The empirical content of Cournot competition*

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Abstract

We consider the testable implications of the Cournot model of market competition. Our approach is nonparametric in that we abstain from imposing any functional specification on market demand and firm cost functions. We derive necessary and sufficient conditions for (reduced form) equilibrium market price and quantity functions to be consistent with the Cournot model. In addition, we present identification results for the corresponding inverse market demand function and the firm cost functions. Finally, we use our approach to derive testable restrictions for the models of perfect competition, collusion and conjectural variations. This identifies the conditions under which these different models are empirically distinguishable from the Cournot model.

Keywords: Cournot competition, perfect competition, collusion, conjectural variations, testable implications, nonparametric

JEL classification numbers: D21, D22, D24

1 Introduction

We present the testable implications of the Cournot model of market competition. Our approach is nonparametric in the sense that it does not require a functional specification for the inverse market demand function and the firm cost functions. Our results allow us to define empirical tests for the model of Cournot competition. Further, we establish identification results for the inverse market demand and the firm cost functions that apply to the Cournot model. Finally, we demonstrate the versatility of our framework by using the same approach to derive

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testable restrictions for alternative models of firm competition, such as perfect competition, perfect collusion (or cartel/monopoly) and conjectural variations models. This also identifies the conditions under which these different models are empirically distinguishable from the Cournot model.

Motivation. We consider a market that trades a homogeneous good. The definition of the market equilibrium then builds on three primitives. Firstly, the inverse market demand defines the market price as a function of the aggregate output and a vector of exogenous variables (covariates), which we refer to as demand shifters; prime examples of demand shifters are the consumers' income, the size of the population, various taste parameters, taxes, expectations of prices for complements/substitutes and future income, etc. Secondly, firm cost functions associate a minimal cost with each producible output. In general, these functions also depend on a vector of supply shifters, such as the factor input prices, production technology parameters, taxes (on input prices), etc. Finally, the specific market structure defines the way in which firms interact with each other (for a given market demand). In this respect, alternative models of firm competition make different assumptions regarding the degree of inter-firm cooperation (from perfect competition to perfect collusion), the time frame (static or dynamic), and the decision variables (prices or quantities) on the basis of which firms compete.

In what follows, our main focus will be on the Cournot model of firm competition. This focus hardly needs any motivation. Historically, the Cournot model was the first theoretical model of modern game theoretic reasoning. In addition, and even more importantly, the model still remains a most important and most widely used model in the literature on industrial organization and international trade. The Cournot model assumes that each firm chooses a profit maximizing output quantity for given inverse market demand and output decisions of the other firms. An appealing feature of the model is that, even though it is fairly simple, it does generate an equilibrium outcome with many attractive features. The model predicts an outcome of prices and aggregate output that is situated between the equilibria predicted by the models of perfect competition and perfect collusion. Moreover, it is able to explain the presence of different firms with strict positive mark-ups and different cost structures, which in turn leads to different market shares.

The theoretical properties of the Cournot equilibrium (such as existence, uniqueness and stability) have been studied extensively and are well understood by now.[1] However, the popularity of the Cournot model in the theoretical literature stands in sharp contrast with the limited attention that went to its empirical implications. Somewhat surprisingly, it turns out that very little is known about the empirically testable restrictions that are imposed by the Cournot model. In this respect, a noteworthy exception is the recent study of Carvajal, Deb, Fenske, and Quah (2010), who use revealed preference techniques (in the tradition of Afriat (1972) and Varian (1984)) to derive testable conditions for a finite data set containing prices and quantities to be consistent with the Cournot model. In the current paper, we complement these authors' work by concentrating on the differential implications of the Cournot model. The difference between our differential approach and the revealed preference approach is that we focus on

[1] See, for example Hahn (1962), Szidarovsky and Yakowitz (1977), Nishimura and Friedman (1981), Novshek (1985), Kolstad and Mathiesen (1987), Gaudet and Salant (1991) and Long and Soubeyran (2000).

properties of (reduced form) equilibrium market price and quantity functions rather than a finite set of prices and quantities.

Contribution. In what follows, we assume an empirical analyst who observes (or knows) the (reduced form) equilibrium market prices and output quantities as a function of some exogenous supply and demand shifters (covariates). We will derive necessary and sufficient conditions for these price and quantity functions to be consistent with the Cournot model (for some inverse market demand and firm cost functions). At this point, it is worth emphasizing that the conditions we develop below are independent of the functional/parametric structure of the underlying inverse demand and cost functions: these conditions apply to each possible specification of this structure if the Cournot model is to hold. In this sense, our approach is nonparametric.

Our specific contributions are the following. First, in Section 2 we characterize the Cournot model by three sets of testable conditions on the equilibrium price and quantity functions. The first set of conditions results from the homogeneous good assumption. As such, these conditions are not specific to the Cournot model per se but apply to any model of market competition that assumes a homogeneous good. Essentially, the conditions express that variation in the supply shifters can only influence the equilibrium prices through the firms' output. The second set of conditions is particular to the Cournot model. These conditions build on the fact that variation in the demand shifters can impact on the marginal cost function only through the firms' output quantities. The way in which this happens depends on the specificity of the Cournot model. The third set of conditions embed the second order conditions for a local optimum. At the end of Section 2, we also show that our framework can be used to identify the underlying structure of the model (i.e. the inverse market demand and firm cost functions) in case the equilibrium price and quantity functions satisfy the three sets of conditions mentioned above.

In Section 3, we demonstrate the versatility of our framework by deriving necessary and sufficient testable implications of other frequently used models of firm competition. Specifically, we consider the models of perfect competition and collusion as well as the conjectural variations model (i.e. a popular model in the literature on new empirical industrial economics). Like before, we define the (necessary and sufficient) conditions on the equilibrium price and quantity functions for consistency with these models. In turn, this makes it possible to empirically distinguish the model of Cournot competition from these other models of firm behavior.

