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On Time-Dependent Spectral Concepts

by
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ABSTRACT

The evolutionary spectrum theory is re-examined and generalized. First, a more general definition of a spectral density matrix is given for a multivariate (purely non-deterministic) nonstationary process. On the basis of that definition, the coherence between two components is time-dependent like the other spectral functions. Several examples show that these functions can be easily computed. Statistical estimation of the spectral density matrix is considered. In particular a new upper bound is given for the bias of the estimators and a simple approximation for the variance. A comparison with the evolutionary spectrum theory is performed. An illustration based on artificial data shows that the spectral functions can be estimated.

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ON TIME-DEPENDENT SPECTRAL CONCEPTS

by

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1. INTRODUCTION

This paper consists in a re-examination of the evolutionary spectrum theory worked out by Priestley (1965, 1966) and his colleagues (Priestley and Tong, 1973, Subba Rao and Tong, 1972, and several other papers). Two aspects are considered : the class of processes under study and the properties of the estimators.

Spectral analysis for nonstationary processes can be used in a large variety of domains : speech and vibrations, meteorology, oceanography, geophysics, economics, demography, ecology and others. The field of application of the present paper is becoming larger every day because of the huge data bases maintained by firms and public organizations and because of the enormous amount of data collected by cheap microprocessors (fluids, energy, pollution, weather, health, e.g. EEG and ECG recordings). The time series data in certain domains are now longer and more reliable. The assumption of stationarity has been previously required because of the short length of the series and the low computing capability. In the time domain, efficient recursive estimation is now routine (e.g. Ljung and Soderström, 1983). In the frequency domain an empirical way of dealing with spectral analysis of nonstationary processes has been advocated by Granger and Hatanaka (1965), Bendat and Piersol (1966), Brillinger and Hatanaka (1968), Otnes and Enochson (1972), Harper et al (1974) and others. There is however a need for a more rigorous treatment of nonstationarity by first defining time-dependent spectral concepts, and then estimating the spectral functions from the data. This has been the major objective of Priestley (1965) and his followers. The resulting evolutionary spectrum theory is nicely summarized in Priestley (1981, chapter 11) where the reader can also find references to some other approaches. Grenier (1983) and Kitagawa and Gersch (1985) propose alternative approaches and provide recent references. Martin (1981) has extended Priestley's results to discrete spectra.

In the evolutionary spectrum theory, the attention is restricted to a class of processes called oscillatory processes. Such a process admits a multitude of oscillatory representations ("family of oscillatory functions" in the original terminology). To each of these representations corresponds a spectrum. The spectral estimates are obtained by complex demodulation and are supposed to approach

more closely the spectrum associated to a so-called natural representation. The definition of natural representation relies on taking a supremum over the multitude of representations. That task has never been undertaken except by Mélard (1985), on a process which is non zero at only two points. In this paper, instead of considering a multitude of representations, we shall restrict ourselves to the one that can be obtained most easily : the innovation or Wold-Cramér representation , as Mélard (1975), Tjøstheim (1976) and others have done in the univariate case. We shall also consider multivariate processes. A feature of the multivariate evolutionary spectrum theory is that the coherence of a bivariate process does not vary with time. The other spectral functions are time-dependent, not the coherence. That problem will be solved by using a wider class of processes than the one of Priestley and Tong (1975). Priestley (1965) assumes the existence of and some properties for a certain Fourier transform. We shall make the same assumption in order to guarantee the physical interpretation of the spectrum and the concept of frequency but other conditions can perhaps be stated. The conditions imposed by Priestley imply that the variance of the process is bounded.

Once the various spectral functions have been properly defined, there remains the statistical problem of estimating them at some frequencies and at some time points, by means of a single time series. Hence the properties of the estimators need to be studied. The method of complex demodulation is known to provide, in an efficient way, approximate estimates of time-dependent spectral density functions. We shall derive similar results within the more general framework stated before and using different data windows for each component of the process. Two causes of bias can arise : 1° the bias due to the frequency leakage, already present in the spectral estimation of stationary processes (e.g. Priestley 1981, chapter 7); 2° the bias which is related to nonstationarity of the process. It is for the latter that we propose some new bounds that are compared to those provided by the evolutionary spectrum theory. The innovation is important for the following reason : in order to be able to derive some bounds, Priestley (1965) restricted the class of processes to define what he calls the characteristic width of the process. The definition makes use of the knowledge of the process from $-\infty$ to ∞. On the contrary our bounds only rely on the local behaviour of the process.

The paper is organized as follows. In Section 2 we shall provide definitions for the population spectral functions called evolutive spectra. The evolutive spectral density is simply the evolutionary spectral density associated with the Wold-Cramér decomposition of the process. These definitions will be illustrated on several examples in Section 3 : the non-stationary white noise, the univariate time-dependent ARMA process, the time-varying regression model and a bivariate time-dependent ARMA(1,1) process. The statistical problem is dealt with in Section 4. A comparison with the evolutionary spectrum theory on some selected points is the object of Section 5. An artificial example adapted from Kiehm and Melard (1982) will be treated in Section 6. It will show that the concepts defined in this paper can be effectively used. Examples on real data sets would not illustrate our argument as well. The interested reader is referred to Grenier (1983) and Kitagawa and Gersch (1985) for other examples. It should be emphasized that, except at the level of some basic definitions, the whole evolutionary spectrum theory remains unchanged. Consequently, all the results obtained are still valid under the alternative assumptions proposed here, with some minor modifications.

2. NOTATIONS AND DEFINITIONS

Let $(\underline{Z}_t;\ t \in \mathbb{Z})$ be a (non-stationary) p-variate second-order, purely non-deterministic stochastic process with (constant) maximum rank. Without restriction to generality, we suppose that $E(\underline{Z}_t) = \underline{0}$. The process can be considered in the Hilbert space spanned by all its components $Z_{i,t}$, for all t. Let H_t^Z be the subspace spanned by all the components up to time t, and P_t^Z the orthogonal projection operator onto H_t^Z. The innovation $\underline{\tilde{\zeta}}_t$ is defined by $\underline{\tilde{\zeta}}_t = \underline{Z}_t - P_{t-1}^Z(\underline{Z}_t)$. It is known (Cramér, 1961) that all pairs of components of $\underline{\tilde{\zeta}}_t$ and $\underline{\tilde{\zeta}}_s$, $s \neq t$, are orthogonal. Let $\underline{S}_t$ be the covariance matrix of $\underline{\tilde{\zeta}}_t$. There exists a one-sided infinite moving average representation or Wold-Cramér decomposition of the process

$$\underline{Z}_t = \sum_{j=0}^{\infty} \underline{\tilde{\psi}}_{tj}\, \underline{\tilde{\zeta}}_{t-j} \qquad (1)$$

whose convergence in the mean is guaranteed, implying that

$$\sum_{j=0}^{\infty} \underline{\tilde{\psi}}_{tj}\, \underline{S}_{t-j}\, \underline{\tilde{\psi}}_{tj}^{T} < \infty$$

(^{T} denotes transposition, * denotes conjugate transposition).
It is assumed that $\underline{S}_t$ has maximum rank for each t. We consider the unique positive definite square root of matrix $\underline{S}_t$, $\underline{S}_t^{1/2}$, and define the normalized innovations $\underline{\zeta}_t = \underline{S}_t^{-1/2}\, \underline{\tilde{\zeta}}_t$ so that (1) can be rewritten as

$$\underline{Z}_t = \sum_{j=0}^{\infty} \underline{\psi}_{tj}\, \underline{\zeta}_{t-j} \quad \text{with} \quad \underline{\psi}_{tj} = \underline{\tilde{\psi}}_{tj}\, \underline{S}_{t-j}^{1/2}\ , \qquad (2)$$

where

$$\sum_{j=0}^{\infty} \underline{\psi}_{tj}\, \underline{\psi}_{tj}^{T} < \infty. \qquad (3)$$

Note that $\underline{\zeta}_t$, the orthonormal principal components of $\underline{\tilde{\zeta}}_t$, is a p-variate (stationary) white noise process with the identity matrix $\underline{I}$ as covariance matrix. Because of the assumption made on $\underline{S}_t$, which slightly restricts the class of processes under consideration, the representations (1) and (2) are uniquely determined. They will

be used to yield a unique definition of spectral functions associated with the p-variate process, after some other concepts are recalled.

