# Nonvariational real Swift-Hohenberg equation for biological, chemical, and optical systems

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We derive asymptotically an order parameter equation in the limit where nascent bistability and long-wavelength modulation instabilities coalesce. This equation is a variant of the Swift-Hohenberg equation that generally contains nonvariational terms of the form  $\psi \nabla^2 \psi$  and  $|\nabla \psi|^2$ . We briefly review some of the properties already derived for this equation and derive it on three examples taken from chemical, biological, and optical contexts. Finally, we derive the equation on a general class of partial differential systems. © 2007 American Institute of Physics. [DOI: 10.1063/1.2759436]

Near the critical point associated with nascent bistability and close to long-wavelength regime, the dynamics of many natural spatially extended systems can be described by a single real partial differential equation. This approximation is very useful for the study of out of equilibrium dissipative structures that can be either periodic or localized in space. In this contribution, we derive a real order parameter equation that includes nonvariational effects and which is capable of describing a very wide class systems. Examples taken from biology, chemistry, and optics are considered together with a general derivation that shows that the obtained real order parameter equation is universal and has a larger spectrum of space-time dynamical behaviors compared with the usual variational Swift-Hohenberg equation.

### I. INTRODUCTION

In 1977, Swift and Hohenberg<sup>1</sup> derived a real order parameter equation that has been widely used to describe convective patterns induced by the Bénard-Marangoni instability (often called non-Boussinesq-Bénard convection<sup>2,3</sup>). Since this pioneering work, the Swift-Hohenberg (SH) model equation has been derived for various nonequilibrium systems, for instance, in optics,<sup>4,5</sup> chemical reactions with diffusion,<sup>6</sup> and in biology.<sup>7</sup> As a result, it is one of the most studied nonlinear equations not only in its field of origin, hydrodynamical systems, but in most domains of the natural sciences. It constitutes a paradigm for the study of pattern formation, localized structures,<sup>8-14</sup> and fronts.<sup>15-19</sup> An important property of this equation is that it admits a Lyapunov functional or "potential" that is minimized by the steady state solutions.

The aim of this paper is to present an order parameter equation that arises in limiting situations where bistability is nascent and the dynamics of long-wavelength modes exhibits critical slowing down. This equation contains the SH model as a particular case. In the double limit above, we show that a wide class of nonlinear systems are governed by<sup>20</sup>

$$\frac{\partial \psi}{\partial t} = Y + C\psi - \psi^3 - (1 \pm \nabla^2)^2 \psi + \alpha \psi \nabla^2 \psi + \beta |\nabla \psi|^2, \quad (1)$$

where  $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ , the ± sign depends on the original problem studied, and the main control parameter is usually *Y*. The above equation is one of the simplest possible nonlinear models of spatial dynamics; it has been derived first for the coherently pumped semiconductor cavity,<sup>21,22</sup> and soon later for a liquid crystal light valve with optical feedback.<sup>23–25</sup> More recently, it was derived for an acoustic resonator containing a viscous medium.<sup>26</sup> A distinctive feature of this equation is the presence of the nonlinear diffusion term  $\psi \nabla^2 \psi$  and  $|\nabla \psi|^2$ , which breaks the  $\psi \rightarrow -\psi$  and  $Y \rightarrow -Y$ symmetry and renders it nonvariational, i.e., it does not possess a Lyapunov functional. An exception is when  $\beta = \alpha/2$ (Ref. 27), in which case (1) does possess the Lyapunov functional

$$\begin{split} \mathcal{F} &= -\int \int \left( Y\psi + (C-1)\frac{\psi^2}{2} - \frac{\psi^4}{4} \pm |\nabla\psi|^2 \right. \\ &- \frac{1}{2}(\nabla^2\psi)^2 - \frac{\alpha}{2}\psi|\nabla\psi|^2 \bigg) dxdy. \end{split}$$

Furthermore, one immediately sees that the above equation reduces to the SH equation if  $\alpha$  and  $\beta$  both vanish and the + sign is assumed in  $(1 \pm \nabla^2)^2$ . Indeed, we will see below that the asymptotic reduction leading to (1) can yield the SH equation for some parameter values. However, the latter is not a robust description in general, as any deviation from these values will generally produce nonzero  $\alpha$  and  $\beta$ .

Before embarking into the derivation of (1), let us make a few remarks on its dynamical properties. First, assuming the + sign, the linear stability of a given steady state yields two instability points at  $Y_{\pm} = \psi_{\pm}^3 + (1-C)\psi_{\pm}$ , where

$$\psi_{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 (1 - C) + 12C}}{6 - \alpha^2 / 2},$$

and with critical wavenumbers  $k_{\pm} = \sqrt{1 - \alpha \psi_{\pm}/2}$ . In the SH model ( $\alpha = 0$ ), this yields two symmetrically located insta-

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bilities  $Y_{-}=-Y_{+}$  with identical critical wave number. In the general case, however, the location and the wavenumbers of the two instabilities do not obey this constraint, and this can have dramatic consequences. In particular, if one lets  $\alpha^2(1-C) + 12C$  be small, an analytic study of the interaction between the two instabilities is possible. A continuous family of branches of periodic solutions can be constructed. By considering the envelope of these branches, the interaction of the two instabilities is shown to give rise to isolated branches of periodic solutions.<sup>21,22</sup> On the other hand, in the case where  $(1-\nabla^2)^2\psi$  holds in (1), corresponding to a stabilizing linear diffusion, modulation instabilities are still possible due to the nonlinear diffusion term  $\psi \nabla^2 \psi$ . Finally, in the absence of a Lyapunov functional, the system does not necessarily relax to a steady state. Indeed, time-dependent dynamics has been reported<sup>25</sup> for this model, although these aspects remain largely unexplored.

