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Dark states of two-mode quantized fields in two-channel models: competing *k*- and *l*-photon processes

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Abstract

We examine the properties of two-mode field states $|\xi\rangle_{kl}$ satisfying the eigenvalue problem $(\hat{a}^k - \xi\hat{b}^l)|\xi\rangle_{kl} = 0, k, l = 1, 2, \ldots,$ subject to the condition that they are also eigenstates of the operator $\hat{N}_{kl} = l\hat{a}^\dagger\hat{a} + k\hat{b}^\dagger\hat{b}$. For the special case where $k = l = 1$, the resultant states are just the $SU(2)$ coherent states of a two-mode field. The states $|\xi\rangle_{kl}$ may arise as population trapping states, also known as dark states, in models involving competitive k- and l-photon processes.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Two-mode field states possessing strong nonclassical properties may be defined in several different ways. For example, the well-known two-mode squeezed vacuum states may be generated from the vacuum by transformations of the form [1]

$$|\zeta\rangle = \exp(\eta\hat{a}^\dagger\hat{b}^\dagger - \eta^*\hat{a}\hat{b})|0\rangle_a|0\rangle_b = (1 - |\zeta|^2)^{1/2}\sum_{n=0}^{\infty}\zeta^n|n\rangle_a|n\rangle_b, \tag{1}$$

where the operator $\eta\hat{a}^\dagger\hat{b}^\dagger - \eta^*\hat{a}\hat{b}$ represents a nonlinear parametric interaction and $\zeta = \exp(\mathrm{i}\phi)\tanh(|\eta|/2)$, $|\zeta| < 1$, and $\eta = |\eta|\,\mathrm{e}^{\mathrm{i}\phi}$. Alternatively, these states can be defined as states satisfying the relation [2]

$$(\hat{a} + \zeta\hat{b}^\dagger)|\zeta\rangle = 0. \tag{2}$$

As the solution to equation (2), the two-mode squeezed vacuum state appears as a coherent population trapping state in a three-level ladder system [3] and in a Raman coupled model with quantized Stokes and anti-Stokes field modes but with a classical pump field [2, 4].

Another type of two-mode field state may be generated from a number state $|0\rangle_a|N\rangle_b$ according to

$$|\xi, N\rangle = \exp(\kappa\hat{a}^\dagger\hat{b} - \kappa^*\hat{a}\hat{b}^\dagger)|0\rangle_a|N\rangle_b$$
$$= (1+|\xi|^2)^{-N/2}\sum_{n=0}^{N}\binom{N}{n}^{1/2}\xi^n|n\rangle_a|N-n\rangle_b, \tag{3}$$

where $\xi = \exp(\mathrm{i}\phi)\tan(|\kappa|/2)$, where $\kappa = |\kappa|\,\mathrm{e}^{\mathrm{i}\phi}$. The interaction $\kappa\hat{a}^\dagger\hat{b} - \kappa^*\hat{a}\hat{b}^\dagger$ may describe a parametric frequency conversion device [5] or a passive beam splitter [6]. The states of equation (3) are sometimes known as the two-mode binomial states[1] [7] and as the $SU(2)$ coherent states [8] for a two-mode field. This latter denotation arises because if we introduce the Schwinger form of the angular momentum operators [9], $\hat{J}_+ = \hat{a}^\dagger\hat{b} = \hat{J}_-^\dagger$, $\hat{J}_3 = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$, and the operator $\hat{J}_0 = (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})/2$ that commutes with all the other operators, then the top line of equation (3) can be written as

$$|\xi, N\rangle = \exp(\kappa\hat{J}_+ - \kappa^*\hat{J}_-)|0\rangle_a|N\rangle_b, \tag{4}$$

which is of the form of the spin coherent state introduced by Radcliffe [10]. The number states $|n\rangle_a|N-n\rangle_b$ map onto the angular momentum states $|j, m\rangle$ via the identifications $j = N/2$ and $m = n - N/2$, as can easily be checked. Alternatively, these states arise as solutions of the equation

$$(\hat{a} - \xi\hat{b})|\xi, N\rangle = 0. \tag{5}$$

Previously, one of us [4] has shown that states satisfying the equation will be trapping states, or dark states, in a model of a two-level atom interacting with two resonant quantized field modes assumed to have different polarizations. They have also been shown to be trapping states in a one-channel Raman coupled model [11].

In [4], another two-level atom model was studied, wherein a three-photon transition competes with a one-photon transition. (A two-photon transition competing with a one-photon transition is ruled out by the selection rules at least on the case of field states, though it might be possible to generate states for a choice of k and l in the case of the two-dimensional vibrational motion of a trapped ion.) The atomic population dynamics of the model were earlier studied by Jyotsna and Agarwal [12]. For this model, dark states must satisfy the equation

$$(\hat{a}^3 - \xi\hat{b})|\xi\rangle = 0. \tag{6}$$

In [4], certain states were found as solutions to equation (6), but their properties were not studied. In this paper, we discuss dark states of a more general form whereby, in an effective two-level atom, l-photon transitions compete with k-photon transitions between the two levels as shown in figure 1. That is, we seek solutions of the equation

$$(\hat{a}^k - \xi\hat{b}^l)|\xi\rangle_{kl} = 0, \qquad k, l = 1, 2, \ldots, \tag{7}$$