Let $(\underline{\tilde{Z}}(B);\ B \in \mathscr{B})$, where $\mathscr{B}$ is the Borel class on $]-\pi, \pi]$, be a p-variate second order random complex measure (Rozanov, 1967) and denote $E\left\{\underline{\tilde{Z}}(B_1)\ \underline{\tilde{Z}}^*(B_2)\right\} = \underline{\mu}(B_1 \cap B_2)$ its pxp covariance matrix, with $B_1, B_2 \in \mathscr{B}$. If we assume that each measure μ_{ij} of $\underline{\mu}$ is absolutely continuous (with respect to the Lebesgue measure), we simply write

$$E\left\{\underline{\tilde{Z}}(d\omega)\ \underline{\tilde{Z}}^*(d\omega)\right\} = \underline{f}(\omega)\ d\omega. \tag{4}$$

It is well known that the stochastic integral representation

$$\underline{Z}_t = \int_{-\pi}^{\pi} e^{i\omega t}\ \underline{\tilde{Z}}(d\omega)$$

provides the suitable definitions and interpretations of spectral concepts for second-order stationary processes. Assuming absolute continuity of $\underline{\mu}$, $\underline{f}(\omega)$ is then the spectral density matrix of process $\underline{Z}_t$.

Similarly, the following representation

$$\underline{Z}_t = \int_{-\pi}^{\pi} \underline{\phi}_t(\omega)\ \underline{z}(d\omega) \tag{5}$$

(Karhunen, 1947), where the elements $\phi_t^{ij}(\omega)$ of $\underline{\phi}_t(\omega)$ are square-integrable with respect to μ_{jj}, has been used in several attempts to extend spectral functions to non-stationary processes. However (5) is not unique without assumptions on $\underline{\phi}_t(\omega)$. A simple set of restrictions will now be proposed in order to extend the spectral concepts.

The normalized innovation process $\underline{\varepsilon}_t$ admits a stochastic integral representation

$$\underline{\varepsilon}_t = \int_{-\pi}^{\pi} e^{i\omega t}\ \underline{z}(d\omega) \tag{6}$$

where $E\left\{\underline{z}(d\omega)\ \underline{z}^*(d\omega)\right\} = (2\pi)^{-1}\ \underline{I}$. Consequently, by putting (6) in (2), we obtain

$$\underline{Z}_t = \int_{-\pi}^{\pi} e^{i\omega t} \underline{A}_t(\omega) \underline{z}(d\omega) \tag{7}$$

where all the elements of

$$\underline{A}_t(\omega) = \sum_{j=0}^{\infty} e^{-i\omega j} \underline{\psi}_{tj} \tag{8}$$

are square-integrable with respect to the Lebesgue measure by (3) and the Riesz-Fisher theorem. The covariance function of the process is thus represented by

$$E\left\{\underline{Z}_t \underline{Z}_s^T\right\} = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i\omega(t-s)} \underline{A}_t(\omega) \underline{A}_s^*(\omega) \, d\omega, \tag{9}$$

which justifies, by analogy to the stationary case, the following definitions :

- The Hermitian positive definite matrix

$$\underline{f}_t(\omega) = (2\pi)^{-1} \underline{A}_t(\omega) \underline{A}_t^*(\omega) \tag{10}$$

is called the evolutive spectral density function provided that some condition on $\underline{A}_t(\omega)$ are satisfied. In some sense, this is a special case of Priestley's evolutionary spectral density matrix in the case of univariate processes. This point will be made clearer in Section 5. In the univariate case, the definition (10) was proposed independently by Tjøstheim (1976) and Mélard (1975) (see the reference in Huyberechts, 1975). However, it had also appeared earlier in a less precise form in Granger and Hatanaka (1974), Miller (1969), Meyer (1972), Subba Rao and Tong (1974), Bowden (1975). In the multivariate case, the defintion is due to De Schutter-Herteleer (1976, 1977).

The dummy variable ω will sometimes be called a frequency, by an abuse of language. Because

$$E\left\{\underline{Z}_t \underline{Z}_t^T\right\} = (2\pi)^{-1} \int_{-\pi}^{\pi} \underline{A}_t(\omega) \underline{A}_t^*(\omega) \, d\omega, \tag{11}$$

$\underline{f}_t(\omega) \, d\omega$ reflects the infinitesimal contribution of ω to the covariance function of the process at time t.

- The j^{th} diagonal element of $\underline{f}_t(\omega)$, $f_t^j(\omega)$, is the (real) evolutive spectral density of the component Z_{tj} in the multivariate process $\underline{Z}_t$.

- The (j,k)th element ($j \neq k$) of $\underline{f}_t(\omega)$, $f_t^{jk}(\omega)$, is the (complex) cross-spectrum between components Z_{tj} and Z_{tk} in the multivariate process $\underline{Z}_t$; furthermore $f_t^{jk}(\omega) = \left\{f_t^{kj}(\omega)\right\}^*$. We denote $c_t^{jk}(\omega) = \mathscr{R}e\left\{f_t^{jk}(\omega)\right\}$ and $q_t^{jk}(\omega) = \mathscr{I}m\left\{f_t^{jk}(\omega)\right\}$.

- $C_t^{jk}(\omega) = \dfrac{f_t^{jk}(\omega)}{\left\{f_t^j(\omega)\ f_t^k(\omega)\right\}^{1/2}}$ is the complex coherence between components Z_{tj} and Z_{tk}

- $K_t^{jk}(\omega) = \dfrac{|f_t^{jk}(\omega)|^2}{f_t^j(\omega)\ f_t^k(\omega)}$ (12)

  is the (square-modulus) coherence between components Z_{tj} and Z_{tk}. It can be interpreted as a squared correlation coefficient between Z_{tj} and Z_{tk} at ω.

- $R_t^{jk}(\omega) = \dfrac{|f_t^{jk}(\omega)|}{f_t^k(\omega)}$ is the gain of Z_{tj} with respect to Z_{tk}. It can be interpreted as the regression coefficient of Z_{tj} with respect to Z_{tk} at the point (t,ω).

- $\varphi_t^{jk}(\omega) = \tan^{-1}\left\{q_t^{jk}(\omega)/c_t^{jk}(\omega)\right\}$ is the phase of Z_{tj} with respect to Z_{tk}. It can be interpreted as a delay, around time t, between the infinitesimal contributions of Z_{tj} and Z_{tk} at ω, expressed in the same unit as ω.

The properties of these spectral functions (Loynes, 1968) can be studied in the same way as Mélard (1978) for the univariate case. Furthermore $|C_t^{jk}(\omega)| \leq 1$ and $0 \leq K_t^{jk}(\omega) \leq 1$. Three features of these definitions must be pointed out :

1° contrarily to the stationary case, ω has in the strict sense no physical interpretation as a frequency. If suitable conditions on $A_t(\omega)$ imply that it is a slowly varying function of time, such as those of Priestley (1965) or alternative conditions, an approximate interpretation of ω as a dominant frequency can be entertained, see the discussion in Priestley (1965, p. 207). We shall always assume that these conditions are fulfilled.

