goes from the study of the one- to the two-period problem, and of Menegatti (2009) and Berger (2010) in the sense that it allows non neutral ambiguity attitudes. In particular, this paper shows that when the effort is undertaken during the first period, ambiguity aversion tends to reinforce risk aversion and has a positive impact on the demand for (self-)insurance and self-protection. However, as for the study of risk attitude in which risk aversion alone is not sufficient to guarantee a higher level of prevention (since *risk* prudence is also needed), I show that the extra condition of ambiguity prudence attitude is also needed to observe this positive impact. The close relationship between prudence and prevention that was achieved in the two-period setting is then re-established in the presence of ambiguity, and an clear answer to the question: *Does ambiguity aversion tend to raise the level of effort?* may be given, contrary to the conflicting results obtained in the one-period settings.

The remaining of this paper is organized as follows. In Section 2, I introduce the simple two-period model under ambiguity by studying the problem of full insurance. I then analyze successively the willingness to pay (Section 3) and optimal effort (Section 4) for self-insurance and self-protection. Section 5 concludes the paper.

2 Model Description and Full Insurance

As mentioned in the Introduction, the model involves both risk and ambiguity: probabilities of second-period final wealth are not objectively known, instead they consist in a set of probabilities, depending on an external parameter θ for which the decision maker (DM) has prior beliefs[5]. Ambiguity may therefore be interpreted as a multi-stage lottery: a first lottery determines the value of parameter θ, and a second

[5]Imagine that parameter θ can take values $\theta_1, \theta_2, ..., \theta_m$ with probabilities $[q_1, q_2, ..., q_m]$, such that the expectation with respect to the parametric uncertainty is written $\mathrm{E}_\theta g(\tilde{\theta}) = \sum_{j=1}^m q_j g(\theta_j)$.

one determines the size of second-period wealth. The second-period wealth distribution $\tilde{w}_2(\theta)$ is represented by the vector $[w_{2,1}, w_{2,2}, ..., w_{2,n}; p_1(\theta), p_2(\theta), ..., p_n(\theta)]$ with $w_{2,1} < w_{2,2} < \cdots < w_{2,n}$.

In the time-separable model, the intertemporal welfare under Klibanoff, Marinacci, and Mukerji (2005, 2009) (KMM) representation is the following:

$$u(w_1) + \beta\phi^{-1}\left\{\mathrm{E}_\theta\phi\left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\}, \tag{1}$$

where w_i is the exogenous wealth in the beginning of period $i = 1, 2$, u represents the period vNM utility functions, $\beta \in [0, 1]$ is the discount factor[6], ϕ represents attitude towards ambiguity, E_θ is the expectation operator taken over the distribution of θ, conditional on all information available during the first period, and E is the expectation operator taken over w_2 conditional on θ. The function ϕ is assumed to be three times differentiable, increasing, and concave under ambiguity aversion, so that the ϕ-certainty equivalent in equation (1) is lower in that case than when the individual is ambiguity neutral characterized by a linear function ϕ[7]:

$$\phi^{-1}\left\{\mathrm{E}_\theta\phi\left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\} \leq \mathrm{E}_\theta\mathrm{E}u(\tilde{w}_2(\tilde{\theta})) = \mathrm{E}u(\tilde{w}_2) \tag{2}$$

In that sense, an ambiguity averse DM dislikes any mean-preserving spread in the space of conditional second period expected utilities.

The right hand side of expression (2) corresponds to the second period welfare obtained by an ambiguity neutral individual who evaluates his welfare by considering the risky second period wealth $\tilde{w}_2$: $[w_{2,1}, w_{2,2}, ..., w_{2,n}; \bar{p}_1, \bar{p}_2, ..., \bar{p}_n]$ with the mean state probabilities $\bar{p}_s = \mathrm{E}_\theta p_s(\tilde{\theta})$, $\forall s = 1, ..., n$. In that sense, an ambiguity neutral individual is nothing but a savagian expected utility agent.

As for the single period model, the study of willingness to pay (WTP) P for risk elimination under ambiguity is straightforward[8]. It corresponds, in this case, to the amount an individual is ready to pay in period 1 to escape the uncertainty in period

[6] In what follows, I assume that $\beta = 1$, an assumption that has no impact on the results obtained.

[7] Notice that for simplicity, I assume that ϕ is only defined for non-negative values. Any value inside the second bracket must therefore be non-negative, which should not be a problem since any positive affine transformation of u represents the same preferences over risky situations. KMM consider for example the unique continuous, strictly increasing function u with $u(0) = 0$ and $u(1) = 1$ that represents any given preferences.

[8] Remark that in the single period model studied by Alary, Gollier, and Treich (2013), this willingness to pay P is nothing but what Berger (2011) or Maccheroni et al. (2013) call the *uncertainty premium*, which is by definition superior to Pratt's risk premium under ambiguity aversion.