and we shall study some of their properties with an emphasis on their nonclassical nature. For the values of k and l respecting the selection rules, the appropriate effective interaction Hamiltonian for the atom–field system would be of the form

$$\hat{H}_{\mathrm{I}} = \hbar g(\hat{a}^k - \xi\hat{b}^l)\hat{\sigma}_+ + H\cdot c, \tag{8}$$

where the $\hat{\sigma}_+ = |e\rangle\langle g|$ is the atomic transition operator, $|e\rangle$ and $|g\rangle$ being the excited and ground atomic states, respectively. The relevant selection rules can be written as $k = l + 2m$ where k and l are positive integers and $m = 0, \pm1, \pm2, \ldots$. The state $|\psi_{\mathrm{dark}}\rangle = |\xi\rangle_{kl}\otimes|g\rangle$ is a dark state in the sense that $\hat{H}_{\mathrm{I}}|\psi_{\mathrm{dark}}\rangle = 0$, i.e., the system does not evolve in such a state [15].

[1] Single-mode binomial states were first considered by Stoler *et al* [7], whereas the two-mode binomial states were first discussed by Dattoli *et al* [7].

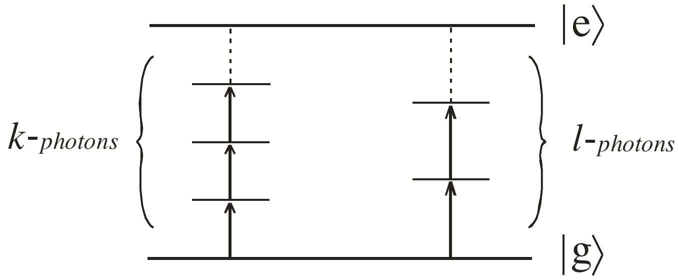

Figure 1. Competing k- and l-photon transitions in a two-level atom.

A rather general solution of equation (7) is of the form

$$|\xi\rangle_{kl} = \int \mathrm{d}^2\alpha\ \Phi(\alpha)|\alpha\rangle_a|(\alpha^k/\xi)^{1/l}\rangle_b, \tag{9}$$

where $|\alpha\rangle$ is a coherent state and $\Phi(\alpha)$ is some weight function. If, for example, the weight function is a delta function, $\Phi(\alpha) = \delta^{(2)}(\alpha-\beta)$, then our solution has the form $|\beta\rangle_a|(\beta^k/\xi)^{1/l}\rangle_b$, a tensor product of the coherent states of the two modes. But in general, the solutions are entangled states. The general density operator of the field modes is given by

$$\hat\rho^{(f)} = |\xi\rangle_{kl\,kl}\langle\xi| = \iint \mathrm{d}^2\alpha\,\mathrm{d}^2\beta\ \Phi(\alpha)\Phi^*(\beta)(|\alpha\rangle_{aa}\langle\beta|)\left(\left|\left(\frac{\alpha^k}{\xi}\right)^{1/l}\right\rangle_{bb}\left\langle\left(\frac{\beta^k}{\xi}\right)^{1/l}\right|\right). \tag{10}$$

The density operators of the individual modes a and b can easily be found to be given by

$$\hat\rho^{(a)} = \iint \mathrm{d}^2\alpha\,\mathrm{d}^2\beta\ \Phi(\alpha)\Phi^*(\beta)|\alpha\rangle_{aa}\langle\beta| \times \exp\left[\left(\frac{\alpha^k}{\xi}\right)^{1/l}\left(\frac{\beta^{*k}}{\xi^*}\right)^{1/l} - \frac{1}{2}\left|\left(\frac{\alpha^k}{\xi}\right)^{1/l}\right|^2 - \frac{1}{2}\left|\left(\frac{\beta^k}{\xi}\right)^{1/l}\right|^2\right], \tag{11}$$

and

$$\hat\rho^{(b)} = \iint \mathrm{d}^2\alpha\,\mathrm{d}^2\beta\ \Phi(\alpha)\Phi^*(\beta)\left|\left(\frac{\alpha^k}{\xi}\right)^{1/l}\right\rangle_{bb}\left\langle\left(\frac{\beta^k}{\xi}\right)^{1/l}\right|\exp\left[\alpha\beta^* - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\right], \tag{12}$$

respectively. Generally, this is a mixed state as $\hat\rho^{(f)} \neq \hat\rho^{(a)}\hat\rho^{(b)}$. Only in the 'symmetric' cases where $k = l$ and $\xi = 1$ do we obtain matching photon statistics in the two modes:

$$\begin{aligned}\hat\rho^{(a)} &= \iint \mathrm{d}^2\alpha\,\mathrm{d}^2\beta\ \Phi(\alpha)\Phi^*(\beta)|\alpha\rangle_{aa}\langle\beta|\exp\left[\alpha\beta^* - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\right],\\ \hat\rho^{(b)} &= \iint \mathrm{d}^2\alpha\,\mathrm{d}^2\beta\ \Phi(\alpha)\Phi^*(\beta)|\alpha\rangle_{bb}\langle\beta|\exp\left[\alpha\beta^* - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\right].\end{aligned} \tag{13}$$

Even under the condition for the matching of statistics we generally still have a mixed state.