2° $\underline{f}_t(\omega)$ is not the Fourier transform of anything making sense; recall that the covariance function (9) depends on t and s, not just t-s;

3° all the spectral functions are defined with respect to the multivariate process; for instance $f_t^j(\omega)$ does not coincide with the evolutive spectral density of the univariate process $(Z_{tj};\ t \in Z)$, except in special cases (see De Schutter-Herteleer, 1976).

Obviously, the fact that these properties hold in the stationary case is a consequence of the simplifying assumption of stationarity.

That procedure can be reproached with entirely relying upon a time-domain representation. However, using results from Hallin (1984, 1986) for pure moving average processes and from Hallin and Ingenbleek (1983) for pure autoregressive processes, it is possible to derive such a representation simply from the knowledge of the covariance function.

3. EXAMPLES

The examples shown in this Section will prove that the theory of Section 2 is extremely simple to apply.

Example 1 - The non-stationary white noise process

Let $(X_t;\ t \in \mathbb{Z})$ be a sequence of uncorrelated random variables with mean 0 and variance 1, and $(g_t;\ t \in \mathbb{Z})$ a finite positive non-random function of time. Then the process $(Z_t = g_t X_t;\ t \in \mathbb{Z})$ is a purely non-deterministic process with evolutive spectral density $f_t(\omega) = g_t^2/(2\pi)$, by direct application of the definition in Section 2.

Example 2 - The evolutive ARMA process

Let $(Z_t;\ t \in \mathbb{Z})$ be a purely non-deterministic process with innovations $\theta_{to}\varepsilon_t$, where $\mathrm{var}(\varepsilon_t) = 1$, satisfying the equation

$$Z_t = \sum_{j=1}^{p} \phi_{tj} Z_{t-j} + \theta_{to}\varepsilon_t - \sum_{j=1}^{q} \theta_{tj}\varepsilon_{t-j}.$$

A necessary condition on the coefficients can be found (Hallin, 1978, Mélard 1985a) in order that the process be purely non-deterministic. Contrary to what is frequently stated (Subba Rao, 1970 p. 313 and more recently Grenier, 1983 p. 908) the evolutive spectral density is not

$$\frac{1}{2\pi}\left|\frac{\theta_{to} - \sum_{j=1}^{q} \theta_{tj} e^{i\omega j}}{1 - \sum_{j=1}^{p} \phi_{tj} e^{i\omega j}}\right|^2 .$$

For instance, for the first order autoregressive process

$$Z_t = \phi_t Z_{t-1} + \theta_t \varepsilon_t$$

we have the following Wold-Cramér decomposition

$$Z_t = \theta_t \varepsilon_t + \phi_t \theta_{t-1} \varepsilon_{t-1} + \phi_t \phi_{t-1} \theta_{t-2} \varepsilon_{t-2} + \cdots$$

so that

$$f_t(\omega) = (2\pi)^{-1} \left| \theta_t + \phi_t \theta_{t-1} e^{i\omega} + \phi_t \phi_{t-1} \theta_{t-2} e^{2i\omega} + \ldots \right|^2$$

instead of

$$\frac{1}{2\pi} \frac{\theta_t^2}{|1 - \phi_t e^{i\omega}|^2} .$$

With the latter formula, (11) is not fulfilled since

$$E(Z_t^2) = \sum_{j=0}^{\infty} \left\{ \left(\prod_{k=0}^{j-1} \phi_{t-k} \right) \theta_{t-j} \right\}^2 .$$

Note also that the necessary condition mentioned above is

$$\sum_{j=1}^{\infty} (\phi_t \phi_{t-1} \cdots \phi_{t-j+1})^2 \theta_{to}^2 < \infty$$

and that the condition that all $|\phi_t| < 1$ in the case where $\theta_{to} = 1$ (Abdrabbo and Priestley, 1967) is too restrictive.

Example 3 - The simple regression model with time-varying regression coefficient

We consider the model $Y_t = \beta_t X_t + d_t \varepsilon_t$ where $(\varepsilon_t; t \in \mathbb{Z})$ is a sequence of uncorrelated random variables with mean 0 and variance 1, $(X_t; t \in \mathbb{Z})$ is a purely non-deterministic process with normalized innovations ε_t and Wold-Cramér decomposition

$$X_t = \sum_{j=0}^{\infty} \psi_{tj} \varepsilon_{t-j} .$$

We assume also that X_t is uncorrelated with ε_s, for all $t, s \in \mathbb{Z}$ and that d_t and β_t are finite non-random functions of time. We want to study the bivariate process $(\underline{Z}_t; t \in \mathbb{Z})$ with $\underline{Z}_t = (X_t, Y_t)^T$. Using the notations of Section 2 and the direct sum operator $\oplus$, we have $H_t^{\underline{Z}} = H_t^X \oplus H_t^\varepsilon$ and $P_{t-1}^{\underline{Z}}(X_t) = P_{t-1}^X(X_t)$ and

$P^{Z}_{t-1}(Y_t) = \beta_t P^{X}_{t-1}(X_t)$. Hence

$$\tilde{\zeta}_t = \begin{pmatrix} \psi_{to}\, \xi_t \\ \beta_t\, \psi_{to}\, \xi_t + d_t\, \varepsilon_t \end{pmatrix}, \qquad S_t = \begin{pmatrix} \psi_{to}^2 & \beta_t\, \psi_{to}^2 \\ \beta_t\, \psi_{to}^2 & \beta_t^2\, \psi_{to}^2 + d_t^2 \end{pmatrix}$$

and

$$S_t^{1/2} = \nu_t \begin{pmatrix} \psi_{to}^2 + \psi_{to}\, d_t & \beta_t\, \psi_{to}^2 \\ \beta_t\, \psi_{to}^2 & \beta_t^2\, \psi_{to}^2 + d_t^2 + \psi_{to}\, d_t \end{pmatrix},$$

where

$$\nu_t = \left\{\beta_t^2\, \psi_{to}^2 + (\psi_{to} + d_t)^2\right\}^{-1/2}. \tag{13}$$

The Wold-Cramér decomposition of $\underline{Z}_t$ is written

$$\underline{Z}_t = \tilde{\zeta}_t + \sum_{j=1}^{\infty} \frac{\psi_{tj}}{\psi_{t-j,0}} \begin{pmatrix} 1 & 0 \\ \beta_t & 0 \end{pmatrix} \tilde{\zeta}_{t-j}$$

or, in terms of the normalized innovations $\underline{\zeta}_t$:

$$\underline{Z}_t = \nu_t \begin{pmatrix} \psi_{to}^2 + \psi_{to}\, d_t & \beta_t\, \psi_{to}^2 \\ \beta_t\, \psi_{to}^2 & \beta_t^2\, \psi_{to}^2 + d_t^2 + \psi_{to}\, d_t \end{pmatrix} \underline{\zeta}_t$$

$$+ \sum_{j=1}^{\infty} \nu_{t-j}\, \psi_{tj} \begin{pmatrix} \psi_{t-j,0} + d_{t-j} & \beta_{t-j}\, \psi_{t-j,0} \\ \beta_t(\psi_{t-j,0} + d_{t-j}) & \beta_t \beta_{t-j}\, \psi_{t-j,0} \end{pmatrix} \underline{\zeta}_{t-j}.$$

