



Precautionary Saving and the Notion of Ambiguity Prudence

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Abstract

This letter develops a set of simple conditions under which an individual is willing to save an extra amount of money due to the presence of ambiguity on its second period wealth. This extra precautionary saving motive is naturally associated to the notion of ambiguity prudence.

Key words: Ambiguity aversion, non-expected utility, uncertainty, saving, prudence

JEL: D81, D91, E21

“It is widely believed that the agents are prudent in the sense that the *uncertainty* affecting future incomes raises current savings” (Gollier (2001)).

1. Introduction

From the early works of Leland (1968), Sandmo (1970), and Drèze and Modigliani (1972), it is well known that the addition of a pure *risk* affecting future incomes may result in an increase in current savings. This precautionary saving motive has been shown to occur if the marginal utility of future consumption is convex, in which case the agent is said, in the words of Kimball (1990), to be prudent, or more precisely *risk prudent*. In this context, risk prudence measures the willingness to accumulate wealth to face future

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risk, which by definition corresponds to a situation in which the probabilities associated with the possible outcomes are perfectly known.

However, if the future wealth of an economic agent is not only risky but is ambiguous in the sense that the probabilities of the possible outcomes depend on an external parameter for which the agent has prior beliefs, under which conditions does the optimal level of saving increase? To answer this question, I consider a simple two-period consumption-saving problem in the presence of ambiguity, and consider non-neutral ambiguity attitudes². Ambiguity takes the form of a second order prior probability distribution over the set of plausible first order distributions of the future income. I use the theory developed by Klibanoff, Marinacci, and Mukerji (2005, 2009) to deal with ambiguity. This model has the advantage to make a distinction between ambiguity aversion and risk aversion, and is tractable enough to allow the application of the well developed machinery of the expected utility sequentially on first and second order probability distributions.

2. The Model

The notion of *ambiguity prudence* is a concept closely related to Kimball's concept of *risk prudence*, but which is due to the presence of ambiguity. To study this concept, I consider a very simple two-period model which is not only risky but is also ambiguous: probabilities of second-period final wealth are not objectively known, instead they consist in a set of probabilities, depending on an external parameter θ for which the agent has prior beliefs³. Ambiguity may therefore be interpreted as a multi-stage lottery: a first lottery determines the value of parameter θ , and a second one determines the size of second-period wealth. The second-period

²Notice that expected utility models may be equivalently seen as model in the absence of ambiguity or as models in the presence of ambiguity but where agents are ambiguity neutral (and therefore behave as subjective expected utility maximizers in the sense of Savage (1954)). However, as first shown by Ellsberg (1961) and later confirmed by numerous experimental studies, ambiguity neutrality is usually inconsistent with individuals' preferences.

³Imagine that parameter θ can take values $\theta_1, \theta_2, \dots, \theta_m$ with probabilities $[q_1, q_2, \dots, q_m]$, such that the expectation with respect to the parametric uncertainty is written $E_{\theta}g(\theta) = \sum_{j=1}^m q_j g(\theta_j)$.

wealth (or labor income⁴) distribution $\tilde{w}_2(\theta)$ is represented by the vector $[w_{2,1}, w_{2,2}, \dots, w_{2,n}; p_1(\theta), p_2(\theta), \dots, p_n(\theta)]$ with $w_{2,1} < w_{2,2} < \dots < w_{2,n}$.

In the simplest version of the model there are only two periods, and the agent has an additively time-separable utility function. In first period, the individual receives a labor income w_1 and decides his levels of consumption c_1 and savings $s = w_1 - c_1$. In second period, the agent disposes of his capital income Rs and of an ambiguous labor income $\tilde{w}_2(\theta)$. The intertemporal saving decision problem under Klibanoff, Marinacci, and Mukerji (2005, 2009) (KMM) representation takes the form:

$$\max_s u(w_1 - s) + \beta \phi^{-1} \left\{ \mathbb{E}_\theta \phi \left\{ \mathbb{E} u(\tilde{w}_2(\tilde{\theta}) + Rs) \right\} \right\} \quad (1)$$