In what follows, we wish to find specific solutions of equation (7) subject to the constraint associated with the constant of motion $\hat N_{\mathrm{E}} = \hat N_{kl} + kl|e\rangle\langle e|$, where $\hat N_{kl} = l\hat a^\dagger\hat a + k\hat b^\dagger\hat b$. It is easy to check that $\hat N_{\mathrm{E}}$ is indeed a constant of the motion. Thus we seek states that trap the atom in the ground state $|g\rangle$ which are, in addition to being solutions of equation (7), also eigenstates of the excitation operator or, as the atom is *in* the ground state, the operator $\hat N_{kl}$. We then study some of the properties of the resulting states.

2. Specific solutions

With a slight change in notation for later convenience, we wish to solve the equation

$$(\hat{a}^k - \xi\hat{b}^l)|\xi, x\rangle_{kl} = 0, \qquad k, l = 1, 2, \ldots, \tag{14}$$

where ξ is a complex number and x is any other quantum number required to specify the state, subject to the condition that $|\xi, x\rangle_{kl}$ is also an eigenstate of the operator $\hat{N}_{kl} = l\hat{a}^\dagger\hat{a} + k\hat{b}^\dagger\hat{b}$ such that $\hat{N}_{kl}|\xi, x\rangle_{kl} = \Lambda_{klx}|\xi, x\rangle_{kl}$, where Λ_{klx} is the eigenvalue. We assume that the required state has the form

$$|\xi, x\rangle_{kl} = \sum_{m,n} C_{m,n}^{(k,l)}|m\rangle_a|n\rangle_b = \sum_{m,n} C_{m,n}^{(k,l)}|m, n\rangle \tag{15}$$

where on the right-hand side we have used the short-hand notation $|m\rangle_a|n\rangle_b \equiv |m, n\rangle$. Substitution into equation (14) leads to the recurrence relation

$$C_{m+k,n}^{(k,l)} = \xi\left[\frac{m!}{n!}\frac{(n+l)!}{(m+k)!}\right]^{1/2} C_{m,n+l}^{(k,l)}. \tag{16}$$

But we have also

$$\hat{N}_{kl}|\xi, x\rangle_{kl} = \Lambda_{klx}|\xi, x\rangle_{kl} = \sum_{m,n}(lm + kn)C_{m,n}^{(l,k)}|m, n\rangle. \tag{17}$$

This equation is satisfied if we introduce the integers p and q, where $q = 0, 1, 2, \ldots, p$, and write $m = kq$ and $n = l(p - q)$ so that we have $\Lambda_{klp} = klp$, where we have set the quantum number $x = p$. (We could have made the choice $m = k(p - q)$ and $n = lq$ though the final result would have been the same.) Our two-mode states $|m, n\rangle$ are then restricted to the form $|kq, l(p - q)\rangle$, and the recurrence relation may be solved to yield

$$C_{kq,l(p-q)} = \xi^q\left[\frac{(lp)!}{(lp - lq)!(kq)!}\right]^{1/2} C_{0,lp}, \tag{18}$$

where $C_{0,lp}$ is determined from normalization to be

$$C_{0,lp} = \left[\sum_{q=0}^{p}|\xi|^{2q}\frac{(lp)!}{(lp - lq)!(kq)!}\right]^{-1/2} \equiv \mathcal{N}_{klp}. \tag{19}$$

Therefore, finally we may write our states as

$$|\xi, p\rangle_{kl} = \mathcal{N}_{klp}\sum_{q=0}^{p}\xi^q\left[\frac{(lp)!}{(lp - lq)!(kq)!}\right]^{1/2}|kq, l(p - q)\rangle. \tag{20}$$

It is perhaps worth pointing out that the factorials inside the square bracket do not constitute a binomial coefficient apart from the special case where $k = l = 1$. However, it is easy to see that, for $l \geqslant k$, we can write

$$\frac{(lp)!}{(lp - lq)!(kq)!} = \binom{lp}{lq}\frac{(lq)!}{(kq)!} = \binom{lp}{lq}\binom{lq}{kq}(lq - kq)!. \tag{21}$$

The expression in the middle is the product of the number of ways of choosing lq photons out of lp times the ratio of number of permutations of lq photons to the number of permutations of kq photons. On the right-hand side, this ratio is rewritten as the number of ways of choosing kq photons out of lq, and converting them to the type of photons in mode a, multiplied by the number of permutations of the discarded $lq - kq$ photons. In the cases where $l = k$, the numbers of equation (21) reduce to those of a single binomial coefficient. For the cases where $k \geqslant l$, we can rewrite the eigenvalue problem as

$$(\hat{b}^l - \eta\hat{a}^k)\widetilde{|\eta\rangle}_{lk} = 0, \tag{22}$$

where $\eta = 1/\xi$, and, using the method discussed above, develop the solution as

$$\begin{aligned}\widetilde{|\eta, p\rangle}_{lk} &= \tilde{\mathcal{N}}_{lkp} \sum_{q=0}^{p} \eta^q \left[\frac{(kp)!}{(kp-kq)!(lq)!}\right]^{1/2} |lq, k(p-q)\rangle \\ &= \tilde{\mathcal{N}}_{lkp} \sum_{q=0}^{p} \eta^q \left[\binom{kp}{kq}\binom{kq}{lq}(kq-lq)!\right]^{1/2} |lq, k(p-q)\rangle,\end{aligned} \tag{23}$$

where

$$\tilde{\mathcal{N}}_{lkp} = \left[\sum_{q=0}^{p} |\eta|^{2q} \frac{(kp)!}{(kp-kq)!(lq)!}\right]^{-1/2}. \tag{24}$$

Of course, we must have the equivalence $\widetilde{|\eta, p\rangle}_{lk} = |\xi, p\rangle_{kl}$. The coefficients in equation (23) have an interpretation similar to the case where $l \geqslant k$: the first binomial coefficient in equation (23) is the number of ways of choosing kq out kp photons, the second is the number of ways of choosing lq photons out of kq, while the factorial is the number of permutations of the discarded $kq - lq$ photons. In what follows, we shall take our states to be written in the form of equation (20). States of this form are evidently entangled. The degree to which they are entangled is examined below.