In Section 4 we illustrate the practical application of our theoretical results. Specifically, we derive the testable implications of the Cournot model for a simple specification of the equilibrium price and quantity functions. For the given specification, we also demonstrate that the Cournot model is empirically distinguishable from the other models of firm competition considered in Section 3.

Summarizing, by deriving the (nonparametric) testable implications of various models of firm behavior on the basis of equilibrium price and quantity functions, this paper takes a natural first step towards a fully integrated approach for testing alternative models of inter-firm competition in real-life settings. In the concluding Section 5, we provide some further discussion of issues related to the practical application of our results. First, we consider the estimation of the equilibrium price and quantity functions from observational data. Next, we discuss the possibility of using our approach to empirically verify specific restrictions on cost and profit functions that are frequently employed in the literature. This will provide a further illustration of the versatility of the framework set out here.

2 Characterizing the Cournot Model

Subsection 2.1 sets the stage by providing a short outline of the Cournot model and the empirical framework we have in mind. Here, we will also introduce some necessary notations, definitions and assumptions. In Subsection 2.2 we move on to the actual characterization of the Cournot model. Finally, in Subsection 2.3 we present (local) identification results.

2.1 The Cournot model

The Cournot model pertains to a market with a single homogeneous good that is produced by N distinct firms. The demand side of the market is determined by a (sufficiently smooth) inverse demand function $P(Q, \mathbf{z})$. The variable Q is the amount of output supplied to the market and $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is an n-dimensional vector of exogenous variables that affect the industry demand, i.e. the demand shifters. We denote by Q_i the output of firm i. By construction, we have $Q = \sum_{i=1}^{N} Q_i$. As usual, we assume that the inverse demand function $P(Q, \mathbf{z})$ is decreasing in Q. Further, each firm $i \leq N$ has a (sufficiently smooth) cost function $C_i(Q_i, \mathbf{w})$, which gives the cost incurred by firm i for producing the output quantity Q_i. The vector $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{R}^m$ is a vector of exogenous variables that influence the firms' costs, i.e. supply shifters.

In general, the vectors $\mathbf{z}$ and $\mathbf{w}$ may have some variables in common. Then, we get that some variables exclusively influence the inverse demand function P (i.e. exclusive demand shifters), while other variables exclusively influence the cost functions C_i (i.e. exclusive supply shifters), and a few variables that influence both the functions P and C_i (both demand and supply shifters). For our results to hold, we merely need to assume that there is at least one exclusive demand shifter and one exclusive supply shifter. However, to keep our following exposition simple, we will assume that the vectors $\mathbf{z}$ and $\mathbf{w}$ have no variables in common (or, no demand shifter is also a supply shifter).[2]

In the Cournot model, each firm i chooses its output Q_i in order to maximize its profit $P(Q, \mathbf{z})\, Q_i - C_i(Q_i, \mathbf{w})$ given the output decisions of all the other firms ($Q_j, j \neq i$). For an interior solution, the Cournot outcome must solve the following set of first order conditions (with $i \leq N$):[3]

$$\frac{\partial P\left(\sum_{j=1}^{N} Q_j, \mathbf{z}\right)}{\partial Q} Q_i + P\left(\sum_{j=1}^{N} Q_j, \mathbf{z}\right) = \frac{\partial C_i(Q_i, \mathbf{w})}{\partial Q_i}. \tag{foc-C}$$

[2] To consider the general case, we only need to introduce a third vector of variables that are both demand and supply shifters. However, because explicitly accounting for this third category of variables does not imply additional testable implications, we choose not to do so.

[3] We exclude corner solutions in what follows. In fact, the only corner solution that makes economic sense is the case where a particular firm chooses to produce nothing. In this case, however, this firm will abstain from entering the market and its behavior is unobservable.

We assume that this system of equations has a unique solution for all values of $(\mathbf{z}, \mathbf{w})$ in an open and connected set $\mathcal{O}$ of $\mathbb{R}^{n+m}$. We can then derive N reduced form functions $q_i(\mathbf{z}, \mathbf{w})$ that determine the equilibrium quantities Q_i as functions of the exogenous variables $(\mathbf{z}, \mathbf{w})$. By substituting these functions in the inverse demand function $P(\sum_{j=1}^{N} Q_j, \mathbf{z})$, we obtain the reduced form equilibrium price function $p(\mathbf{z}, \mathbf{w}) = P\left(\sum_{i=1}^{N} q_i(\mathbf{z}, \mathbf{w}), \mathbf{z}\right)$, which defines the equilibrium prices in terms of the exogenous variables $(\mathbf{z}, \mathbf{w})$.

The second order conditions for a local maximum give the following additional set of conditions (with $i \leq N$):

$$2\frac{\partial P\left(\sum_{j=1}^{N} Q_j, \mathbf{z}\right)}{\partial Q} + \frac{\partial^2 P\left(\sum_{j=1}^{N} Q_j, \mathbf{z}\right)}{\partial Q^2} Q_i \leq \frac{\partial^2 C_i(Q_i, \mathbf{w})}{\partial Q_i^2}. \tag{soc-C}$$

In practice, the empirical analyst observes neither the inverse demand function $P(Q, \mathbf{z})$ nor the cost functions $C_i(Q_i, \mathbf{w})$, which makes it impossible to directly verify the conditions (foc-C) and (soc-C). However, as indicated in the Introduction, we assume that the analyst does know the (reduced form) equilibrium market price and quantity functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ for all values of $(\mathbf{z}, \mathbf{w})$ in the set $\mathcal{O}$. In principle, this only requires knowledge of the equilibrium price and outputs at the prevailing values of the exogenous variables $(\mathbf{z}, \mathbf{w})$. (We will return to identification and estimation of the functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ in the concluding section.) The next definition formally states when the equilibrium price and quantity functions are consistent with the model of Cournot competition.

