Abstract. A universal symmetric truncation of the bosonic string Hilbert space yields all known closed fermionic string theories in ten dimensions, their D-branes and their open descendants. We highlight the crucial role played by group theory and two-dimensional conformal field theory in the construction and emphasize the predictive power of the truncation. Such circumstantial evidence points towards the existence of a mechanism which generates space-time fermions out of bosons dynamically within the framework of bosonic string theory.
1. Introduction and conclusion

It is well-known that ten-dimensional fermionic strings can be analyzed in terms of bosonic operators, a consequence of the boson-fermion equivalence in two dimensions. The approach taken here is different. We show that the Hilbert space of the bosonic string compactified on suitable sixteen dimensional tori contain subspaces with fermionic degrees of freedom. This programme was initiated in 1986 in the framework of closed strings \[1\]. We revisited the approach in the last two years \[2, 3\] and extended the construction to the open string sectors. The recent developments in the Conformal Field Theory description of open strings \[4\] are instrumental to our results.

We determine the fermionic subspaces by performing a truncation of the bosonic Hilbert space. To ensure consistency of the truncation for open string sectors (and hence for D-branes) we impose a symmetric truncation in both closed string sectors. This consistency condition lifts all ambiguities about the fermionic subspaces found by truncation and, most importantly, allows the approach to be predictive. We can classify all the fermionic subspaces and exhibit how this classification is related to global properties of the group \(SO(16)\). All fermionic strings live in subspaces of the bosonic string compactified on sublattices of the \(E_8 \times SO(16)\) weight lattice and the sublattice \(E_8 \times E_8\) of \(E_8 \times SO(16)\) contains the supersymmetric theories IIA and IIB as well as the heterotic superstrings. All non-heterotic strings and all their (stable and unstable) D-branes are classified by the discrete subgroups of the centre of \(SO(16)\). Significantly, the characteristic properties of fermionic D-branes (tension, charge conjugation, chirality changing T-dualities) are predicted from purely bosonic considerations. Furthermore, the Chan-Paton groups of tadpole-free open fermionic strings are also correctly obtained via truncation, and in particular, the anomalies in type I do cancel.

Truncation provides a dictionary translating all fermionic string properties to bosonic string ones. If a non-perturbative mechanism exists which isolates the fermionic subspaces, the scope of the M-theory quest would be considerably enlarged: there would be no elementary fermions at a fundamental level and supersymmetry would have a dynamical origin.

2. Symmetric truncation

The truncation of the bosonic string Hilbert space which yields all its ten-dimensional fermionic subspaces is highly constrained. A first constraint originates in the closed string sector, where coherence of the theory imposes that modular invariance be preserved by truncation, while a second constraint emerges from the open string sector, where one must require the truncation to be consistent with boundary conditions relating the left and right moving closed strings, i.e. with the introduction of D-branes. The resulting truncation must be symmetric and we now review how it works.

We perform a toroidal compactification of the 26-dimensional closed bosonic string theory at an enhanced symmetry point with gauge group \(\mathcal{G}_L \times \mathcal{G}_R\), with \(\mathcal{G}_L, \mathcal{G}_R\) two
semi-simple, simply laced Lie groups. With both groups of rank \( d = 24 - s \), \( 0 \leq s \leq 24 \), the compactified bosonic theory lives in \( s + 2 \) dimensions, and the original transverse Lorentz group \( SO(24)_{tr} \) becomes the Lorentz group \( SO(s)_{tr} \), which does not possess the spinorial representations needed to accommodate space-time fermions, and which cannot therefore play the role of the transverse Lorentz group of a fermionic theory in \( s + 2 \) dimensions. In order to manufacture an appropriate Lorentz group, one uses a stringy analog of the field theoretical construction which turns isospin into spin in four dimensional gauge theory. Namely, one requires that \( G_L \) and \( G_R \) (in the heterotic case only \( G_R \)) admit an \( SO(s)_{int} \) subgroup, and one takes as new transverse Lorentz group the diagonal \( SO(s)_{diag} \) group with algebra

\[
so(s)_{diag} = \text{diag}[so(s)_{tr} \times so(s)_{int}]
\]  

