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A new proposition on the martingale representation theorem and on the approximate hedging of contingent claim in mean-variance criterion

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Abstract:
In this work we revisit the problem of the hedging of contingent claim using mean-square criterion. We prove that in incomplete market, some probability measure $\mathbf{Q} \sim \mathbf{P}$ *can be identified so that* $\{S_n\}$ *becomes* $\{F_n\}$*-martingale under* $\mathbf{Q}$. *This is in fact a new proposition on the martingale representation theorem. The new results also identify a weight function that serves to be an approximation to the Radon-Nikodým derivative of the unique neutral martingale measure* $\mathbf{Q}$.

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1. Introduction

The activity of a stock market takes place usually in discrete time. Unfortunately such markets with discrete time are incomplete, so the traditional pricing and hedging of contingent claim are usually not applicable.

The purpose of this work is to propose a simple method for hedging a contingent claim or an option in mean-variance criterion.

(a) Let $\{S_n, n = 0,1,\ldots,N\}, S_n \in \mathbf{R}^d$, be a sequence of discounted stock prices defined on a probability space $\{\Omega, F, \mathbf{P}\}$, and $\{F_n, n = 0,1,\ldots,N\}$ be a sequence of sigma-algebras of information available up to the time n.

(b) An $\{F_n\}$-measurable random variable H is called a contingent claim that in the case of a standard call option $H = \max(S_n - K, 0)$.

(c) The portfolio $\gamma = \{\gamma_n, n = 1,2,\ldots.N\}$ with $\gamma_n = (\gamma_n^1, \gamma_n^2, \ldots \gamma_n^j)$, where γ_n^j is the number of securities of type j kept by the investor in the time interval $[n-1, n)$. γ_n is $F_{(n-1)}$-measurable (based on the information available up to the time $n-1$). Thus, $\{\gamma_n\}$ is said to be predictable .

(d) Suppose that $\Delta S_n = S_n - S_{n-1}, H \in L_2(\mathbf{P})$,

(e) $G_n(\gamma) = \sum_{k=1}^{n} \gamma_k \Delta S_k$ is gain with $\gamma_k \Delta S_k = \sum_{j=1}^{d} \gamma_k^j V S_k^j$

The traditional problem is to find constant c and $\gamma = \{\gamma_n, n = 1,2,\ldots.N\}$, such that

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$$E_{\mathbf{P}}\{H-c-G_N(\gamma)\}^2 \to \min \tag{1.1}$$

Definition 1. $(\gamma_n^*)=(\gamma_n^*(c))$ minimizes the expectation in (1.1) is called an optimal strategy in the mean square criterion corresponding to initial capital c.

Problem (1.1) has been investigated in a number of works such as Föllmer and Schweiser (1991), Schweiser (1995, 1996), Schäl (1994), and Nechaev (1998). However, the solution for (1.1) has been very complicated as $\{S_n\}$ is not a $\{F_n\}$-martingale under **P**.

When $\{S_n\}$ is $\{F_n\}$-martingale under some measure $\mathbf{Q}\sim\mathbf{P}$, we can find c,γ such that:

$$E_{\mathbf{Q}}\{H-c-G_N(\gamma)\}^2 \to \min \tag{1.2}$$

The solution of this problem may be simple enough, and the construction of an optimal strategy is much easier in practice.

We notice that if $L_N = d\mathbf{Q}/d\mathbf{P}$ then

$$E_{\mathbf{Q}}\{H-c-G_N(\gamma)\}^2 = E_{\mathbf{P}}\left\{\{H-c-G_N(\gamma)\}^2 L_N\right\} \tag{1.3}$$

can be considered a weighted expectation under **P** of $(H-c-G_N)^2$ with the weight L_N. This is similar to the pricing of asset based on a neutral martingale measure.

In this work we give a solution of the problem (1.3) and a martingale representation theorem in the case of discrete time.

2. Defining the optimal portfolio

Let **Q** be a probability measure such that **Q** is equivalent to **P**, and under **Q**, $\{S_N, n=1,2,\ldots,N\}$ is a martingale, then

$$E_n(X)=E_{\mathbf{Q}}(X|F_N), H_N=H, H_n=E_n(H)$$

Theorem 1. If $\{S_n, n=0,1,\ldots,N\}, S_n\in\mathbf{R}^d$ is a $\{F_n\}$ **Q** then

$$E_{\mathbf{Q}}(H-H_0-G_N(\gamma^*))^2 = \min E_{\mathbf{Q}}(H-c-G_N(\gamma))^2, \tag{2.1}$$

Where

$$\begin{aligned}\gamma_n^* &= E_{n-1}\{(\Delta H_n\Delta S_n)[\mathrm{var}(\Delta S_n)]^{-1}\}\\ &= E_{n-1}\{(H\Delta S_n)[\mathrm{var}(\Delta S_n)]^{-1}\} \qquad \mathbf{P}-a.s.\end{aligned} \tag{2.2}$$

with the convention that 0/0 = 0.

