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White and Coloured External Noise and Transition Phenomena in Nonlinear Systems

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It is shown that in a system whose phenomenological description does not present any instability a transition can be induced by external noise. The class of systems in which such a phenomenon can occur is determined.

1. Introduction

The stability properties of non-equilibrium systems have become over the last decade a subject of widespread interest, primarily because of their implications for the understanding of cooperative phenomena in physics, chemistry, biology and even more remote fields [1–3]. Well-known examples of cooperative non-equilibrium systems are the laser [4, 5], the Belousov-Zhabotinskii reaction [6], the Bénard instability [7], the glycolytic oscillations [8, 9] and the current instabilities of semi-conductors [10]. The variety of dynamic and steady state behaviors in such systems is very broad. It has been explored essentially by perturbative methods derived from bifurcation theory [11], singular perturbation theory [12], group theory [13] and also by more qualitative approaches based on catastrophe theory [14, 15].

The common property of non-equilibrium cooperative phenomena is that they require some energy dissipation and occur via points of branching where the stability properties of various regimes change abruptly. Hence, it is to be expected that fluctuations play an important role in their mechnism of onset. Usually, it is assumed that the environment, i.e. the ensemble of external constraints acting on the system, does not fluctuate. The only fluctuations which are allowed and are susceptible to trigger the transitions between different regimes, are internal fluctuations. The latter originate from the statistical nature of the processes taking place inside the system and which involve many degrees of freedom. For example, the evolution of a chemical system can be seen as a succession of discrete random jumps in the numbers of reacting particles. The effect of these internal fluctuations is usually taken into account by assuming that the evolution of the system can be described by a Markovian birth and death process in the space of the total number of reacting particles [2, 3]. This leads to the so-called Master equation. Using this theory the influence of internal fluctuations has been extensively studied [16, 17] especially in the thermodynamic limit. It is found that in this limit, the stochastic description reduces to the deterministic one, except at first order transition points where macroscopic equations lose their validity [18–20].

In the phenomenological treatments as well as in the Master equation description, it is supposed that all appearing parameters are non-fluctuating quantities. However, a certain subset of them characterizes in general the influence of the environment which in many cases should be considered as randomly varying. Thus, to take the effect of this environmental noise into account, the phenomenological equations should be regarded as differential equations with random parameters. This approach can in principle be extended to the Master equation. However as mentioned above, internal fluctuations can be neglected in the thermodynamic limit (except at a first order transition point). It is therefore quite appropri-

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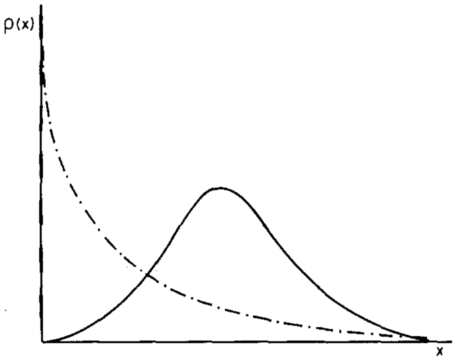

Fig. 1. Sketch of the probability density in the case of model (1.1). The continuous curve refers to small values of the variance. The mixte curve is obtained when the variance exceeds some critical value $\sigma^2/2>\lambda>0$. For $\lambda<0$ the density is a δ-function at zero (dashed curve)

ate in a first stage to analyze the influence of external noise at the level of phenomenological equations. A first example of non-linear macroscopic system corresponding to the following kinetic equation

$$\dot{x}=\tilde{\lambda}x-x^2 \tag{1.1}$$

has been investigated from this point of view recently [21]. $\tilde{\lambda}$ is supposed to be Gaussian white noise with a mean value λ and a variance σ^2*. The striking effect observed in this system and which is sketched in Figure 1, is that external noise may induce a *new* transition point in addition to the deterministic one at $\lambda=0$. This *qualitative* change in the macroscopic behaviour of the system was recently confirmed experimentally by Kabashima et al. [22] with an equivalent electrical circuit system. The results demonstrate the soundness of the above approach and particularly of the way to take into account the external noise. Thus it seems extremely worthwhile to pursue the study of external noise effects, especially in view of the qualitative new phenomena which can be expected.