(2.1)
generated by \( J^{ij} = L^{ij} + K^{ij}_0 \), \( i < j \), \( i, j = 1, \ldots, s \). The algebraic set-up Eq.(2.1) is a first step in creating spin from isospin. The second step is to ensure the closure of the full Lorentz algebra in \( s + 2 \) dimensions. This can be done only if all states corresponding to 12 compact dimensions are removed, except for some zero-modes, and the maximum value of \( s \) accommodating fermions turns out to be 8. Although these facts follow from the highly non-trivial closure condition of the Lorentz algebra, they can be understood in simpler qualitative terms. Indeed, the existence of space-time fermions in a covariant formalism is rooted in the existence of worldsheet supersymmetry. The central charge of the superghost is 11 and the timelike and longitudinal fermions contribute 1 to the central charge. The hidden superconformal invariance in the light-cone gauge thus requires the removal of 12 bosonic fields. The zero-modes kept in the 12 dimensions account for the oscillator zero-point energy which is equal to \((-1/24) \times 12\) (in units \( \alpha' = 1/2 \)) and which is to be removed. Therefore zero-modes kept in 12 dimensions must contribute an energy 1/2. In this way, space-time fermions can be obtained provided a truncation of the Hilbert space is performed. At this stage truncation is done by hand, but the group theory classification of fermionic D-branes and the host of correct predictions resulting from truncation strongly suggest the existence of an underlying dynamics.

We now specify the truncation and restrict hereafter to \( s = 8 \), i.e. to 10-dimensional fermionic strings. The toroidal compactification should therefore be performed on the lattice of a Lie group of rank \( d = 24 - 8 = 16 \) with subgroup \( SO(8)_{int} \). The compactification lattice in both sectors (or in the right sector only for the heterotic strings) is taken to be a sublattice of the \( E_8 \times SO(16) \) weight lattice which preserves the modular invariance of the partition function, whose \( G_L \times G_R \) lattice contribution \( P(\tau, \bar{\tau}) \) is separately modular invariant and given by,

\[
P(\tau, \bar{\tau}) = \sum_{\alpha, \beta} N_{\alpha \beta} \bar{\alpha}_L(\bar{\tau}) \beta_R(\tau),
\]  

(2.2)

where

\[
\beta_R(\tau) = \sum_{\sqrt{2\alpha'} \mathbf{p}_R \in \langle 0 \rangle} \exp\{2\pi i \tau \left[ \frac{(\mathbf{p}_R + \mathbf{p}_{BR})^2}{2} + N^{(c)}_R - \frac{\delta}{24} \right] \}.
\]  

(2.3)
Here $\beta$ is a partition function for a sublattice $(\beta)$ of the $G_R = E_8 \times SO(16)$ weight lattice (i.e. $(\beta) = (o)_E \oplus (i)_{16}$, $i = o, v, s, c$) and $p_{\beta R}$ is a fixed vector, arbitrarily chosen, of the sublattice $(\beta)$. $N_R^{(c)}$ is the oscillator number in the $\delta = 16$ compact dimensions. A similar expression holds for $\bar{\alpha}_L(\bar{\tau})$, $\bar{\alpha}$ labeling a partition function for a sublattice of the weight lattice of $G_L$. The coefficients $N_{\alpha\beta}$ are 0 or 1 and are chosen in such a way that $P(\tau, \bar{\tau})$ is modular invariant.

In order to proceed with the truncation (exemplified here in the right sector of the theory), we decompose the $SO(16)$ factor of $G_R$ in $SO'(8) \times SO(8)$ and first truncate all states created by oscillators in the 12 dimensions defined by the $E_8 \times SO'(8)$ root lattice. The group $SO(8)$ is identified with the internal symmetry group $SO(8)_{\text{int}}$. As discussed above, the closure of the new Lorentz algebra dictates we keep zero-modes in the 16 compact dimensions in such a way that

$$\frac{1}{2} p_R^2 [E_8 \times SO(16)] = \frac{1}{2} p_R^2 [SO(8)] + \frac{1}{2}, \quad (2.4)$$

with $p_R(\mathcal{G})$ a vector of the weight lattice of the group $\mathcal{G}$. The zero-mode contribution $1/2$ in Eq.(2.4) comes from $SO'(8)$ as there are no vectors of norm squared one in $E_8$. The only zero-mode contributions from $E_8 \times SO(8)'$ we keep are two fixed $SO(8)'$ 4-vectors $p'_v$ and $p'_s$, so that we truncate the lattice partition functions according to,

$$o_{16} \to v_8, \quad v_{16} \to o_8,$$

$$s_{16} \to -s_8, \quad c_{16} \to -c_8. \quad (2.5)$$

It follows from the closure of the Lorentz algebra that states belonging to $v_8$ or $o_8$ are bosons while those belonging to the spinor partition functions $s_8$ and $c_8$ are space-time fermions. In accordance with the spin-statistic theorem we have flipped the sign in the partition function of the space-time spinor partition functions.

