CONSISTENT ESTIMATION OF THE ASYMPTOTIC COVARIANCE STRUCTURE OF MULTIVARIATE SERIAL CORRELATIONS (*)

by

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Abstract

A method is proposed for estimating, in a consistent way, the asymptotic covariance structure of serial correlations for a multivariate second order stationary process. To obtain a consistent estimator of this structure, which is also of the non-negative definite type, results relative to the scalar case are generalized. The method consists in weighting appropriately the elements of the sample autocovariance matrices in a generalization of Bartlett's formula so that the estimators converge in the L_1-norm. Several useful applications of the results of the paper are discussed: statistical inference in the corner method for transfer function models, inference about vector autocorrelation coefficients, improvements in other specification procedures for univariate and multivariate processes, a test of homogeneity of several independent time series and a test of independence of two multivariate time series.

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Key words: multivariate time series, consistent estimator, asymptotic covariance, serial correlation, model specification, independance, test of independence.

1. Introduction

Let $\{\underline{X}_t ; t \in \mathbb{Z}\}$ be a multivariate second order stationary process of dimension p : $\underline{X}_t^{\mathrm{T}} = (X_{1t}, \ldots, X_{pt})$. Without loss of generality, we can assume that $\mathrm{E}[\underline{X}_t] \equiv \underline{0}$ and let

$$\mathrm{E}[\underline{X}_t \underline{X}_{t+k}^{\mathrm{T}}] = \underline{\Gamma}(k) = (\gamma_{ij}(k)), \tag{1}$$

$k \in \mathbb{Z}$. Further, let

$$\rho_{ij}(k) = \frac{\gamma_{ij}(k)}{\{\gamma_{ii}(0)\gamma_{jj}(0)\}^{1/2}} \tag{2}$$

and the correlation matrix at lag k will be denoted by $\underline{\rho}(k) = (\rho_{ij}(k)), k \in \mathbb{Z}$. Note that $\underline{\Gamma}(-k) = \underline{\Gamma}(k)^{\mathrm{T}}$ and $\underline{\rho}(-k) = \underline{\rho}(k)^{\mathrm{T}}$, $k \in \mathbb{Z}$.

Let $\{\underline{X}_1, \ldots, \underline{X}_n\}$ be a realization of length n of the process $\{\underline{X}_t\}$. The sample covariance matrix at lag k, $0 \le k \le n - 1$, is defined by

$$\underline{C}(k) = \frac{1}{n} \sum_{t=1}^{n-k} (\underline{X}_t - \overline{\underline{X}})(\underline{X}_{t+k} - \overline{\underline{X}})^{\mathrm{T}} = (c_{ij}(k)), \tag{3}$$

where $\overline{\underline{X}} = (1/n) \sum_{t=1}^{n} \underline{X}_t$ is the vector sample mean. For $-n + 1 \le k \le 0$, we let $\underline{C}(k) = \underline{C}(-k)^{\mathrm{T}}$ and for $|k| \ge n$, $\underline{C}(k) \equiv \underline{0}$. The sample correlation matrix at lag k, $0 \le |k| \le n - 1$, is denoted by $\underline{R}(k) = (r_{ij}(k))$, where

$$r_{ij}(k) = \frac{c_{ij}(k)}{\{c_{ii}(0)c_{jj}(0)\}^{1/2}}. \tag{4}$$

In the sequel, we will make use of the asymptotic distribution of the multivariate serial correlations. Before stating the result, let us introduce some more notations. Let

$$s_{ij}(k) = n^{1/2}\{r_{ij}(k) - \rho_{ij}(k)\} \tag{5}$$

and the convolution square of the covariance (matrix) function

$$\theta_k(i,j,l,m) = \sum_{u=-\infty}^{\infty} \gamma_{ij}(u)\gamma_{lm}(u+k). \tag{6}$$

The following theorem which is a generalization of well known results of Bartlett (1946, 1966) is proved in Roy (1989).

Theorem 1

Let $\{\underline{X}_t\}$ be a multivariate second order stationary process satisfying the assumptions of the central limit theorem of Hannan (1976). Further, suppose that all cumulants of fourth order are zero and that the spectral density of each component of $\{\underline{X}_t\}$ is square integrable. Then, any finite subset of the $s_{ij}(k)$'s are jointly asymptotically normal with zero mean and covariance structure given by