Proof. We shall prove the theorem only for the case $d=1$. We note that:

$H_N = H_0+\Delta H_1+\ldots+\Delta H_N$ and

$E_{n-1}(\Delta H_N-\gamma_n\Delta S_n)^2 = E_{n-1}(\Delta H_N)^2-2\gamma_n E_{n-1}(\Delta H_N\Delta S_n)+\gamma_n^2 E_{n-1}(\Delta S_n)^2$

This expression takes the minimum value when $\gamma_n=\gamma_n^*$.

Furthermore, we have:

$$E_{\mathbf{Q}}(H_N - c - G_N(\gamma))^2 = E_{\mathbf{Q}}\left\{H_0 - c - \sum_{n-1}^{N}(\Delta H_N - \gamma_n \Delta S_n)\right\}^2 =$$

$$= (H_0 - c)^2 + \sum_{n-1}^{N} E_{\mathbf{Q}}\{\Delta H_n - \gamma_n \Delta S_n\}^2 = (H_0 - c)^2 + E_{\mathbf{Q}} \sum_{n-1}^{N} E_{n-1}\{\Delta H_n - \gamma_n \Delta S_n\}^2$$

$$= (H_0 - c)^2 + E_{\mathbf{Q}} \sum_{n-1}^{N} E_{n-1}\{\Delta H_n - \gamma_n^* \Delta S_n\}^2$$

$$= E_Q\{H_N - H_0 - G_N(\gamma^*)\}^2$$

3. The martingale representation theorem

Theorem 2. Let $\{H_n, n = 0,1,\ldots\}$, $\{S_n, n = 0,1,\ldots\}$ be random variables defined on the same probability space $\{\Omega, F, \mathbf{P}\}$, $F_n^S = \sigma(S_0,\ldots,S_n)$. We denote $\Pi(S,\mathbf{P})$ a set of the probability measure $\mathbf{Q}$ such that $\mathbf{Q} \sim \mathbf{P}$, and that $\{S_n\}$ is $\{F_n^S\}$-martingale under $\mathbf{Q}$. Thus, if $F = \vee_{n=0}^{\infty} F_n^S, H_n, S_n \in L_2(\mathbf{P})$ and if $\{H_n\}$ is also a martingale under $\mathbf{Q}$, we have:

1. $H_n = H_0 + \sum_{k=1}^{n} \gamma_k \Delta S_k + C_n$, a.s. (3.1)

where $\{C_n\}$ is $\{F_n^S\}$-$\mathbf{Q}$-martingale orthogonal to the martingale $\{S_n\}$, that is $E_{n-1}\{\Delta C_n \Delta S_n\} = 0, \forall n = 0,1,2,\ldots$, whereas $\{\gamma_n\}$ is $\{F_n^S\}$-predictable.

2. $H_n = H_0 + \sum_{k=1}^{n} \gamma_k \Delta S_k := H_0 + G_N(\gamma)$, $\mathbf{P}$-a.s. (3.2)

for all n finite iff the set $\Pi(S,\mathbf{P})$ consists of only one element.

Remark 1. By the fundamental theorem of mathematical finance, a stock market has no arbitrage opportunity and is complete iff $\Pi(S,\mathbf{P})$ consists of only one element and in this case we have (3.2) with γ being defined by (2.2). Furthermore, in this case *the conditional probability distribution of* $\{S_n\}$ *given* $\{F_{n-1}^S\}$ *concentrates at* $d+1$ *points of* $\mathbf{R}^d$ (see [2]).

4. Examples

Example 1. Let us consider a stock with the discounted price S_0 at $t = 0$, S_1 at $t = 1$, where:

$$S_1 = \begin{cases} \frac{3}{2}S_0 & \text{with prob. } p_1 \\ S_0 & \text{with prob. } p_2 \quad p_1, p_2, p_3 > 0, p_1 + p_2 + p_3 = 1. \\ \frac{1}{2}S_0 & \text{with prob. } p_3 \end{cases}$$

Suppose that there is an option on the above stock with the maturity at $t = 1$ and with strike price $K = S_0$. We shall show that there are several probability measures $\mathbf{Q} \sim \mathbf{P}$ such that under $\mathbf{Q}$, S_0, S_1 is a martingale, or equivalently $E_{\mathbf{Q}}(\Delta S_1) = 0$.