An analysis of models exhibiting a first order transition between two simultaneously stable steady states [23–25] has revealed the existence of noise-induced phase transitions. It has been found that even above the deterministic critical transition point, the stationary probability density admits two maxima and one minimum if the variance of the fluctuating parameter lies within a certain interval. Our objective is to study the mechanism of such transitions more deeply and to characterize mathematically the class of systems in which they occur. In Section 2, we consider a minimal representative model system. After a brief description of its deterministic features, we study the effect of Gaussian white noise for which a complete analytical treatment is possible. In particular, the results illustrate the fact that transitions under the influence of external noise may occur in systems whose phenomenological equation *never* presents an instability point whatever the values of the parameters. In such cases, the breakdown of deterministic predictions is complete. An analysis of the influence of coloured noise follows in Section 3. It is argued that qualitatively the same phenomenon is observed, proving that we are not dealing with artefacts due to the idealisation of white noise. In Section 4, we discuss the general implications of noise in non-linear systems and establish a criterion to determine when transitions can be induced solely by external noise.

* Contrary to the internal fluctuations case where the variance is inversely proportional to the system's size, σ is taken here to be size independent. For a particular realization see [22]

2. Transitions Induced by External White Noise

a) We consider the phenomenological equation:

$$\dot{x}=\alpha-x+\tilde{\beta}x(1-x)=f(x) \tag{2.1}$$

where x can take values in the interval $[0,1]$. One possible realisation of this kinetic equation is the reaction scheme:

$$X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} Y$$

$$A+X+Y \xrightarrow{l_1} 2Y+A$$

$$B+X+Y \xrightarrow{l_2} 2X+B. \tag{2.2}$$

Obviously, the reactions conserve the total number of X and Y particles:

$$X+Y=N=\text{constant}. \tag{2.3}$$

Using this relation, it can easily be seen that (2.1) gives the time evolution of the fraction $x=X/N$ with:

$$\alpha=\frac{k_2}{k_1+k_2} \quad \text{and} \quad \tilde{\beta}=\frac{(l_2B-l_1A)/N}{k_1+k_2}. \tag{2.4}$$

Without restricting the generality of the following analysis the symmetrical case will be chosen:

$$\dot{x}=\tfrac{1}{2}-x+\tilde{\beta}x(1-x)=f(x). \tag{2.1a}$$

For the stationary states of (2.1a) we have the relation:

$$\tilde{\beta}=\frac{x_s-\frac{1}{2}}{x_s(1-x_s)} \tag{2.5}$$

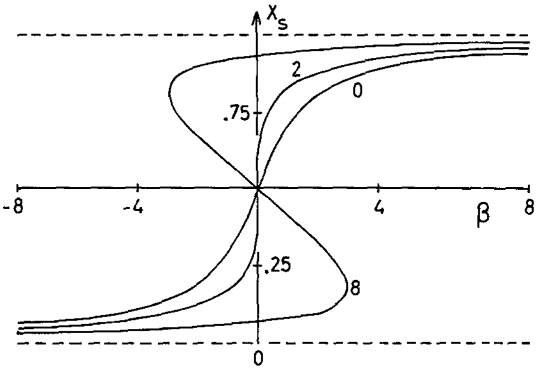

Fig. 2. Extrema of the probability density (2.12) as a function of β and for three values of the variance

which is depicted in Figure 2 (the curve labelled 0). The stability properties of x_s are determined from the relation:

$$\omega(x_s) = f'(x)|_{x_s} \tag{2.6}$$

x_s is stable, if $\omega(x_s) < 0$. From (2.1a), using (2.5), we have:

$$\omega(x_s) = -\frac{x_s^2 - x_s + \frac{1}{2}}{x_s(1-x_s)}. \tag{2.7}$$

It is easily verified that ω is negative definite for any $x_s \in [0,1]$. Thus the above model does not display any instability.