$$\lim_{n\to\infty} n\,\mathrm{cov}(s_{ab}(k), s_{de}(h)) =$$

$$\begin{aligned}
&\frac{1}{2}\rho_{ab}(k)\rho_{de}(h)\left\{\frac{\theta_0(a,d,a,d)}{\gamma_{aa}(0)\gamma_{dd}(0)} + \frac{\theta_0(b,d,b,d)}{\gamma_{bb}(0)\gamma_{dd}(0)} + \frac{\theta_0(a,e,a,e)}{\gamma_{aa}(0)\gamma_{ee}(0)} + \frac{\theta_0(b,e,b,e)}{\gamma_{bb}(0)\gamma_{ee}(0)}\right\} \\
&-\rho_{ab}(k)\left\{\frac{\theta_h(a,d,a,e)}{\gamma_{aa}(0)\,\{\gamma_{dd}(0)\gamma_{ee}(0)\}^{1/2}} + \frac{\theta_h(b,d,b,e)}{\gamma_{bb}(0)\,\{\gamma_{dd}(0)\gamma_{ee}(0)\}^{1/2}}\right\} \\
&-\rho_{de}(h)\left\{\frac{\theta_k(b,d,a,d)}{\gamma_{dd}(0)\,\{\gamma_{aa}(0)\gamma_{bb}(0)\}^{1/2}} + \frac{\theta_k(b,e,a,e)}{\gamma_{ee}(0)\,\{\gamma_{aa}(0)\gamma_{bb}(0)\}^{1/2}}\right\} \\
&+\frac{\theta_{h-k}(a,d,b,e) + \theta_{h+k}(b,d,a,e)}{\{\gamma_{aa}(0)\gamma_{bb}(0)\gamma_{dd}(0)\gamma_{ee}(0)\}^{1/2}},
\end{aligned} \tag{7}$$

where k, $h \geq 0$.

Given (2), in the sequel (7) will be seen as a function of the $\gamma_{ij}(k)$'s and of the θ_k's. Let $\underline{r}$ be an arbitrary finite subset of the $r_{ij}(k)$'s stored in a vector and let $\underline{\Sigma}$ be its asymptotic covariance matrix whose elements are given by (7). That matrix is non-negative definite. Very often (see Section 4) statistical problems require an estimate of (7) since the $\gamma_{ij}(k)$'s (and consequently the θ_k's) are unknown.

There is no guarantee, however, that mere substitution of the $\gamma_{ij}(k)$'s by the $c_{ij}(k)$'s in (7) provides a consistent estimator, because of the infinite sums contained in (6). That problem also arises in spectral analysis (Priestley, 1981). The usual remedy consists in weighting appropriately the terms in (6). Robinson (1977) has proposed such a solution in the univariate case. However, his method does not necessarily lead to a non-negative definite estimator of $\underline{\Sigma}$.

In order to insure non-negative definiteness of $\hat{\underline{\Sigma}}$, Mélard and Roy (1984, 1987) have proposed, in the univariate case, to replace the $\gamma(k)$'s by weighted estimators $w(k)c(k)$. The main purpose of this paper is to generalize to the multivariate case the method of Mélard and Roy. The resulting estimator of $\underline{\Sigma}$ is consistent and non-negative definite.

The non-negative definiteness of $\hat{\underline{\Sigma}}$ is addressed in Section 2. The weighting scheme so that $\hat{\underline{\Sigma}}$ converges in L_1-norm to $\underline{\Sigma}$, hence in probability, is specified more completely in Section 3. Finally, several applications of the procedure are discussed in Section 4.

2. A non-negative definite estimator of the asymptotic covariance structure

We first prove

Theorem 2

Let

$$\hat{\theta}_k(i,j,l,m) = \sum_{u=-\infty}^{\infty} w(u)c_{ij}(u)w(u+k)c_{lm}(u+k), \qquad (8)$$

where $\{w(u), u \in \mathbb{Z}\}$ is a non-negative definite function. Then, under the assumptions of Theorem 1, the matrix $\hat{\underline{\Sigma}}$, obtained by replacing θ_k by $\hat{\theta}_k$ and $\gamma_{ij}(k)$ by $w(k)c_{ij}(k)$ in $\underline{\Sigma}$, is non-negative definite.

The proof rests on two lemmas.

Lemma 3

Let $\underline{\Gamma}(t,s)$ be a $p \times p$ matrix function on $\mathbb{Z} \times \mathbb{Z}$. It is the covariance function of a p-dimensional second order stochastic process $\{\underline{X}_t; t \in \mathbb{Z}\}$ if, and only if, it is of non-negative definite type.