2, and is defined as follows:

$$u(w_1 - P) + u(\mathrm{E}\tilde{w}_2) = u(w_1) + \phi^{-1}\left\{\mathrm{E}_\theta\phi\left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\}.$$

If the individual were ambiguity neutral, he would be ready to pay P_0 defined by $u(w_1 - P_0) + u(\mathrm{E}\tilde{w}_2) = u(w_1) + \mathrm{E}u(\tilde{w}_2)$ to eliminate the same risk. It is then easy to see, using inequality (2), that P is always higher than P_0 under ambiguity aversion in the two-period model. As in the single period model, ambiguity averse individuals are therefore ready to pay a higher premium for risk elimination, since the elimination of the risk automatically eliminates the ambiguity attached to this risk. This extra premium is the two-period version of the *ambiguity premium* defined in Berger (2011).

3 Willingness to Pay under Ambiguity

I now reexamine the willingness to pay (WTP) for infinitesimal insurance or protection in the context of a two-period model. To to highlight the differences between the one and two-period models, I remain in the framework developed by Alary, Gollier, and Treich (2013) (AGT hereafter), and assume that ambiguity is concentrated on a state i. In this case, the ambiguous probability to be in state i is $p_i(\theta)$, while the probability to be in any other state $s \neq i$ is given by

$$p_s(\theta) = (1 - p_i(\theta))\pi_s,$$

where π_s is the *unambiguous* probability of being in state s[9] conditional on the information that the state is not i. Remark that if there are only two states of nature, this structure simply reduces to the case with ambiguous probabilities $p(\theta)$ and $1 - p(\theta)$. From now on, I also assume, without loss of generality, that θ may be ranked in such a way that p_i is increasing in θ.

3.1 Willingness to Pay for Self-insurance

Self-insurance in a two-period world is a risk management tool thanks to which an individual has the opportunity to exert an effort today to reduce a cost in a specific state i tomorrow. By letting $P(\epsilon)$ denote the willingness to furnish this effort to increase marginally the wealth in state i and such that the level of welfare

[9] An implicit assumption is that $\sum_{s\neq i}\pi_s = 1$.

is not altered, we have:

$$u(w_1 - P(\epsilon)) + \phi^{-1}\left\{\mathrm{E}_\theta\phi\left\{p_i(\tilde{\theta})u(w_{2,i}+\epsilon) + [1-p_i(\tilde{\theta})]\sum_{s\neq i}\pi_s u(w_{2,s})\right\}\right\}$$
$$= u(w_1) + \phi^{-1}\left\{\mathrm{E}_\theta\phi\left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\}.$$

Totally differentiating this equation with respect to ϵ and evaluating it at $\epsilon = 0$ leads to

$$P'(0) = \frac{\mathrm{E}_\theta\phi'\left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\left[p_i(\tilde{\theta})u'(w_{2,i})\right]}{\phi'\left\{\phi^{-1}\left\{\mathrm{E}_\theta\phi\left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\}\right\}u'(w_1)}. \quad (3)$$

It is easy to see that the marginal WTP for self-insurance of an ambiguity neutral individual ($\phi' \equiv$ constant) is:

$$P'_N(0) = \frac{u'(w_{2,i})\mathrm{E}_\theta p_i(\tilde{\theta})}{u'(w_1)}. \quad (4)$$

An ambiguity averse individual has thus a higher marginal WTP to insure state i if $P'(0) > P'_N(0)$. To compare equations (5) and (4) in the most common case of non increasing ambiguity aversion, I use the following lemma and its corollary:

Lemma 1 *Let ϕ be a three times differentiable function reflecting ambiguity aversion. If ϕ exhibits DAAA (Decreasing Absolute Ambiguity Aversion) then* $\mathrm{E}\phi'\{\tilde{x}\} > \phi'\left\{\phi^{-1}\left\{\mathrm{E}\phi\{\tilde{x}\}\right\}\right\}$.

Proof ϕ is DAAA is equivalent to saying that $-\phi'$ is more concave than ϕ, or equivalently that $-\phi'''/\phi'' > -\phi''/\phi'$. Since $(\phi')^{-1}$ is a decreasing function, the proof follows from the fact that the certainty equivalent of function ϕ is larger than that of function $-\phi'$. ∎

Corollary 1 *If ϕ exhibits CAAA (Constant Absolute Ambiguity Aversion), then* $\mathrm{E}\phi'\{\tilde{x}\} = \phi'\left\{\phi^{-1}\left\{\mathrm{E}\phi\{\tilde{x}\}\right\}\right\}$.