As an example of our states, let us consider the special case where $k = l = 1$. In this case, the normalization factor sums to the closed form $\mathcal{N}_{11p} = (1 + |\xi|^2)^{-p/2}$ and thus we have

$$|\xi, p\rangle_{kl} = (1 + |\xi|^2)^{-p/2} \sum_{q=0}^{p} \xi^q \left[\frac{p!}{(p-q)!q!}\right]^{1/2} |q, p-q\rangle. \tag{25}$$

This, as expected, is nothing but the $SU(2)$ coherent state where a total of p photons is binomially distributed over the two modes of the field. Note that $\hat{N}_{11} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$ is now just the total photon number so that $\hat{N}_{11}|\xi, p\rangle_{11} = p|\xi, p\rangle_{11}$. Also note that the product number states $|q, p-q\rangle = |q\rangle_a|p-q\rangle_b$ are eigenstates of the number difference operator $\hat{\Delta}_{11} = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$ such that $\hat{\Delta}_{11}|q, p-q\rangle = (2q-p)|q, p-q\rangle$. For the more general case, we introduce the operator $\hat{\Delta}_{kl} = l\hat{a}^\dagger\hat{a} - k\hat{b}^\dagger\hat{b}$ such that

$$\hat{\Delta}_{kl}|kq, l(p-q)\rangle = kl(2q-p)|kq, l(p-q)\rangle. \tag{26}$$

3. Properties of the states

We now study some of the properties of our states with the particular goal of exposing their nonclassical nature. But first we examine the joint photon number distributions given by $P_{n,m} = |\langle n, m|\xi, p\rangle_{kl}|^2$, this being the probability of finding n photons in mode a and m photons in mode b. The only nonzero joint probabilities are those given by

$$P_{kq,l(p-q)} = |\mathcal{N}_{klp}|^2 \frac{(lp)!}{(lp-lq)!(kq)!} |\xi|^{2q}. \tag{27}$$

To establish a baseline for the sake of later comparison, we first take the case for $k = l = 1$ as per the $SU(2)$ coherent states. The joint probability versus n and m is plotted in figure 2 for $p = 20$ and for (a) $\xi = 1$ and (b) for $\xi = 2$. We note that the populated states are

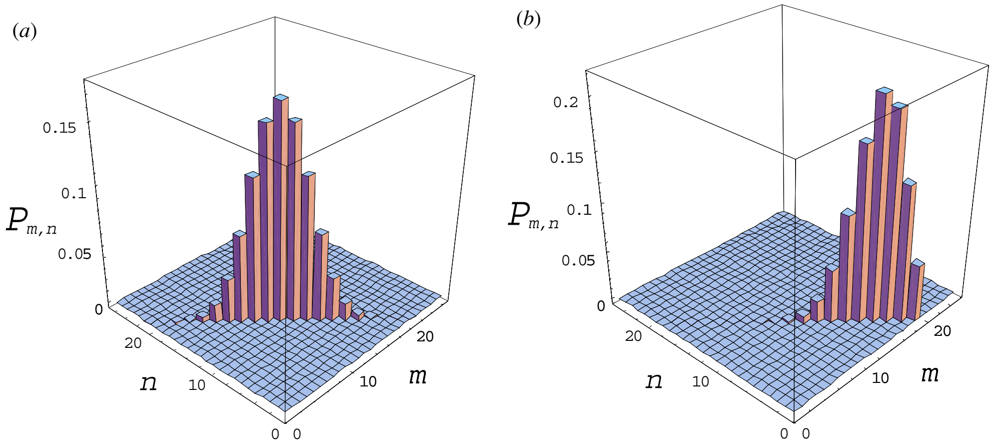

Figure 2. Joint probability distribution $P_{m,n}$ versus n and m for $k = l = 1$, $p = 20$ and for (*a*) $|\xi| = 1$, (*b*) $|\xi| = 2$.

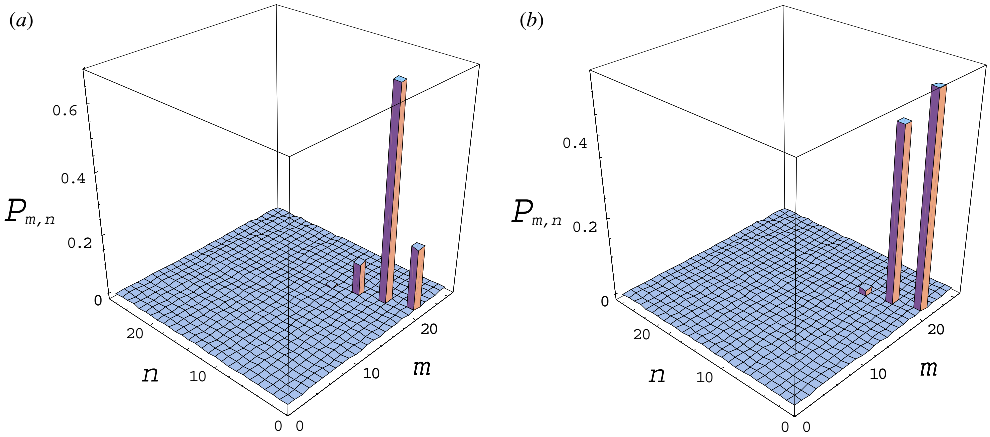

Figure 3. Joint probability distribution $P_{m,n}$ versus n and m for $k = 3$, $l = 1$, $p = 20$ and for (*a*) $|\xi| = 1$, (*b*) $|\xi| = 2$.