Proof

The proof results as a generalization of Section 37 of Loève (1978). Without restriction to generality, it can be supposed that the mean vector is equal to $\underline{0}$. The proof of the "only if" part is completed as follows. For every finite subset S_m of $\mathbb{Z}$ and every set of $p \times 1$ vectors $\underline{a}_t = (a_{t1}, \ldots, a_{tp})^{\mathrm{T}}, t \in S_m$, we have

$$0 \le \mathrm{var}\left(\sum_{t \in S_m} \underline{a}_t^{\mathrm{T}} \underline{X}_t \right) = \mathrm{E}\left\{ \left(\sum_{t \in S_m} \underline{a}_t^{\mathrm{T}} \underline{X}_t \right) \left(\sum_{s \in S_m} \underline{a}_s^{\mathrm{T}} \underline{X}_s \right)^{\mathrm{T}} \right\}.$$

Hence :

$$\sum_{t,s \in S_m} \underline{a}_t^{\mathrm{T}} \underline{\Gamma}(t,s) \underline{a}_s \ge 0. \tag{9}$$

Conversely, suppose that $\underline{\Gamma}(t,s)$ satisfies (9). We shall build a Gaussian stochastic process such that

$$\mathrm{E}(\underline{X}_t \underline{X}_s^{\mathrm{T}}) = \underline{\Gamma}(t,s). \tag{10}$$

Let us consider

$$\exp\left\{ -\frac{1}{2} \sum_{t,s \in S_m} \underline{a}_t^{\mathrm{T}} \underline{\Gamma}(t,s) \underline{a}_s \right\}$$

as a function of the elements of the vectors $\underline{a}_t, t \in S_m$. By (9), this is the characteristic function of a normal distribution of pm zero-mean random variables. The normal laws of finite subsets of random variables among the elements of $\underline{X}_t$ for $t \in S_m$ are consistent since their law coincides with the marginal law obtained by setting $a_{ti} \equiv 0$ for the omitted corresponding random variables X_{ti} . Thus, by Kolmogorov's fundamental theorem on stochastic processes, there exists a Gaussian stochastic process $\{\underline{X}_t; t \in \mathbb{Z}\}$ which satisfies (10).

Lemma 4

Let $\{\underline{\gamma}(k); k \in \mathbb{Z}\}$ be the autocovariance matrix function of a zero mean second order stationary multivariate process $\{\underline{X}_t; t \in \mathbb{Z}\}$. Let $\{w(k); k \in \mathbb{Z}\}$ be a non-negative definite function on $\mathbb{Z}$. Then $\{w(k)\underline{\gamma}(k), k \in \mathbb{Z}\}$ is the autocovariance matrix function of a second order stationary multivariate process.

Proof

The process being second order stationary, we have $\underline{\Gamma}(t,s) = \underline{\gamma}(t-s)$. By the "only if" part of Lemma 3, for every finite subset S_m of $\mathbb{Z}$ and every set of $p\times 1$ vectors $\underline{a}_t$, $t \in S_m$, we have

$$\sum_{t,s \in S_m} \underline{a}_t^{\mathrm{T}} \underline{\gamma}(t-s) \underline{a}_s \geq 0.$$

By the "if" part of Lemma 3, $\{w(k); k \in \mathbb{Z}\}$ is the autocovariance function of a zero-mean second order stationary process $\{Y_t; t \in \mathbb{Z}\}$, that we may select to be independent of $\{\underline{X}_t; t \in \mathbb{Z}\}$. The autocovariance matrix function of $\{Y_t \underline{X}_t; t \in \mathbb{Z}\}$ is given by

$$\mathrm{E}\{Y_t \underline{X}_t \underline{X}_{t+k}^{\mathrm{T}} Y_{t+k}\} = \mathrm{E}\{Y_t Y_{t+k}\}\ \mathrm{E}\{\underline{X}_t \underline{X}_{t+k}^{\mathrm{T}}\} = w(k)\underline{\gamma}(k).$$

This completes the proof.

Proof of Theorem 2

Following arguments given by Loève for univariate processes, we show that the matrix function $\{\underline{C}(k), k \in \mathbb{Z}\}$ defined by (3) is of non-negative definite type . Let $\underline{Y}_t = \underline{X}_t - \overline{\underline{X}}, t = 1, \ldots, n$, and $\underline{Y}_t = \underline{0}$, otherwise. We have

$$\sum_{t,s \in S_m} \underline{a}_t^{\mathrm{T}} \underline{C}_{t-s} \underline{a}_s = \sum_{t,s \in S_m} \underline{a}_t^{\mathrm{T}} \left(\frac{1}{n} \sum_{r=-\infty}^{\infty} \underline{Y}_{r+s} \underline{Y}_{r+t}^{\mathrm{T}} \right) \underline{a}_s$$

$$= \frac{1}{n} \sum_{r=-\infty}^{\infty} \sum_{t,s \in S_m} \underline{a}_t^{\mathrm{T}} \underline{Y}_{r+s} \underline{Y}_{r+t}^{\mathrm{T}} \underline{a}_s$$

$$= \frac{1}{n} \sum_{r=-\infty}^{\infty} \left(\sum_{t \in S_m} \underline{a}_t^{\mathrm{T}} \underline{Y}_{r+t} \right)^2 \geq 0$$

for every subset S_m of $\mathbb{Z}$ and every set of $p \times 1$ vectors $\underline{a}_t = (a_{t1}, \ldots, a_{tp})^{\mathrm{T}}, t \in S_m$. Hence, by Lemma 4, $\{w(k)\underline{C}(k), k \in \mathbb{Z}\}$ is an autocovariance function.