Non increasing absolute ambiguity aversion refers to the notion of *ambiguity prudence attitude*, an individual characteristic necessary to observe an extra precautionary saving motive due to the presence of ambiguity on future wealth. This characteristic, which is stronger than requiring $\phi''' > 0$, has furthermore been shown to be sufficient for ambiguity prudence if ambiguity is concentrated on a particular state i and the agent is risk prudent ($u''' > 0$) (see Berger (2013) for details).

Ambiguity aversion therefore raises the marginal WTP for insurance in state i if $\text{cov}_\theta(\phi'\{\text{E}u(\tilde{w}_2(\tilde{\theta})\}, p_i(\tilde{\theta})) > 0$ and the individual has an ambiguity prudent attitude. Since p_i is assumed to be increasing in θ, by the covariance rule, and because ϕ' is decreasing under ambiguity aversion, we only need $Eu(\tilde{w}_2(\theta))$ to be decreasing in θ. Decomposing this expression enables to see that the condition needed is similar to the one in AGT:

$$\text{E}u(\tilde{w}_2(\theta)) = -p_i(\theta)\left[\sum_{s\neq i} u(w_{2,s}) - u(w_{2,i})\right] + \sum_{s\neq i} u(w_{2,s})$$

and $\text{E}u(\tilde{w}_2(\theta))$ is therefore decreasing in θ if ψ defined as the certainty equivalent of second period wealth conditional on the state not being i: $\sum_{s\neq i} \pi_s u(w_{2,s}) = u(\psi)$, is higher than second period wealth in state i: $w_{2,i}$. This leads to the following proposition:

Proposition 1 *In the two-period model of self-insurance in which ambiguity is concentrated on the insured state i, ambiguity aversion raises the marginal WTP to self-insure state i if the individual manifests ambiguity prudence attitude and second period wealth in state i is smaller than the second period certainty equivalent* ψ.

The intuition to this result is analogous to the one resulting from the study of willingness to pay for an increase in second period wealth in a Kreps and Porteus (1978)/Selden (1978) model. When the second period wealth in state i is considered as *unfavorable* in the sense that the utility obtained in that state is smaller than his expected utility in the others states, raising $w_{2,i}$ has a positive impact on the the conditional second period expected utilities $\text{E}u(\tilde{w}_2(\theta))$, something which valuable for any individual with $\phi' > 0$. However, this raise in $w_{2,i}$ has a cost: an effort that has to be furnished in advance (period 1). In the Kreps-Porteus/Selden model, we know that risk aversion raises the marginal WTP for an increase in second period wealth only if the individual is prudent, a condition which is satisfied in that context if the individual manifest decreasing absolute risk aversion (DARA). Given the similarity between Kreps-Porteus/Selden and KMM models, it is therefore not surprising that ambiguity aversion is not anymore sufficient to guarantee that the marginal WTP to self-insure state i increases. An additional condition analogous to prudence is needed. Non increasing absolute ambiguity aversion – or ambiguity prudence attitude – is this extra condition in the presence of ambiguity

3.2 Willingness to Pay for Self-protection

Another tool that may be used to deal with the presence of uncertainty in second period is self-protection: an individual has the opportunity to furnish an effort today to alter the probability of a specific state i tomorrow. In this subsection, I examine the effect of ambiguity aversion on the marginal willingness to furnish a self-protection effort in the context of a two-period model.
Proceeding as before, I denote by $P(\epsilon)$ the WTP today for a reduction ϵ in the probability of state i tomorrow, such that the intertemporal level of welfare is not modified. Furthermore, following AGT, I assume that the degree of ambiguity[10] is not altered by the change of p_i: p_i is equally affected for any value of θ, and the distribution of second period wealth conditional on the state not being i remain identical. Mathematically, $P(\epsilon)$ is defined as follows:

$$u(w_1 - P(\epsilon)) + \phi^{-1}\left\{\mathrm{E}_\theta \phi \left\{ \left[p_i(\tilde{\theta}) - \epsilon\right] u(w_{2,i}) + \left[1 - p_i(\tilde{\theta}) + \epsilon\right] \sum_{s \neq i} \pi_s u(w_{2,s}) \right\}\right\}$$
$$= u(w_1) + \phi^{-1}\left\{\mathrm{E}_\theta \phi \left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\}.$$

Totally differentiating this expression with respect ϵ and evaluating it at $\epsilon = 0$ yields:

$$P'(0) = \frac{\left[\sum_{s \neq i} u(w_{2,s}) - u(w_{2,i})\right] \mathrm{E}_\theta \phi' \left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}}{\phi' \left\{\phi^{-1}\left\{\mathrm{E}_\theta \phi \left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\}\right\} u'(w_1)}. \tag{5}$$