Consequently

$$A_t^{11}(\omega) = \sum_{j=0}^{\infty} \nu_{t-j}\, \psi_{tj}\, (\psi_{t-j,0} + d_{t-j})\, e^{-i\omega j}$$

$$A_t^{12}(\omega) = \sum_{j=0}^{\infty} \nu_{t-j}\, \psi_{tj}\, \psi_{t-j,0}\, \beta_{t-j}\, e^{-i\omega j}$$

$$A_t^{21}(\omega) = \beta_t\, A_t^{11}(\omega) - \nu_t\, \psi_{to}\, d_t\, \beta_t$$

$$A_t^{22}(\omega) = \beta_t\, A_t^{12}(\omega) + \nu_t\, d_t(\psi_{to} + d_t).$$

Then

$$f_t^{11}(\omega) = |A_t^{11}(\omega)|^2 + |A_t^{12}(\omega)|^2$$

$$f_t^{12}(\omega) = f_t^{21*}(\omega) = \beta_t\, f_t^{11}(\omega) + \nu_t\, d_t \left\{(\psi_{to} + d_t)\, A_t^{12}(\omega) - \psi_{to}\, \beta_t\, A_t^{11}(\omega)\right\}$$

$$f_t^{22}(\omega) = \beta_t^2\, f_t^{11}(\omega) + d_t^2 + 2\,\nu_t\, d_t\, \beta_t \left[(\psi_{to} + d_t)\ \mathrm{Re}\left\{A_t^{12}(\omega)\right\} - \psi_{to}\, \beta_t\ \mathrm{Re}\left\{A_t^{11}(\omega)\right\}\right]. \qquad (14)$$

To compute the coherence (12) we need

$$|f_t^{12}(\omega)|^2 = f_t^{11}(\omega)\, f_t^{22}(\omega) - d_t^2\, f_t^{11}(\omega) + \nu_t^2\, d_t^2\, |(\psi_{to} + d_t)\, A_t^{12}(\omega) - \psi_{to}\beta_t A_t^{11}(\omega)|^2$$

$$= f_t^{11}(\omega)\, f_t^{22}(\omega) - f_t^{\varepsilon}(\omega)$$

by using (13), where we have let

$$f_t^{\varepsilon}(\omega) = \nu_t^2\, d_t^2\, |(\psi_{to} + d_t)\, A_t^{11}(\omega) + \psi_{to}\, \beta_t\, A_t^{12}(\omega)|^2.$$

Hence

$$K_t^{12}(\omega) = 1 - \frac{f_t^{\varepsilon}(\omega)}{f_t^{11}(\omega)\, f_t^{22}(\omega)}\ .$$

Since $K_t^{12}(\omega)$ depends on t, the process is not oscillatory.

In the stationary case, $\psi_{tj} = \psi_j$ for all j, $\beta_t = \beta$ and $d_t = d$, for all t, and if we define

$$\psi(\omega) = \sum_{j=0}^{\infty} \psi_j\, e^{i\omega j}$$

the preceding formulae are specialized as follows :

$$A^{11}(\omega) = \nu(\psi_o + d)\ \psi(\omega)$$

$$A^{12}(\omega) = \nu\psi_o\ \beta\ \psi(\omega),$$

where

$$\nu^2 = \left\{\beta^2\ \psi_o^2 + (\psi_o + d)^2\right\}^{-1} ,$$

$$f^{11}(\omega) = |\psi(\omega)|^2$$

$$f^{22}(\omega) = \beta^2\ f^{11}(\omega) + d^2$$

$$f^{\varepsilon}(\omega) = \nu^2\ d^2\ |(\psi_o + d)^2\ \psi(\omega) + \beta^2\ \psi_o^2\ \psi(\omega)|^2 = d^2\ |\psi(\omega)|^2$$

$$K^{12}(\omega) = 1 - \frac{d^2}{f^{22}(\omega)} .$$

in agreement with e.g. Priestley (1981, chapter 9).

This example is due to De Schutter-Herteleer (1976). A generalization to the model

$$Y_t = \sum_{j=0}^{r} \beta_{tj}\ X_{t-j} + \sum_{j=0}^{s} \theta_{tj}\ \varepsilon_{t-j}$$

where X_t is a moving average process with time-dependent coefficients is considered by Mélard and Wybouw (1984).

Example 4 - An evolutive bivariate ARMA process

Let's consider the bivariate ARMA(1,1) process with time-dependent coefficients

$$\underline{Z}_t - \underline{\phi}_t\ \underline{Z}_{t-1} = \underline{\zeta}_t - \underline{\theta}_t\ \underline{\zeta}_{t-1}$$

where the $\underline{\zeta}_t$ are the normalized innovations, $\underline{\phi}_t$ and $\underline{\theta}_t$ are 2x2 matrices with time-dependent elements, and

$$\underline{S}_t = \begin{pmatrix} 1 & \rho_t \\ \rho_t & 1 \end{pmatrix} .$$

Appropriate conditions on $\underline{\phi}_t$ and $\underline{\theta}_t$ are given by Hallin (1978) and Mélard (1985a). It is easily seen that

$$\underline{\tilde{\psi}}_{t1} = \underline{\phi}_t - \underline{\theta}_t$$

$$\underline{\tilde{\psi}}_{tj} = \left(\prod_{k=0}^{j-2} \underline{\phi}_{t-k} \right) (\underline{\phi}_{t-j+1} - \underline{\theta}_{t-j+1}), \quad j \geqslant 2.$$

On the other hand

$$\underline{S}_t^{1/2} = (2\ \nu_t)^{-1/2} \begin{pmatrix} \nu_t & \rho_t \\ \rho_t & \nu_t \end{pmatrix} \text{ with } \nu_t = 1 + (1 - \rho_t^2)^{1/2}.$$

Given these preliminaries, $\underline{f}_t(\omega)$ can be calculated in the same way as before. In particular, if $\underline{\phi}_t = \underline{\theta}_t = \underline{0}$, then $\underline{f}_t(\omega) = (2\pi)^{-1}\ \underline{S}_t$ and $K_t(\omega) = \rho_t^2$. This very simple process has a time-dependent coherence. Therefore it is not an oscillatory process, according to the definition proposed by Priestley and Tong (1975). We shall comment on this in Section 5.

4. THE STATISTICAL ESTIMATION OF EVOLUTIVE SPECTRA

Granger (1964), Priestley (1965) and Brillinger and Hatanaka (1969) have described the estimation method which makes use of complex demodulation (e.g. Priestley 1981, chapter 11). Priestley (1965, 1966) and Subba Rao and Tong (1972) have given approximate results on the bias and the covariance matrix of the estimators. Goodman and Dubman (1969) and Droesbeke (1977) have attempted to provide asymptotic distributions. Our contribution is new by the following aspects : 1° each component will be demodulated using the most appropriate filter, not necessarily the same one for all components; 2° we shall give a new bound for the bias of the estimators; 3° some computational aspects will be emphasized, especially in the formula for the approximate variances.