Note that other generalizations of the SH model have been found in the literature. In particular, let us note the case of optical parametric oscillators,<sup>28–30</sup> for which the spatial differential operator  $(1 + \nu\psi^2 + \nabla^2)^2$  is found. The nonlinear diffusive term appears in the limit of large detunings. Other examples are proposed but not derived in Ref. 31. In these two instances, however, the equation still has the  $\psi \rightarrow -\psi$ symmetry, which strongly constraints the dynamics. Let us also mention the abundance of examples in optics of the complex SH equation with cubic<sup>32–35</sup> and quintic<sup>36</sup> nonlinearities. Another variation was also found in plant ecology.<sup>37,38</sup>

Previous studies have already focused on Eq. (1). In Ref. 39, the term  $\psi \nabla^2 \psi$  was heuristically added to the SH equation in order to take account of the non-Boussinesq effects in CO<sub>2</sub> convection instabilities, while in Ref. 40 the term  $|\nabla \psi|^2$  was added to stabilize hexagonal patterns in the SH equation without quadratic nonlinearity and study fronts between various patterns. The present derivation gives an *a posteriori* justification for these approaches.

The rest of the paper is organized as follows. We first derive (1) on three examples chosen across various fields of the natural sciences: the Edblom, Órban, and Epstein (EOE) model in Sec. II, the Fitzhugh-Nagumo equation in Sec. III, and a semiconductor cavity model in Sec. IV. In a second step, we sketch the general derivation of the order parameter equation by considering a general class of nonlinear models, with only some minor assumptions to simplify the algebra.

### II. CHEMISTRY: THE EDBLOM, ÓRBAN, AND EPSTEIN (EOE) MODEL

Edblom, Órban, and Epstein have shown the occurrence of oscillatory behavior in the iodate oxidation of sulfite in a continuous flow stirred tank reactor.<sup>41</sup> A description of the reaction in terms of component processes and the associated empirical rate laws was given by Gáspár and Showalter.<sup>42</sup> Because the resulting model is very stiff and difficult to integrate, a simplified EOE model was proposed,<sup>43</sup>

$$\frac{\partial v}{\partial t} = -(1+a)v + u + F + uv^2 + d\nabla^2 v, \qquad (3)$$

where the dimensionless variables u and v correspond to HSO<sub>3</sub> and H<sup>+</sup> concentrations, respectively; a and b are reduced reaction rates; F is a flow rate; and d is the ratio of the diffusion constants of v and u, respectively.

Denoting the steady state with a subscript s, Eq. (2) yields

$$u_s = \frac{av_s}{1+b+v_s^2},\tag{4}$$

and, substituting in (3), we obtain an implicit relation between the control parameters a and  $v_s$ ,

$$a = a_s(v_s) = \frac{(F - v_s)(1 + b + v_s^2)}{bv_s}.$$
(5)

Two limit points coalesce if  $da_s/dv_s=0$  and  $d^2a_s/dv_s^2=0$ . This yields the critical values

$$a_c = 8 \frac{1+b}{b}, \quad F_c = \sqrt{27(1+b)}, \quad u_c = \frac{\sqrt{12(1+b)}}{b},$$
  
 $v_c = \sqrt{3(1+b)}.$ 

Let us now study the linear space dynamics of the EOE model in the neighborhood of that critical point. The linear stability analysis with respect to finite wavelength perturbations, i.e., perturbations of the form  $\exp(i\mathbf{k}\cdot\mathbf{x}+\sigma t)$ , yield the characteristic equation,

$$p_0(k) + p_1(k)\sigma + \sigma^2 = 0.$$

A Turing instability (also called a modulational instability) happens if  $\sigma$ =0 with nonzero *k* and  $\partial\sigma/\partial k$ =0 (Refs. 44 and 45). This value of *k* determines the wavelength  $2\pi/k$  of the emerging dissipative periodic structures, as was found experimentally in chemical open reactors.<sup>46,47</sup> Differentiating the polynomial with respect to *k*, we have

$$\frac{\partial p_0}{\partial k} + p_1 \frac{\partial \sigma}{\partial k} + \frac{\partial p_1}{\partial k} \sigma + 2\sigma \frac{\partial \sigma}{\partial k} = 0,$$

and imposing  $\sigma = \partial \sigma / \partial k = 0$ , we find the condition,

$$\frac{\partial p_0}{\partial k} = 2d(2+2b+k^2) - 3 - \frac{4}{b} = 0.$$

In the limit of very large wavelengths, i.e.,  $k \rightarrow 0$ , this yields the constraint,

$$d = \frac{4+3b}{4b(1+b)}.$$
 (6)