The matrix $\underline{\Sigma}$ whose elements are given by (7), and the θ_k's by (6), is non-negative definite whatever the process, hence whatever the autocovariance matrix function $\underline{\Gamma}(k)$. Replacing θ_k by $\hat{\theta}_k$ and $\gamma_{ij}(k)$ by $w(k)c_{ij}(k)$ is equivalent to replacing $\underline{\Gamma}(k)$ by $w(k)\underline{C}(k)$, which proves the theorem.

3. A consistent estimator of the asymptotic covariance structure

Generalizing Mélard and Roy (1984), we point out a particular weighting scheme which ensures consistency. In (8), the sum reduces to a finite one and we use the notation $\sum_u^*$ to represent a sum for u going from $-n + 1$ to $n - 1 - k$. More precisely, we have

Theorem 5

Let

$$\hat{\theta}_k(i, j, l, m) = \sum_u{}^{*} w(ub_n) c_{ij}(u) w((u+k)b_n) c_{lm}(u+k),$$

where $w : \mathbb{R} \to \mathbb{R}$ has the following properties : w is continuous at the origin, such that $w(0) = 1$, is bounded, square integrable, and has at most a finite number of discontinuity points. The sequence of real numbers $\{b_n; n \geq 1\}$ is such that $b_n \to 0$ and $n b_n \to \infty$ as $n \to \infty$. Then under the assumptions of Theorem 1, $\hat{\theta}_k(i, j, l, m)$ converges to $\theta_k(i, j, l, m)$ in the L_1-norm sense.

We first prove

Lemma 6

Let

$$\tilde{\gamma}_{ij}(k) = \left(1 - \frac{|k|}{n}\right)\gamma_{ij}(k), \qquad |k| \leq n - 1. \tag{11}$$

Then,

$$\mathbf{E}[\{c_{ij}(k) - \tilde{\gamma}_{ij}(k)\}^2] = O\left(\frac{1}{n}\right),$$

uniformly in *k*.

Proof

The proof is based on arguments provided by Mélard and Roy (1984). Let

$$\tilde{c}_{ij}(k) = \frac{1}{n}\sum_{t=1}^{n-k} X_{it}X_{j,t+k} \tag{12}$$

so that

$$c_{ij}(k) - \tilde{c}_{ij}(k) = -\overline{X}_i\overline{X}_j\left(1 + \frac{k}{n}\right) + \frac{\overline{X}_i}{n}\sum_{t=1}^{k} X_{jt} + \frac{\overline{X}_j}{n}\sum_{t=n-k+1}^{n} X_{it}.$$

Hence, $\mathrm{E}[\{c_{ij}(k) - \tilde{c}_{ij}(k)\}^2]$ is bounded by a sum of terms of the form

$$\frac{1}{n^4}\sum_{t_1}\sum_{t_2}\sum_{t_3}\sum_{t_4}\mathbf{E}\{X_{i_1t_1}X_{i_2t_2}X_{i_3t_3}X_{i_4t_4}\} \tag{13}$$

where $i_1, \ldots, i_4 \in \{i, j\}$ and each sum is carried on some subset of $\{1, 2, ..., n\}$. By Isserlis formula (Priestley, 1981, vol. 1, p. 325), (13) is equal to a sum of three terms

$$\frac{1}{n^4}\sum_{t_1}\sum_{t_2}\sum_{t_3}\sum_{t_4}\{\gamma_{i_1i_2}(t_1-t_2)\gamma_{i_3i_4}(t_3-t_4)+\gamma_{i_1i_3}(t_1-t_3)\gamma_{i_2i_4}(t_2-t_4)$$
$$+\gamma_{i_1i_4}(t_1-t_4)\gamma_{i_2i_3}(t_2-t_3)\}. \qquad (14)$$

The first term of (14), for example, is bounded in absolute value by

$$\left\{\frac{1}{n^2}\sum_{u=-n+1}^{n-1}(n-|u|)\,|\gamma_{i_1i_2}(u)|\right\}.\left\{\frac{1}{n^2}\sum_{u=-n+1}^{n-1}(n-|u|)\,|\gamma_{i_3i_4}(u)|\right\}. \qquad (15)$$

Using Cauchy-Schwarz inequality, the first factor of (15) is itself bounded by

$$\frac{1}{n}\sum_{u=-n+1}^{n-1}|\gamma_{i_1i_2}(u)| \le \frac{1}{n}\left\{(2n-1)\sum_{-\infty}^{\infty}\gamma_{i_1i_2}^2(u)\right\}^{1/2}$$

which is $O(n^{-1/2})$ since the series in (6) converges, by hypothesis. Hence (15) is $O(n^{-1})$, and also (14) and (13), by similar arguments. The upper bounds do not depend on k and therefore the convergence is uniform in k.