Definition 1 (Cournot consistency) *Consider equilibrium price and quantity functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ ($i \leq N$). These functions are Cournot consistent if there exist an inverse demand function $P(Q, \mathbf{z})$ and cost functions $C_i(Q_i, \mathbf{w})$ such that for all $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$:*

$$P\left(\sum_{i=1}^{N} q_i(\mathbf{z}, \mathbf{w}), \mathbf{z}\right) = p(\mathbf{z}, \mathbf{w}), \tag{CC.1}$$

$$\frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right)}{\partial Q} q_i(\mathbf{z}, \mathbf{w}) + P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right) = \frac{\partial C_i(q_i(\mathbf{z}, \mathbf{w}), \mathbf{w})}{\partial \partial Q_i} \quad \textit{and} \tag{CC.2}$$

$$2\frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right)}{\partial Q} + \frac{\partial^2 P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right)}{\partial Q^2} q_i(\mathbf{z}, \mathbf{w}) \leq \frac{\partial^2 C_i(q_i(\mathbf{z}, \mathbf{w}), \mathbf{w})}{\partial Q_i^2}. \tag{CC.3}$$

Requirement (CC.1) relates the observed equilibrium prices $p(\mathbf{z}, \mathbf{w})$ to the unobserved inverse demand function $P(Q, \mathbf{z})$ evaluated at the equilibrium quantities $q_i(\mathbf{z}, \mathbf{w})$. Condition (CC.2) states that the observed equilibrium quantities $q_i(\mathbf{z}, \mathbf{w})$ must solve the first order conditions for the Cournot equilibrium. The condition is obtained by substituting $q_i(\mathbf{z}, \mathbf{w})$ into (foc-C). Finally, condition (CC.3) requires that the second order conditions (soc-C) are satisfied at equilibrium.

Before we discuss the characterization of Cournot consistency, we impose the following mild assumption to ensure non-triviality of the functions $q_i(\mathbf{z}, \mathbf{w})$:

Assumption 1 *For all* $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$ *and all firms* $i \leq N$ *there is at least one* $k \leq n$ *and one* $\ell \leq m$ *such that:*

$$\frac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial z_k} \neq 0 \qquad \textit{and} \qquad \sum_{i=1}^{N} \frac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial w_\ell} \neq 0.$$

This assumption is always satisfied if, for example, $q_i(\mathbf{z}, \mathbf{w})$ is strictly monotone in one demand shifter in $\mathbf{z}$ and one supply shifter in $\mathbf{w}$. Clearly, Assumption 1 is verifiable for given functions $q_i(\mathbf{z}, \mathbf{w})$ $(i \leq N)$.

2.2 Testable implications of the Cournot model

We are now in a position to establish necessary and sufficient conditions on $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ such that these functions satisfy Cournot consistency as defined in Definition 1. Our main focus will be on the case with both the number of demand shifters in $\mathbf{z}$ and the number supply shifters in $\mathbf{w}$ larger or equal than two, i.e. $n, m \geq 2$.[4] In what follows, we will provide an intuitive introduction to our testable conditions as necessary conditions for Cournot consistency. As we will explain, these conditions are threefold and correspond to (CC.1), (CC.2) and (CC.3) in Definition 1. In the Appendix, we prove that these necessary conditions are also sufficient (but this argument is more technical and less intuitive).

To obtain the first set of necessary conditions, we start from the requirement (CC.1) in Definition 1. We recall that this requirement equates the equilibrium price function with the inverse demand function. Here, we exploit the fact that variation of any supply shifter in $\mathbf{w}$ influences the equilibrium price only through its impact on the quantity functions $q_i(\mathbf{z}, \mathbf{w})$. Then, if we take the partial derivatives of condition (CC.1) with respect to any two shifters w_k and w_ℓ in $\mathbf{w}$ $(k, l \leq m)$, we get:

$$\frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial w_k} = \frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right)}{\partial Q} \sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_k} \qquad \text{and}$$

$$\frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial w_\ell} = \frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right)}{\partial Q} \sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_\ell}.$$

If we multiply the first equation by $\sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_\ell}$ and the second by $\sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_k}$, we obtain the following condition:

$$\frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial w_k} \sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_\ell} = \frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial w_\ell} \sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_k}. \qquad \text{(nec1-CC.1)}$$

Condition (nec1-CC.1), must hold for all pairs $k, \ell \leq m$. This condition does not only give us a set of necessary conditions for the existence of an inverse demand function. It also allows

[4] In the proof of Theorem 1 we argue that we get much simpler (necessary and sufficient) conditions if $n = 1$ and/or $m = 1$.

us to identify the slope of the inverse demand function, $\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right)\Big/\partial Q$, which we will denote by the (reduced form) function $\tau(\mathbf{z},\mathbf{w})$. Indeed, let the supply shifter $k \leq m$ satisfy Assumption 1. Then it follows that:

$$\frac{\partial P\left(\sum_{i=1}^{N} q_i(\mathbf{z},\mathbf{w}),\mathbf{z}\right)}{\partial Q} = \frac{\dfrac{\partial p(\mathbf{z},\mathbf{w})}{\partial w_k}}{\displaystyle\sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial w_k}} \equiv \tau(\mathbf{z},\mathbf{w}) \leq 0. \tag{nec2-CC.1}$$

Given the above, $\tau(\mathbf{z},\mathbf{w})$ is well-defined as it does not depend on the identity of k. The inequality restriction in condition (nec2-CC.1) follows from our assumption that the function $P(Q,\mathbf{z})$ is decreasing in Q. Conditions (nec1-CC.1) and (nec2-CC.1) constitute our first set of necessary conditions for Cournot consistency. Clearly, these conditions are not specific to the Cournot model but apply to any market trading a homogeneous good.