Assuming *d* is fixed, this is an implicit equation for a critical value  $b_c$  of *b*. In the vicinity of this critical point, we now seek a solution of the form

$$u(x,t) = \frac{\sqrt{12(1+b_c)}}{b_c} (1+\varepsilon u_1(\boldsymbol{\xi},\tau)+\varepsilon^2 u_2(\boldsymbol{\xi},\tau)+\cdots),$$

$$v(x,t) = \sqrt{3}(1+b_c)(1+\varepsilon v_1(\boldsymbol{\xi},\tau)+\varepsilon^2 v_2(\boldsymbol{\xi},\tau)+\cdots),$$

with the development,

$$\begin{split} &a=8\frac{1+b_c}{b_c}(1+\varepsilon^2a_2+\varepsilon^3a_3), \quad b=b_c(1+\varepsilon b_1),\\ &F=\sqrt{27(1+b_c)}(1+\varepsilon^2F_2), \end{split}$$

where  $\varepsilon$  is a small parameter and we have introduced the slow space and time variables  $\xi = \varepsilon^{1/2} x$  and  $\tau = \varepsilon^2 t$ . Substituting into (2) and (3), we find at  $O(\varepsilon)$  that  $u_1 = -\frac{1}{2}v_1$ . At  $O(\varepsilon^2)$ , we have

$$0 = -2(1+b_c)(2u_2+v_2-2a_2) - \frac{1}{2}\nabla^2 v_1,$$
(7)

$$0 = \frac{4+3b_c}{b_c}(2u_2+v_2) - 8\frac{1+b_c}{b_c}a_2 + 3F_2 + d\nabla^2 v_1.$$
(8)

The first equation yields

$$2u_2 + v_2 = 2a_2 - \frac{1}{4(1+b_c)}\nabla^2 v_1$$

and, substituting the result into (8), this gives

$$0 = -2a_2 + 3F_2 + \left[d - \frac{4 + 3b_c}{4b_c(1 + b_c)}\right] \nabla^2 v_1$$

Hence, either  $\nabla^2 v_1$  is a constant or

$$a_2 = \frac{3}{2}F_2$$
 and  $d = \frac{4+3b_c}{4b_c(1+b_c)}$ , (9)

which we assume here. Note that the second condition above is identical to (6). Finally, at  $O(\varepsilon^3)$ , Eqs. (2) and (3) yield, respectively,

$$\begin{aligned} \frac{\partial u_1}{\partial \tau} &= -2(1+b_c) \bigg( 2u_3 + v_3 - 2a_3 - \frac{3}{4}v_1^3 + 3(a_2 - F_2)v_1 \bigg) \\ &+ \nabla^2 u_2 + \bigg( \frac{b_c b_1}{2+2b_c} + \frac{3}{4}v_1 \bigg) \nabla^2 v_1, \end{aligned} \tag{10}$$

$$\frac{\partial v_1}{\partial \tau} = \frac{4+3b_c}{b_c} (2u_3+v_3) - \frac{1+b_c}{b_c} (3v_1^3+8a_3-12(a_2-F_2)v_1) + \frac{1}{b_c} \left(\frac{b_1}{1+b_c} - \frac{3v_1}{2}\right) \nabla^2 v_1 + d\nabla^2 v_2.$$
(11)

Solving (10) for  $u_3$  and substituting into (11), the latter equation eventually simplifies to

$$\begin{split} \left(1 - \frac{4}{b_c} + 4b_c\right) &\frac{\partial v_1}{\partial \tau} = (1 + b_c)(-8a_3 + 6F_2v_1 - 3v_1^3) \\ &+ \left[\frac{(2 + b_c)(2 + 3b_c)b_1}{b_c(1 + b_c)} - \frac{3}{2}v_1\right] \nabla^2 v_1 \\ &- d\nabla^4 v_1. \end{split}$$

In terms of the original variables, and setting  $v_1(\boldsymbol{\xi}, \tau) = \psi(\boldsymbol{x}, t)/\varepsilon$ , this equation writes

$$1 - \frac{4}{b_c} + 4b_c \bigg) \frac{\partial \psi}{\partial t} = (1 + b_c) \bigg( 12 \frac{F - F_c}{F_c} - 8 \frac{a - a_c}{a_c} + 6 \frac{F - F_c}{F_c} \psi - 3 \psi^3 \bigg) + \bigg[ \frac{(2 + b_c)(2 + 3b_c)}{b_c(1 + b_c)} \frac{b - b_c}{b_c} - \frac{3}{2} \psi \bigg] \nabla^2 \psi - d\nabla^4 \psi.$$
(12)

Let us rewrite this equation as

$$c_1 \frac{\partial \psi}{\partial t} = c_2 + c_3 \psi - c_4 \psi^3 + c_5 \nabla^2 \psi - c_6 \nabla^4 \psi + c_7 \psi \nabla^2 \psi.$$

Then with the rescaling

$$\psi' = \frac{2\sqrt{c_4c_6}}{c_5}\psi, \quad \mathbf{x}' = \sqrt{|c_5/2c_6|}\mathbf{x}, \quad t' = \frac{c_5^2}{4c_1c_6}t$$

it becomes

$$\begin{aligned} \frac{\partial \psi'}{\partial t'} &= Y + C\psi' - \psi'^3 - (1 \pm \nabla^2)^2 \psi' + \alpha \psi' \nabla^2 \psi', \\ Y &= \frac{8c_2 c_6 \sqrt{c_4 c_6}}{c_5^3}, \quad C &= \frac{4c_3 c_6}{c_5^2} + 1, \quad \alpha = \pm \frac{-c_7}{\sqrt{c_4 c_6}}, \\ &\pm &= \frac{-c_5}{|c_5|}. \end{aligned}$$

We thus obtain Eq. (1) with  $\beta = 0$ . Note that for (12) to be physically acceptable, one must have

$$1 - \frac{4}{b_c} + 4b_c > 0 \to b_c > 0.88 \to d < 1;$$

otherwise, any inhomogeneous perturbation of the homogeneous steady state is unstable and the growth rate of a perturbation is unbounded as the wavenumber tends to infinity. Should such a situation arise, it would be necessary to proceed to higher orders of the perturbation expansion until regularizing terms of the form  $\nabla^6 \psi$  appear in (12).