On the other hand

$$\mathrm{E}[\{\tilde{c}_{ij}(k)-\tilde{\gamma}_{ij}(k)\}^2]=\frac{1}{n^2}\sum_{t_1=1}^{n-k}\sum_{t_2=1}^{n-k}\mathrm{E}\{X_{it_1}X_{j,t_1+k}X_{it_2}X_{j,t_2+k}\}-\tilde{\gamma}_{ij}^2(k)$$
$$\le\frac{1}{n}\sum_{u=-n+k+1}^{n-k-1}\{\gamma_{ij}(u+k)\gamma_{ij}(k-u)+\gamma_{ii}(u)\gamma_{jj}(u)\},$$

using once more Isserlis formula. Let us show that the last expression is $O(n^{-1})$ uniformly in k. By Cauchy-Schwarz inequality, the first sum is bounded by

$$\left|\sum_{u=-n+k+1}^{n-k-1}\gamma_{ij}(u+k)\gamma_{ij}(k-u)\right|\le\left\{\sum_{u=-\infty}^{\infty}\gamma_{ij}^2(u+k)\sum_{u=-\infty}^{\infty}\gamma_{ij}^2(k-u)\right\}^{1/2}=\theta_0(i,j,i,j).$$

A similar argument holds for the second term. The final result then follows from the inequality

$$\mathrm{E}[\{c_{ij}(k)-\tilde{\gamma}_{ij}(k)\}^2]\le 2(\mathrm{E}[\{c_{ij}(k)-\tilde{c}_{ij}(k)\}^2]+\mathrm{E}[\{\tilde{c}_{ij}(k)-\tilde{\gamma}_{ij}(k)\}^2]).$$

Proof of Theorem 5

We have to show that

$$\lim_{n\to\infty} \mathrm{E}\{|\,\hat{\theta}_k(i,j,l,m) - \theta_k(i,j,l,m)|\} = 0 \qquad (16)$$

for any k, $k \geq 0$. Denote $w_{nu} = w(ub_n)$. Following arguments from Robinson (1977) and adapted by Mélard and Roy (1984), we decompose the difference in (16) as

$$\sum\nolimits_u^* w_{nu} w_{n,u+k}[\{c_{ij}(u) - \tilde{\gamma}_{ij}(u)\}\, c_{lm}(u+k) + \{c_{lm}(u+k) - \tilde{\gamma}_{lm}(u+k)\}\,\tilde{\gamma}_{ij}(u)]$$
$$+\sum\nolimits_u^*[\{w_{nu} w_{n,u+k} - 1\}\,\tilde{\gamma}_{ij}(u)\tilde{\gamma}_{lm}(u+k)] + \sum\nolimits_u^*[\tilde{\gamma}_{ij}(u)\tilde{\gamma}_{lm}(u+k)] - \theta_k(i,j,l,m).$$

Cauchy-Schwarz inequality implies that

$$\mathrm{E}\{|\,\hat{\theta}_k(i,j,l,m) - \theta_k(i,j,l,m)|\} \leq \{\mathrm{E}(e_1)\mathbf{E}(e_2)\}^{1/2} + \{e_3\,\mathrm{E}(e_4)\}^{1/2} + \{e_5 e_6\}^{1/2} + |\,e_7|\,, \qquad (17)$$

where

$$e_1 = \sum\nolimits_u^* w_{nu}^2\{c_{ij}(u) - \tilde{\gamma}_{ij}(u)\}^2, \qquad e_2 = \sum\nolimits_u^* w_{n,u+k}^2 c_{lm}^2(u+k),$$

$$e_3 = \sum\nolimits_u^* w_{nu}^2 \tilde{\gamma}_{ij}^2(u), \qquad e_4 = \sum\nolimits_u^* w_{n,u+k}^2\{c_{lm}(u+k) - \tilde{\gamma}_{lm}(u+k)\}^2,$$

$$e_5 = \sum\nolimits_u^* |\,w_{nu} w_{n,u+k} - 1|\,\tilde{\gamma}_{ij}^2(u), \qquad e_6 = \sum\nolimits_u^* |\,w_{nu} w_{n,u+k} - 1|\,\tilde{\gamma}_{lm}^2(u+k),$$

$$e_7 = \sum\nolimits_u^*[\tilde{\gamma}_{ij}(u)\tilde{\gamma}_{lm}(u+k)] - \theta_k(i,j,l,m).$$