Let us then consider our second set of conditions, which are particular to the Cournot model. To obtain these conditions, we first substitute the function $\tau(\mathbf{z},\mathbf{w})$ into condition (CC.2):

$$p(\mathbf{z},\mathbf{w}) + \tau(\mathbf{z},\mathbf{w}) q_i(\mathbf{z},\mathbf{w}) = \frac{\partial C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})}{\partial Q_i}.$$

Next, we use the fact that the demand shifters in $\mathbf{z}$ only influence the marginal costs of a firm through their effect on $q_i(\mathbf{z},\mathbf{w})$. Differentiating our last equation with respect to any two shifters z_k and z_ℓ in $\mathbf{z}$ ($k,\ell \leq n$), we obtain:

$$\begin{aligned} &\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k} + \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k} q_i(\mathbf{z},\mathbf{w}) + \tau(\mathbf{z},\mathbf{w}) \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k} = \frac{\partial^2 C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})}{\partial Q_i^2} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k} \quad \text{and} \\ &\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_\ell} + \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_\ell} q_i(\mathbf{z},\mathbf{w}) + \tau(\mathbf{z},\mathbf{w}) \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} = \frac{\partial^2 C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})}{\partial Q_i^2} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell}. \end{aligned} \tag{1}$$

Multiplying the first equation by $\dfrac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell}$ and the second one by $\dfrac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}$ leads to:

$$
\begin{aligned}
\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} + \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} q_i(\mathbf{z},\mathbf{w}) & \\
= \frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_\ell}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k} + \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_\ell}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k} q_i(\mathbf{z},\mathbf{w}) &
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
\left[\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} - \frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_\ell}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}\right] + & \\
q_i(\mathbf{z},\mathbf{w})\left[\frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} - \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_\ell}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}\right] &= 0
\end{aligned}
\tag{nec-CC.2}
$$

Thus, the model of Cournot competition holds only if condition (nec-CC.2) holds for all $k, \ell \leq n$ and $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$. This yields our second set of conditions for Cournot consistency.

Finally, we focus on the third condition (CC.3). From (nec2-CC.1) we know that $\partial P(\sum_j q_j(\mathbf{z},\mathbf{w}),\mathbf{z}) \big/ \partial Q$ is identified by the function $\tau(\mathbf{z},\mathbf{w})$. Then, if we differentiate the same condition (nec2-CC.1) with respect to a variable w_ℓ that satisfies the condition of Assumption 1, we get

$$
\frac{\partial^2 P(\sum_j q_j(\mathbf{z},\mathbf{w}),\mathbf{z})}{\partial Q^2}\sum_{j=1}^{N}\frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial w_\ell} = \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial w_\ell}.
$$

Equivalently,

$$
\frac{\partial^2 P(\sum_j q_j(\mathbf{z},\mathbf{w}),\mathbf{z})}{\partial Q^2} = \frac{\dfrac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial w_\ell}}{\displaystyle\sum_{j=1}^{N}\frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial w_\ell}}.
$$

Next, from (1) we can obtain the value of $\partial^2 C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w}) \big/ \partial Q_i^2$. Substituting these values in (CC.3), we obtain the following condition for all variables z_k and w_ℓ that satisfy Assumption 1,

$$
\tau(\mathbf{z},\mathbf{w}) + \left(\frac{\dfrac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial w_\ell}}{\displaystyle\sum_{j=1}^{N}\frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial w_\ell}} - \frac{\dfrac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k}}{\dfrac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}}\right) q_i(\mathbf{z},\mathbf{w}) \leq \frac{\dfrac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}}{\dfrac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}} \tag{nec-CC.3}
$$

Our main result states that the conditions (nec1-CC.1), (nec2-CC.1), (nec-CC.2) and (nec-CC.3) are not only necessary but also sufficient for Cournot consistency.

Theorem 1 *Consider equilibrium price and quantity functions $p(\mathbf{z},\mathbf{w})$ and $q_i(\mathbf{z},\mathbf{w})$ ($i \leq N$) that are sufficiently smooth on $\mathcal{O}$ and satisfy Assumption 1. These functions are Cournot consistent if and only if:*

- *for all* $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$ *and all* $k, \ell \leq m$*:*

$$\frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial w_k} \sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_\ell} = \frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial w_\ell} \sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_k}, \tag{nec1-CC.1}$$

- *for all* $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$ *and all* w_ℓ *(*$\ell \leq m$*) that satisfy Assumption 1:*

$$\frac{\dfrac{\partial p(\mathbf{z}, \mathbf{w})}{\partial w_\ell}}{\displaystyle\sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_\ell}} \equiv \tau(\mathbf{z}, \mathbf{w}) \leq 0, \tag{nec2-CC.1}$$

- *for all* $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$ *and all* $k, \ell \leq n$*:*

$$\left[\frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial z_k} \frac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial z_\ell} - \frac{\partial p(\mathbf{z}, \mathbf{w})}{\partial z_\ell} \frac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial z_k}\right] + \\ q_i(\mathbf{z}, \mathbf{w}) \left[\frac{\partial \tau(\mathbf{z}, \mathbf{w})}{\partial z_k} \frac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial z_\ell} - \frac{\partial \tau(\mathbf{z}, \mathbf{w})}{\partial z_\ell} \frac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial z_k}\right] = 0, \tag{nec-CC.2}$$

- *for all* $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$ *and all* w_ℓ *(*$\ell \leq m$*) and* z_k *(*$k \leq n$*) that satisfy Assumption 1:*

$$\tau(\mathbf{z}, \mathbf{w}) + \left(\frac{\dfrac{\partial \tau(\mathbf{z}, \mathbf{w})}{\partial w_\ell}}{\displaystyle\sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z}, \mathbf{w})}{\partial w_\ell}} - \frac{\dfrac{\partial \tau(\mathbf{z}, \mathbf{w})}{\partial z_k}}{\dfrac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial z_k}} \right) q_i(\mathbf{z}, \mathbf{w}) \leq \frac{\dfrac{\partial p(\mathbf{z}, \mathbf{w})}{\partial z_k}}{\dfrac{\partial q_i(\mathbf{z}, \mathbf{w})}{\partial z_k}}. \tag{nec-CC.3}$$

As a final note, we observe that, if there is only one supply shifter and one demand shifter (i.e. $n = m = 1$), then the only testable implications left are (nec2-CC.1) and (nec-CC.3).