### **III. BIOLOGY: THE FITZHUGH-NAGUMO MODEL**

The FitzHugh-Nagumo equations<sup>48</sup> were proposed as a simplified version of the Hodgkin-Huxley model<sup>49</sup> to describe electric excitations in nervous membranes. The excitation is mediated by an electrochemical reaction involving sodium and potassium ion flow. The model involves a voltage-like variable u that allows regenerative self-excitation, and a recovery variable v. Presently, we consider a slightly modified version of this model by adding a quadratic nonlinear term ( $\mu u^2$ ) to the kinetic equation of the recovery variable and allowing it to diffuse,

$$\frac{\partial u}{\partial t} = u - u^3 - v + \nabla^2 u, \qquad (13)$$

$$\frac{\partial v}{\partial t} = \kappa (\gamma u - \mu u^2 - v - a) + \delta \nabla^2 v.$$
(14)

In these equations,  $\gamma$  and *a* are positive parameters,  $\delta$  is the ratio of the diffusion coefficients for *v* and *u*, and  $\kappa$  is the ratio of the characteristic of the characteristic times. The homogeneous steady state is given by

$$u_{s}^{3} + (\gamma - 1)u_{s} - \mu u_{s}^{2} = a$$

and  $v_s = \gamma u_s - \mu u_s^2 - a$ . As before, nascent bistability happens when the curve  $(a(u_s), u_s)$  is vertical, i.e., when

$$u_s = u_c = \frac{1}{3}\mu, \quad \gamma = \gamma_c = 1 + \frac{\mu^3}{3}.$$

This yields  $v_c = \mu(9 - \mu^2)/27$  and  $a_c = \mu^2/27$ . Consider next linear perturbations of the form  $\exp(i\mathbf{k}\cdot\mathbf{x} + \sigma t)$ . Following the same procedure as in Sec. II, the condition to have  $\sigma = 0$  and  $\partial\sigma/\partial k = 0$  for small but finite k is to have

$$\kappa = \kappa_c = \delta(1 - \mu^2/3).$$

To summarize, the conditions for nascent bistability and long-wavelength dynamics are

$$u_c = \frac{\mu}{3}, \quad \gamma_c = 1 + \frac{\mu^3}{3}, \quad \kappa_c = \delta\left(1 - \frac{\mu^2}{3}\right)$$

To study finite-size perturbation around this critical point, we introduce a small parameter  $\varepsilon$  and let  $z=z_c+\varepsilon z_1+\varepsilon^2 z_2+\cdots$  for  $z=u, v, a, \gamma$ , and  $\kappa$  in Eqs. (13) and (14). In addition, we rescale space and time as  $\xi = \varepsilon^{1/2} x$  and  $\tau = \varepsilon^2 t$ . Solving order by order in  $\varepsilon$ , we find at  $O(\varepsilon)$  that

$$u_1 = \frac{3}{3 - \mu^3} v_1,$$

and that  $a_1$  and  $\gamma_1$  must satisfy the solvability condition

$$a_1 = \mu \gamma_1 / 3.$$

Next, at  $O(\varepsilon^2)$ , we obtain

$$u_2 = \frac{3}{3 - \mu^2} v_2 + \frac{27\mu}{(3 - \mu^2)^3} v_1^2 - \frac{9}{(3 - \mu^2)^2} \nabla^2 v_1,$$

together with the solvability condition

$$\frac{\delta}{9}(3-\mu^2)(3a_2-\mu\gamma_2)-\frac{3\,\delta a_1}{\mu}v_1=0.$$

For this to hold for any function  $v_1$ , we must have

$$a_2 = \mu \gamma_2 / 3, \quad a_1 = 0.$$

Finally, at  $O(\varepsilon^3)$  the solvability condition is

$$\begin{split} \left(1 - \frac{1}{\delta}\right) &\frac{\partial v_1}{\partial \tau} = \frac{3 - \mu^2}{9} (3a_3 - \mu\gamma_3) - \gamma_2 v_1 - \frac{9}{(3 - \mu^2)^2} v_1^3 \\ &+ \frac{18\mu}{(3 - \mu^2)^2} |\nabla v_1|^2 + \frac{3}{3 - \mu^2} \left(\frac{\kappa_1}{\delta} \right. \\ &+ \frac{6\mu}{3 - \mu^2} v_1 \left(\nabla^2 v_1 - \frac{3}{3 - \mu^2} \nabla^4 v_1\right) \end{split}$$

In terms of the original variables, with  $v_1(\boldsymbol{\xi}, \tau) = (3 - \mu^2) \psi(\boldsymbol{x}, t) / \varepsilon$ , we thus find

$$(1 - \delta^{-1})(3 - \mu^2) \frac{\partial \psi}{\partial t}$$
  
=  $(3 - \mu^2) \left[ \frac{a - a_c}{3} - (\gamma - \gamma_c) \left( \frac{\mu}{9} + \psi \right) - 9 \psi^3 \right]$   
+  $18\mu |\nabla \psi|^2 + 3 \left( \frac{\kappa - \kappa_c}{\delta} + 6\mu \psi \right) \nabla^2 \psi - 3 \nabla^4 \psi.$  (15)