Assuming again that the second period wealth in state i: $w_{2,i}$ is smaller than the certainty equivalent ψ defined above (i.e that self-protection aims to reduce the probability of an *unfavorable* state) so that the marginal WTP is positive, it is easy to show that the marginal WTP to self-protect state i is higher under ambiguity aversion if:

$$\mathrm{E}_\theta \phi' \left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\} > \phi' \left\{\phi^{-1}\left\{\mathrm{E}_\theta \phi \left\{\mathrm{E}u(\tilde{w}_2(\tilde{\theta}))\right\}\right\}\right\}. \tag{6}$$

According to Lemma 1, this will be the case if individual manifests DAAA. Under CAAA, it is easy to see that the marginal WTP for self-protection in state i remains the same under ambiguity aversion. Alternatively remark also that if $w_{2,i} > \psi$ (i.e if the state to self-protect is a favorable state and the marginal WTP is negative)

[10]The *degree of ambiguity* is here defined in a very specific sense. It is kept constant if, for all states $s = 1, ..., n$, $\mathrm{Var}_\theta(p_s(\tilde{\theta}))$ is unaltered. Remark that in general this definition is not appropriate to study the degree of ambiguity, which is a much more compelex issue, as discussed in Jewitt and Mukerji (2011) and Berger (2011).

results are reversed. These results prove the following proposition and its corollary.

Proposition 2 *In the two-period model of self-protection in which ambiguity is concentrated on state* i*, ambiguity aversion raises (reduces) the marginal WTP to self-protect state* i *under DAAA (IAAA) if second period wealth in state* i *is smaller than the second period certainty equivalent* ψ*, and reduces (raises) it otherwise.*

Corollary 2 *In the two-period model of self-protection in which ambiguity is concentrated on state* i*, ambiguity aversion does not modify the marginal WTP to self-protect state* i *under CAAA.*

These results are different than in the single period model in which under DARA, ambiguity aversion *reduces* the marginal WTP to self-protect state i if wealth in state i is smaller than the precautionary equivalent wealth level conditional on state being not i (Proposition 3 in AGT).

The intuition here is similar as before. Since the effect of self-protection on probability of state i is identical for any value of θ, and since the distribution of other states conditional on $s \neq i$ is not modified, raising p_i has a positive and equal impact on conditional second period expected utility $\mathrm{E}u(\tilde{w}_2(\theta))$ for all values of θ. The cost of this increase is paid in first period so that the extra condition of DAAA is needed to observe a raise in the marginal WTP to self-protect state i due to the introduction of ambiguity aversion.

4 Optimal Effort under Ambiguity

In this section, I examine successively the impact of ambiguity aversion on the optimal insurance and protection in favor of state i in a two-period model. The general form of the decision maker's problem is given by:

$$\max_e u(w_1 - e) + \phi^{-1}\left\{\mathrm{E}_\theta \phi\left\{U(e, \tilde{\theta})\right\}\right\}, \tag{7}$$

where $U(e, \theta) = p_i(e, \theta)u(w_{2,i}(e)) + [1 - p_i(e, \theta)]\sum_{s \neq i} \pi_s u(w_{2,s})$ is the second period expected utility, conditional on the parameter θ. Notations are kept the same as before[11] and e represents the level of effort furnished to self-insure or self-protect state i. Problem (7) is a problem of self-insurance when $p_i(e, \theta) = p_i(\theta)$ for all levels of effort e, and a problem of self-protection when $w_{2,i}(e) = w_{2,i}$ for all e. I assume that $p_i(e, \theta)$ and $w_{2,i}(e)$ are differentiable in e and that when state i is *unfavorable*,

[11] Remember that β is fixed to unity for simplicity and without altering the final result.

$p_{ie}(e,\theta) \equiv \frac{\partial p_i(e,\theta)}{\partial e} \leq 0$ for all θ, and that $\frac{\partial w_{2,i}(e)}{\partial e} \geq 0$. Notice that under KMM specification, the concavity of u and ϕ does not guarantee that the maximization problem (7) is convex, so additional assumptions are needed for the solution of this program to be unique. These conditions are summarized in the following proposition.

Proposition 3 *The maximization program of a two-period self-insurance or self-protection problem under ambiguity as described by* (7) *is convex if:*

- *function ϕ has a concave absolute ambiguity tolerance: $-\phi'(U)/\phi''(U)$ is concave in U,*

and

- *$w_{2,i}(e)$ is concave in e in the self-insurance case: $\partial^2 w_{2,i}(e)/\partial e^2 \leq 0$, or*
- *$p_i(e,\theta)$ is convex in e in the self-protection case: $\partial^2 p_i(e,\theta)/\partial e^2 \geq 0$ for all θ.*

Proof Relegated to the Appendix. ∎

Analogously to the risk theory literature, concave absolute ambiguity tolerance is a property satisfied by the most widely-used specifications in the literature. In particular, it is satisfied for the families of constant relative ambiguity aversion (CRAA): logarithmic and power functions, of constant absolute ambiguity aversion (CAAA): exponential functions, and of quadratic functions.