All heterotic strings were obtained, using Eq.(2.5), in reference [6]. To obtain all fermionic D-branes in the non-heterotic theories, we must truncate both sectors of the modular invariant partition functions Eq.(2.2) according to Eq.(2.5). As the $E_8$ lattice is Euclidean even self-dual, we concentrate on the $SO(16)$ weight lattice. Their are four even self-dual Lorentzian $SO(16)$ lattices. The corresponding modular invariant partition functions are (modulo the contribution from the $E_8$ lattice and from the non-compact dimensions),

$$OB_b = \bar{o}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} s_{16} + \bar{c}_{16} c_{16}, \quad (2.6)$$

$$OA_b = \bar{o}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} c_{16} + \bar{c}_{16} s_{16}, \quad (2.7)$$

$$II B_b = \bar{o}_{16} o_{16} + \bar{s}_{16} o_{16} + \bar{o}_{16} s_{16} + \bar{s}_{16} s_{16}, \quad (2.8)$$

$$II A_b = \bar{o}_{16} o_{16} + \bar{c}_{16} o_{16} + \bar{o}_{16} s_{16} + \bar{c}_{16} s_{16}. \quad (2.9)$$

They yield, after symmetric truncation, the four consistent non-heterotic ten-dimensional fermionic string partition functions, namely,

† Hence the terminology ‘symmetric truncation’.
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Table 1. Bosonic D9-brane amplitudes.

<table>
<thead>
<tr>
<th></th>
<th>$2^5 \mathcal{A}_{\text{tree}}$</th>
<th>$\mathcal{A}_{\text{loop}}$</th>
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<tbody>
<tr>
<td>$OB_b$</td>
<td>$(1/2)(o_{16} + v_{16} + s_{16} + c_{16})$</td>
<td>$o_{16}$</td>
</tr>
<tr>
<td>$OA_b$</td>
<td>$o_{16} + v_{16}$</td>
<td>$o_{16} + v_{16}$</td>
</tr>
<tr>
<td>$IIB_b$</td>
<td>$o_{16} + s_{16}$</td>
<td>$o_{16} + s_{16}$</td>
</tr>
<tr>
<td>$IIA_b$</td>
<td>$2 o_{16}$</td>
<td>$o_{16} + v_{16} + s_{16} + c_{16}$</td>
</tr>
</tbody>
</table>

\[ OB_b \rightarrow \bar{v}_8 \ v_8 + \bar{o}_8 \ o_8 + \bar{s}_8 \ s_8 + \bar{c}_8 \ c_8 \equiv OB, \] \hspace{1cm} (2.10)

\[ OA_b \rightarrow \bar{v}_8 \ v_8 + \bar{o}_8 \ o_8 + \bar{s}_8 \ c_8 + \bar{c}_8 \ s_8 \equiv OA, \] \hspace{1cm} (2.11)

\[ IIB_b \rightarrow \bar{v}_8 \ v_8 - \bar{s}_8 \ v_8 - \bar{v}_8 \ s_8 + \bar{s}_8 \ s_8 \equiv IIB, \] \hspace{1cm} (2.12)

\[ IIA_b \rightarrow \bar{v}_8 \ v_8 - \bar{c}_8 \ v_8 - \bar{v}_8 \ s_8 + \bar{c}_8 \ s_8 \equiv IIA. \] \hspace{1cm} (2.13)

3. Fermionic D-branes and torus geometry

The properties of the bosonic D9-branes pertaining to the four different theories compactified on $E_8 \times SO(16)$ lattices can be related to the geometry of the configuration space torus characterizing each compactification. These tori are linked to each other through global properties of the universal covering group $\tilde{SO}(16)$ as we shall now show.

The amplitudes $\mathcal{A}_{\text{tree}}$ describing the D9-branes in the tree channel are obtained from the torus partition functions Eqs.(2.6)-(2.9) by imposing Dirichlet boundary conditions on the compact space. In the tree channel, the latter consists in the following relation between compactified momenta,

\[ p_L - p_R = 0, \] \hspace{1cm} (3.1)

as well as in a match between left and right oscillators. The amplitudes of elementary bosonic D9-branes are given in Table 1, both in the tree channel and in its S-dual loop channel.