This time [using the same kind of final rescaling as with (12)] we obtain Eq. (1), with both  $\alpha, \beta \neq 0$ . For the same reasons as with (12), this equation is physically acceptable only if  $(\delta-1)(3-\mu^2)>0$ . Note that in the case where  $\mu=0$ , which corresponds to the usual version of the Fitzhugh–Nagumo equations, then (15) becomes the SH equation.<sup>50</sup> However, the latter is not a robust description with respect to small changes of the original model, since, as soon as  $\mu \neq 0$ , nonlinear diffusion terms appear.

# IV. OPTICS: THE SEMICONDUCTOR MICRORESONATOR

The final example concerns the semiconductor cavity driven by a coherent optical field. This setup has been intensely studied in recent years for both its fundamental and practical interests. Indeed, in the plane transverse to the cavity axis, localized structures (often called "cavity solitons") have been excited experimentally.<sup>51,52</sup> Moreover, it has been shown that they can subsequently be manipulated <sup>53–55</sup> as optical bits of information. The derivation of (1) for this problem was done in Refs. 21 and 22. The starting model<sup>56</sup> involves the complex amplitude *F* of the electromagnetic field in the cavity and the electric carrier density in the semiconductor material, *Z*,

$$\frac{\partial F}{\partial t} = (1 + i\alpha)ZF - i\nabla^2 F + Y, \qquad (16)$$

$$\frac{\partial Z}{\partial t} = \gamma [P - Z - (1 + 2Z)|F|^2 + D\nabla^2 Z].$$
(17)

In these equations, time and space have been rescaled with the decay rate of the electromagnetic field and the diffraction length, respectively; Y is the (real) amplitude of the driving field, which is assumed here to be resonant with one mode of the cavity;  $\alpha$  is a phenomenological parameter that is a characteristic of the semiconductor material;  $\gamma$  is the decay rate of electric carriers density. Finally, P describes the electric current flowing into the semiconductor junction. We consider negative values of P, which means that the device is below lasing threshold.

The homogeneous stationary solution is given implicitly by

$$Y = -(1 + i\alpha)(P - |F|^2)F/(1 + 2|F|^2)$$

Nascent bistability is reached for the critical values  $F_c = (1 - i\alpha)\sqrt{3/2(1 + \alpha^2)}$ ,  $Z_c = -3/2$ ,  $P_c = -9/2$ , and

 $Y_c = \sqrt{27(1 + \alpha^2)/8}$ . At this critical point, the cubic characteristic polynomial has a zero root when

$$\gamma k^2 \{ D[9 + (2k^2 - 3\alpha)^2] + 8(2k^2 - 3\alpha) \} = 0,$$
(18)

where k is the wavenumber of the plane wave pertubation considered in the linear analysis of the homogeneous steady states. Equation (18) admits the solution k=0, which corresponds to the neutral mode, and also a very large wavelength, i.e.,  $0 < k \ll 1$ , if  $D \simeq D_c = 8\alpha/3(1+\alpha^2)$ . We now explore the system dynamics in the neighborhood of this operation point. To this end, we set  $F = F_c(1+\varepsilon f+\cdots)$ ,  $Z = Z_c(1+\varepsilon z+\cdots)$ ,  $P = P_c + \varepsilon^2 p_2$ ,  $Y = Y_c(1+\varepsilon^2 y_2 + \varepsilon^3 y_3 + \cdots)$ , and  $D = D_c + \varepsilon d_1 + \cdots$ . As usual, we normalize the time and space scales by the transformation  $\tau \sim \varepsilon^2 t$  and  $\boldsymbol{\xi} \sim \varepsilon^{1/2} \mathbf{x}$ , and it is convenient to set

$$\tau = \frac{\varepsilon^2 t}{1/\gamma + D_c/\alpha}, \quad \boldsymbol{\xi} = \frac{\varepsilon^{1/2} \mathbf{x}}{\sqrt{D_c}}.$$

Proceeding in exactly the same fashion as for the two previous examples, we find that z=-f in the leading order problem. The next order yields the solvability condition  $y_2 = -p_2/2$ . Finally, at the third order of the perturbation development, we obtain

$$\frac{\partial f}{\partial \tau} = 4y_3 - f\left(\frac{p_2}{3} + f^2\right) + (d_1 - 5f/2)\nabla^2 f - a\nabla^4 f - 2|\nabla f|^2,$$
(19)

where  $a = (1 - \alpha^2)/4\alpha^2$ , and a should be positive.