b) We will now analyze the influence of fluctuations in the concentrations of the catalysts A and B. In this section we will assume that the changes occur on a much shorter time scale than the evolution of the macroscopic system. Therefore we can make the idealization of white noise and consider that the parameter $\tilde{\beta}$ in (2.1a), as a fluctuating quantity given by a Gaussian white noise with mean β and variance σ^2. Thus we arrive at the following stochastic differential equation:

$$dx_t = \{\tfrac{1}{2} - x_t + \beta x_t(1-x_t)\}\, dt + \sigma x_t(1-x_t)\, dW_t \tag{2.8}$$

W_t denotes the Wiener process (Brownian motion) the derivative of which, in the sense of generalized functions, is Gaussian white noise: $\dot{W}_t = \xi_t$, $E\xi_t = 0$, $E\xi_t\xi_{t'} = \delta(t-t')$. We will interpret the stochastic differential equation (2.8) as an Ito-equation [26]. It is easily verified that the results of the following analysis do not change qualitatively if (2.8) would be interpreted in the sense of Stratonovic.

To the Ito-equation (2.8) corresponds the Fokker-Planck equation:

$$\partial_t p(x) = -\partial_x\{\tfrac{1}{2} - x + \beta x(1-x)\}\, p(x) + \tfrac{1}{2}\sigma^2\, \partial_{xx} x^2(1-x)^2\, p(x). \tag{2.9}$$

The stationary solution of this Fokker-Planck equation, which is further written in the general form:

$$\partial_t p = -\partial_x f(x)\, p + \tfrac{1}{2}\sigma^2\, \partial_{xx} G^2(x)\, p \tag{2.10}$$

is given by:

$$p_s(x) = \frac{1}{G^2(x)} \exp \frac{2}{\sigma^2} \int \frac{f(x)}{G^2(x)}\, dx \tag{2.11}$$

provided $\int_{r_1}^{r_2} p_s(x)\, dx$ is finite and the boundaries r_1 and r_2 be natural boundaries. These conditions are fulfilled for (2.9) and we obtain ($\mathcal{N}$: norm):

$$p_s(x) = \frac{\mathcal{N}}{x^2(1-x)^2} \cdot \exp \frac{2}{\sigma^2}\left(-\frac{1}{2x(1-x)} - \beta \ln\left(\frac{1-x}{x}\right)\right). \tag{2.12}$$

The extrema x_m of $p_s(x)$ can be calculated from the relation:

$$f(x_m) - \sigma^2 G(x_m)\, G'(x_m) = 0. \tag{2.13}$$

We obtain:

$$\tfrac{1}{2} - x_m + \beta x_m(1-x_m) - \sigma^2 x_m(1-x_m)(1-2x_m) = 0. \tag{2.14}$$

For simplicity let us discuss the case that $\beta = 0$, i.e. the stationary solution of the phenomenological equation (2.1a) is $x_s = 1/2$. Equation (2.14) yields:

$$x_{m_1} = 1/2 \quad \text{and} \quad x_{m_\pm} = \{1 \pm \sqrt{1 - 2/\sigma^2}\}/2. \tag{2.15}$$

