On Conditions in Central Limit Theorems for Martingale Difference Arrays Long Version

Abdelkamel Alj
Département de Mathématique, Université Libre de Bruxelles

Rajae Azrac
Université Mohammed V-Souissi

Guy Mélard
SBS-EM, ECARES, Université Libre de Bruxelles

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50, F.D. Roosevelt Ave., B-1050 Brussels BELGIUM
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On Conditions in Central Limit Theorems for Martingale Difference Arrays Long Version

Abdelkamel Alj*, Rajae Azrak†, and Guy Mélard‡.

Abstract

An alternative central limit theorem for martingale difference arrays is presented. It can be deduced from the literature but it is not stated as such. It can be very useful for statisticians and econometricians. An illustration is given in the context of ARMA models with time-dependent coefficients. This note ends with a discussion about the conditions.

Keywords: Unconditional Lyapunov condition, Conditional Lindeberg condition.

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*Dept of Mathematics, Université libre de Bruxelles CP112, Blv du Triomphe, B-1050 Bruxelles, Belgium (e-mail: abdelalj@ulb.ac.be).

†Université Mohammed V-Souissi, Faculté des Sciences Juridiques Economiques et Sociales, Salé, Morocco (e-mail: razrak96@yahoo.fr).

‡ECARES, Université libre de Bruxelles CP114/4, Avenue Franklin Roosevelt 50, B-1050 Bruxelles, Belgium (e-mail: gmelard@ulb.ac.be).

1 Introduction and main results

Although martingale sequences are still dominant, martingale arrays arise in a large number of statistical, economic and financial areas. Let us mention estimation for observation driven models for Poisson counts (Davis et al., 2005), the analysis of non stationary time series (Dahlhaus, 1997, Azrak and Mélard, 2006), asymptotics of quadratic forms, consistency of bootstrap methods and nonparametric density estimation under a uniform (ϕ-) mixing condition (Neumann, 2013), the analysis of matching estimators (Abadie and Imbens, 2009), non-linear cointegrating regression and fractionally integrated processes (Wang, 2011), nonlinear nonstationary processes (Wang and Phillips, 2012), explosive cointegrated systems (Magdalinos and Phillips, 2009, Philipps and Magdalinos, 2007), efficiency and robustness (Davidson, 2000), panel data models with sequential exogeneity (Kuersteiner and Prucha, 2012), asymptotics of the principal components estimator of large factor models (Onatski, 2012a), consistency of the Eicker-White estimator in the context of unbounded heteroscedasticity (Kourogenis and Pittis, 2010), tests against nonstationary volatility (Xu 2008a and 2008b), improved pre-averaging estimator of integrated volatility (Lee, 2010), convergence of stochastic integrals for dependent heterogeneous processes (see the references in Hansen, 1992). We are interested in central limit theorems (CLTs) for martingale difference arrays. See Hall and Heyde (1980) and Gaenssler and Haeusler (1986) for reviews. However, we have not encountered the statement of Theorem 1.3 below in the main reference books, e.g. Davidson (1994), Hall and Heyde (1980), Jacod and Shiryaev (1987), Kallenberg (1997), Pollard (1984), Shorack (2000), Shorack and Wellner (1986), and Williams (1991).

There exists at least a CLT for martingale difference arrays with an unconditional Lyapunov condition. It is stated in Hamilton (1994) who refers to White (1984). But that theorem, Theorem 1.2 below, assumes that a limit in probability of a series of random variables does exist and is equal to a given constant. The problem is that there is a priori no evidence that this assumption holds in some cases, for example in Azrak and Mélard (2006).

Let $\{(Y_{k,T}, F_{k,T}), 1 \leq k \leq T\}, T \in \mathbb{N}$, be a martingale difference sequence for each $T > 1$, i.e. $\{(Y_{k,T}, F_{k,T}), 1 \leq k \leq T, T \in \mathbb{N}\}$ is a martingale difference array. $I\{A\}$ is the indicator function of the set A. All limits in the paper are when $T \to \infty$. We denote convergence in probability and convergence in distribution, respectively, by $\xrightarrow{\text{P}}$ and $\xrightarrow{\text{d}}$. The custom when dealing with arrays is to assume that the normalization is subsumed under the definition. Here we need to leave the factor $1/T$ not only for the purpose of the statistical application in Section 2 but also because subsumed normalization would complicate (a) in Theorems 1.2 and 1.3. The following result is due to Brown (1971).

Theorem 1.1. *If*
(i) for every $\epsilon > 0$, $\frac{1}{T}\sum_{k=1}^{T} E\left[Y_{k,T}^2 I\{Y_{k,T}^2 > \epsilon T\}|F_{k-1,T}\right] \xrightarrow{P} 0$;
(ii) $\frac{1}{T}\sum_{k=1}^{T} E\left[Y_{k,T}^2|F_{k-1,T}\right] \xrightarrow{P} 1$;
then $\frac{1}{\sqrt{T}}\sum_{k=1}^{T} Y_{k,T} \xrightarrow{d} N(0,1)$, *i.e. the CLT holds.*

There exist alternative CLT's for martingale differences arrays but few with conditional Lyapunov conditions. One of them is stated by Hamilton (1994, p. 193) as a corollary of an (unproven) CLT for martingale differences sequences. There is a reference to White (1984, p. 130) but where only sequences are treated, not arrays. We have checked that White (2000), the most recent edition of White (1984), which is an improvement on many aspects doesn't improve on this point. Here is the statement with the above notations.