### **V. GENERAL DERIVATION**

We now generalize the derivations above by considering systems of the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = (\boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_\lambda \cdot \boldsymbol{\lambda}) \cdot \boldsymbol{u} + (\boldsymbol{\mathcal{Q}}_0 + \boldsymbol{\mathcal{Q}}_\lambda \cdot \boldsymbol{\lambda}) \boldsymbol{u} \boldsymbol{u} + \boldsymbol{\mathcal{C}} \boldsymbol{u} \boldsymbol{u} \boldsymbol{u}$$
$$+ \cdots + \boldsymbol{\mathcal{D}} \cdot \nabla^2 \boldsymbol{u} + \boldsymbol{\mathcal{D}}_2 \cdot \nabla^4 \boldsymbol{u} + \cdots .$$
(20)

Without loss of generality, we assume that u=0 is the reference state. For notational simplicity, the system is assumed to depend linearly on the vector parameter  $\lambda$ , although from the examples we have seen before, this is by no means necessary; the terms  $(\mathcal{Q}_0 + \mathcal{Q}_\lambda \cdot \lambda)uu$  and *Cuuu* represent quadratic and cubic nonlinearities, respectively; finally,  $\mathcal{D}$  is a diffusion matrix and the term  $\mathcal{D}_2 \cdot \nabla^4 u$  has been added for generality. The latter can be relevant, for instance, to diffraction problems involving the propagation operator  $e^{ia\nabla^2} = 1 + ia\nabla^2 - \frac{1}{2}a^2\nabla^4 + \cdots$  (Ref. 24). Once again, this last term is not necessary for the presence of a bi-Laplacian in the order parameter equation.

Being only separated from a limit point by an  $O(\varepsilon)$  distance, critical slowing down brings the relevant time to  $\tau = \varepsilon^2 t$ . On the other hand, spatial dynamics is assumed to occur on the slow scale  $\xi = \varepsilon^{1/2} x$ . The variables and parameters are thus expanded as

$$\mathbf{\lambda} = \mathbf{\lambda}_0 + \varepsilon \mathbf{\lambda}_1 + \varepsilon^2 \mathbf{\lambda}_2 + \cdots .$$
 (22)

From the proximity of the limit point, the linear operator  $\mathcal{L}_0 + \mathcal{L}_\lambda \cdot \lambda_0$  possesses a zero eigenvalue, and we shall assume that all other eigenvalues are negative. We shall denote the left and right eigenvectors as

$$\boldsymbol{w}^{\boldsymbol{\alpha}} \cdot (\boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_\lambda \cdot \boldsymbol{\lambda}_0 - \alpha \boldsymbol{I}) = (\boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_\lambda \cdot \boldsymbol{\lambda}_0 - \alpha \boldsymbol{I}) \cdot \boldsymbol{v}^{\boldsymbol{\alpha}} = \boldsymbol{\theta}.$$
(23)

To first order in  $\varepsilon$ , we obviously get

$$\boldsymbol{\theta} = (\boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_\lambda \cdot \boldsymbol{\lambda}_0) \cdot \boldsymbol{u}_1, \tag{24}$$

and we therefore have

$$\boldsymbol{u}_1 = \boldsymbol{\psi}(\boldsymbol{\xi}, \tau) \boldsymbol{v}^0. \tag{25}$$

The function  $\psi(\boldsymbol{\xi}, \tau)$  is the order parameter we are looking for. The second order problem is

$$\boldsymbol{\theta} = (\boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_\lambda \cdot \boldsymbol{\lambda}_0) \cdot \boldsymbol{u}_2 + (\boldsymbol{\mathcal{L}}_\lambda \cdot \boldsymbol{\lambda}_1) \cdot \boldsymbol{u}_1 + (\boldsymbol{\mathcal{Q}}_0 + \boldsymbol{\mathcal{Q}}_\lambda \cdot \boldsymbol{\lambda}_0) \boldsymbol{u}_1 \boldsymbol{u}_1 + \boldsymbol{\mathcal{D}} \cdot \nabla^2 \boldsymbol{u}_1,$$
(26)

$$= (\mathcal{L}_{0} + \mathcal{L}_{\lambda} \cdot \boldsymbol{\lambda}_{0}) \cdot \boldsymbol{u}_{2} + (\mathcal{L}_{\lambda} \cdot \boldsymbol{\lambda}_{1}) \cdot \boldsymbol{v}^{0} \boldsymbol{\psi} + (\mathcal{Q}_{0} + \mathcal{Q}_{\lambda} \cdot \boldsymbol{\lambda}_{0}) \boldsymbol{v}^{0} \boldsymbol{v}^{0} \boldsymbol{\psi}^{2} + \mathcal{D} \cdot \boldsymbol{v}^{0} \nabla^{2} \boldsymbol{\psi}.$$
(27)

In order to solve this equation for  $u_2$ , a solvability condition is that the right-hand side be orthogonal to  $w^0$ , i.e., that

$$0 = \boldsymbol{w}^{\mathbf{0}} \cdot [(\boldsymbol{\mathcal{L}}_{\lambda} \cdot \boldsymbol{\lambda}_{1}) \cdot \boldsymbol{v}^{\mathbf{0}} \boldsymbol{\psi} + (\boldsymbol{\mathcal{Q}}_{0} + \boldsymbol{\mathcal{Q}}_{\lambda} \cdot \boldsymbol{\lambda}_{0}) \boldsymbol{v}^{\mathbf{0}} \boldsymbol{v}^{\mathbf{0}} \boldsymbol{\psi}^{2} + \boldsymbol{\mathcal{D}} \cdot \boldsymbol{v}^{\mathbf{0}} \nabla^{2} \boldsymbol{\psi}].$$
(28)