In order to identify the configuration space torus on which each theory is defined, recall that in the conformal $\sigma$-model description of these theories in presence of torsion $b_{ab}$, the left and right momenta are given by

\[ p_R = \left[ \frac{1}{2} m_b + n^a (b_{ab} + g_{ab}) \right] e^b, \]

\[ p_L = \left[ \frac{1}{2} m_b + n^a (b_{ab} - g_{ab}) \right] e^b, \] \hspace{1cm} (3.2)

where \{e^a\} is the lattice-dual basis of the basis \{e_a\} defining the configuration space torus

\[ x \equiv x + 2\pi n^a e_a \quad n^a \in \mathbb{Z}, \] \hspace{1cm} (3.3)

and the lattice metric is given by $g_{ab} = e_a.e_b$. The weight vectors $2e_a$ generate four sublattices of the weight lattice of SO(16). They can be read off from the second column in Table 1, as $\mathcal{A}_{\text{loop}}$ yields the winding lattice $n^a e_a$. The classification of bosonic
Fermionic Subspaces of the Bosonic String

Figure 1. Projected weight lattice of $SO(16)$ in the plane of the two orthogonal simple roots $r_7$ and $r_8$. The volumes $\xi_{OB_b} = 2, \xi_{OA_b} = \xi_{IIB_b} = 1, \xi_{IIA_b} = \frac{1}{2}$ of the unit cells, exhibited in shaded areas, must be multiplied by $(2\pi)^8.2^{-8}$ to yield the $SO(16)$ compactification space torus volume of the four bosonic theories (in units where $\alpha' = 1/2$). The theories IIB and IIB' are isomorphic and differ by the interchange of $s_{16}$ and $c_{16}$.

D-branes can be visualized by the volume-preserving projection of configuration space tori in Figure 1, and provides, after truncation, the classification of the ten-dimensional fermionic subspaces.

The tori $\tilde{t}$ of the four bosonic theories are, as group spaces, the maximal toroids $\tilde{T}/Z_c$ of the locally isomorphic groups $E_8 \times \tilde{SO}(16)/Z_c$ where $Z_c$ is a subgroup of the centre $Z_2 \times Z_2$ of the universal covering group $\tilde{SO}(16)$. We write

$$\tilde{t}(OB_b) = \tilde{T},$$
$$\tilde{t}(OA_b) = \tilde{T}/Z_2^d,$$
$$\tilde{t}(IIB_b) = \tilde{T}/Z_2^2 \text{ or } \tilde{T}/Z_2,$$
$$\tilde{t}(IIA_b) = \tilde{T}/(Z_2 \times Z_2),$$

(3.4)
where $Z_2^d = \text{diag}(Z_2 \times Z_2)$ and the superscripts $\pm$ label the two isomorphic $IIB_b$ theories obtained by interchanging $(s)_{16}$ and $(c)_{16}$. There is thus a unified picture for the four theories related to the global properties of the $SO(16)$ group.

**Tension of an elementary D9-brane:** Since tree amplitudes are proportional to the square of the D-brane tension $T$, Table 1 provides the following relations between the tensions of the elementary D9-branes of the different theories

$$\sqrt{2} T_{OB_b} = T_{OA_b} = T_{IIB_b} = (1/\sqrt{2}) T_{IIA_b}.$$ (3.5)

To get their values, we recall that the tension $T_{Dp}^{\text{bosonic}}$ of a $Dp$-brane in the 26-dimensional uncompactified theory is \[ \sqrt{\frac{\pi}{2^4 \kappa_{26}}}(2\pi \alpha'^{1/2})^{11-p}, \] (3.6)
where $\kappa_{26}^2 = 8\pi G_{26}$ and $G_{26}$ is the Newtonian constant in 26 dimensions. The tensions of the Dirichlet D9-branes of the four compactified theories are obtained from Eq.(3.6) by expressing $\kappa_{26}$ in term of the 10-dimensional coupling constant $\kappa_{10}$. Recalling that $\kappa_{26} = \sqrt{V \kappa_{10}}$ where $V$ is the volume of the configuration space torus, one finds, using Figure 1,

$$T_{OB_b} = \frac{\sqrt{\pi}}{\sqrt{2} \kappa_{10}} (2\pi \alpha'^{1/2})^{11-6},$$ (3.7)

$$T_{OA_b} = T_{IIB_b} = \frac{\sqrt{\pi}}{\kappa_{10}} (2\pi \alpha'^{1/2})^{11-6},$$ (3.8)

$$T_{IIA_b} = \frac{\sqrt{2} \sqrt{\pi}}{\kappa_{10}} (2\pi \alpha'^{1/2})^{11-6}.$$ (3.9)

These are consistent with Eq.(3.5).

Truncation on the loop amplitudes of Table 1 yields the amplitudes describing the fermionic D9-branes of respectively $OB, OA, IIB$ and $IIA$. Furthermore tension is conserved in the truncation as proven in reference [2]. The tensions of the different bosonic D9-branes given in Eqs.(3.7)-(3.9) are thus equal, when measured with the same gravitational constant $\kappa_{10}$, to the tensions of the corresponding fermionic D9-branes [7, 8]. This is indeed a correct prediction.