along a diagonal perpendicular to the line $n = m$. For $\xi = 1$, the population is symmetrically distributed about the diagonal. For $\xi = 2$, the distribution will be nonsymmetrical but, of course, still along the line perpendicular to the diagonal of the line $n = m$.

We next consider the case where $k = 3$ and $l = 1$, one of the systems studied in [4]. The joint photon number distribution $P_{n,m}$ for these states is plotted in figure 3. The distribution is, of course, still along a line, though the line is no longer perpendicular to the diagonal, and furthermore 'holes' in the distribution are evident.

As a last example, we present the joint photon number distribution for the case where $k = l = 2$ in figure 4. The fields in this case could be degenerate in frequency but have different polarizations.

To reveal the nonclassical nature of our states, we first examine the Mandel Q parameter defined for each mode as

$$Q_i = \frac{\langle n_i^2 \rangle - \langle n_i \rangle^2 - \langle n_i \rangle}{\langle n_i \rangle}, \qquad i = a, b, \tag{28}$$

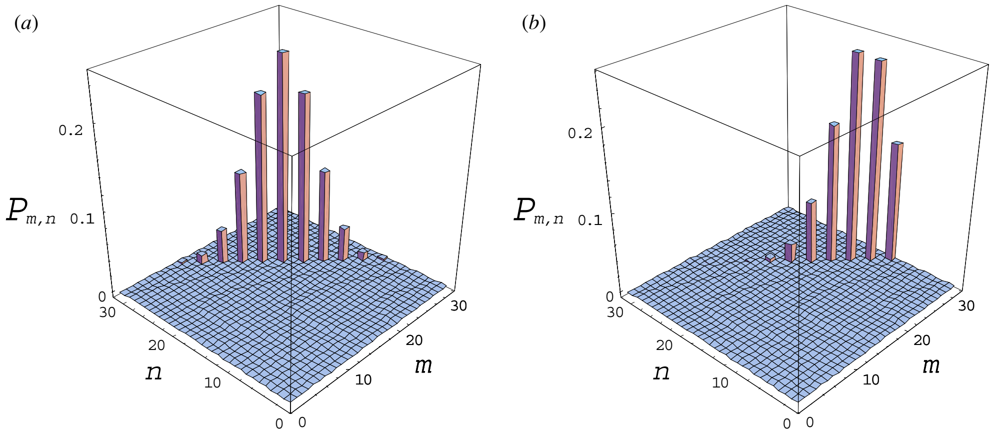

Figure 4. Joint probability distribution $P_{m,n}$ versus n and m for $k = l = 2$, $p = 20$ and for (*a*) $|\xi| = 1$, (*b*) $|\xi| = 2$.

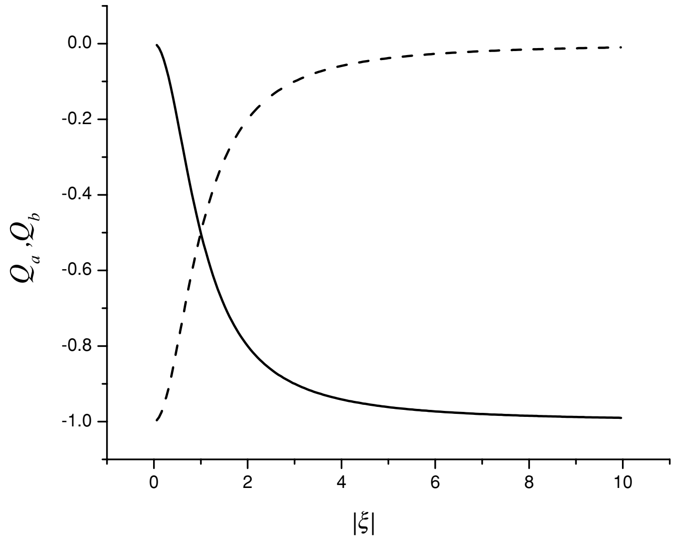

Figure 5. Q parameters for $k = l = 1$ and $p = 10$ as functions of $|\xi|$. Q_a is the solid line and Q_b is the dashed line.

where $n_a = a^\dagger a$ and $n_b = b^\dagger b$ are the number operators for each mode. For $-1 \leqslant Q_i < 0$, the photon statistics are sub-Poissonian, or amplitude, or number, squeezed, and hence are nonclassical. It is well known that for the $SU(2)$ coherent states, both modes are sub-Poissonian for the entire range of the parameter $|\xi|$ [8]. In fact, the Q parameters in that case can be calculated in closed form as [8]

$$Q_a = -\frac{|\xi|^2}{1 + |\xi|^2}, \qquad Q_b = -\frac{1}{1 + |\xi|^2}, \tag{29}$$

results that are independent of the total number of photons p (in our notation). In figures 5–7, we present our results for $k = l = 1$ and then $k = 3, l = 1$, and for $k = l = 2$, for $p = 10$ and for $p = 20$ as a function $|\xi|$. We note a striking difference between the first set of states and the second. In the former, the Q parameter for the a mode is always negative, going