We wish this to hold for any  $\psi$ . The solvability condition thus splits into three parts. One the one hand, the condition

$$0 = \boldsymbol{w}^{\boldsymbol{0}} \cdot (\boldsymbol{\mathcal{L}}_{\lambda} \cdot \boldsymbol{\lambda}_{1}) \cdot \boldsymbol{v}^{\boldsymbol{0}}$$
<sup>(29)</sup>

imposes that we stay sufficiently close in the parameter space to the locus of the limit point. Next, the condition

$$0 = \boldsymbol{w}^{\boldsymbol{0}} \cdot (\boldsymbol{\mathcal{Q}}_0 + \boldsymbol{\mathcal{Q}}_\lambda \cdot \boldsymbol{\lambda}_0) \boldsymbol{v}^{\boldsymbol{0}} \boldsymbol{v}^{\boldsymbol{0}}$$
(30)

is the condition for nascent bistability. Finally,

$$0 = \boldsymbol{w}^{0} \cdot \boldsymbol{\mathcal{D}} \cdot \boldsymbol{v}^{0} \tag{31}$$

is the necessary condition for Turing bifurcations or modulational instabilities to occur on a sufficiently slow spatial scale. Assuming (29)–(31), we can now solve for  $u_2$  and find

$$\boldsymbol{u}_{2} = \sum_{\alpha < 0} \frac{\boldsymbol{v}^{\alpha}}{|\alpha|} \boldsymbol{w}^{\alpha} \cdot \left[ (\boldsymbol{\mathcal{L}}_{\lambda} \cdot \boldsymbol{\lambda}_{1}) \cdot \boldsymbol{v}^{0} \boldsymbol{\psi} + (\boldsymbol{\mathcal{Q}}_{0} + \boldsymbol{\mathcal{Q}}_{\lambda} \cdot \boldsymbol{\lambda}_{0}) \boldsymbol{v}^{0} \boldsymbol{v}^{0} \boldsymbol{\psi}^{2} + \boldsymbol{\mathcal{D}} \cdot \boldsymbol{v}^{0} \nabla^{2} \boldsymbol{\psi} \right],$$
(32)

or, more compactly,

$$\boldsymbol{u}_{2} = \boldsymbol{u}_{21}\boldsymbol{\psi} + \boldsymbol{u}_{22}\boldsymbol{\psi}^{2} + \boldsymbol{u}_{23}\nabla^{2}\boldsymbol{\psi}.$$
 (33)

Proceeding finally to third order, the problem to solve for  $u_3$  is

$$\frac{\partial \psi}{\partial \tau} \boldsymbol{v}^{0} = (\boldsymbol{\mathcal{L}}_{0} + \boldsymbol{\mathcal{L}}_{\lambda} \cdot \boldsymbol{\lambda}_{0}) \cdot \boldsymbol{u}_{3} + (\boldsymbol{\mathcal{L}}_{\lambda} \cdot \boldsymbol{\lambda}_{1}) \cdot \boldsymbol{u}_{2}$$
$$+ 2(\boldsymbol{\mathcal{Q}}_{0} + \boldsymbol{\mathcal{Q}}_{\lambda} \cdot \boldsymbol{\lambda}_{0})\boldsymbol{u}_{1}\boldsymbol{u}_{2} + \boldsymbol{\mathcal{Q}}_{\lambda} \cdot \boldsymbol{\lambda}_{1}\boldsymbol{u}_{1}\boldsymbol{u}_{1}$$
$$+ \boldsymbol{\mathcal{C}}\boldsymbol{u}_{1}\boldsymbol{u}_{1}\boldsymbol{u}_{1} + \boldsymbol{\mathcal{D}} \cdot \nabla^{2}\boldsymbol{u}_{2} + \boldsymbol{\mathcal{D}}_{2} \cdot \nabla^{4}\boldsymbol{u}_{1}.$$
(34)

The solvability condition is, this time,

$$\boldsymbol{w}^{\boldsymbol{0}} \cdot \boldsymbol{v}^{\boldsymbol{0}} \frac{\partial \boldsymbol{\psi}}{\partial \tau} = \boldsymbol{w}^{\boldsymbol{0}} \cdot \left[ (\boldsymbol{\mathcal{L}}_{\lambda} \cdot \boldsymbol{\lambda}_{1}) \cdot \boldsymbol{u}_{2} + 2(\boldsymbol{\mathcal{Q}}_{0} + \boldsymbol{\mathcal{Q}}_{\lambda} \cdot \boldsymbol{\lambda}_{0}) \boldsymbol{u}_{1} \boldsymbol{u}_{2} + \boldsymbol{\mathcal{Q}}_{\lambda} \cdot \boldsymbol{\lambda}_{1} \boldsymbol{u}_{1} \boldsymbol{u}_{1} + \boldsymbol{\mathcal{C}} \boldsymbol{u}_{1} \boldsymbol{u}_{1} \boldsymbol{u}_{1} + \boldsymbol{\mathcal{D}} \cdot \nabla^{2} \boldsymbol{u}_{2} + \boldsymbol{\mathcal{D}}_{2} \cdot \nabla^{4} \boldsymbol{u}_{1} \right].$$
(35)