2.3 Identification

If the equilibrium price and quantity functions are found to satisfy the conditions for Cournot consistency in Theorem 1, then a natural next question asks for identifying the underlying structure of the model. In this subsection, we present a brief discussion of such identification. Like before, we assume an empirical analyst who knows the equilibrium market price and quantity functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ for all values of $(\mathbf{z}, \mathbf{w})$ in the set $\mathcal{O}$.

As for the Cournot model, identification pertains to the inverse demand function $P(Q, \mathbf{z})$ and the cost functions $C_i(Q_i, \mathbf{w})$. In general, these functions cannot be globally identified because we are unable to retrieve their value for Q, $\mathbf{z}$ and $\mathbf{w}$ that are not part of the observed equilibrium outcome. As such, our following discussion focuses on local identification (i.e. defined in a neighborhood of equilibrium price-quantity points). In fact, as we will explain, such local identification is fairly easily obtained.

To begin, we consider identification of $P(Q, \mathbf{z})$. We first look at point identification and, subsequently, we extend our reasoning to local identification. As a starting point, we note that $P(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z})$ is identical to the value of $p(\mathbf{z}, \mathbf{w})$. In other words, if there exist vectors $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$ with $\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}) = Q$, we have that $P(Q, \mathbf{z}) = P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right) = p(\mathbf{z}, \mathbf{w})$, which is known. This shows that $P(Q, z)$ is point identified on the equilibrium path. In the Appendix we show that we can extend this result to show local identification around the equilibrium path.

Corollary 1 *Consider vectors $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$. If $\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}) = Q$, then there exists a neighborhood of $(Q, \mathbf{z})$ such that $P(Q', \mathbf{z}')$ is identified for all $(Q', \mathbf{z}')$ in this neighborhood.*

Next, identification of the cost functions $C_i(Q_i, \mathbf{w})$ is a bit more involved. These functions can only be recovered up to an additive constant. This follows from the fact that the first order conditions (foc-C) only involve the marginal cost functions $\partial C_i(Q_i, \mathbf{w})/\partial Q_i$, which remain unaffected if we add a fixed number to $C_i(Q_i, \mathbf{w})$. Now, as for the marginal cost functions $\partial C_i(Q_i, \mathbf{w})/\partial Q_i$, we can follow a similar reasoning as before. Specifically, to obtain point identification, we note that, if $Q_i = q_i(\mathbf{z}, \mathbf{w})$ for $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$, then the marginal cost $\partial C_i(Q_i, \mathbf{w})/\partial Q_i$ can be recovered. This follows from the requirement:

$$\begin{aligned} \frac{\partial C_i(Q_i, \mathbf{w})}{\partial Q_i} &= \frac{\partial C_i(q_i(\mathbf{z}, \mathbf{w}), \mathbf{w})}{\partial Q_i} \\ &= p(\mathbf{z}, \mathbf{w}) + \tau(\mathbf{z}, \mathbf{w}) q_i(\mathbf{z}, \mathbf{w}); \end{aligned}$$

which is known because $\tau(\mathbf{z}, \mathbf{w})$ is identified on the equilibrium path. Again, we can extend this result to obtain local identification.

Corollary 2 *Consider vectors $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$. If $q_i(\mathbf{z}, \mathbf{w}) = Q_i$, then there exists a neighborhood of $(Q_i, \mathbf{w})$ such that $\partial C_i(Q_i', \mathbf{w}')/\partial Q_i$ is identified for all $(Q_i', \mathbf{w}')$ in this neighborhood.*

3 Other models of firm competition

In this section, we compare the testable restrictions of the Cournot model (in Theorem 1) with the ones that apply to other popular models of market competition. Specifically, we consider the models of perfect competition, perfect collusion and conjectural variation. We begin by providing a brief description of these three models. Subsequently, we present their characterization. For compactness, we will not explicitly consider identification in this section. However, the reasoning is directly analogous to the one in Subsection 2.3.

Perfect competition. The perfect competition model assumes that each firm maximizes its total profit for exogenously given prices. This model has a long tradition in economic theory and in general equilibrium theory, where price taking behavior entails a Pareto optimal market allocation. Given this theoretical relevance of the model, it seems particularly interesting to derive its testable implications, and to compare these implications with the ones of the Cournot model.

Under price taking behavior, we get the the following set of first order conditions (with $i \leq N$):

$$P(Q, \mathbf{z}) = \frac{\partial C_i(Q_i, \mathbf{w})}{\partial Q_i}. \tag{foc-PC}$$

Like before, we assume this system of equations has a unique solution for all values of $(\mathbf{z}, \mathbf{w})$ in an open and connected set $\mathcal{O}$ of $\mathbb{R}^{n+m}$. The second order conditions requires that the cost function is (locally) convex (with $i \leq N$):

$$\frac{\partial^2 C_i(Q_i, \mathbf{w})}{\partial Q_i^2} \geq 0 \tag{soc-PC}$$

Then, we can derive N (reduced form) equilibrium price and quantity functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$, with the vectors $\mathbf{z}$ and $\mathbf{w}$ containing demand and supply shifters, respectively. Analogous to before, we can define when these functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ are consistent with the model of perfect competition (or competition consistent).

Definition 2 (competition consistency) *Consider equilibrium price and quantity functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ ($i \leq N$). These functions are competition consistent if there exist an inverse demand function $P(Q, \mathbf{z})$ and cost functions $C_i(Q_i, \mathbf{w})$ such that for all $(\mathbf{z}, \mathbf{w}) \in \mathcal{O}$: condition* (CC.1) *is satisfied and, in addition,*

$$P\left(\sum_{j=1}^{N} q_j(\mathbf{z}, \mathbf{w}), \mathbf{z}\right) = \frac{\partial C_i(q_i(\mathbf{z}, \mathbf{w}), \mathbf{w})}{\partial Q_i} \text{ and,}$$

$$\frac{\partial^2 C_i(q_i(\mathbf{z}, \mathbf{w}), \mathbf{w})}{\partial Q_i^2} \geq 0.$$

Thus, the functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ must again meet three requirements. The condition (CC.1) is the same as for the Cournot model and results from the homogeneous good assumption. The second condition is specific to the perfect competition model, and expresses that the equilibrium quantity functions must solve the first order conditions (foc-PC). The third condition corresponds to (soc-PC).