Under the special case of ambiguity neutrality, problem (7) becomes a simple two-period problem in the expected utility framework. It consists in finding the level of effort e that maximizes:

$$u(w_1 - e) + \mathrm{E}_\theta U(e, \tilde{\theta}).$$

The optimal level of effort e^* chosen by an ambiguity averse individual is the solution of the first-order condition (FOC):

$$-u'(w_1 - e^*) + \mathrm{E}_\theta U_e(e^*, \tilde{\theta}) = 0, \tag{8}$$

where $U_e(e,\theta) = \partial U(e,\theta)/\partial e$. The first term of this expression represents the marginal cost of effort and the second represents the marginal benefits of self-protection or of self-insurance.

Ambiguity aversion therefore raises the optimal level of effort if the FOC of problem (7) evaluated at e^* is positive. This is the case if:

$$\frac{\mathrm{E}_\theta\left[\phi'\{U(e^*,\tilde{\theta})\}U_e(e^*,\tilde{\theta})\right]}{\phi'\left\{\phi^{-1}\left\{\mathrm{E}_\theta\phi\{U(e^*,\tilde{\theta})\}\right\}\right\}} \geq \mathrm{E}_\theta U_e(e^*,\tilde{\theta}). \tag{9}$$

The interpretation of this condition is simple: since ambiguity only affects variables during the second period, the marginal cost of effort, which takes place in first period, is unaffected and the condition indicates that the marginal benefit of protection or insurance must be higher under ambiguity aversion.
Using Lemma 1 and its corollary, it is easy to see that under CAAA, condition (9) is equivalent to:

$$\mathrm{cov}_\theta\left(\phi'\{U(e^*,\tilde{\theta})\}, U_e(e^*,\tilde{\theta})\right) \geq 0. \tag{10}$$

Moreover, condition (9) is always satisfied under DAAA if condition (10) holds. Since ϕ' is decreasing under ambiguity aversion, using the covariance rule the condition therefore becomes:

Proposition 4 *Ambiguity aversion raises the optimal level of effort in a two-period model as the one described by* (7) *if* $U(e^*,\theta)$ *and* $U_e(e^*,\theta)$ *are anti-comonotonic and if the individual manifests ambiguity prudence attitude, where* e^* *is defined by* (8).

4.1 Optimal Level of Self-insurance

I now investigate the conditions under which this proposition holds in the case of self-insurance. In that case, remember that the individual has the opportunity to furnish an effort e in first-period to increase his wealth to $w_{2,i}(e)$ in the insurable state i in second period. Conditional second period expected in the case of self-insurance is therefore given by:

$$U(e,\theta) = p_i(\theta)u(w_{2,i}(e)) + [1-p_i(\theta)]\sum_{s\neq i}\pi_s u(w_{2,s}).$$

Since p_i is assumed to be increasing in θ, it is easy to see that $U(e^*,\theta)$ decreases with θ if $w_{2,i}(e^*) < \psi$, and increases with θ otherwise, while $U_e(e^*,\theta) = p_i(\theta)u'(w_{2,i}(e^*))\frac{\partial w_{2,i}(e^*)}{\partial e}$ is increasing in θ. Combining theses results with condition (9) and using Lemma 1 and its corollary proves the following proposition:

Proposition 5 *In a two-period model of self-insurance of a state* i *in which ambiguity is concentrated, ambiguity aversion raises the optimal level of self-insurance* *under DAAA if second period wealth in state i is smaller than the second period certainty equivalent ψ.*

Remark that when $w_{2,i}(e^*) > \psi$, no general conclusion may be drawn. In that case, DAAA may increase or decrease the optimal level of effort.

Corollary 3 *In a two-period model of self-insurance of a state i in which ambiguity is concentrated, ambiguity aversion raises the optimal level of self-insurance under CAAA if second period wealth in state i is smaller than the second period certainty equivalent ψ, and decreases it otherwise.*

Example This result extends to a two period framework the results obtained by Snow (2011) in the particular case of a world with two states: a loss and a no-loss state. Under this assumption, if an insurable loss L occurs the second period wealth is $w_{2,i}(e^*) = w_2 - L(e^*)$, and is $\psi = w_2$ in the no-loss state. Snow (2011)'s result showing that ambiguity aversion increases the optimal level of self-insurance is then easily extended to a two-period world if the individual manifests CAAA or DAAA. Finally, if the loss function has the particular form: $L(e) = L - ke$, it is also possible to interpret the results in the context of a standard coinsurance problem where the premium e is paid in first period and for each dollar of which the insured agent receives an indemnity k if the loss occurs. In this case, ambiguity aversion raises the insurance coverage rate if the individual manifests non increasing ambiguity aversion. This result is the two-period version of Corollary 1 in Alary, Gollier, and Treich (2013) and is synthesized in the following corollary:

Corollary 4 *In the standard coinsurance problem with two states in which the insurance premium is paid in first period and uncertainty realizes in second period, ambiguity aversion raises the insurance coverage rate if the individual has an ambiguity prudence attitude.*

4.2 Optimal Level of Self-protection

I now consider the problem of self-protection: the effect of effort is to reduce the probability $p_i(e, \theta)$ of an unfavorable state i in which ambiguity is concentrated. Conditional second period expected takes the form:

$$U(e, \theta) = p_i(e, \theta)u(w_{2,i}) + [1 - p_i(e, \theta)] \sum_{s \neq i} \pi_s u(w_{2,s}).$$

As before and without loss of generality, I assume that $p_i(e^*, \theta)$ is increasing in θ so that $U(e^*, \theta)$ is a decreasing function of θ when state i is unfavorable. From Proposition 4, a sufficient condition to observe a higher level of effort under CAAA or DAAA than under ambiguity neutrality in the self-protection model, therefore simply becomes that the marginal benefit of effort $U_e(e^*, \theta) = -p_{ie}(e^*, \theta)\left[\sum_{s \neq i} \pi_s u(w_{2,s}) - u(w_{2,i})\right]$ is increasing in θ. The key element is how $-p_{ie}(e^*, \theta)$ evolves with θ, or alternatively how the degree of ambiguity is affected by a change in the level of effort[12]. If the degree of ambiguity is not altered by a change in the level of effort, as it was the case in the section studying the willingness to pay, $p_{ie}(e^*, \theta)$ is independent of θ and the covariance in (10) is equal to zero. In this case, an individual manifesting strictly DAAA will always choose a higher level of self-protection under ambiguity aversion, while an individual manifesting CAAA will self-protect exactly the same way as an ambiguity neutral agent. If on the contrary, the degree of ambiguity decreases with the level of effort exerted as it seems natural in many situations, $p_{ie}(e^*, \theta)$ is decreasing in θ so that there exists an additional incentive for an ambiguity averse decision maker to raise self-protection. It is therefore clear that in this situation, non increasing absolute ambiguity aversion – or ambiguity prudence attitude – raises the optimal level of effort. Finally, in the more implausible case where effort increases the level of ambiguity as in AGT, $p_{ie}(e^*, \theta)$ is increasing in θ and the ambiguity prudence attitude effect is not anymore sufficient to raise optimal self-protection. The following proposition and its corollary summarize theses results:

Proposition 6 *In the two-period problem of self-protection of an unfavorable state i in which ambiguity is concentrated, ambiguity prudence attitude is sufficient to raise the optimal self-protection effort under ambiguity aversion if effort decreases the degree of ambiguity of state i.*

Corollary 5 *In the two-period problem of self-protection of an unfavorable state i in which ambiguity is concentrated, an agent manifesting DAAA (resp. CAAA) chooses a higher (similar) level of self-protection than (as) an ambiguity neutral agent if effort does not affect the ambiguity of state i.*

To illustrate what precedes, consider the following examples.

Examples Imagine there are only two states of the world: a loss and a no-loss state in which second period wealth is respectively $w_2 - L$ with conditional probability

[12] To see the link with the degree of ambiguity defined above, remark that $\frac{\partial \text{Var}_\theta p_i(e,\tilde{\theta})}{\partial e} = 2\text{cov}_\theta(p_i(e,\tilde{\theta}); p_{ie}(e,\tilde{\theta}))$.

$p(e,\theta)$, or w_2 with probability $1-p(e,\theta)$. Consider two particular forms of loss probability functions that are both linear in the ambiguity parameter θ: $p^1(e,\theta) = p(e)+\theta$, and $p^2(e,\theta) = \theta p(e)$.

It is easy to see that in the additive case, $U_e(e^*,\theta) = -p'(e^*)\big[u(w_2) - u(w_2 - L)\big]$ so that an increase in θ has no effect on $U_e(e^*,\theta)$. The level of self-protection is therefore exactly the same for any individual manifesting constant ambiguity attitude[13]. In particular, an ambiguity neutral individual and a maxmin expected utility maximizer à la Gilboa and Schmeidler (1989) both choose to self-protect precisely the same way. If the individual manifests DAAA, he will always choose a higher level of protection under ambiguity aversion.

Imagine now that the degree of ambiguity is made smaller when the effort increases in the neighborhood of e^*. This is the case with the multiplicative form described above, where $U(e^*,\theta) = u(w_2) - \theta p(e^*)\big[u(w_2) - u(w_2 - L)\big]$ and $U_e(e^*,\theta) = -\theta p'(e^*)\big[u(w_2) - u(w_2 - L)\big]$. It is easy to check that an increase in θ will have a negative impact on U and a positive impact on U_e so that condition (10) is respected. Figure 1 illustrates the situation when there are two possible values of θ: θ_1 and θ_2, and when the ambiguous loss probability is linear in θ.