Let us consider the estimation of a component $f_t^{jk}(\omega)$ of $\underline{f}_t(\omega)$ at time $t = \tau$ and in a band centered on $\omega = \lambda$. We have a realization $(\underline{Z}_t;\ t \in T)$ of the process, where the interval of observation is $T = \{1, \ldots, n\}$, say. For the jth component of $\underline{Z}_t$ we select a low-pass symmetric filter with coefficients g_u^j such that : 1° $g_u^j = 0$, $u \notin U_j$, where U_j is a set of integers surrounding 0, and 2° $g_u^j = g_{-u}^j$, with $u, -u \in U_j$. For example, the g_u^j can be defined by the product of convolution of several simple moving averages. We define

$$Z_t^j(\lambda) = \sum_{u \in U_j} g_u^j \, Z_{t-u}^j \, e^{-i\lambda(t-u)}. \tag{15}$$

For the components (j,k) of $\underline{Z}_t$, we select a low-pass filter with coefficients w_m^{jk} such that $w_m^{jk} = 0$, $m \notin M_{jk}$, where M_{jk} is a set of integers surrounding 0. We define

$$\hat{f}_\tau^{jk}(\lambda) = H_{jk} \sum_{m \in M_{jk}} w_m^{jk} \, Z_{\tau-m}^j(\lambda) \, Z_{\tau-m}^{k*}(\lambda), \tag{16}$$

where the factor H_{jk} will be determined later.

We shall need the following assumption : for each t, $A_t^{jk}(\omega)$ is a bounded function of ω. Let $T_\tau = U_{j,k} \{\tau-m;\ m \in M_{jk}\}$. Consider the smallest non-negative constant B_τ^{jk} such that

$$|A_{t-u}^{jk}(\omega) - A_t^{jk}(\omega)| \leq |u| \; B_\tau^{jk} \tag{17}$$

for all $u \in U_j$, $t \in T_\tau$, $\omega \in]-\pi, \pi]$. It should be noted that the B_τ^{jk}'s provide bounds on the non-stationarity of the process around time τ (Mélard, 1978). In particular, if $A_t^{jk}(\omega)$ does not depend on t in T_τ, we have $B_\tau^{jk} = 0$. Let $B_\tau^j = \max_k \{B_\tau^{jk}\}$. We introduce the following notations :

$$b_j = \left(\sum_{u \in U_j} |u| |g_u^j| \right)^{-1}$$

$$b_j' = \left(\sum_{u \in U_j} (g_u^j)^2 \right)^{-1/2}$$

$$G_j(\theta) = \sum_{u \in U_j} g_u^j \, e^{-iu\theta}$$

$$r_t^{jk}(\omega,\theta) = \sum_{u \in U_j} g_u^j \left\{ A_{t-u}^{jk}(\omega) - A_t^{jk}(\omega) \right\} e^{-i\theta t}. \tag{18}$$

Both b_j and b_j' are related to the "width" of the filter g_u^j. The following upper bound of (18), for $t \in T_\tau$, will be used :

$$|r_t^{jk}(\omega,\theta)| \leq \left\{ \sum_{u \in U_j} |g_u^j| \; |u| \right\} B_\tau^{jk} \leq B_\tau^j / b_j. \tag{19}$$

By substituting (7) in (15), we obtain

$$Z_t^j(\lambda) = \sum_{\ell=1}^{p} \int_{-\pi}^{\pi} \left\{ G_j(\omega-\lambda) \; A_t^{j\ell}(\omega) + r_t^{j\ell}(\omega, \omega-\lambda) \right\} e^{i(\omega-\lambda)t} \; z_\ell(d\omega).$$

Hence, with the notation $U_{jk} = U_j \cup U_k$, we have similarly to Eq. ( ) of Priestley (1965) :

$$E\left\{ Z_t^j(\lambda) \; Z_t^{k*}(\lambda) \right\} = \left(\sum_{u \in U_{jk}} g_u^j \, g_u^k \right) \int_{-\pi}^{\pi} G_{jk}(\omega) f_t^{jk}(\lambda+\omega) d\omega + I_1 + I_2 + I_3 \tag{20}$$

where

$$G_{jk}(\omega) = \frac{G_j(\omega) \; G_k(\omega)}{\int_{-\pi}^{\pi} G_j(\theta) \; G_k(\theta) \; d\theta} \quad ,$$

$$I_1 = \int_{-\pi}^{\pi} G_j(\omega-\lambda) \left\{ \sum_{\ell=1}^{p} A_t^{j\ell}(\omega)\ r_t^{k\ell *}(\omega,\omega-\lambda) \right\} d\omega \ ,$$

$$I_3 = \int_{-\pi}^{\pi} \sum_{\ell=1}^{p} r_t^{j\ell}(\omega,\omega-\lambda)\ r_t^{k\ell *}(\omega,\omega-\lambda)\ d\omega$$

and for I_2 an expression similar to I_1, j being interchanged with k.

We can bound $|I_1|$ by using (19), Schwarz inequality, the fact that $G_j(\omega)$ is real and periodic with period 2π, Parseval's identity and (11) :

$$|I_1| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |G_j(\omega-\lambda)| \left\{ \sum_{\ell=1}^{p} |A_t^{j\ell}(\omega)| \right\} (B_\tau^k/b_k)\ d\omega$$

$$\leqslant \frac{p^{1/2}}{2\pi}\ (B_\tau^k/b_k) \left[\int_{-\pi}^{\pi} \left\{ G_j(\omega) \right\}^2 d\omega \ . \int_{-\pi}^{\pi} \sum_{\ell=1}^{p} |A_t^{j\ell}(\omega)|^2\ d\omega \right]^{1/2}$$

$$= p^{1/2}(B_\tau^k/b_k\ b_j') \left\{ \mathrm{var}(Z_t^j) \right\}^{1/2} \ . \qquad (21)$$

Similarly

$$|I_3| \leqslant p(B_\tau^j\ B_\tau^k/b_j\ b_k). \qquad (22)$$

Going back to (15), we see that

$$E\left\{ \hat{f}_\tau^{jk}(\lambda) \right\} = H_{jk} \left\{ \left(\sum_{u\in U_{jk}} g_u^j\ g_u^k \right) \sum_{m\in M_{jk}} w_m^{jk} \int_{-\pi}^{\pi} G_{jk}(\omega)\ f_{\tau-m}^{jk}\ (\lambda+\omega)\ d\omega + \text{bias} \right\}. \qquad (23)$$

In order that $\hat{f}_\tau^{jk}(\lambda)$ be an estimate of averaged values of $f_t^{jk}(\omega)$ around $t = \tau$ and $\omega = \lambda$, the sum of weights should be 1. Given (20), there remains to impose that

$$\sum_{m\in M_{jk}} w_m^{jk} = 1 \qquad \text{and} \qquad H_{jk} = \left(\sum_{u\in U_{jk}} g_u^j\ g_u^k \right)^{-1}$$

so that the first term of (23) becomes

$$\sum_{m\in M_{jk}} w_m^{jk} \int_{-\pi}^{\pi} G_{jk}(\omega)\ f_{\tau-m}^{jk}\ (\lambda+\omega). \qquad (24)$$

Besides the bias has the following upper bound, derived from (21-22) :

$$H_{jk} \left\{ p^{1/2} \left[(B_\tau^k / b_k b'_j)\ V_\tau^j + (B_\tau^j / b_j b'_k)\ V_\tau^k \right] + p\ (B_\tau^j\ B_\tau^k / b_j b_k) \right\} \qquad (25)$$

where

$$V_\tau^j = \sup_{t \in T_\tau} \left\{ \mathrm{var}(Z_t^j) \right\}^{1/2}.$$

For a stationary process, $A_t^{jk}(\omega)$ defined by (8) does not depend on t. Hence $B_\tau^j = B_\tau^k = 0$, and (25) is equal to zero.