In fact, suppose that $\mathbf{Q}$ is a probability measure such that under $\mathbf{Q}$, S_1 takes the values of $\frac{3}{2}S_0$, S_0, $\frac{1}{2}S_0$, with the positive probabilities q_1, q_2, q_3, respectively, then:

$$E_{\mathbf{Q}}(\Delta S_1) = 0 \Leftrightarrow S_0(q_1 - q_3)/2 = 0 \Leftrightarrow q_1 = q_3.$$

Therefore, $\mathbf{Q}$ is defined by $(q_1, 1-2q_1, q_1)$, $0 < q_1 < \frac{1}{2}$.

In the above market, the payoff of the option is:

$$H = (S_1 - K)_+ = (\Delta S_1)_+ = \max(\Delta S_1, 0).$$

Apparently, it is feasible to construct an optimal portfolio with:

$$\gamma^* = \frac{E_{\mathbf{Q}}(H\Delta S_1)}{E_{\mathbf{Q}}(\Delta S_1)^2} = \frac{1}{2}$$

$$E_{\mathbf{Q}}(H) = \frac{q_1 S_0}{2}.$$

Example 2. A semi-continuous market model, which is discrete in time, but continuous in state. Now, let us consider a financial market with two assets:

(a) A risk-less asset $\{B_n, n = 0,1,\ldots,N\}$ which exhibits a dynamics given by (4.1):

$$B_n = \exp\left\{\sum_{k=1}^{n} r_k\right\}, 0 < r_k < 1 \tag{4.1}$$

(b) A risky asset $\{S_n, n = 0,1,\ldots,N\}$ given by the following dynamics

$$S_n = S_0 \exp\left\{\sum_{k=1}^{n} [\mu(S_{k-1}) + \sigma(S_{k-1})g_k]\right\} \tag{4.2}$$

where $\{g_n, n = 0,1,\ldots,N\}$ is a sequence of an $NIID \sim N(0,1)$ random variable. It follows directly from (4.2) that

$$\begin{aligned} S_n &= S_{n-1}\exp\{\mu(S_{n-1}) + \sigma(S_{n-1})g_n\}, \\ \mu(S_{n-1}) &= a(S_{n-1}) - \sigma^2(S_{n-1})/2, \end{aligned} \tag{4.3}$$

with S_0 given, and $a(x), \sigma(x)$ being some functions defined on $[0,\infty)$.

The discounted price of risky asset $\tilde{S}_n = S_n / B_n$ is:

$$\tilde{S}_n = S_0 \exp\left\{\sum_{k=1}^{n} [\mu(S_{k-1}) - r_k + \sigma(S_{k-1})g_k]\right\} \tag{4.4}$$

We now find a martingale measure $\mathbf{Q}$ for this model.

It is easy to see that $E_{\mathbf{P}}\{\exp(\lambda g_k)\} = \exp(\lambda^2/2)$, for $g_k \sim N(0,1)$, hence

$$E\left\{\exp\left(\sum_{k=1}^{n}\left[\beta_k(S_{k-1})g_k - \beta_k^2(S_{k-1})\right]\right)\right\} = 1 \tag{4.5}$$

Thus, putting

$$L_n = \exp\left(\sum_{k=1}^{n}\left[\beta_k(S_{k-1})g_k - \beta_k^2(S_{k-1})\right]\right), n = 1,\ldots,N, \tag{4.6}$$

and if $\tilde{S}_n/\tilde{S}_{n-1}$ is a measure such that $d\mathbf{Q} = L_N d\mathbf{P}$, then $\mathbf{Q}$ is also a probability measure. In addition, we see that

$$\tilde{S}_n/\tilde{S}_{n-1} = \exp\{\mu(S_{n-1}) - r_n + \sigma(S_{n-1})g_n\}. \tag{4.7}$$

Denoting by E°, E expectations corresponding to $\mathbf{P}, \mathbf{Q}$, and $E_n(\cdot) = E\{(\cdot)|F_n^S\}$, then choosing

$$\beta_n = \frac{a(S_{n-1}) - r_n}{\sigma(S_{n-1})} , \tag{4.8}$$

then $E_{n-1}\left\{\tilde{S}_n / \tilde{S}_{n-1}\right\} = E^{\circ}\left\{L_n \tilde{S}_n / \tilde{S}_{n-1} \middle| F_n^S\right\} / L_{n-1} = 1$, which implies $\left\{\tilde{S}_n\right\}$ is a martingale under $\mathbf{Q}$. Also, under $\mathbf{Q}$, S_n can be represented in the form

$$S_n = S_{n-1} \exp\{\mu^*(S_{n-1}) + \sigma(S_{n-1}) g_n^*\} \tag{4.9}$$

Where $\mu^*(S_{n-1}) = r_n - \sigma^2(S_{n-1})/2, g_n^* = -\beta_n + g_n$ is a Gaussian $N(0,1)$. It is not easy to show the structure of $\Pi(S, \mathbf{P})$ for this model. We can choose the probability measure $\mathbf{Q}$ or the weight function L_N to find the optimal portfolio.