where w_i is the exogenous wealth (or labor income) in the beginning of period $i = 1, 2$, u represents the period vNM utility functions, R is the risk-free gross interest rate, $\beta \in [0, 1]$ is the discount factor, ϕ represents attitude towards ambiguity, \mathbb{E}_θ is the expectation operator taken over the distribution of θ conditional on all information available during the first period, and \mathbb{E} is the expectation operator taken over w_2 conditional on θ . The function ϕ is assumed to be three times differentiable, increasing, and concave under ambiguity aversion, so that the ϕ -certainty equivalent in equation 1 is lower in that case than when the individual is ambiguity neutral characterized by a linear function ϕ :

$$\phi^{-1} \left\{ \mathbb{E}_\theta \phi \left\{ \mathbb{E} u(\tilde{w}_2(\tilde{\theta}) + Rs) \right\} \right\} \leq \mathbb{E}_\theta \mathbb{E} u(\tilde{w}_2(\tilde{\theta}) + Rs) = \mathbb{E} u(\tilde{w}_2 + Rs). \quad (2)$$

An ambiguity averse DM therefore dislikes any mean-preserving spread in the space of conditional second period expected utilities. The right hand side of expression 2 corresponds to the second period expected utility obtained by an ambiguity neutral individual who evaluates his welfare by considering the risky second period wealth $\tilde{w}_2: [w_{2,1}, w_{2,2}, \dots, w_{2,n}; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_n]$ with the mean state probabilities $\bar{p}_s = \mathbb{E}_\theta p_s(\tilde{\theta})$, $\forall s = 1, \dots, n$. In that sense, an ambiguity neutral individual is nothing but a savagian expected utility agent, so that problem 1 simplifies to the well-known saving decision problem studied by Kimball (1990).

⁴If labor supply is assumed to be inelastic so that labor income corresponds to net endowment wealth.

For convenience, I write $\tilde{w}_2(\theta) = w_2 + \tilde{z}(\theta)$, so that second period labor income may be separated into its expected value $w_2 = E_\theta E \tilde{w}_2(\tilde{\theta})$ and the ambiguous part $\tilde{z}(\tilde{\theta})$. For ease of comparison, I denote by \tilde{z} the *corresponding risk*: $[z_1, z_2, \dots, z_{2,n}; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_n]$ which would be considered by an ambiguity neutral individual⁵.

As a reference point, I first consider the case of ambiguity neutrality. The optimal level of saving s^* chosen by an ambiguity-neutral agent is implicitly given by

$$u'(w_1 - s^*) = \beta E_\theta E u'(w_2 + Rs^* + \tilde{z}(\tilde{\theta}))R \quad (3)$$

which is assumed to hold at some interior value s^* ⁶. Remark that this is exactly the problem studied by Kimball (1990), since by assumption we have: $E_\theta E u'(w_2 + Rs^* + \tilde{z}(\tilde{\theta})) = E u'(w_2 + Rs^* + \tilde{z})$. In that case, we know that the level of saving s^* is higher than the level of saving optimally chosen in the absence of risk if and only if $u''' \geq 0$.

If ϕ is not linear, the first order condition (FOC) of problem 1 is:

$$-u'(w_1 - s) + \beta \frac{E_\theta \phi' \{E u(w_2 + Rs + \tilde{z}(\tilde{\theta}))\} E u'(w_2 + Rs + \tilde{z}(\tilde{\theta}))R}{\phi' \left\{ \phi^{-1} \left\{ E_\theta \phi \{E u(w_2 + Rs + \tilde{z}(\tilde{\theta}))\} \right\} \right\}} = 0 \quad (4)$$

so that the level of saving due to ambiguity aversion increases if the left hand side of equation 4 computed at s^* is positive. This is the case if and only if

$$\frac{E_\theta \phi' \{E u(w_2^* + \tilde{z}(\tilde{\theta}))\} E u'(w_2^* + \tilde{z}(\tilde{\theta}))}{\phi' \left\{ \phi^{-1} \left\{ E_\theta \phi \{E u(w_2^* + \tilde{z}(\tilde{\theta}))\} \right\} \right\}} \geq E_\theta E u'(w_2^* + \tilde{z}(\tilde{\theta})) \quad (5)$$

where $w_2^* \equiv w_2 + Rs^*$. If condition 5 is satisfied, the individual is said to be *ambiguity prudent* in the spirit of Kimball. Ambiguity prudence is then defined as follows:

Definition An agent is *ambiguity prudent* if the introduction of ambiguity through a mean-preserving spread in the space of first order distributions of his future wealth raises his optimal level of saving.