In the following discussion, it will be assumed that all the components are analyzed by using the same filters, hence $U_{jk} = U_j = U$, $M_{jk} = M$, $g_u^j = g_u$, $w_m^{jk} = w_m$, $G_j(\omega) = G(\omega)$, $G_{jk}(\omega) = \tilde{G}(\omega)$, $b_j = b$, $H_{jk} = b_j'^2 = b'^2$, for all j and k. Furthermore, (24) and (25) can be written respectively :

$$b'^2 \sum_{m \in M} w_m \int_{-\pi}^{\pi} \tilde{G}(\omega)\ f_{\tau-m}^{jk}(\lambda+\omega)\ d\omega, \qquad (26)$$

$$p^{1/2}\ (B_\tau^k\ V_\tau^j + B_\tau^j\ V_\tau^k)(b'/b) + p\ B_\tau^j\ B_\tau^k\ (b'/b)^2.$$

For a non-stationary process, $B_\tau^j \neq 0$ in general. The bias as well as the quality of approximation of $f_\tau^{jk}(\lambda)$ reached by (26) will depend mainly on the form of the coefficients g_u. If $\tilde{G}(\omega)$ is highly concentrated in the region of 0, so that the degree of resolution in the frequency domain is high, then the ratio b'/b will be relatively large, suggesting a certain amount of bias due to non-stationarity and low resolution in the time domain. Conversely, if the resolution in the frequency domain is low, the resolution in the time domain will be higher. This is the *uncertainty principle* given by Priestley (1965).

Priestley (1966) has obtained an approximate formula for the variance of the evolutionary spectrum estimator within the framework of semi-stationary process. Subba Rao and Tong (1972) have generalized these results to multivariate processes. The derivation is heuristic. It can be adapted to the evolutive spectral density function $\hat{f}_\tau(\lambda)$ using a unique modification. The assumptions are :

(a) $(Z_t;\ t \in \mathbb{Z})$ is a process with the same first four moments as a Gaussian process;

(b) all the functions $G_j(\omega)$ are more concentrated around $\omega = 0$ than the $A_t^{jk}(\omega)$ are;

(c) let

$$W_{jk}(\omega) = \sum_{m \in M_{jk}} w_m^{jk} e^{-i\omega m};$$

it is assumed that all the functions $W_{jk}(\omega)$ are more concentrated around $\omega = 0$ than the $G_j(\omega)$ are;

(d) the matrices $\underline{A}_{t-m}^{jk}(\omega)$ do no vary too much when $m \in M_{jk}$, for all $t \in T_\tau$.

The last assumption is invoked at one place in the derivation where the Fourier-Stieltjes transform of $A_t(\omega)$ was used by Priestley (1966, p. 233). Its role is to support approximations like

$$\sum_{m \in M_{jk}} \sum_{m' \in M_{jk}} |A_{t-m}(\omega)|^2 \, |A_{t-m'}(\omega)|^2 \, w_m^{jk} \, w_{m'}^{jk} \, e^{i\omega'(m-m')} \sim |A_t(\omega)|^4 \, |W_{jk}(\omega')|^2$$

The final approximations give

$$\begin{aligned}
&\operatorname{var}\left\{\hat{f}_\tau^j(\lambda)\right\} \sim L_{jj} \left\{f_\tau^j(\lambda)\right\}^2 \\
&\operatorname{var}\left\{\hat{c}_\tau^{jk}(\lambda)\right\} \sim \frac{L_{jk}}{2} \left[f_\tau^j(\lambda) \, f_\tau^k(\lambda) + \left\{c_\tau^{jk}(\lambda)\right\}^2 - \left\{q_\tau^{jk}(\lambda)\right\}^2\right] \\
&\operatorname{var}\left\{\hat{q}_\tau^{jk}(\lambda)\right\} \sim \frac{L_{jk}}{2} \left[f_\tau^j(\lambda) \, f_\tau^k(\lambda) - \left\{c_\tau^{jk}(\lambda)\right\}^2 + \left\{q_\tau^{jk}(\lambda)\right\}^2\right] \\
&\operatorname{cov}\left\{\hat{c}_\tau^{jk}(\lambda), \hat{q}_\tau^{jk}(\lambda)\right\} \sim L_{jk} \, c_\tau^{jk}(\lambda) \, q_\tau^{jk}(\lambda),
\end{aligned} \tag{27}$$

where

$$L_{jk} = \frac{\left[\sum_{m \in M_{jk}} \left\{w_m^{jk}\right\}^2\right] \left[\sum_{u \in U_{jk}} \left\{(g^j * g^k)_u\right\}^2\right]}{(2\pi)^2 \left[\sum_{u \in U_j} (g_u^j)^2\right] \left[\sum_{u \in U_k} (g_u^k)^2\right]}, \quad (g^j * g^k)_u = \sum_{v \in U_{jk}} g_v^j \, g_{u-v}^k.$$

Note that these formulae are computationnally more convenient than those given by Priestley (1966) and Subba Rao and Tong (1972), which involved several integrals. Less efficient but equivalent formulae were used by De Schutter-Herteleer (1976, 1977) who showed through Monte Carlo experiments that they are more accurate than could have been feared because of the numerous approximations leading to (27). When two different time points τ_1 and τ_2 (or two different frequencies λ_1 and λ_2) are considered, not too close to each other, the covariance between two estimators, e.g.

$$\text{cov}\left\{\hat{f}^j_{\tau_1}(\lambda),\ \hat{f}^j_{\tau_2}(\lambda)\right\} \qquad \left(\text{or cov}\left\{\hat{f}^j_{\tau}(\lambda_1),\ \hat{f}^j_{\tau}(\lambda_2)\right\}\right)$$

is approximately equal to zero.

5. COMPARISON WITH THE EVOLUTIONARY SPECTRUM THEORY

The comparison between the approach described in Sections 2 and 4 and the evolutionary spectrum theory can be made at two levels : the assumptions about the class of processes and the bounds of the bias due to nonstationarity. We shall first explain why the restrictions put in the definition of an oscillatory process lead to a concept of coherence which is time invariant and why the evolutive coherence (12) can be time-varying.

Let consider (4) and (5) in the following special case :

1° the measures μ_{jk} are absolutely continuous;

2° $\underline{f}(\omega)$ is a full rank pxp matrix almost everywhere;

3° $\underline{\phi}_t(\omega) = \exp(i\omega t)\ \underline{\theta}_t(\omega)$, where $\underline{\theta}_t(\omega)$ is a diagonal matrix whose elements $\theta_t^j(\omega)$ are square-integrable with respect to the Lebesgue measure.

Almost everywhere in $]-\pi,\pi]$, $\underline{f}(\omega)$ is a Hermitian positive definite matrix with elements $f^{jk}(\omega)$, so that we can define $\underline{g}(\omega) = \left\{\underline{f}(\omega)\right\}^{1/2}$ and $\underline{z}(d\omega)$ $(2\pi)^{-1/2}\ g^{-1}(\omega)\ \underline{\tilde{z}}(d\omega)$. Note that

$$E\left\{\underline{z}(d\omega)\ \underline{z}^*(d\omega)\right\} = (2\pi)^{-1}\ \underline{g}^{-1}(\omega)\ \underline{f}(\omega)\ \underline{g}^{-1}(\omega) = (2\pi)^{-1}\ \underline{I}.$$

The stochastic integral representation of the process is thus

$$\underline{Z}_t = (2\pi)^{1/2} \int_{-\pi}^{\pi} e^{i\omega t}\ \underline{\theta}_t(\omega)\ \underline{g}(\omega)\ \underline{z}(d\omega) \qquad (28)$$

or (8) where we have let $\underline{A}_t(\omega) = (2\pi)^{1/2}\ \underline{\theta}_t(\omega)\ \underline{g}(\omega)$. In accordance with (10) we define

$$\underline{f}_t(\omega) = \underline{\theta}_t(\omega)\ \underline{f}(\omega)\ \underline{\theta}_t^*(\omega).$$