Thus for $\sigma^2 > 2$ the stationary probability distribution possesses three extrema. Since $p_s(x)$ tends to zero for $x \to 0$ and $x \to 1$ we have the following situation, see Figure 2: For $\sigma^2 < 2$, $x_{m_1} = 1/2$ is a maximum. For $\sigma^2 > 2$, $x_{m_1} = 1/2$ becomes a minimum and two maxima appear at $x_{m_\pm}$ which tend to zero resp. to one as σ^2 tends to infinity. The situation is qualitatively the same for the unsymmetric case of $\beta \neq 0$. Even if the deterministic steady state solution lies close to either one of the boundaries, nevertheless the probability distribution will always become bimodal once σ^2 becomes larger than some critical value. The latter increases with $|\beta|$. This model thus always exhibits a transition solely triggered by external noise. For $\beta = 0$, this transition is a soft one. At $\sigma_c^2 = 2$, $x_{m_1} = 1/2$ is a triple root and the distance between x_{m+} and x_{m-} tends to zero like $(\sigma^2 - \sigma_c^2)^{1/2}/\sigma$ for $\sigma^2 \downarrow \sigma_c^2$. For $\beta \neq 0$ the transition is a hard one as can be seen from Figure 2. For $\beta > 0$ ($\beta < 0$) the peak corresponding to the steady state of the deterministic equation (2.1a) moves towards 1 (towards 0) with growing σ^2 and if σ^2 depasses $\sigma_c^2(|\beta|) > 2 = \sigma_c^2(0)$ a second peak appears at a finite distance from the original one, near the other boundary of the state space. If we keep σ^2 fixed and bigger than 2 and vary β along the real line, the situation ressembles a first order transition as is clear from the sigmoidal form of the curve for the extrema of $p_s(x)$, e.g. $\sigma^2=8$. The above facts can be summarized in the statement that, as to the extrema of $p_s(x)$, in the (β, σ^2) half plane we have a cusp catastrophe with critical point at (0, 2).

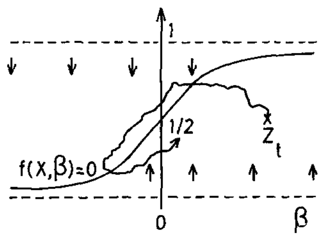

Fig. 3. State space with switching curve for the process $z_t=(\beta_t, x_t)$ satisfying equations (3.5), (3.6)

3. Transition Induced by External Coloured Noise

To study the case of white noise is useful mainly because the solution of a differential equation:

$$\dot{x}=f(x, \beta_t) \tag{3.1}$$

with β_t being white noise, is a Markov process for which powerful analytical methods exist. However, in some cases this idealization might be inadequate and it might be necessary to take into account the effect of the finite correlation time of the fluctuations of the surrounding.* In such cases, the external noise can often be modelled as the output of a "filter" having white noise as input. Thus β_t can then be assumed to be coloured noise, i.e. a stationary ergodic Markov process described by the Ito-stochastic differential equation:

$$d\beta_t=\alpha(\beta_t)\,dt+\sigma(\beta_t)\,dW_t \tag{3.2}$$

or, more generally, real noise by which we mean any stationary, non-markovian process.

In this section, we treat the coloured noise case. Although x_t is not Markovian anymore, it turns out that the pair $z_t=(\beta_t, x_t)$ is a Markov diffusion process satisfying the system of Equations (3.1) and (3.2). In Section 2, we gave the necessary and sufficient condition under which (3.2) has a stationary and ergodic solution, as well as the explicit form of the stationary density $\bar{p}_s(\beta)$ (cf. Eq. (2.11)). We assume that this condition is satisfied. Since there is no diffusion term in (3.1), the 2-dimensional diffusion process z_t is *degenerate*. Its state space Z is a part of the (β, x)-plane $\mathbb{R}\times\mathbb{R}$. The curves $f(x, \beta)=0$ are called *switching curves* for the drift term in the x-direction. The drift indeed changes sign on these curves. The process z_t can cross these curves only horizontally since there is no noise term in the x direction. This situation is illustrated in the Figure 3.

We would like to study the long term behavior of the non-Markovian x_t, but we are forced to do it via the investigation of the long term behavior of the Markovian pair $z_t=(\beta_t, x_t)$, since only the *pair* is amenable for analysis. The Fokker-Planck equation, in the stationary case, for the pair is:

$$\partial_x\{f(x,\beta)\,p(\beta,x)\}+\partial_\beta\{\alpha(\beta)\,p(\beta,x)\}$$
$$=\tfrac{1}{2}\partial_{\beta\beta}\{\sigma^2(\beta)\,p(\beta,x)\}. \tag{3.3}$$

It is in general extremely difficult to decide whether this equation has a unique 2-dimensional probability density $p_s(\beta, x)$ as solution, i.e. whether the system (3.1), (3.2) has a unique stationary process $z_t^0=(\beta_t, x_t^0)$ as solution. If so, then x_t^0 is stationary with a density $q_s(x)$, and:

$$\int_{x\in\mathbb{R}} p_s(\beta,x)\,dx=p_s(\beta), \qquad \int_{\beta\in\mathbb{R}} p_s(\beta,x)\,d\beta=q_s(x). \tag{3.4}$$

Our task is complicated by the fact that z_t is degenerate and thus the ergodic theory by Hasminski [27] for non-degenerate multidimensional diffusion is not directly applicable. However in some cases, this theory can be shown to hold even for systems with degenerate diffusion [28, 29].