Theorem 1.2. *Assume that* $E\left[Y_{k,T}^2\right] > 0$ *for all* k *and* T. *If*
(a) for some real constant $B > 0$, $E\left[Y_{k,T}^{2+\delta}\right] < B < \infty$ *for all* k, T, *and for some* $\delta > 0$;
(b) $\frac{1}{T}\sum_{k=1}^{T} Y_{k,T}^2 \xrightarrow{P} 1$;
(c) $\frac{1}{T}\sum_{k=1}^{T} E\left[Y_{k,T}^2\right] \rightarrow 1$;
then $\frac{1}{\sqrt{T}}\sum_{k=1}^{T} Y_{k,T} \xrightarrow{d} N(0,1)$.

The difficulty may be to check (b), and it is the case at least in the context of Section 2. Our goal is to show the following alternative theorem.

Theorem 1.3. *Assume that* $E\left[Y_{k,T}^2\right] > 0$ *for all* k *and* T. *If (a) and (ii) hold, then* $\frac{1}{\sqrt{T}}\sum_{k=1}^{T} Y_{k,T} \xrightarrow{d} N(0,1)$.

Proof (of Theorem 1.3). (a) is an unconditional Lyapunov condition. Let us first show, by using standard arguments (e.g. White, 1984, p. 112, in the independent identically distributed case) that it implies the following unconditional Lindeberg condition (ULC)
(i') for every $\epsilon > 0$, $\frac{1}{T}\sum_{k=1}^{T} E\left[Y_{k,T}^2 I\{Y_{k,T}^2 > \epsilon T\}\right] \rightarrow 0$.
Indeed, for every $\delta > 0$

$$Y_{k,T}^2 I\{Y_{k,T}^2 > \epsilon T\} \leq \frac{|Y_{k,T}|^{2+\delta}}{(\sqrt{\epsilon T})^{\delta}}.$$

Hence, by using (a),

$$\frac{1}{T}\sum_{k=1}^{T} E\left[Y_{k,T}^2 I\{Y_{k,T}^2 > \epsilon T\}\right] \leq (\epsilon T)^{-\delta/2}\frac{1}{T}\sum_{k=1}^{T} E\left[|Y_{k,T}|^{2+\delta}\right]$$
$$\leq (\epsilon T)^{-\delta/2} B \to 0.$$

But, as shown by Gaenssler et al. (1978), this in turn implies the conditional Lindeberg condition (CLC). Indeed, for any adapted random array $\{(g_{k,T}, F_{k,T}), 1 \leq k \leq T\}$ such that $g_{k,T} \geq 0$,

$$\sum_{k=1}^{T} E\left[g_{k,T}\right] = \sum_{k=1}^{T} E\left[E\left(g_{k,T}|F_{k-1,T}\right)\right] = E\left[\sum_{k=1}^{T} E\left(g_{k,T}|F_{k-1,T}\right)\right] \to 0$$

implies that $E\left(g_{k,T}|F_{k-1,T}\right) \xrightarrow{\text{P}} 0$, for every $k = 1, ..., T$. Hence we apply this result with $g_{k,T} = (1/T)Y_{k,T}^2 I\{Y_{k,T}^2 > \epsilon T\} \geq 0$, proving that (i) holds. Hence the two assumptions of Brown's Theorem 1.1 hold, proving the CLT.

2 Application

Azrak and Mélard (2006) have studied the asymptotic properties of ARMA models with coefficients depending on time. They refer to a CLT for martingale *sequences* from Basawa and Prakasa Rao (1980, p. 388). One assumption of the latter is an *unconditional Lyapunov* condition that Azrak and Mélard (2006) can check in their context. They have also considered ARMA models where the coefficients depend on time k but also on the series length T. The asymptotic properties are only sketched in this case, noticing that a CLT for martingale difference *arrays*, with a reference to Hall and Heyde (1980, Theorem 2.23, p. 44, and Corollary 3.1, p. 58). One assumption of the latter is a *conditional Lindeberg* condition that Azrak and Mélard (2006) didn't check in their context. They said replacing it by a *Lyapunov* condition but it should be a *conditional Lyapunov* condition. Using Theorem 1.3 above completes the proof since the *unconditional Lyapunov* condition implies the *conditional Lindeberg* condition.