Perfect collusion. Let us now turn to the model of perfect collusion. This model assumes that all firms in the market cooperate, so as to maximize their joint profit. From a normative perspective, collusion has a strongly negative welfare effect on the demand side of the market, which makes it relevant to derive the testable implications of this model. Specifically, these implications enable us to empirically verify whether the model effectively holds and, even more interestingly, to analyze whether it is empirically distinguishable from other models of firm behavior (with less negative welfare effects).

Formally, perfect collusion means that firms choose the outputs that maximize the joint profit, $P(Q,\mathbf{z})Q - \sum_{i=1}^{N} C_i(Q_i,\mathbf{w})$, which obtains the following set of first order conditions (with $i \leq N$):

$$\frac{\partial P\left(Q,\mathbf{z}\right)}{\partial Q}Q + P(Q,\mathbf{z}) = \frac{\partial C_i(Q_i,\mathbf{w})}{\partial Q_i}. \qquad \text{(foc-ColC)}$$

Again, we assume this system has a unique solution for all values of $(\mathbf{z},\mathbf{w})$ in an open and connected set $\mathcal{O}$ of $\mathbb{R}^{n+m}$. The second order condition are the following (with $i \leq N$):

$$2\frac{\partial P(Q,\mathbf{z})}{\partial Q} + \frac{\partial^2 P(Q,\mathbf{z})}{\partial Q^2}Q \leq \frac{\partial^2 C_i(Q_i,\mathbf{w})}{\partial Q_i^2}. \qquad \text{(soc-ColC)}$$

Directly similar to before, we then obtain the following conditions for the equilibrium price and quantity functions $p(\mathbf{z},\mathbf{w})$ and $q_i(\mathbf{z},\mathbf{w})$ to be consistent with the model of perfect collusion (or collusion consistent).

Definition 3 (collusion consistency) *Consider equilibrium price and quantity functions $p(\mathbf{z},\mathbf{w})$ and $q_i(\mathbf{z},\mathbf{w})$ ($i \leq N$). These functions are collusion consistent if there exist an inverse demand function $P(Q,\mathbf{z})$ and cost functions $C_i(Q_i,\mathbf{w})$ such that for all $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$: condition* (CC.1) *is satisfied and, in addition,*

$$\frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right)}{\partial Q}\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}) + P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right) = \frac{\partial C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})}{\partial Q_i} \text{ and,}$$

$$2\frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right)}{\partial Q} + \frac{\partial^2 P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right)}{\partial Q^2}\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}) \leq \frac{\partial^2 C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})}{\partial Q_i^2}.$$

The conjectural variations model. Lastly, we consider the conjectural variations model. This model is widely used in the new empirical industrial organizations literature, to assess the degree of competition within a given market. The conjectural variations model relates the markup of price over marginal cost to a parameter λ_i that measures the degree to which the firms in the market behave competitively.[5] A parameter value equal to zero then means that there is no market power, or, the firms behave as in the case of perfect competition. Alternatively, if this conjectural variations parameter equals one, then the firms behave like in the Cournot model. Values of λ_i between zero and one, capture the models situated between these two benchmark cases. Finally, a value of the parameter above one indicates collusive behavior. Like for the perfect collusion model, the relevance of measuring the conjectural variations parameter is that increased market power implies strongly negative welfare effects on the demand side of the market. As such, if we are capable of estimating the value of this parameter, then we can -at least in principle- decide whether or not certain firms abuse their market power.

As a theoretical construct, the conjectural variations parameter is usually interpreted as the change in aggregate output in response to an infinitesimal increase in the output of a single

[5] Following Corts (1999), this parameter is also known as the conduct parameter.

firm (i.e. the conjectural variation). Although this interpretation is controversial from a theoretical point of view,[6] the conjectural variations model still remains widely employed in the literature. Indeed, an attractive property of the model is that it provides an easily implemented set of conditions that are sufficient to establish econometric identification of the degree of competition. Focusing on a linear demand function, Bresnahan (1982) showed that identification is guaranteed if one introduces a rotation variable in the aggregate demand equation, i.e. it suffices to introduce an exogenous variable that shifts the slope of the demand function. Lau (1982) extended this result by showing identification even without assuming a particular functional structure for the equilibrium price and quantity functions. He finds that the conjectural variations parameter is identified as long as aggregate demand is non-separable in at least one exogenous variable.

Although these results allow one to identify the conjectural variations parameter if the conjectural variations model is the appropriate one, they do not provide any guidance as to whether this model effectively corresponds to the true underlying data generating process. Interestingly, we can again fairly easily adapt our above framework to provide (necessary and sufficient) testable conditions for the equilibrium price and quantity functions to be consistent with the conjectural variations model.

Formally, the model assumes the existence of (a fixed set of) conjectural variations parameters λ_i ($i \leq N$) such that the equilibrium quantities satisfy the following set of first order conditions (with $i \leq N$):

$$P(Q, \mathbf{z}) + \lambda_i \frac{\partial P(Q, \mathbf{z})}{\partial Q} Q_i = \frac{\partial C_i(Q_i, \mathbf{w})}{\partial Q_i}. \qquad \text{(foc-CvC)}$$

Clearly, $\lambda_i = 0$ gives the first order conditions for the perfect competition model, while $\lambda_i = 1$ obtains the first order conditions for the Cournot model. Similar to before, we assume the system (foc-CvC) has a unique solution for all values of $(\mathbf{z}, \mathbf{w})$ in an open and connected set $\mathcal{O}$ of $\mathbb{R}^{n+m}$. The second order conditions associated with the conjectural variations model are given by (with $i \leq N$):

$$(1 + \lambda_i) \frac{\partial P(Q, \mathbf{z})}{\partial Q} + \lambda_i \frac{\partial^2 P(Q, \mathbf{z})}{\partial Q^2} Q_i \leq \frac{\partial^2 C_i(Q_i, \mathbf{w})}{\partial Q_i^2}. \qquad \text{(soc-CvC)}$$

Given this, we can define the following conditions for the functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ to be consistent with the conjectural variations model (or conjectural variations consistent).