To be more specific, let us assume that β_t is the Ornstein-Uhlenbeck process, i.e. a stationary ergodic Gaussian process with mean 0 and covariance $E\beta_s\beta_t=(\sigma^2/2\alpha)\exp(-\alpha|t-s|)$, $\alpha>0$, and f is the function (2.1), so that (3.1) and (3.2) become:

$$d\beta_t=-\alpha\beta_t\,dt+\sigma\,dW_t, \qquad \alpha, \sigma>0 \tag{3.5}$$

$$dx_t=\{(\tfrac{1}{2}-x_t)-\beta_t x_t(1-x_t)\}\,dt. \tag{3.6}$$

The process $z_t=(\beta_t, x_t)$ remains forever in the set $Z=\mathbb{R}\times[0,1]$ once we start there because $f(0,\beta)=1/2>0$ and $f(1,\beta)=-\frac{1}{2}<0$ so that the lines $x=0$ and $x=1$ can never be crossed. Thus, Z is an invariant set of states and can be interpreted without any difficulty as the state space which for physical reason should be restricted to this set.

We now sketch the proof that there is a unique stationary probability distribution

$$\mu(A,B)=\int_A\int_B p_s(\beta,x)\,dx\,d\beta \tag{3.7}$$

* If f is nonlinear in β_t the concept of white noise does not make sense.

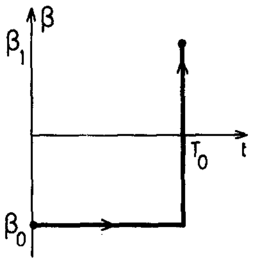

Fig. 4

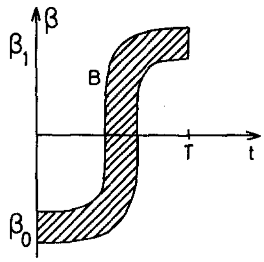

Fig. 5

in Z, so that, if we start with initial distribution μ, the process z_t^0 and thus x_t^0 become stationary.

That there is at least one stationary distribution μ in Z follows from the fact that any solution of (3.6) with $x_0 \in [0, 1]$ is bounded (Ref. 27, p. 74). It is the uniqueness of μ which is non-trivial.

The uniqueness of μ follows from the fact that z_t^0 runs from any neighborhood U_0 in Z to any other neighborhood U_1 infinitely often with probability one, so that the mean waiting time is finite, i.e. z_t^0 is positive recurrent. Then Z is a *minimal* invariant set of states, i.e. there is no smaller closed set inside Z such that μ vanishes outside, and z_t^0 spends in each neighborhood in Z a positive portion of time.

To see this, fix the following situation (without loss of generality): an ideal transition from the neighborhood of z_0 to the neighborhood of z_1 would be the curve going from $z_0 = (\beta_0, x_0)$ straight to (β_0, x_1) and from there straight to $z_1 = (\beta_1, x_1)$. This path would amount to a noise trajectory β_t as shown in Figure 4. Here T_0 is finite, non-random and depends only on β_0, x_0, and x_1. Such a β_t is impossible. But we can find by a continuity argument in the vicinity of the idealized trajectory a whole tube B of β_t's taking the 2-dimensional process from U_0 in time $T > T_0$ to U_1. Consider this tube B as a subset of the space of continuous functions $C[0, T]$ endowed with the probability measure μ_β resulting from $p_s(\beta)$. This set has positive probability, $\mu_\beta(B) > 0$, since μ_β is equivalent to Wiener measure (Ref. 30, p. 86) giving positive probability to every open set. Since μ_β is invariant with respect to time shifts in function space, it will happen infinitely often for almost every trajectory of β_t that a piece of it fits into the tube B (cf. Fig. 5) and that the mean waiting time between such events is finite since β_t is positive recurrent. Thus z_t^0 will run infinitely often from U_0 to U_1 with finite mean waiting time.