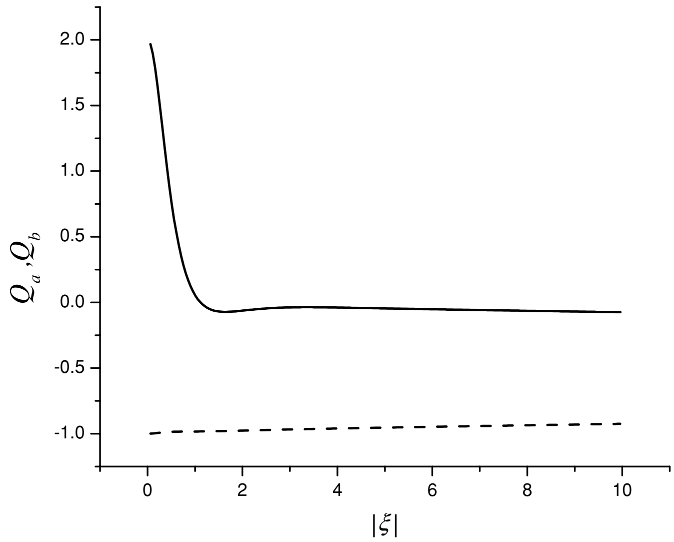

Figure 6. Q parameters for $k = 3$, $l = 1$ and $p = 10$ as functions of $|\xi|$. Q_a is the solid line and Q_b is the dashed line.

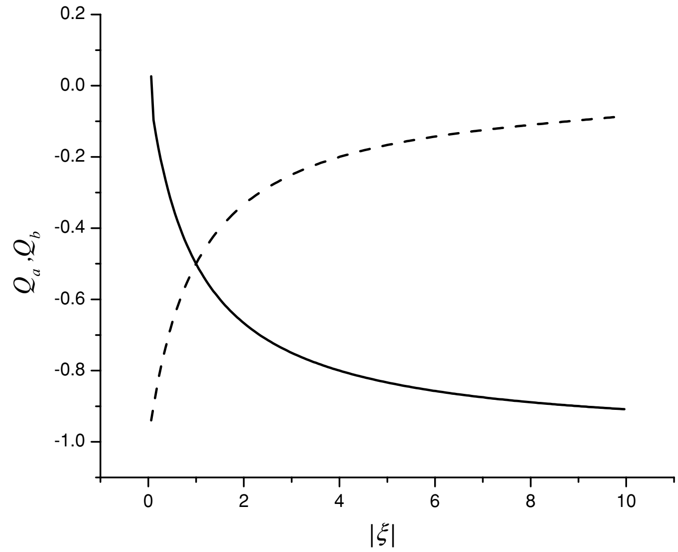

Figure 7. Q parameters for $k = l = 2$ and $p = 10$ as functions of $|\xi|$. Q_a is the solid line and Q_b is the dashed line.

asymptotically to -1 for large $|\xi|$, whereas in the latter we see that the parameter for mode a is positive for small $|\xi|$, dips slightly negative at $|\xi| = 1.3$ and levels off at $Q = 0$ for large $|\xi|$. For the b mode, the Q parameters both start from -1 but increase at different rates, the one for the $k = l = 1$ state going to zero much more rapidly with increasing $|\xi|$ than the state for $k = 3, l = 1$. For the third case where $k = l = 2$, we note similarities with the case $k = l = 1$ where, as in that case, the photon statistics of each mode are sub-Poissonian over the same range of ζ considered.