Definition 4 (conjectural variations consistency) *Consider equilibrium price and quantity functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$ ($i \leq N$). These functions are conjectural variations consistent if there exist an inverse demand function $P(Q, \mathbf{z})$ and cost functions $C_i(Q_i, \mathbf{w})$ such that for all*

[6] However, see d'Aspremont, Dos Santos Ferreira, and Gérard-Varet (2007), and d'Aspremont and Dos Santos Ferreira (2009), who provide several rationales for this conduct parameter using a game theoretic approach.

$(\mathbf{z},\mathbf{w}) \in \mathcal{O}$: *condition* (CC.1) *is satisfied and, in addition,*

$$\frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right)}{\partial Q}\lambda_i q_i(\mathbf{z},\mathbf{w}) + P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right) = \frac{\partial C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})}{\partial Q_i} \text{ and,}$$

$$(1+\lambda_i)\frac{\partial P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right)}{\partial Q} + \lambda_i \frac{\partial^2 P\left(\sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}),\mathbf{z}\right)}{\partial Q^2} q_i(\mathbf{z},\mathbf{w}) \le \frac{\partial^2 C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})}{\partial Q_i^2}.$$

Characterization. Starting from Definitions 2, 3 and 4, a similar argument as for Theorem 1 yields the following result.

Theorem 2 *Consider equilibrium price and quantity functions* $p(\mathbf{z},\mathbf{w})$ *and* $q_i(\mathbf{z},\mathbf{w})$ *($i \le N$) that are sufficiently smooth on* $\mathcal{O}$ *and satisfy Assumption 1. These functions are*

- *competition consistent if and only if*
  - *(i) conditions* (nec1-CC.1) *and* (nec2-CC.1) *are satisfied,*
  - *(ii) for all* $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ *and all* $k,\ell \le n$*:*

$$\left[\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} - \frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_\ell}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}\right] = 0, \qquad \text{(nec-PC.2)}$$

  - *(iii) for all* $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ *and all* z_k *($k \le n$) that satisfy Assumption 1:*

$$\frac{\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}}{\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}} \ge 0. \qquad \text{(nec-PC.3)}$$

- *collusion consistent if and only if*
  - *(i) conditions* (nec1-CC.1) *and* (nec2-CC.1) *are satisfied,*
  - *(ii) for all* $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ *and all* $k,\ell \le n$*:*

$$\left[\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} - \frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_\ell}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}\right] + \sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w})\left[\frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} - \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_\ell}\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}\right] + \tau(\mathbf{z},\mathbf{w})\left[\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell}\sum_{j=1}^{N}\frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial z_k} - \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}\sum_{j=1}^{N}\frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial z_\ell}\right] = 0, \qquad \text{(nec-ColC.2)}$$

(iii) for all $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ and all z_k ($k \leq n$) and all w_ℓ ($\ell \leq m$) that satisfy Assumption 1:

$$\tau(\mathbf{z},\mathbf{w}) \left(2 - \frac{\sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial z_k}}{\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}} \right) + \sum_{j=1}^{N} q_j(\mathbf{z},\mathbf{w}) \left(\frac{\frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial w_\ell}}{\sum_{j=1}^{n} \frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial w_\ell}} - \frac{\frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k}}{\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}} \right) \leq \frac{\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}}{\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}}. \quad \text{(nec-ColC.3)}$$

- *conjectural variations consistent if and only if there exist a set of fixed numbers $\{\lambda_i\}_{i \leq N}$ such that,*

  (i) conditions (nec1-CC.1) *and* (nec2-CC.1) *are satisfied,*

  (ii) for all $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ and all $k, \ell \leq n$:

$$\left[\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} - \frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_\ell} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k} \right] + \lambda_i q_i(\mathbf{z},\mathbf{w}) \left[\frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} - \frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_\ell} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k} \right] = 0, \quad \text{(nec-CvC.2)}$$

  (iii) for all $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ and all z_k ($k \leq n$) and w_ℓ ($\ell \leq m$) that satisfy Assumption 1:

$$\tau(\mathbf{z},\mathbf{w}) + \lambda_i \left(\frac{\frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial w_\ell}}{\sum_{j=1}^{N} \frac{\partial q_j(\mathbf{z},\mathbf{w})}{\partial w_\ell}} - \frac{\frac{\partial \tau(\mathbf{z},\mathbf{w})}{\partial z_k}}{\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}} \right) q_i(\mathbf{z},\mathbf{w}) \leq \frac{\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial z_k}}{\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}}. \quad \text{(nec-CvC.3)}$$

4 An illustration

As a final exercise, let us demonstrate the application of our theoretical results. Specifically, we illustrate the testable implications derived above for a simple specification of the (reduced form) equilibrium price and quantity functions. This also shows that the Cournot model is empirically distinguishable from other models of firm competition even for this simple specification.

To ease our exposition, we assume that all N firms have the same quantity function, i.e. $q_i(\mathbf{z}, w) = q(\mathbf{z}, w)$ for each i. We then consider the following equilibrium price and quantity functions:

$$\ln(p(\mathbf{z}, w)) = a_1 z_1 + a_2 z_2 + a_3 w,$$
$$\ln(q(\mathbf{z}, w)) = b_1 z_1 + b_2 z_2 + b_3 w,$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are real-valued parameters. We note that these functions are sufficiently smooth for our results to apply. Furthermore, our set-up is simple in that the functions only depend on two demand shifters and one supply shifter. To guarantee that Assumption 1 holds, we assume that b_1, b_2 and b_3 are all different from zero.