Knowing this, we can completely carry over Hasminski's construction (Ref. 27, p. 153–180) giving us in our case the following result: there is a unique stationary 2-dimensional probability measure μ in Z having a density $p_s(\beta, x)$. This density is the unique non-negative bounded solution of the Fokker-Planck

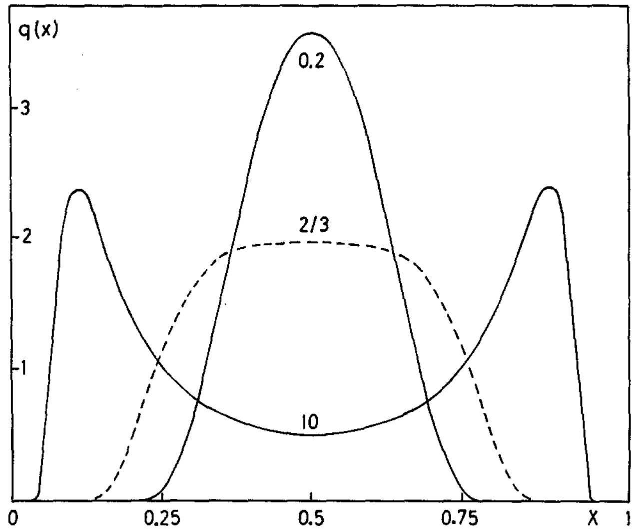

Fig. 6. Plot of $q(x)$ demonstrating the appearance of a bimodal distribution as a result of the effect of increasing the variance of a source of coloured noise. The critical variance is equal to 2/3

equation (3.3). We have for any solution z_t of (3.5) and (3.6) and any integrable g the law of large numbers (ergodic theorem)

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g(z_t)\, dt = \iint_Z g(\beta, x)\, p_s(\beta, x)\, d\beta\, dx \tag{3.8}$$

with probability one, and for the transition probability density of z_t $\lim_{t \to \infty} p(z_1, t, z) = p_s(\beta, x)$, $z = (\beta, x)$.

The ergodic theorem (3.8) enables us to determine all quantities related to the stationary density p_s, e.g. p_s itself, just by observing an arbitrary single trajectory of the 2-dimensional process long enough.

Equation (3.6) thus has exactly one stationary solution x_t^0 which is stationarily connected with the noise β_t, and

$$q_s(x) = \int_{\beta = -\infty}^{\infty} p_s(\beta, x)\, d\beta, \qquad 0 \leqq x \leqq 1.$$

The form of q_s can be found by the following arguments. The process (β_t, x_t^0) will stay a large amount of time around the switching curve $f(x, \beta) = 0$ or $x = h(\beta)$ with $\beta = h^{-1}(x)$ given by (2.5). Therefore, a first approximation of $q_s(x)$ will be the density $q(x)$ of the variable $h(\beta_t)$: $q(x) = p_s(h^{-1}(x))\, |h^{-1}(x)'|$. Since β_t is $\mathfrak{N}(0, \sigma^2)$, we obtain:

$$q(x) = \frac{(1/2 - x)^2 + 1/4}{\sqrt{2\pi}\, \sigma x^2 (1 - x)^2} \exp\left\{ -\left(\frac{1/2 - x}{x(1 - x)} \right) \Big/ (2\sigma^2) \right\}$$

with $0 \leqq x \leqq 1$. This density satisfies $q(0) = q(1) = 0$, $q(1/2) = 4/(\sqrt{2\pi}\, \sigma)$ and looks as represented in Figure 6.

This behavior can be intuitively explained as follows: the Gaussian distribution $\mathfrak{N}(0, \sigma^2)$ of β_t is being transformed by h into a distribution on $[0,1]$. For small σ^2 the probability mass is concentrated around 1/2, while for large σ^2 the curve h maps most of the mass into the neighborhood of 0 and 1.