The theory in Azrak and Mélard (2006) is too complex to be reported here but we can consider a simple example of a first-order time-dependent autoregressive process, defined by

$$y_k = \phi_{k,T}(\beta) y_{k-1} + e_k,$$

where the e_k's are zero-mean independent random variables with a known constant variance σ^2, and the initial condition is e.g. $y_0 = 0$. Suppose that the coefficient $\phi_{k,T}(\beta)$ is a function of k and T (possibly, but not exclusively, of k/T) and depends on a parameter β, for example $\phi_{k,T}(\beta) = \beta k/T$, with $\beta < 1$ so that the autoregressive coefficient doesn't become too large when $T \to \infty$, preventing the process from being explosive. Then, up to an additive constant, $-2\sigma^2$ times the log-likelihood is $\ell_T(\beta) = \sum_{k=1}^{T} \alpha_{k,T}(\beta)$, where $\alpha_{k,T}(\beta) = \{y_k - \phi_{k,T}(\beta)y_{k-1}\}^2$. Denote β_0 the true value of β.

To prove asymptotic normality of the estimator, it is required to prove that, for $\beta = \beta_0$,

$$\frac{1}{\sqrt{T}}\frac{d\ell_T}{d\beta} = \frac{-2}{\sqrt{T}}\sum_{k=1}^{T}\{y_k - \phi_{k,T}(\beta)y_{k-1}\}\frac{d\phi_{k,T}(\beta)}{d\beta}y_{k-1} \tag{2.1}$$

converges in law to a normal distribution and it is here that a central limit theorem for martingales plays a role. In that context, the derivative of $\alpha_{k,T}(\beta)$ is definitively not a martingale difference sequence unless the derivative of $\phi_{k,T}(\beta)$ does not depend on T. For example, if $\phi_{k,T}(\beta) = \beta k/T$, the term of the sum in (2.1) for $k = 2$ is equal to $(y_2 - 2\beta y_1/T)(2/T)y_1$. Hence the second term of the sum in (2.1) is $(y_2 - \beta y_1)y_1$ for $T = 2$ but $2(y_2 - (2/3)\beta y_1)y_1/3$ for $T = 3$, so that they differ. But it is easy to show that, for every T, the derivative of $\alpha_{k,T}(\beta)$ is a martingale difference sequence, and hence we have a martingale difference array.

To discuss the choice between a conditional or an unconditional Lyapunov condition, it suffices to take again the term for $k = 2$ and $\beta = \beta_0$. With $\delta = 2$ in Theorem 1.2, a conditional Lyapunov condition would be based on

$$E_{\beta_0}\left[\left((y_2 - \phi_{2,T}(\beta)y_1)\frac{d\phi_{2,T}(\beta)}{d\beta}y_1\right)^4 \middle| F_{1,T}\right] = E(e_2^4)\left(\left.\frac{d\phi_{2,T}(\beta)}{d\beta}\right|_{\beta=\beta_0}\right)^4 y_1^4$$

which is not bounded if e.g. the process is Gaussian. On the other hand, the use of Theorem 1.3 is allowed if

$$E_{\beta_0}\left[\left((y_2 - \phi_{2,T}(\beta)y_1)\frac{d\phi_{2,T}(\beta)}{d\beta}y_1\right)^4\right] = E(e_2^4)\left(\left.\frac{d\phi_{2,T}(\beta)}{d\beta}\right|_{\beta=\beta_0}\right)^4 E(y_1^4)$$

is bounded, and this can be checked under assumptions of boundedness on the coefficients and their derivatives with respect to β and existence of moments.

3 Comments and conclusions

Theorem 1.3 can be considered as a straightforward consequence of existing results but it may be worth to state it explicitly. We have found part of the arguments in Onatski (2012b),

Figure 1: Implications among conditional and unconditional Lindeberg and Lyapunov conditions

$$\begin{array}{ccc} \text{Conditional Lindeberg} & \Longleftarrow & \text{Unconditional Lindeberg} \\ \Uparrow & & \Uparrow \\ \text{Conditional Lyapunov} & \Longrightarrow & \text{Unconditional Lyapunov} \end{array}$$

a technical appendix to Onatski (2012a) and in Lee (2010). The argumentation is however different and doesn't use Gaenssler et al. (1978). Instead Onatski (2012b, p. 28) says that the unconditional Lindeberg condition implies the conditional Lindeberg condition by using the fact that convergence in the mean implies convergence in probability, which is similar to the argumentation in the proof of Theorem 1.3. Lee (2010) does not provide a justification for that step.

Gaenssler et al. (1978) show an example satisfying CLC when the ULC condition is not satisfied. Assuming (c), they also give a condition, which coincides with (ii), under which CLC and ULC are equivalent. Hall and Heyde (1980, p. 45) say that, under a condition of uniform integrability on $\{V_{k,T}^2, k \geq 1\}$, where $V_{k,T}^2$ is the left hand side of (ii), CLC is equivalent to ULC. But they do not specify that CLC is weaker than ULC and that proving ULC is enough for obtaining a CLT. The alternative Theorem 1.3 is more general than the one stated by Hamilton (1994) on the basis of White (1984) and its assumptions may be easier to check.

Finally it is worth noticing that the *conditional Lyapunov* condition implies the *conditional Lindeberg* condition (by the same arguments as in the proof of Theorem 1.3) and also that the *conditional Lyapunov* condition implies the *unconditional Lyapunov* condition because $E(E(|Y_{k,T}|^{2+\delta}|F_{k-1,T})) = E(|Y_{k,T}|^{2+\delta})$ for every $\delta > 0$. Fig. 1 shows all the implications.

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