**Charge conjugation:** Different D9-branes of a given bosonic theory are joined by strings of minimal size $e_a = \frac{1}{2} w_a$. Hence the number of distinct fermionic branes is equal to the number of lattice points in the unit cell of the bosonic torus. Charge conjugation for fermionic branes arises from a lattice shift by $(v)$ in the bosonic string. A glance at Figure 1 provides us with a correct prediction of four charged stable fermionic D9-branes in $OB$ (two D-branes and their corresponding antibranes), two neutral unstable D9-branes in $OA$, two charged stable D9-brane in $IIB$ (one D-brane and its antibrane) and one neutral unstable D9-brane in $IIA$.

**Fermionic Dp-branes** ($p < 9$): Up to now, we have considered D9-branes. The determination of fermionic D8-branes (and more generally of lower dimensional even Dp-branes) by truncation appears at first sight impossible because there seem to be no bosonic counterpart to the fermionic chirality changing T-duality which relates fermionic
D9-branes and D8-branes. The switching between $OA$ and $OB$ (or $IIA$ and $IIB$) theories by T-duality is imposed in fermionic strings by worldsheet supersymmetry in the covariant formalism, or equivalently by the closure of the Lorentz algebra in the light-cone gauge. Hence truncation for even dimensional branes can be consistent with Lorentz invariance only if there exists an involution $I$ in the parent bosonic theories relating $OA_b$ and $OB_b$ (or $IIA_b$ and $IIB_b$) which would interchange the lattice partition functions of their D-branes. Fermionic D8-branes would then be related by truncation, in agreement with the requirement of Lorentz invariance, to parent bosonic branes obtained by submitting bosonic D9-branes to both $I$ and to a bosonic T-duality. Remarkably such an involution does exist as we now explain.

In the description of toroidal compactification of bosonic strings, E-duality maps a D-brane localized on a torus onto a D-brane completely wrapped on it or vice versa [9]. At an enhanced symmetry point E-duality does not necessarily map a given lattice onto itself. Even more strikingly, starting from any Lagrangian realization of $OA_b$ and $IIA_b$, E-duality always maps $OA_b$ onto $OB_b$ and $IIA_b$ onto $IIB_b$ [3]. Such a duality we call odd E-duality to distinguish it from even E-dualities which map a lattice onto itself. Lagrangian realizations connected by odd E-dualities are called odd realizations. While for $OA_b$ and $IIA_b$ there are only odd realizations, for $OB_b$ and $IIB_b$ one may have both even and odd realizations. These differ by inequivalent antisymmetric fields $b_{ab}$. Thus the required involution is realized by the odd E-duality and the correct partition functions of even fermionic Dp-branes are obtained, in accordance with Lorentz invariance, from the truncation of the D(p+8)-brane resulting from combining an odd E-duality on the $SO(16)$ torus with a bosonic T-duality.

We thus see that, as a consequence of the existence of odd E-dualities, chirality changing T-dualities in fermionic strings are encoded in the bosonic string!

4. Tadpole-free open descendants

Space-filling D-branes may be used to define open string theories. These are plagued by massless tadpoles which give rise to divergences. In the uncompactified 26-dimensional bosonic string, these divergences can be eliminated by a restriction to unoriented strings and introducing a Chan-Paton group $SO(2^{13})$. Geometrically this amounts to take $2^{12}$ D25-branes (+images) to cancel the negative tension of an $O25$ orientifold. We now explain how truncation yields all the tadpole-free fermionic open descendants.

Compactification of the unoriented bosonic string at an enhanced symmetry point generically reduces the rank of the Chan-Paton group ensuring tadpole cancellation. In addition, symmetry breaking may occur because D9-branes may sit at different locations in the lattice. The explicit computation of the Chan-Paton multiplicities is done by requiring the cancellation of the tadpoles arising in the tree channel of the annulus, the Möbius strip and the Klein bottle. Furthermore, one proves that the Chan-Paton group is conserved by truncation. We obtain in this way the Chan-Paton groups listed in Table 2 [2, 3].
Table 2. Open descendants, \( N \) is the number of D-branes (+images).

<table>
<thead>
<tr>
<th>Chan-Paton group</th>
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<tr>
<td>( OB_b \to OB \to B )</td>
</tr>
<tr>
<td>( OA_b \to OA \to A )</td>
</tr>
<tr>
<td>( IIB_b \to IIB \to I )</td>
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It is particularly remarkable that for type I where the tadpole would induce a genuine anomaly, the correct Chan-Paton group follows from bosonic considerations only.

Acknowledgments

This work was supported in part by the NATO grant PST.CLG.979008.