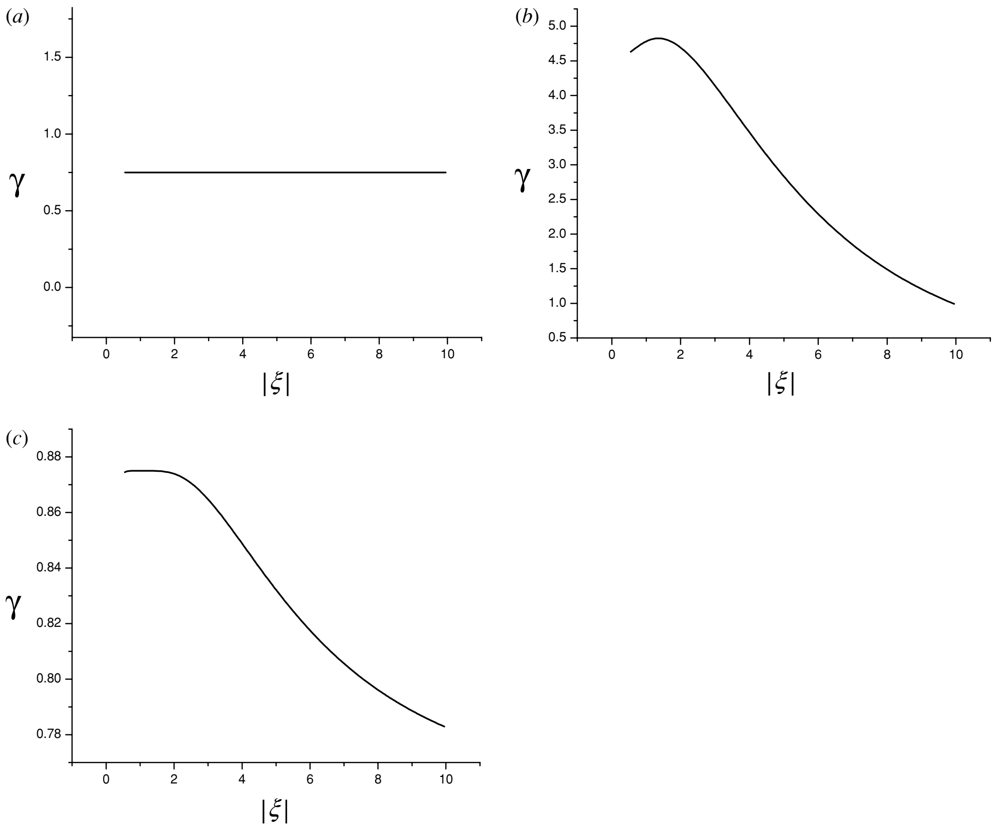

Figure 8. γ versus $|\xi|$ for (*a*) $k = l = 1, p = 4$; (*b*) $k = 3, l = 1, p = 4$; and (*c*) $k = l = 2, p = 4$.

Next we study the anticorrelation properties of these field states. To that end we consider the normalized cross-correlation function

$$\gamma = \frac{\langle a^\dagger b^\dagger ba\rangle}{\langle a^\dagger a\rangle\langle b^\dagger b\rangle}. \tag{30}$$

If $0 \leqslant \gamma < 1$, the field states are anticorrelated. Again, for the case of the $SU(2)$ coherent a closed form result may be obtained [8]:

$$\gamma = \frac{p-1}{p}, \tag{31}$$

which is less than 1 for any p but obviously approaches 1 as $p \to \infty$. Evidently, only for low values of p is there significant anti-correlation. The result is independent of $|\xi|$. In figures 8 and 9, we plot γ versus $|\xi|$ for the cases $k = l = 1$, $k = 3$, $l = 1$ and $k = l = 2$ for $p = 4$ and for $p = 20$, respectively. We use here a smaller value of p in order to compare with what happens in the case of larger values. As can be seen from figure 8(*a*), for $k = l = 1$, the parameter γ takes the value $\gamma = 3/4 = 0.75$, whereas for $p = 20$ one has $\gamma = 19/20 = 0.95$ in figure 9(*a*) meaning that the two modes are anti-correlated, but less so for higher values of p. For the case $k = 3, l = 1$, the states are not anti-correlated but are, in fact, highly correlated, becoming less so for higher $|\xi|$ for the case of $p = 4$ (figure 8(*b*)), whereas for $p = 20$ (figure 9(*b*)) there appear very strong correlations for high $|\xi|$. And finally, for the case $k = l = 2$, we see

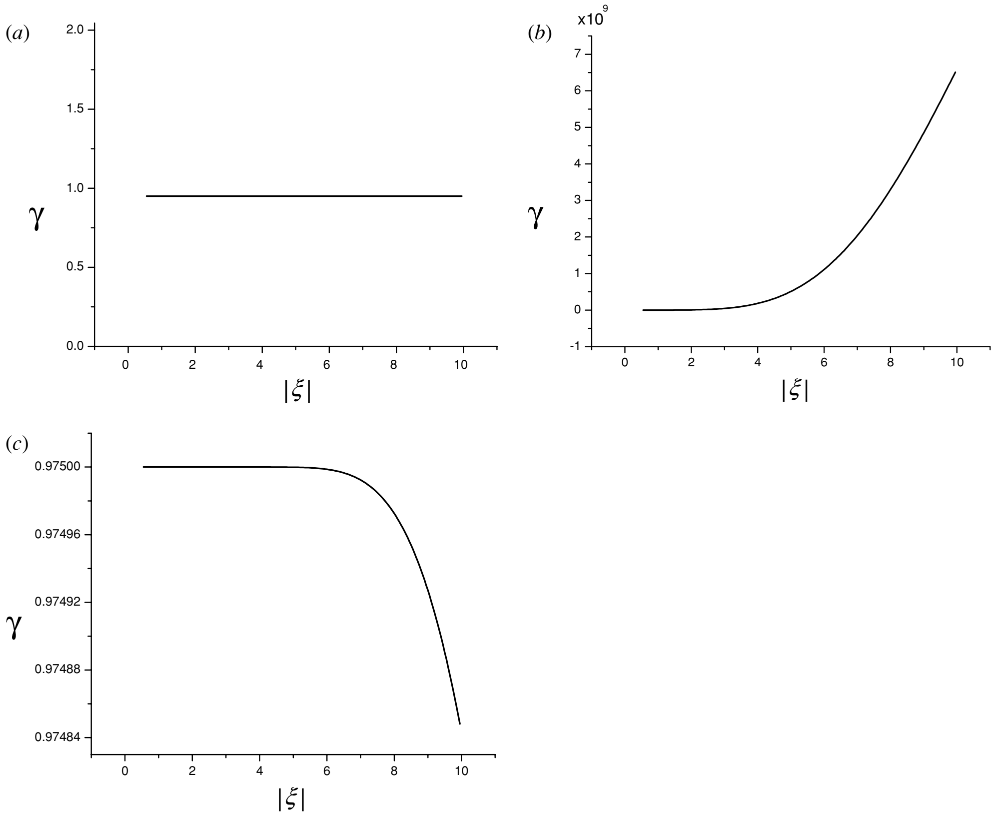

Figure 9. γ versus $|\xi|$ for (a) $k = l = 1, p = 20$; (b) $k = 3, l = 1, p = 20$; and (c) $k = l = 2, p = 20$.

that the states are anti-correlated, more so for the case with $p = 4$ (figure 8(c)) than for the case $p = 20$ (figure 9(c)) where the anti-correlations are practically nonexistent.