The q_s will essentially show the same behavior as the approximation q, with the difference that separation for large σ^2 will be less extreme, since if the fluctuation of β_t is very strong, the x_t^0-component will have less time to approach the switching curve $x = h(\beta)$.

4. Conclusions

The system studied in Sections 2 and 3, belongs to a class of systems which can be characterized as follows. The (x, β) phase plane contains a switching curve $f(x, \beta) = 0$ or equivalently $x = h(\beta)$ which has finite asymptotic values:

$$\lim_{\beta \to -\infty} h(\beta) = x_1, \qquad \lim_{\beta \to -\infty} h(\beta) = x_0.$$

Supposing that $x_1 > x_0$ with $f(x_0, \beta) > 0$ and $f(x_1, \beta) < 0$, this situation may be represented as in Figure 7. If the switching curve $h(\beta)$ is monotone, it can be interpreted as a distribution function of some probability density $\tilde{p}(\beta)$, i.e.

$$h(\beta) = \int_{-\infty}^{\beta} \tilde{p}(\beta')\, d\beta'. \tag{4.1}$$

If β_t is chosen to be distributed according to this probability density, then the first approximation $q(x)$ to the stationary density of x, is the uniform distribution on $[x_0, x_1]$. Thus the onset of a double peaked distribution will occur for that σ^2 for which the actually chosen distribution of β_t best resembles the distribution function related to h.

These arguments can readily be applied to the class of systems whose switching curve presents only one asymptote, e.g. if:

$$\lim_{\beta \to -\infty} h(\beta) \text{ diverges}, \qquad \lim_{\beta \to +\infty} h(\beta) = 0.$$

The probability distribution always becomes single peaked on increasing σ^2. Indeed some finite part of the probability mass is mapped into a narrowing domain near zero, accounting for the growth of a peak in this neighborhood; the remaining part of the probability mass being spread over some x interval $[a, \infty]$, where a depends on the mean value $\tilde{\beta}$. This spreading results in the disappearance of any peak which could exist in this interval $[a, \infty]$ for small values of σ^2. Thus, these systems have the interesting property that on increasing σ^2, the distribution may

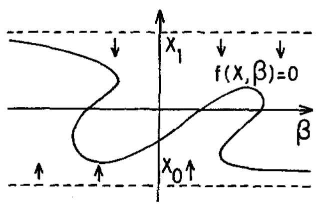

Fig. 7

successively become single peaked around some $x_1 \in [a, \infty]$, double peaked around $x_0 \in [0, a]$, $x_1 \in [a, \infty]$ and finally single peaked around $x_0 \in [0, a]$. Such systems as Schlögl's chemical reaction scheme [31] or some recently investigated predator-prey systems [23, 24] belong to this class of systems.

From a physical point of view, these results also permit to stress that the thermodynamic conditions compatible with the occurrence of dissipative structures and non-equilibrium cooperative processes is remarkably wider than the conditions currently investigated. We already mentioned in the introduction that these phenomena are critically dependent on a minimal level of energy dissipation: they occur via instabilities due to far from equilibrium environmental constraints. This result underlies the whole of our understanding of the relation between order and energy dissipation in macroscopic systems. It is however important to realize that strictly speaking this thermodynamic requirement does not imply, as it is usually and conveniently done, that external constraints must be constant in time and correspond *at each instant* to large deviations from equilibrium. Self-organization is possible under much weaker environmental constraints once the latter are fluctuating quantities. Remarkably, in such situations their mean value is less important than their variance. Consequently, non-equilibrium order-disorder transitions become possible *even if on the average the environment is at equilibrium* (for a simple chemical example see Ref. 24). Furthermore our results also examplify that such transitions are not critically dependent on the choice of a particular kind of noise source. This suggests that these noise induced mechanisms of self-organization may have a wide range of applicability. Particularly they may provide a physical basis for the understanding of many ordering processes taking place in complex biological systems and in which it is well-known that fluctuating environmental factors play an important role.

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