Remark 2. The models (4.1), (4.2) are that of discretization of the following diffusion model. Let us consider a financial market with continuous time of two assets :

(a) A risk-less asset: $B_t = \exp\left\{\int_o^t r(u) du\right\}$, and

(b) A risky asset: $dS_t = S_t\left(a(S_t)dt + \sigma(S_t)dW_t\right)$, S_0 is given, or

$$S_t = \exp\left\{\int_o^t \left[a(S_u) - \sigma^2(S_u)/2\right] du + \int_0^t \sigma(S_u) dW_u\right\}, \quad 0 \le t \le T. \tag{4.10}$$

Putting

$$\mu(S) = a(S) - \sigma^2(S)/2 \tag{4.11}$$

and dividing $[0,T]$ into N equal intervals $\{0, \Delta, 2\Delta, \ldots, N\Delta\}$, where $N = T/\Delta$ sufficiently large, it follows from (4.10),(4.11) that

$$\begin{aligned} S_{n\Delta} &= S_{(n-1)\Delta} \exp\left\{\int_{(n-1)\Delta}^{n\Delta} \mu(S_u) du + \int_{(n-1)\Delta}^{n\Delta} \sigma(S_u) dW_u\right\} \\ &\cong S_{(n-1)\Delta} \exp\{\mu(S_{(n-1)\Delta})\Delta + \sigma(S_{(n-1)\Delta})[W_{n\Delta} - W_{(-1)\Delta}]\} \\ &\cong S_{(n-1)\Delta} \exp\{\mu(S_{(n-1)\Delta})\Delta + \sigma(S_{(n-1)\Delta})\Delta^{1/2} g_n\} \end{aligned}$$

where $\{g_n, n = 1, \ldots, N\}$ is a sequence of the $NIID \sim N(0,1)$ random variables. Thus, we obtain the model:

$$S_{n\Delta} = S_{(n-1)\Delta} \exp\{\mu(S_{(n-1)\Delta})\Delta + \sigma(S_{(n-1)\Delta})\Delta^{1/2} g_n\}. \tag{4.12}$$

Similarly we have

$$B_n = B_{(n-1)} \exp\{\Delta r_n\}. \tag{4.13}$$

According to (4.10) ,the discounted price of the stock S_t is

$$\tilde{S}_n = S_t / B_t = S_0 \exp\left\{\int_0^t [\mu(S_u) - r_u] du + \int_0^t \sigma(S_u) dW_u\right\}.$$

The unique probability measure $\mathbf{Q}$ under which $\left\{\tilde{S}_t, F_t^S, \mathbf{Q}\right\}$ is a martingale is defined by

$$\left(d\mathbf{Q}/d\mathbf{P}\right)\left|F_t^S = \exp\left\{\int_0^T \beta_u dW_u - \tfrac{1}{2}\int_0^T \beta_u^2 du\right\} := L_T(\omega)\right., \tag{4.14}$$

where $\beta_s = (a(S_s) - r_s)/\sigma(S_s)$, and under **Q**, then: $W_t^* = W_t + \int_0^t \beta_u du$ is a Wiener process. It is obvious that L_T can be approximated by:

$$L_N = \exp\left\{\sum_{k=1}^{N}\left[\beta_k \Delta^{1/2} g_k - \Delta\beta_k^2/2\right]\right\} \tag{4.15}$$

where

$$\beta_n = \left(a(S_{(n-1)\Delta}) - r_{n\Delta}\right)/\sigma(S_{(n-1)\Delta}) \tag{4.16}$$

Therefore, the weight function (4.14) is an approximation to a Radon-Nikodým derivative of the unique neutral martingale measure $\mathbf{Q}$ to $\mathbf{P}$, where $\mathbf{Q}$ can be used to price such derivatives.

4. Further problems to be investigated

We realize that further problems that could arise in these models are the following:

1) We have to show that for the weight function (4.15)

$$E_{\mathbf{Q}}(H - H_0 - G_N(\gamma^*))^2 \to 0 \;\; as \;\; N \to \infty \;\; or \;\; \Delta \to 0$$

2) Which neutral martingale measure $\mathbf{Q}$ is the nearest one with the subjective measure $\mathbf{P}$ in the semi-continuous model?

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