It can be shown that each term in the left-hand side of (17) goes to zero as $n \to \infty$. By Lemma 6, we have : $\mathrm{E}\{e_1\} \leq 0(n^{-1})\sum\nolimits_u^* w_{nu}^2 \leq O(n^{-1}b_n^{-1})\sum\nolimits_u^* w_{nu}^2 b_n$. The limit of the last sum is $\int_{-\infty}^{\infty} w^2(z)dz$ which is finite; hence $\mathbf{E}\{e_1\} \to 0$. Similarly, we have $\mathbf{E}\{e_4\} \to 0$. The term e_3 is bounded by

$$\left(\sup_{z\in\Re} w^2(z)\right)\sum\nolimits_u^* \tilde{\gamma}_{ij}^2(u) \leq \left(\sup_{z\in\Re} w^2(z)\right)\theta_0(i,j,i,j) < \infty.$$

Further, we have

$$\mathrm{E}\{e_2\} \leq 2\sum\nolimits^* w_{n,u+k}^2[\mathrm{E}\{(c_{lm}(u+k) - \tilde{\gamma}_{lm}(u+k))^2\} + \tilde{\gamma}_{lm}^2(u+k)] \leq 2(\mathrm{E}\{e_4\} + e_3^*) < \infty$$

where e_3^* is a term similar to e_3.

Let r be a fixed integer and write $e_5 = s_1 + s_2$ where s_1 is the sum of the terms of e_5 from $-r$ to r and s_2 the sum of the terms from $-n + 1$ to $-r - 1$ and from $r + 1$ to $n - 1 - k$. For $|u| \leq r$, ub_n and $(u+k)b_n$ tend to 0. Then, by continuity of w and since $w(0) = 1$, $s_1 \to 0$ as $n \to \infty$. The term s_2 is bounded by

$$\left(\sup_{z \in \Re} |w(z)| + 1\right)^2 \left(\sum_{u=-\infty}^{-r-1} \gamma_{ij}^2(u) + \sum_{u=r+1}^{\infty} \gamma_{ij}^2(u)\right)$$

where the two sums can be made arbitrarily small by choosing r sufficiently large, since $\theta_0(i,j,i,j)$ is finite. Hence $s_2 \to 0$ and $e_5 \to 0$. The proof that $e_6 \to 0$ is similar. To show that e_7 tends to zero, we will make use of the following classical result on convergent series. For any sequence of real or complex numbers $\{a_u; u \in \mathbb{Z}\}$, if $\sum_{u=-\infty}^{\infty} a_u$ converges, then

$$\lim_{n \to \infty} \sum_{u=1}^{n} \left(1 - \frac{u}{n}\right) a_u = \sum_{u=-\infty}^{\infty} a_u. \tag{18}$$

The term $|e_7|$ is bounded by :

$$\left| \sum\nolimits_u^* \tilde{\gamma}_{ij}(u) \tilde{\gamma}_{lm}(u+k) - \sum\nolimits_u^* \tilde{\gamma}_{ij}(u) \gamma_{lm}(u+k) \right| +$$

$$\left| \sum\nolimits_u^* \tilde{\gamma}_{ij}(u) \gamma_{lm}(u+k) - \sum_{u=-\infty}^{\infty} \gamma_{ij}(u) \gamma_{lm}(u+k) \right|.$$

Then, by (18), we have :

$$\lim_{n \to \infty} \sum\nolimits_u^* \tilde{\gamma}_{ij}(u) \gamma_{lm}(u+k) = \sum_{u=-\infty}^{\infty} \gamma_{ij}(u) \gamma_{lm}(u+k)$$

and

$$\lim_{n \to \infty} \sum\nolimits_u^* \tilde{\gamma}_{ij}(u) \tilde{\gamma}_{lm}(u+k) = \lim_{n \to \infty} \sum\nolimits_u^* \tilde{\gamma}_{ij}(u) \gamma_{lm}(u+k) = \sum_{-\infty}^{\infty} \gamma_{ij}(u) \gamma_{lm}(u+k).$$

Thus $\lim_{n \to \infty} |e_7| = 0$ and the proof is completed.

The result of the present section and Theorem 2 can be summarized as follows.

Theorem 7

Under the assumptions of Theorems 1 and 5, suppose that the function w is non-negative, and let the matrix $\hat{\underline{\Sigma}}$ obtained by replacing in $\underline{\Sigma}$, θ_k by $\hat{\theta}_k$ of Theorem 5 and $\gamma_{ij}(k)$ by $w(kb_n)c_{ij}(k)$. Then $\hat{\underline{\Sigma}}$ is (i) a non-negative definite, and (ii) a consistent estimator of $\underline{\Sigma}$.