⁵From what precedes, it should be clear that \tilde{z} is a pure (i.e. zero-mean) risk.

⁶I assume that problem 1 is convex. A sufficient condition for this is ϕ to exhibit concave absolute ambiguity tolerance (see Gierlinger and Gollier (2008) for details).

The uncertainty affecting future income may therefore have different impacts on current saving. It increases the willingness to save in the standard expected utility framework if future income is risky and the agent is *risk* prudent, and it further raises current saving if future income is ambiguous and the agent is *ambiguity* prudent.

Unfortunately expression 5 is not easy to handle. Intuitively, it tells us that the consumer's willingness to save increases if the expected marginal utility of wealth is higher under ambiguity aversion. To simplify this condition, I make use of the following lemma:

Lemma 1. *Let ϕ be a three times differentiable function reflecting ambiguity aversion. If ϕ exhibits DAAA (Decreasing Absolute Ambiguity Aversion) then $E\phi'\{\tilde{x}\} > \phi'\{\phi^{-1}\{E\phi\{\tilde{x}\}\}\}$.*

Proof ϕ is DAAA is equivalent to saying that $-\phi'$ is more concave than ϕ , or equivalently that $-\phi'''/\phi'' > -\phi''/\phi'$. Since $(\phi')^{-1}$ is a decreasing function, the proof follows from the fact that the certainty equivalent of function ϕ is larger than that of function $-\phi'$. \square

Corollary 1. *If ϕ exhibits CAAA (Constant Absolute Ambiguity Aversion), then $E\phi'\{\tilde{x}\} = \phi'\{\phi^{-1}\{E\phi\{\tilde{x}\}\}\}$.*

Using Lemma 1 and its corollary, it is easy to see that under CAAA, condition 5 may be rewritten as the covariance condition:

$$\text{cov}_\theta \left(\phi' \{Eu(w_2^* + \tilde{z}(\tilde{\theta}))\}, Eu'(w_2^* + \tilde{z}(\tilde{\theta})) \right) \geq 0. \quad (6)$$

In the DAAA case, condition 5 for ambiguity prudence is automatically satisfied under condition 6. Hence in most usual situations⁷, consumers exhibit ambiguity prudence if condition 6 is satisfied. Using the covariance rule, this will be the case under ambiguity aversion if $Eu(w_2^* + \tilde{z}(\theta))$ and $Eu'(w_2^* + \tilde{z}(\theta))$ are anti-comonotone in θ .

Notice that this condition strongly depends on the vNM utility function considered and on the prior distributions of $\tilde{z}(\theta)$. As a consequence, non-increasing absolute ambiguity aversion is, in general, not a sufficient condition for ambiguity prudence. There are still a number of special cases in

⁷Gierlinger and Gollier (2008) for example mention that DAAA is a reasonable property of uncertainty preferences.

which CAAA and DAAA are sufficient to observe an increase in the level of saving due to the presence of ambiguity. These cases are listed in the proposition below.

Proposition 1. *Non increasing absolute ambiguity aversion is a necessary and sufficient condition for ambiguity prudence in any of the following situations:*

- (i.) *the DM is risk neutral,*
- (ii.) *the DM has a vNM utility function of the CARA form,*
- (iii.) *the DM is risk averse and there are only two states of the world,*
- (iv.) *the DM is risk prudent and ambiguity is concentrated on a particular state i .*

Proof (i.) If u is linear, the covariance in 6 is zero, such that an individual manifesting CAAA chooses to save exactly the same amount as in the absence of ambiguity, and it is easy to see, using a chain of inequalities, that condition 5 is respected under DAAA.

(ii.) If u is CARA, we have $A Eu(w_2^* + \tilde{z}(\theta)) = -Eu'(w_2^* + \tilde{z}(\theta))$ where A is the coefficient of absolute risk aversion. Since ϕ' is decreasing under ambiguity aversion, condition 6 always holds true.