Quadrature squeezing does not exist in either of the modes of states. Two-mode states for which there are anti-correlations (or correlations, for that matter) between the modes cannot exhibit quadrature squeezing within the individual modes.

Finally, we examine the degree to which our states are entangled. Because the state vector of the two-mode system of equation (20) has already the form of a Schmidt decomposition [13], which, for any bipartite system in a pure state, is given by

$$|\psi\rangle = \sum_i g_i |u_i\rangle \otimes |v_i\rangle, \tag{32}$$

where the $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are orthogonal vectors from different Hilbert spaces $\mathcal{H}_u$ and $\mathcal{H}_v$, the corresponding reduced density operators for each of the subsystems are

$$\hat{\rho}_u = \sum_i |g_i|^2 |u_i\rangle\langle u_i|, \qquad \hat{\rho}_v = \sum_i |g_i|^2 |v_i\rangle\langle v_i|. \tag{33}$$

The von Neumann entropy

$$S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho}\ln\hat{\rho}] \tag{34}$$

for each of the subsystems will be the same:

$$S(\hat{\rho}_u) = S(\hat{\rho}_v) = -\sum_i |g_i|^2 \ln |g_i|^2. \tag{35}$$

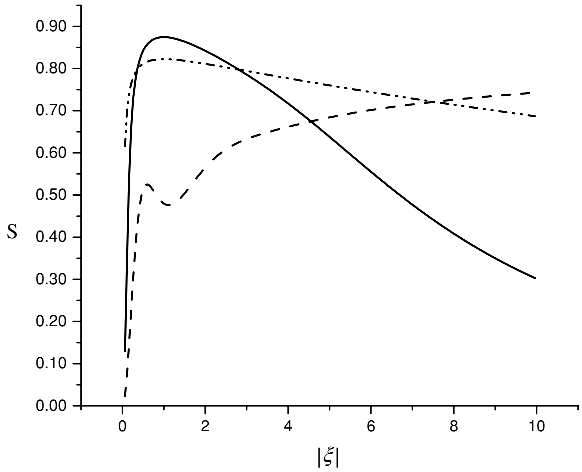

Figure 10. Entropy S versus $|\xi|$ for $p = 20$ for (*a*) $k = l = 1$ (solid line), (*b*) $k = 3, l = 1$ (dashed line), and (*c*) $k = l = 2$ (dot-dashed line).

For a pure state, $S = 0$, whereas for an entangled state of the entire system one has $S > 0$. For our states of equation (20), the von Neumann entropy is given by

$$S = -\sum_{q=0}^{p} |g_q|^2 \ln |g_q|^2, \tag{36}$$

where

$$g_q = \mathcal{N}_{klp}\xi^q \left[\frac{(lp)!}{(lp - lq)!(kq)!}\right]^{1/2}. \tag{37}$$

The results for the states we have considered above are presented in figure 10 for the case where $p = 20$. We find that the states for $k = l = 1$ tend to become entangled to a lesser degree for increasing $|\xi|$. For the case $k = l = 2$, the states exhibit a similar decrease in the degree of entanglement but at a much slower rate than for the previous case. And finally, for the case $k = 3, l = 1$, we note that the degree of entanglement increases for increasing $|\xi|$.

4. Conclusions

In this paper, we have studied a class of two-mode field states that may be considered as extensions of the already familiar $SU(2)$ coherent states of a two-mode field having a binomial photon number distribution. The states have been shown to possess strong nonclassical properties, particularly with respect to sub-Poissonian statistics in each of the modes and with entanglement. We have been entirely concerned with two-mode optical states, but it should be possible to generate such states in the two-dimensional vibrational motion of a trapped ion. This prospect is being investigated.

Our study should be considered as an extension of previous work by many authors on so-called higher power coherent states, as for example the work by Nieto and Truax [14] for a single-mode field and that of Jex *et al* in the case of two-mode fields [15]. In contrast to our work, the states studied in [15] are eigenstates of the *product* of powers of the annihilation operators, i.e., eigenstates of $\hat{a}^k\hat{b}^l$. The special case where $k = l = 1$ are the state known as the pair coherent states (they must also be eigenstates of the number difference operator $\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$) [16]. These states, unlike ours, are a superposition of an infinite number of product states where the product states are themselves correlated between each mode, whereas the product states for our states are anti-correlated. A possible extension to this work would be to study the generalization of equation (2) to the eigenvalue problem

$$[\hat{a}^k + \zeta(\hat{b}^\dagger)^l]|\zeta\rangle_{kl} = 0. \tag{38}$$

Such states could act as trapping states in a k-photon versus l-photon hyper-Raman system. This problem is under consideration and will be reported on elsewhere.

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