Proof

For each n, the sequence $\{w_{nk} = w(kb_n), k\in \mathbb{Z}\}$ is non-negative definite and Part (i) follows from Theorem 2. Also, part (ii) follows directly from Theorem 5 and the fact that convergence in L_1-norm implies convergence in probability.

4. Applications

This Section is devoted to several useful applications of the results of the paper.

4.1 The corner method

The corner method (Beguin et al., 1980) is one of the methods used for specifying univariate autoregressive moving average (ARMA) models (e.g. de Gooijer et al., 1986). It has been extended partially to cover transfer function models (Hanssens and Liu, 1982). Let us consider the bivariate case. In order to identify the transfer function relating X_2 in function of X_1, we consider, for $i\geq 0$ and $j\geq 1$ the determinant $\hat{\Delta}(i,j)$ of the Toeplitz matrix whose (r,s)-th element is $r_{12}(i + r - s)$, which is the estimator of $\Delta(i,j)$ built similarly with $\rho_{12}(i + j - 1)$. Paesmans (1988) proved a corner characterization theorem for transfer function models, generalizing the theorem in Beguin et al. (1980), so that a corner of zeros in the Δ array identifies the orders of the polynomials in the numerator and denominator of the tranfer function.

For ARMA models, Mareschal and Mélard (1988) developed a fast algorithm for the consistent estimation of the asymptotic standard error of $\hat{\Delta}(i,j)$, based on results of Mélard and Roy (1987), improving thereby the potentially nonconsistent estimator of Beguin et al. (1980). That algorithm can be generalized (Paesmans 1988) to transfer function models by noting that, as $n \to \infty$:

$$\mathrm{var}(\hat{\Delta}(i,j)) = \underline{d}^{\mathrm{T}}(i,j)\underline{\Sigma}\underline{d}(i,j), \tag{19}$$

where $\underline{d}(i,j)$ is the derivative of $\Delta(i,j)$ with respect to the vector $\underline{\rho} = (\rho_{12}(0),...,\rho_{12}(K))^{\mathrm{T}}$, $K = i + j - 1$, and $\underline{\Sigma}$ is the asymptotic covariance matrix of the corresponding vector $\underline{r}$, whose elements are given by (7). The use of the method of this paper to obtain a consistent nonnegative definite estimator $\hat{\underline{\Sigma}}$ of $\underline{\Sigma}$ ensures that the estimate of (19) will never be negative. Note that the approach of Hanssens and Liu (1982) lacks a statistical yardstick for comparing the determinants $\hat{\Delta}(i,j)$ to zero.

4.2 Other specification methods

The last remark can be extended to a large number of specification or identification methods in time series modeling. Among the few methods which involve a statistical appraisal of the identification statistics, the unified approach based on the extended sample autocorrelation function (Tsay and Tiao 1984) is probably the most powerful in the sense that it can cope with nonstationarity. A crude approximation of n^{-1} is however used as an approximation of Bartlett's formula, although a study is said to be needed on the subject. Jeon and Park (1986) go a little further in this respect but restrict themselves to Bartlett's formula for moving average models. The results of Mélard and Roy (1984) can be exploited in these contexts to provide more accurate significance limits.

For multivariate series, the most usual procedure in applied work consists in displaying cross-correlation matrices in terms of indicator symbols (Tiao and Box, 1981): a plus sign is used to indicate a value greater than $2n^{-1/2}$, a minus sign a value less than $-2n^{-1/2}$ and a dot to indicate a value between $-2n^{-1/2}$ and $2n^{-1/2}$. Athough the authors do not interpret these indicator symbols in the sense of formal significance tests, standard error estimates better suited than $n^{-1/2}$ can be a bonus. Davies et al. (1985) described a procedure based on numerical integration for the computation of the exact moments of the sample correlation matrix at lag k, for moving average processes only. Mélard and Roy (1987) have argued, in the univariate case, in favor of model-free standard errors which can be used to implement confidence regions and tests for one or several lags. The latter approach can easily be extended to the multivariate case, using the results of this paper.