Using (25) and (33), the right-hand side of this equation can be rearranged as

$$\begin{split} \boldsymbol{w}^{\boldsymbol{\theta}} \cdot ([(\mathcal{L}_{\lambda} \cdot \lambda_{1})\boldsymbol{u}_{21} + (\mathcal{L}_{\lambda} \cdot \lambda_{2})\boldsymbol{v}^{\boldsymbol{\theta}}]\boldsymbol{\psi} \\ &+ [(\mathcal{L}_{\lambda} \cdot \lambda_{1})\boldsymbol{u}_{22} + (\mathcal{Q}_{\lambda} \cdot \lambda_{1})\boldsymbol{v}^{\boldsymbol{\theta}}\boldsymbol{v}^{\boldsymbol{\theta}} + 2(\mathcal{Q}_{0} \\ &+ \mathcal{Q}_{\lambda} \cdot \lambda_{0})\boldsymbol{v}^{\boldsymbol{\theta}}\boldsymbol{u}_{21}]\boldsymbol{\psi}^{2} \\ &+ [2(\mathcal{Q}_{0} + \mathcal{Q}_{\lambda} \cdot \lambda_{0})\boldsymbol{v}^{\boldsymbol{\theta}}\boldsymbol{u}_{22} + \mathcal{C}\boldsymbol{v}^{\boldsymbol{\theta}}\boldsymbol{v}^{\boldsymbol{\theta}}\boldsymbol{v}^{\boldsymbol{\theta}}]\boldsymbol{\psi}^{3} + [(\mathcal{L}_{\lambda} \cdot \lambda_{1})\boldsymbol{u}_{23} \\ &+ \mathcal{D} \cdot \boldsymbol{u}_{21}]\nabla^{2}\boldsymbol{\psi} + 2(\mathcal{Q}_{0} + \mathcal{Q}_{\lambda} \cdot \lambda_{0})\boldsymbol{v}^{\boldsymbol{\theta}}\boldsymbol{u}_{23}\boldsymbol{\psi}\nabla^{2}\boldsymbol{\psi} \\ &+ \mathcal{D} \cdot \boldsymbol{u}_{22}\nabla^{2}(\boldsymbol{\psi}^{2}) + (\mathcal{D} \cdot \boldsymbol{u}_{23} + \mathcal{D}_{2} \cdot \boldsymbol{v}^{\boldsymbol{\theta}})\nabla^{4}\boldsymbol{\psi}). \end{split}$$

Equation (35) is therefore of the form

$$c_1 \frac{\partial \psi}{\partial \tau} = c_2 \psi + c_3 \psi^2 - c_4 \psi^3 + c_5 \nabla^2 \psi - c_6 \nabla^4 \psi + c_7 \psi \nabla^2 \psi$$
$$+ c_8 \nabla^2 (\psi^2), \qquad (36)$$

where  $c_4$  and  $c_6$  should be positive to avoid nonphysical blowup. Finally, after rescaling space, time, and  $\psi$ , this equation can be recast as

$$\frac{\partial \psi}{\partial \tau} = r\psi + s\psi^2 - \psi^3 - (1 \pm \nabla^2)^2 \psi + \alpha' \psi \nabla^2 \psi + \beta' |\nabla \psi|^2, \quad (37)$$

which is equivalent to (1). The equation contains only four independent parameters and the  $\pm$  sign is actually  $-c_5/|c_5|$ .

#### **VI. CONCLUSIONS**

We have presented and derived the real order parameter equation (1) for different examples of nonequilibrium systems of the reaction-diffusion type for chemistry and biology and of the reaction-diffusion-diffraction type for optics. We further extended the applicability of this description by deriving (1) for a general class of nonlinear models. The universality of this equation resides in the fact that one can formulate the general conditions under which it can be derived. Let the steady state be implicitly given by  $\lambda(u_s)$ , where  $\lambda$  is the main control parameter and  $u_s$  characterizes the homogeneous steady state. The systems should be

- (1) Close to the nascent bistability where the phenomenon of slowing down occurs:  $d\lambda/du_s$ ,  $d^2\lambda/du_s^2=0$ ;
- (2) Close to a large wavelength symmetry-breaking instability: lim<sub>k→0</sub> ∂p<sub>0</sub>/∂k=0, where p<sub>0</sub>(k)+p<sub>1</sub>(k)σ+p<sub>2</sub>(k)σ<sup>2</sup> +··· is the characteristic polynomial for perturbations of the form exp(ik ⋅ x + σt).

These conditions make the critical point near which (1) is valid a codimension 3 point. In practice, therefore, at least three parameters [for example, a, b, and F in (12)] are necessary in order to approach this point. While this is a strong restriction to the use of this model, we note that the same comment applies to the usual Swift-Hohenberg equation as well. Still, this is a very useful model to study some complex nonlinear behaviors and if we are unable to understand such behaviors in (1) or in the Swift-Hohenberg equation, then there is little hope to understand them at all.

The model (1) differs from the usual Swift-Hohenberg equation in many aspects. The fundamental difference comes from the nonvariational terms ( $\psi \nabla^2 \psi$  and/or  $|\nabla \psi|^2$ ). Some consequences of these have already been identified:

- Localized structures can move with a constant speed.<sup>25</sup> This behavior is clearly attributed to the absence of any Lyapunov or potential to minimize for Eq. (1);
- (2) Two modulational instabilities can exist with different wavelengths. The interaction between these instabilities can lead to the formation of isolated branches of dissipative structures in the bifurcation diagram.<sup>21</sup>

The spectrum of space-time dynamical behaviors of Eq. (1) is thus wider than that of the usual variational Swift-Hohenberg. Owing to its general character, Eq. (1) has a larger domain of application in various natural systems and may be regarded as a minimal mathematical model for investigating nonvariational effects that are observed experimentally in pattern-forming systems.

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