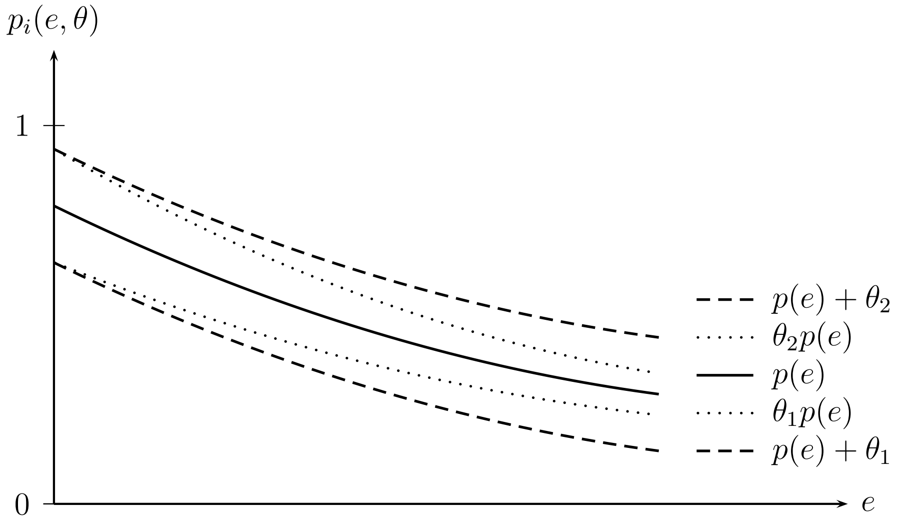

Figure 1: Linear ambiguous loss probability

As can be seen in Figure 1, when θ increases, from θ_1 to θ_2[14], different scenarii are

[13]Remember that according to Klibanoff, Marinacci, and Mukerji (2005), constant ambiguity attitude is characterized either by linear or exponential function ϕ.

[14]Remark that in this example, $p(e)$ is the loss probability law considered by an ambiguity neutral agent and that the ambiguity averse DM associates the same prior belief to each value of θ, in such a way that $\theta_2 = -\theta_1$ in the additive case, and $\theta_2 = 2 - \theta_1$ in the multiplicative case.

possible. In the additive case, the slopes of the two dashed lines are exactly the same for any given level of effort. Ambiguity in this case is therefore constant. On the contrary, with the multiplicative form it is easy to see that the dotted curve for any given level of effort is steeper with θ_2 than with θ_1. Intuitively, this corresponds to a situation in which ambiguity decreases with the effort furnished and condition (10) is therefore respected.
The intuition behind these two examples is simple. In the absence of ambiguity, we know that a key determinant of the optimal level of self-protection is the slope of $p(e)$ (which determines the marginal benefit of effort). When ambiguity is introduced, the DM does not know exactly in which situation he is: if his prior beliefs are equal, he considers he has one chance out of two to be confronted to a loss with probability $p(e, \theta_1)$, and one chance out of two to have $p(e, \theta_2)$. If the individual is ambiguity neutral, this situation does not affect him and the decision is taken by considering the expected law $p(e)$. However, if the agent is ambiguity averse, he will over-evaluate the less desirable outcome (i.e. the law $p(e, \theta_2)$) and hence take a decision by considering a law *somewhere above* the line $p(e)$. In the special case of infinite ambiguity aversion, corresponding to the maxmin model of Gilboa and Schmeidler (1989), the DM takes his decision by considering the worst scenario $p(e, \theta_2)$.

The study of these two particular cases in which the probability is linear in parameter θ emphasizes the differences there are between the single and the two-period models. In the single period model, when both the marginal cost and the marginal benefit of self-protection are affected by the introduction of ambiguity, it is indeed impossible to sign the effect ambiguity aversion has on the optimal prevention, even when the probabilities are linear in the ambiguity parameter. In particular, in that situation, the DM will always choose to reduce his demand of self-protection if the probability law is additive, while he will choose a higher level of protection if the probability law is multiplicative (Snow (2011)). This inability to obtain a general result is due to the fact that both the marginal cost and the marginal benefit of self-protection increase under ambiguity aversion. The net effect therefore depends on which one is more affected. In the two-period model analyzed in this paper however, ambiguity aversion only affects the marginal benefit, making it possible to draw general conclusions.

5 Conclusion

In this paper, I show that ambiguity aversion alone is not sufficient to sign the effect ambiguity has on the decision to (self-)insure or self-protect when two periods are considered. An additional condition defined as *ambiguity prudence attitude* – or non increasing absolute ambiguity aversion – is then studied, and it is shown that in most usual situations this condition tends to raise the incentive to undertake an effort (insurance or prevention) in first period when non neutral attitude towards ambiguity is considered.