4.3 Vector autocorrelation

By analogy to Escoufier (1973) who introduced a coefficient of correlation between two random vectors, Cléroux and Roy (1987) proposed the following coefficient of vector autocorrelation at lag k for a multivariate second order stationary process:

$$\lambda(k) = \frac{\mathrm{tr}\{\underline{\Gamma}(k)\underline{\Gamma}(k)^{\mathrm{T}}\}}{\mathrm{tr}\{\underline{\Gamma}(0)^2\}}, \quad k \in Z, \tag{20}$$

where tr{•} denotes the trace operator. A consistent estimator of $\lambda(k)$ is given by

$$\hat{\lambda}(k) = \frac{\mathrm{tr}\{\underline{C}(k)\underline{C}(k)^{\mathrm{T}}\}}{\mathrm{tr}\{\underline{C}(0)^2\}}, \quad |k| < n. \tag{21}$$

The coefficients of vector autocorrelation provide a much simpler image of the correlation structure of a multivariate time series than the correlation matrices $\underline{R}(k)$. They can also serve to test the hypothesis of white noise and to detect nonzero lag autocorrelations.

However, the coefficients defined by (20) and (21) are not fully scale invariant. Alternate coefficients of vector autocorrelation $\delta(k)$ and $\hat{\delta}(k)$ that are scale invariant are obtained by replacing in (20) and (21) $\underline{\Gamma}(k)$ and $\underline{C}(k)$ by $\underline{\rho}(k)$ and $\underline{R}(k)$, respectively. A test for $\underline{\rho}(k) = \underline{0}$ ($\delta(k) = 0$) requires a non-negative definite and consistent estimator of the asymptotic covariance structure of the elements of $\underline{R}(k)$. Similarly, when testing the hypothesis of white noise against autocorrelation at a given lag, the power of the test based on $\hat{\delta}(k)$ can be evaluated as long as we have a suitable estimator of the asymptotic covariance structure of the elements of $\underline{R}(k)$ and $\underline{R}(0)$.

4.4 Test of homogeneity

Mélard and Roy (1984) have already considered testing the equality of the autocovariances of two univariate time series, using a quadratic form statistic of the form

$$\underline{z}^{\mathrm{T}}\underline{\hat{\Sigma}}^{-1}\underline{z},$$

where $\underline{z}$ is the difference between the vectors $\underline{c}_j = (c(0), c(1), \ldots, c(K))^{\mathrm{T}}$ for j = 1, 2, and $\underline{\hat{\Sigma}}$ is a pooled estimator of the asymptotic covariance matrices of both vectors. A test of homogeneity and stability was derived from that work (Mélard and Roy, 1983). The method has been extended to a sequential procedure for testing the homogeneity across time of a sequence of independent time series (Barone and Mélard, 1989). Using the results of this paper, the previous method has been extended to testing the equality of the serial correlations (instead of covariances) of multivariate (instead of univariate) time series.

4.5 Test of independence

Haugh (1976) proposed a procedure to test the independence between two time series X_{1t} and X_{2t}. First, the individual series are filtered with appropriate univariate ARMA models, leading to residual series u_{1t} and u_{2t}, respectively. Second, the $(2K + 1)$ residual cross-correlations $\tilde{r}_{12}(k)$, $k = -K, -K + 1, ..., K$, are computed. To test $\rho_{12}(k) = 0$ for all k, the following test statistic is used:

$$n \sum_{k=-K}^{K} \tilde{r}_{12}^{2}(k) \cdot$$

An alternative procedure aimed at being more powerful for certain alternatives was proposed by Koch and Yang (1986). Like the method of Haugh, it can suffer from the approximation induced by preliminary filtering. Using the method described in this paper, the filtering step can be avoided. Indeed, let us consider the statistic

$$Q = n\underline{r}^{\mathrm{T}}\underline{\hat{\Sigma}}^{-1}\underline{r}, \qquad (22)$$

where the vector $\underline{r}$ is composed of $r_{12}(-K)$, $r_{12}(-K + 1)$, ..., $r_{12}(K)$. Under the null hypothesis, Q is distributed asymptotically as a chi square random variable with $2K + 1$ degrees of freedom.

The method can be further extended to testing the independence between two sets of variables. Let $\underline{X}_t$ be partitioned into $\underline{Y}_t$ and $\underline{Z}_t$, with q and p - q variables respectively. To test the null hypothesis $\underline{\rho}_{YZ}(k) = \underline{0}$ for all k, we consider the $q(p - q)\times 1$ vector $\underline{r}(k) = \text{vec}\ \underline{r}_{YZ}(k)$ and the vector $\underline{r}$ composed of $\underline{r}(-K)$, $\underline{r}(-K + 1)$, ..., $\underline{r}(K)$. The test statistic has then the same form as (22), but the number of degrees of freedom is now $q(p - q)(2K + 1)$. Note that, following Mareschal and Mélard (1988), it is recommended to use a pseudo-inversion algorithm like the one of Healy (1968), in order to avoid numerical problems due to near singularity of the covariance matrix, in the case of highly autocorrelated time series. The number of degrees of freedom is then adjusted appropriately.

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