For instance $f_t^{jk}(\omega) = \theta_t^j(\omega)\ \theta_t^{k*}(\omega)\ f^{jk}(\omega)$. This is precisely the definition of the cross-spectral density given by Priestley and Tong (1973, Eqs (4.4)-(4.8)). In this case

$$K_t^{jk}(\omega) = \frac{|f^{jk}(\omega)|^2}{f^{jj}(\omega)\ f^{kk}(\omega)}$$

which is actually time-invariant. The reason is that $\underline{A}_t(\omega)$ depends on t through a diagonal matrix factor. The more general definition (8) makes use of a matrix $\underline{A}_t(\omega)$ without this restrictive property, although $\underline{f}(\omega)$ is replaced by a scalar matrix. A good illustration of the restrictions induced by (28) is given by Examples 3 and 4 of Section 2. Clearly, $\underline{A}_t(\omega)$ cannot be put under the form $(2\pi)^{1/2}\ \underline{\theta}_t(\omega)\ \underline{g}(\omega)$ where $\underline{\theta}_t(\omega)$ is diagonal, and, as an obvious consequence, the coherence is time-dependent.

Another aspect of the evolutionary spectrum theory are the assumptions made on $\underline{A}_t(\omega)$ in order to express that it is a slowly-varying function of time. Restricting ourselves to univariate processes for the sake of clear exposition, a general representation (5) is first assumed such that $\phi_t(\omega) = \exp(i\omega t)\ A_t(\omega)$ and $A_t(\omega)$, as a function of t, admits a Fourier-Stieltjes transform $D_\omega(\theta)$ such that $|D_\omega(d\theta)|$ has an absolute maximum at $\theta = 0$. The characteristic width of $A_t(\omega)$ is then defined by

$$B(A) = \left[\sup_\omega \int_{-\pi}^{\pi} |\theta|\ |D_\omega(d\theta)|\right]^{-1}. \qquad (29)$$

An oscillatory process is termed semi-stationary if there exists at least one such function $A_t(\omega)$. There may exist many others, each inducing a different integral representation (5) and a different spectral density function of the same form as (10). A so-called natural representation is characterized by a function $A_t^N(\omega)$ such that $B(A^N)$ reaches (or is arbitrarily close to) the supremun B_Z of $B(A)$ over all functions $A_t(\omega)$ with the properties mentioned above: For an example of a derivation of B_Z, the characteristic width of the process, we refer the reader to Mélard (1985b).

It should be noted that the assumptions and definitions, such as (29), are ad hoc. Instead of the integral of $|\theta|\ |D_\omega(d\theta)|$ we could have used $|\theta|^2 |D_\omega(d\theta)|$ for instance. It was shown by Mélard (1975) that the results are simpler. Priestley (1965, p. 212) admits that there are various ways of defining a slowly varying function. In order to guarantee the interpretation of the spectrum we shall assume that such a condition holds even if we do not use that assumption explicitely.

Priestley (1965, 1966) argues that, among the multiple spectral density functions, the estimation procedure leads automatically to the evolutionary spectrum with respect to the natural representation, with characteristic width B_Z. Indeed, the expression he has found for an upper bound of the bias is proportional to $\{B(A)\}^{-1}$. Clearly that bound is minimized when the natural representation is used. Before going any further, it should be pointed out that we cannot rely on properties of convergence. The best we can do is to evaluate an upper bound of the bias (Priestley, 1966) but the bias itself cannot be determined and it cannot be made smaller by taking longer realizations. Let $f_\tau^N(\lambda)$ be the spectrum associated with the natural representation and $f_\tau^C(\lambda)$, the evolutive spectrum, associated to the Wold-Cramér decomposition. First, it is not because an upper bound of $b_\tau^N(\lambda) = E\{|\hat{f}_\tau(\lambda) - f_\tau^N(\lambda)|\}$ is smaller than an upper bound of $b_\tau^C(\lambda) = E\{|\hat{f}_\tau(\lambda) - f_\tau^C(\lambda)|\}$, that we can conclude that $b_\tau^N(\lambda) < b_\lambda^C(\lambda)$. The smallest upper bound on the bias does not imply the smallest bias. Furthermore there is no reason for believing that the upper bound based on the characteristic width is the smallest. Indeed, use is made of the whole behaviour of the process, from $-\infty$ to ∞. On the contrary, the upper bound obtained in Section 4 is purely local and does not require knowledge of the process outside the observation interval [1,n]. It may happen that B_τ^{jk} is small but that $B_{\tau'}^{jk}$ is higher. This means that non-stationarity can be more pronounced around τ' than around τ. Since it is practically impossible to compute B_Z exactly except in some simple cases (Mélard, 1985b) we cannot proceed any further with the comparison without additional assumptions.

Suppose that the natural representation exists and coincides with the representation induced by the Wold-Cramér decomposition. Given that

$$e^{-i\theta u} - 1 = -\, i\theta u\, e^{-i\theta u\eta}$$

with $\eta \in [0,1]$, (17) can be written

$$|A_{t-u}(\omega) - A_t(\omega)| = \left|\int_{-\pi}^{\pi} e^{i\theta t}(e^{-i\theta u} - 1)\, D_\omega(d\theta)\right|$$

$$= \left|\int_{-\pi}^{\pi} e^{i\theta t}\left\{-\, i\theta u \exp(-i\theta u\eta)\right\} D_\omega(d\theta)\right|$$

$$\leq |u| \int_{-\pi}^{\pi} |\theta|\, |D_\omega(d\theta)|$$

$$\leq |u| \sup_\omega \left\{\int_{-\pi}^{\pi} |\theta|\, |D_\omega(d\theta)\right\} = |u|\, B_A^{-1}.$$

Hence we can take $B_\tau = B_A^{-1}$ in (17). Presumably there exists a smaller upper bound which may depend on τ, does not use the process from $-\infty$ to ∞ but reflects only the amount of non-stationarity around $t = \tau$. The following example will illustrate the point. Let

$$Z_t = \begin{cases} \varepsilon_t & t \neq 10^4 \\ 1000001\ \varepsilon_t & t = 10^4 \end{cases}$$

where $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is a white noise process, so that

$$A_t(\omega) = \begin{cases} 1 & t \neq 10^4 \\ 1000001 & t = 10^4 \end{cases}$$

and

$$D_\omega(d\theta) = (2\pi)^{-1} \left\{\delta(0) + 10^6\ e^{i10^4\theta}\right\} d\theta$$

using Dirac's δ function. Then

$$\left\{B(A)\right\}^{-1} = \int_{-\pi}^{\pi} |\theta|\ |D_\omega(d\theta)| = 10^6(2\pi)^{-1} \int_{-\pi}^{\pi} |\theta|\ d\theta = \frac{\pi}{2}\ 10^6,$$

If we have data in the interval [1,1000], we can take $B_\tau = 0$ which results in an absence of bias due to non-stationarity. Obviously $B_\tau < \{B(A)\}^{-1}$. Only if we are interested in $f_\tau(\lambda)$ in the neighbourhood of $\tau = 10^4$ must we take non-stationarity into account. There

$$|A_{t-u}(\omega) - A_t(\omega)| \leqslant 10^6$$

and the bias will be much higher.