(iii.) If there are only two states z_1 and z_2 , we have: $Eu(w_2^* + \tilde{z}(\theta)) = p(\theta)u(w_2^* + z_1) + (1-p(\theta))u(w_2^* + z_2) = u(w_2^* + z_2) + p(\theta)[u(w_2^* + z_1) - u(w_2^* + z_2)]$ and $Eu'(w_2^* + \tilde{z}(\theta)) = u'(w_2^* + z_2) + p(\theta)[u'(w_2^* + z_1) - u'(w_2^* + z_2)]$. Unsurprisingly, the two terms $[u(w_2^* + z_1) - u(w_2^* + z_2)]$ and $[u'(w_2^* + z_1) - u'(w_2^* + z_2)]$ are necessarily of opposite sign if u is concave.

(iv.) If ambiguity is concentrated on a specific state i , we have: $Eu(w_2^* + \tilde{z}(\theta)) = p_i(\theta)u(w_2^* + z_i) + (1-p_i(\theta))\sum_{s \neq i} \pi_j u(w_2^* + z_j) = \sum_{s \neq i} \pi_j u(w_2^* + z_j) + p_i(\theta)\left[u(w_2^* + z_i) - \sum_{s \neq i} \pi_j u(w_2^* + z_j)\right]$ where π_j is the probability of being in state j conditional on the state not being i . Moreover $Eu'(w_2^* + \tilde{z}(\theta)) = \sum_{s \neq i} \pi_j u'(w_2^* + z_j) + p_i(\theta)\left[u'(w_2^* + z_i) - \sum_{s \neq i} \pi_j u'(w_2^* + z_j)\right]$. If $z_i \geq E\tilde{z}_{-i} = \sum_{j \neq i} \pi_j z_j$, then we have⁸: $u(w_2^* + z_i) \geq u(w_2^* + E\tilde{z}_{-i}) \geq$

⁸Note that if $z_i \leq E\tilde{z}_{-i}$ all inequalities are reversed.

$Eu(w_2^* + \tilde{z}_{-i})$ where the first inequality results from $u' > 0$ and the second from $u'' < 0$. Moreover, if $u'' < 0$ and $u''' > 0$, we also have:

$$u'(w_2^* + z_i) \leq u'(w_2^* + E\tilde{z}_{-i}) \leq Eu'(w_2^* + \tilde{z}_{-i}).$$

It is then easy to see that the two terms between brackets $u(w_2^* + z_i) - \sum_{s \neq i} \pi_j u(w_2^* + z_j)$ and $u'(w_2^* + z_i) - \sum_{s \neq i} \pi_j u'(w_2^* + z_j)$ are of opposite sign. By the concavity of ϕ , the two terms in condition 6 therefore covary positively. \square

The intuition to this result is comparable to the one resulting from the study of precautionary saving in a Kreps and Porteus (1978)/Selden (1978) model. In this model, we know that the addition of a pure risk in second period wealth raises the optimal saving if the individual is prudent, a condition which is satisfied in that context if the individual manifest decreasing absolute risk aversion (DARA). Given the similarity between Kreps-Porteus/Selden and KMM models, it is therefore not surprising that the conditions guaranteeing an increase in saving due to the presence of ambiguity include non increasing absolute ambiguity aversion. Remark that DAAA is weaker than requiring $\phi''' \geq 0$. This is because the future utility is represented by the ϕ -certainty equivalent of the expected utilities, rather than by the expected ϕ -valuation of conditional expected utilities.

In general however, DAAA and CAAA are not sufficient for ambiguity prudence. To understand why, consider one of the major features KMM model claims to achieve: “a separation between ambiguity, identified as a characteristic of the decision maker’s subjective beliefs, and ambiguity attitude, a characteristic of the decision maker’s tastes”. As argued in Berger (2011), since subjective beliefs are the result of a combination of available information and personal subjective treatment of this information, they consequently also constitute a representation of the individual’s attitude towards ambiguity. Beliefs therefore represent a part of individual’s preferences that I refer to *ambiguity perception*. This ambiguity perception attitude plays a role in the determination of precautionary saving via inequality 6. To distinguish the two effects, remark that condition 5 necessary to observe extra precautionary saving due the non neutral attitude towards ambiguity may

be rewritten as⁹:

$$aEu'(w_2^* + \tilde{z}^\circ) \geq Eu'(w_2^* + \tilde{z}), \quad (7)$$

where $a = \frac{E_\theta \phi' \{Eu(w_2^* + \tilde{z}(\theta))\}}{\phi' \{ \phi^{-1} \{ E_\theta \phi \{ Eu(w_2^* + \tilde{z}(\theta)) \} \} \}}$, and where \tilde{z}° only differs from \tilde{z} by the probabilities associated to each z_i . In particular: $\tilde{z}^\circ \sim [z_1, z_2, \dots, z_{2,n}; \bar{p}_1^\circ, \bar{p}_2^\circ, \dots, \bar{p}_n^\circ]$ where $\bar{p}_i^\circ = \sum_{j=1}^m \frac{q_j \phi' \{Eu(w_2^* + \tilde{z}(\theta_j))\}}{\sum_{t=1}^m q_t \phi' \{Eu(w_2^* + \tilde{z}(\theta_t))\}} p_i(\theta_j)$ is the distorted mean probability of state i .

With these notations it is easy to see that $a \geq 1$ corresponds to non increasing ambiguity aversion. From now on, I call this the *ambiguity prudence attitude effect* so that an individual has an ambiguity prudent attitude if $a \geq 1$. As already noted before, remark that under risk neutrality, this effect would be sufficient to observe precautionary saving, but that this is generally not the case under risk aversion. This is due to the presence of a second effect: the *ambiguity prudence perception effect*. An individual has an ambiguity prudent perception if:

$$Eu'(w_2^* + \tilde{z}^\circ) \geq Eu'(w_2^* + \tilde{z}), \quad (8)$$

meaning that the expected marginal utility of saving using the distorted random variable \tilde{z}° must be higher than the expected marginal utility of saving using the original \tilde{z} considered by an ambiguity neutral agent. It can be shown that this condition is equivalent to the covariance condition 6. It should be clear that the ambiguity prudence perception effect depends on individual's preferences towards ambiguity: first because \tilde{z}° is constructed in such a way that the probabilities directly depend on function ϕ , and second because these probabilities are functions of prior beliefs that, as argued before, also incorporate a part of individual's preferences. Remark for example that \bar{p}_i° would be equal to \bar{p}_i for all i under ambiguity neutrality, so that the ambiguity prudence perception effect would be null. Gierlinger and Gollier (2008) call this the "pessimism effect" and analyze the conditions under which this effect increases the efficient discount rate. In the context of this letter, their results could be used to analyze the effect of ambiguity aversion on the precautionary saving. In that case, their proposition would be formulated as:

⁹This decomposition was first proposed by Gierlinger and Gollier (2008) when studying social efficient discount rate.

Proposition 2. [Gierlinger and Gollier (2008)] *An individual has an ambiguity prudent perception if one of the following conditions holds:*

- (i.) *The set of first order distributions $(\tilde{z}(\theta_1), \tilde{z}(\theta_2), \dots, \tilde{z}(\theta_n))$ can be ranked according to first-degree stochastic dominance and u is increasing and concave.*
- (ii.) *The set of first order distributions $(\tilde{z}(\theta_1), \tilde{z}(\theta_2), \dots, \tilde{z}(\theta_n))$ can be ranked according to second-degree stochastic dominance and u is increasing and concave and exhibits prudence.*
- (iii.) *The set of first order distributions $(\tilde{z}(\theta_1), \tilde{z}(\theta_2), \dots, \tilde{z}(\theta_n))$ can be ranked according to Jewitt (1989) and u is increasing and concave, and exhibits DARA.*

Proof Refer to Propositions 4 and 5 in Gierlinger and Gollier (2008). \square

This enlarges the cases listed in Proposition 1 for which non increasing absolute ambiguity aversion (or ambiguity prudent attitude) is sufficient to observe extra precautionary saving due to the presence of ambiguity under ambiguity aversion.

In general however, it is possible that the set of first order distributions cannot be ranked according to any of the above mentioned orders and that the situation do not correspond to any of the listed situations of Proposition 1. In that case, ambiguity prudence will result from the combination of the ambiguity prudence attitude and perception effects.

3. Conclusion

Analogously to what Kimball (1990) did in the risk theory literature, this letter defines the notion of ambiguity prudence as an individual characteristic leading him to save an extra amount of money if his future income is ambiguous. Necessary and sufficient conditions to observe ambiguity prudence are proposed considering both individual's tastes and perception of ambiguity.

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