This paper thus enables to sign the effect of ambiguity aversion on (self-)insurance and self-protection under a plausible set of conditions. It distinguishes from the recent papers of Snow (2011) and Alary, Gollier, and Treich (2013) in which the marginal cost of effort is also affected by ambiguity, and that are therefore not able to draw general conclusions because of the conflicting effect ambiguity aversion has on marginal benefit and marginal cost.

Appendix

Proof of Proposition 3. This proof[15] is based on the following Lemma, that can be found in Gollier (2001).

Lemma 2 *Let ϕ be a twice differentiable, increasing and concave function: $\mathbb{R} \rightarrow \mathbb{R}$. Consider a probability vector $(q_1, ..., q_m) \in \mathbb{R}_+^m$ with $\sum_{\theta=1}^m q_\theta = 1$, and a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, defined as*

$$f(U_1, ..., U_m) = \phi^{-1}\left\{\sum_{\theta=1}^{m} q_\theta \phi\left\{U_\theta\right\}\right\}.$$

Let T be the function such that $T(U) = -\frac{\phi'\{U\}}{\phi''\{U\}}$. Function f is concave in $\mathbb{R}^m$ if and only if T is weakly concave in $\mathbb{R}$.

First, remark that program (7) is convex if

$$V(e) = \phi^{-1}\left\{\mathrm{E}_\theta \phi\left\{p_i(e,\tilde{\theta})u(w_{2,i}(e)) + \left[1 - p_i(e,\tilde{\theta})\right]\sum_{s\neq i}\pi_s u(w_{2,s})\right\}\right\}$$

is concave in e.

[15] This proof is adapted from Gierlinger and Gollier (2008).

Self-insurance ($p_i(e,\theta) = p_i(\theta)$ for all levels of e): Consider two scalars e_1 and e_2, and let $U_{j\theta}$ denote the second period expected utility conditional on θ, for a level of effort e_j: $U_{j\theta} = p_i(\theta)u(w_{2,i}(e_j)) + [1 - p_i(\theta)] \sum_{s\neq i} \pi_s u(w_{2,s})$. Under the notations above, $V(e_j) = f(U_{j1}, ..., U_{jm})$. Then, under concavity of u and $w_{2,i}$, and for any $(\lambda_1, \lambda_2) \in \mathbb{R}^2_+$ such that $\lambda_1 + \lambda_2 = 1$, we have:

$$\lambda_1 u(w_{2,i}(e_1)) + \lambda_2 u(w_{2,i}(e_2)) \leq u(\lambda_1 w_{2,i}(e_1) + \lambda_2 w_{2,i}(e_2)) \leq u(w_{2,i}(\lambda_1 e_1 + \lambda_2 e_2)).$$

Multiplying the first and the third parts of this chain of inequalities by $p_i(\theta)$ and adding $[1 - p_i(\theta)] \sum_{s\neq i} \pi_s u(w_{2,s})$ yields:

$$\lambda_1 U_{1\theta} + \lambda_2 U_{2\theta} \leq U_{\lambda\theta} \equiv p_i(\theta)u(w_{2,i}(e_\lambda)) + [1 - p_i(\theta)] \sum_{s\neq i} \pi_s u(w_{2,s})$$

for all θ, where $e_\lambda = \lambda_1 e_1 + \lambda_2 e_2$. Because f is increasing in $\mathbb{R}^m$ if ϕ is increasing, this implies:

$$V(e_\lambda) = f(U_{\lambda 1}, ..., U_{\lambda m}) \geq f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1m} + \lambda_2 U_{2m}).$$

On the other side, if $-\phi'/\phi''$ is concave, by Lemma 2 we have:

$$\begin{aligned} f(\lambda_1 U_{11} + \lambda_2 U_{21}, ..., \lambda_1 U_{1m} + \lambda_2 U_{2m}) &\geq \lambda_1 f(U_{11}, ..., U_{1m}) + \lambda_2 f(U_{21}, ..., U_{2m}) \\ &= \lambda_1 V(e_1) + \lambda_2 V(e_2). \end{aligned}$$

Combining these two results yields $V(\lambda_1 e_1 + \lambda_2 e_2) \geq \lambda_1 V(e_1) + \lambda_2 V(e_2)$ implying that V is concave in e.

Self-protection ($w_{2,i}(e) = w_{2,i}$ for all levels of e): In this case, the proof is similar but $U_{j\theta}$ is now given by $U_{j\theta} = p_i(e_j, \theta)u(w_{2,i}) + [1 - p_i(e_j, \theta)] \sum_{s\neq i} \pi_s u(w_{2,s})$, and we exploit the convexity of $p_i(e, \theta)$ in e to obtain $\lambda_1 U_{1\theta} + \lambda_2 U_{2\theta} \leq U_{\lambda\theta}$. ∎

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