6. ILLUSTRATION

This Section is partly based on Kiehm and Mélard (1981). Let consider two special cases of the evolutive bivariate ARMA(1,1) process discussed in example 4 of Section 3 :

(A) $\underline{\phi}_t = \underline{\theta}_t = 0$

(B) $$\underline{\phi}_t = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix} + \left(\frac{t}{401}\right) \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}$$

$$\underline{\theta}_t = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix} + \left(\frac{t}{401}\right) \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} .$$

In both cases $\rho_t = (2t/401)-1$. These relations are used for t such that $1 \leqslant t \leqslant 401$. For $t \leqslant 0$, we take the value at t = 1. Since we shall use time series of length 400, values for $t \geqslant 401$ are not needed. Note that (A) is a bivariate non-stationary white noise so that $\underline{f}_t(\omega)$ does not depend on ω. The coherence $K_t(\omega) = \rho_t^2$ is a quadratic function of t decreasing from 1 (at t = 0) to 0 (at t = 200.5), then increasing to 1 (at t = 401).

For case (B), the theoretical evolutive spectral density function has been calculated using the Wold-Cramér decomposition (A) and neglecting of the terms for $j \geqslant 500$. From that, $f_t^1(\omega)$, $f_t^2(\omega)$ and $K_t^{12}(\omega)$ have been obtained.

One hundred time series of length n = 400 have been generated from the Gaussian processes (A) and (B), using the algorithm of Marsaglia, Maclaren and Bray (1964) and the RANF generator of the C.D.C. Cyber system. The method of Section 4 has then been applied under the following conditions

$$\tau_i = 40 + 64(i-1), \qquad (i = 1, \ldots, 6);$$
$$\lambda_i = (i-1)\pi/6, \qquad (i = 1, \ldots, 7);$$
$$g_u^1 = g_u^2 = (g * g)_u, \text{ where } g_u = \begin{cases} 1/17, & |u| \leqslant 8, \\ 0, & |u| > 8; \end{cases}$$

$$w_m^{11} = w_m^{22} = w_m^{12} = w_m^{21} = \begin{cases} \frac{1}{47}, & |m| \leqslant 23, \\ 0, & |m| > 23; \end{cases}$$

As discussed by Kiehm and Mélard (1981), the estimates of the coherence should be biased when the true value is close to 0. Therefore, we have estimated the complex coherence $C_t^{12}(\omega)$ on the 100 series, computed the arithmetic mean of the estimates ave $\{\hat{c}_t^{12}(\omega)\}$ = ave $\{\hat{c}_t^{12}(\omega)\}$ + i ave $\{\hat{q}_t^{12}(\omega)\}$ and taken the square modulus [ave $\{\hat{c}_t^{12}(\omega)\}]^2$ + [ave $\{\hat{q}_t^{12}(\omega)\}]^2$ denoted by $\bar{K}_t^{12}(\omega)$.

Table I contains the values of $K_t^{12}(\omega)$ and $\bar{K}_t^{12}(\omega)$ for process (A). Despite the fact that $K_t^{12}(\omega)$ varies considerably in function of t, it can be estimated fairly well.

Table II and III refer to process (B). It can be observed that $f_t^2(\omega)$ varies much more in function of t and ω than $f_t^1(\omega)$ does. Table II shows the values of $f_t^2(\omega)$, $\bar{f}_t^2(\omega)$ = ave $\{\hat{f}_t^2(\omega)\}$, the arithmetic mean of the estimates, and SE $f_t^2(\omega)$, the standard error of the estimates of $f_t^2(\omega)$ over the simulations. It can be seen that the estimates are very good in spite of the large variations in function of t and ω. As indicated by the theory of Section 4, the standard error seem to be proportional to the evolutive spectral density. Here $K_t^{12}(\omega)$ varies also with respect to ω but the results are as good as before.

These examples show that the theoretical evolutive spectral elements can be effectively computed from the process and that they can be estimated very well, even with short, highly nonstationary time series, and also outside the framework stated by Priestley and Tong (1973). Some other simulation results are reported by De Schutter-Herteleer (1976), by Kiehm and Mélard (1981), and by Mélard and Wybouw (1984).

Table I : Values of the true coherence $K_t^{12}(\omega)$ on the first line and the square modulus of the average of complex coherence estimates $\bar{K}_t^{12}(\omega)$ on the second line, for process (A).

t	40	104	168	232	296	360
$K_t^{12}(\omega)$, all ω	.642	.234	.028	.023	.220	.619
$\bar{K}_t^{12}(\omega)$ ω						
0	.608	.215	.029	.008	.236	.510
$\pi/6$	.631	.192	.014	.024	.206	.631
$2\pi/6$	.632	.199	.034	.011	.201	.575
$3\pi/6$	.610	.137	.025	.056	.222	.500
$4\pi/6$	.613	.235	.019	.015	.163	.590
$5\pi/6$	.605	.178	.012	.033	.181	.586
π	.591	.168	.025	.023	.246	.542

Table II: Value of the true evolutive density function on the first line, $\bar{f}_t^2(\omega)$ on the second line and S.E. $f_t^2(\omega)$ on the third line, for process (B).

ω	t	40	104	168	232	296	360
0	$f_t^2(\omega)$	.19	.64	1.23	1.94	2.56	2.52
	$\bar{f}_t^2(\omega)$	.20	.62	1.39	1.83	2.70	2.19
	SE $f_t^2(\omega)$	(.02)	(.05)	(.11)	(.15)	(.25)	(.16)
$\pi/6$		.55	.89	1.34	1.82	2.17	2.11
		.55	.79	1.37	1.82	2.04	2.01
		(.03)	(.05)	(.07)	(.11)	(.12)	(.13)
$2\pi/6$		1.48	1.44	1.53	1.66	1.77	1.75
		1.51	1.39	1.55	1.70	1.58	1.76
		(.07)	(.08)	(.08)	(.09)	(.10)	(.11)
$3\pi/6$		2.62	2.00	1.67	1.55	1.55	1.58
		2.94	1.88	1.74	1.69	1.51	1.54
		(.19)	(.12)	(.11)	(.09)	(.09)	(.09)
$4\pi/6$		3.66	2.44	1.78	1.49	1.43	1.50
		3.57	2.33	1.87	1.47	1.34	1.48
		(.20)	(.12)	(.12)	(.09)	(.08)	(.09)
$5\pi/6$		4.37	2.73	1.84	1.45	1.36	1.47
		4.70	2.75	1.86	1.41	1.44	1.50
		(.29)	(.17)	(.11)	(.09)	(.08)	(.09)
π		4.62	2.83	1.86	1.43	1.34	1.45
		4.57	2.90	1.82	1.27	1.50	1.55
		(.35)	(.26)	(.14)	(.12)	(.12)	(.13)

Table III : Value of $K_t^{12}(\omega)$ on the first line and $\bar{K}_t^{12}(\omega)$ on the second line for process (B).

ω	t	40	104	168	232	296	390
0	$K_t^{12}(\omega)$	.016	.019	.005	.008	.141	.554
	$\bar{K}_t^{12}(\omega)$	.004	.015	.009	.001	.173	.497
$\pi/6$		.537	.160	.019	.016	.158	.462
		.514	.147	.013	.009	.131	.442
$2\pi/6$		.697	.259	.028	.020	.167	.482
		.673	.231	.027	.004	.123	.418
$3\pi/6$		.729	.293	.034	.026	.220	.590
		.680	.187	.020	.062	.221	.486
$4\pi/6$		.750	.334	.044	.037	.307	.710
		.721	.328	.030	.030	.294	.654
$5\pi/6$		.767	.371	.054	.048	.392	.802
		.728	.298	.031	.056	.327	.765
π		.774	.385	.057	.053	.427	.837
		.716	.300	.050	.045	.404	.753

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