Because we have only a single supply shifter, (nec1-CC.1) automatically holds. Next, we get

$$\tau(\mathbf{z}, \mathbf{w}) = \frac{a_3 p(\mathbf{z}, w)}{N b_3 q(\mathbf{z}, w)}.$$

Therefore, it suffices that $\frac{a_3}{b_3} \leq 0$ for (nec2-CC.1) to hold.

To show the possibility to empirically distinguish the four models of market competition discussed above, we consider the different conditions in Theorems 1 and 2. For the given specification of the price and quantity functions, we obtain

$$\text{(nec-ColC.2)} \quad : \quad p(\mathbf{z}, w) q(\mathbf{z}, w)(1 + \frac{a_3}{b_3})(a_1 b_2 - a_2 b_1) = 0,$$
$$\text{(nec-CvC.2)} \quad : \quad p(\mathbf{z}, w) q(\mathbf{z}, w)(1 + \frac{\lambda_i a_3}{N b_3})(a_1 b_2 - a_2 b_1) = 0.$$

We recall that (nec-CvC.2) complies with (nec-CC.2) if $\lambda_i = 1$ and with (nec-PC.2) if $\lambda_i = 0$.

From these equations it is clear that we cannot disentangle the four models on the basis of the above conditions if $a_1 b_2 - a_2 b_1 = 0$. In fact, we need $a_1 b_2 - a_2 b_1 = 0$ to obtain consistency with the perfect competition condition (nec-PC.2) (which complies with $\lambda_i = 0$). In case $\lambda_i \neq 0$ and $a_1 b_2 - a_2 b_1 \neq 0$, the above equations reduce to

$$\text{(nec-ColC.2)} \quad : \quad a_3 = -b_3,$$
$$\text{(nec-CvC.2)} \quad : \quad a_3 = -\frac{N}{\lambda_i} b_3.$$

Clearly, for $N > 1$ and $\lambda_i > 0$ such that $N \neq \lambda_i$, this obtains mutually distinguishable conditions for (nec-CC.2) (Cournot model), (nec-ColC.2) (perfect collusion) and (nec-CvC.2) (conjectural variations model). Straightforward (but tedious) calculations show that the conditions (nec-CC.3), (nec-ColC.3) and (nec-CvC.3) are satisfied as soon as the above conditions for the corresponding models are also satisfied.

5 Concluding discussion

We established necessary and sufficient conditions for (reduced form) equilibrium price and quantity functions to be consistent with the Cournot model of market competition. Our conditions are nonparametric, i.e. they do not rely on a particular functional specification of these price and quantity functions. The conditions show that the Cournot model has strong testable implications, which can be verified as soon as the specification of the price and quantity functions is given. Next, we have presented identification results for the inverse market demand function and the firm cost functions that underlie firm behavior that is consistent with the Cournot model. Furthermore, we have demonstrated the versatility of our framework by using the same approach to derive testable restrictions for the perfect competition, perfect collusion and conjectural variations models. Using these results, we have shown that the different models are empirically distinguishable even for a simple specification of the equilibrium price and quantity functions.

Given all this, the next crucial step consists of bringing our theoretical results to empirical data. Throughout, we have assumed that the empirical analyst knows the price and quantity functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$. In practice, these functions must be retrieved from a finite data set, which involves identification as well as estimation issues. As for identification, an important concern pertains to appropriately accounting for measurement errors and/or omitted variables (or unobserved heterogeneity). Interestingly, in the recent literature there has been a surge of papers that define the conditions under which such identification is possible. See, for example, Matzkin (2007) for an overview.

Next, as for the estimation of the price and quantity functions, the empirical analyst may use either standard parametric techniques (using flexible functional forms) or more recently developed nonparametric techniques (such as Kernel estimation, local linear regression, sieve estimation, etc.). Our results may actually provide useful guidance in selecting the functional form that is used for dealing with particular empirical questions. Generally, for a given estimation of the price and quantity functions, our testable conditions can be verified by using appropriate statistical techniques.

As a concluding remark, we indicate that our approach also provides a flexible framework for empirically verifying frequently used restrictions on cost and/or profit functions. As a most notable example, Novshek (1985) showed that (under some regularity conditions) a Cournot equilibrium exists if the marginal revenue of every firm is a decreasing function of the aggregate output of all other firms in the market (which can also be formulated as a submodularity condition for the profit function of each firm); and Gaudet and Salant (1991), Szidarovsky and Yakowitz (1977), Kolstad and Mathiesen (1987), Long and Soubeyran (2000) established related conditions for uniqueness of this equilibrium. Interestingly, following a similar reasoning as above it is actually fairly simple to derive testable implications of these conditions for a given specification of the functions $p(\mathbf{z}, \mathbf{w})$ and $q_i(\mathbf{z}, \mathbf{w})$.[7]

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[7] For compactness, we do not include a formal argument here. But it is available from the authors upon request.

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A Appendix

We will only prove Theorem 1. The proof of Theorem 2 is readily analogous. Similarly, we only consider the proof of Corollary 1. The proof of the other corollary is again analogous.

A.1 Proof of Theorem 1

Necessity for $n, m \geq 2$ was demonstrated above, so here we restrict ourselves to sufficiency. Our proof relies to a large extent on a lemma of Goldman and Uzawa (1964):

Lemma 1 *Consider two sufficiently smooth functions $f(\mathbf{x})$ and $g(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^t$. Then, if there exists a function η such that for all $\mathbf{x}$ and $j \leq t$:*

$$\frac{\partial f(\mathbf{x})}{\partial x_j} = \eta(\mathbf{x}) \frac{\partial g(\mathbf{x})}{\partial x_j},$$

then there exist a function F such that:

$$f(x) = F(g(x)).$$

Condition nec1-CC.1 implies that $\dfrac{\partial p(\mathbf{z},\mathbf{w})}{\partial w_k} = 0$ if $\sum_{i=1}^N \dfrac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial w_k} = 0$. Thus, we have that conditions nec1-CC.1 and nec2-CC.1 imply,

$$\frac{\partial p(\mathbf{z},\mathbf{w})}{\partial w_k} = \tau(\mathbf{z},\mathbf{w}) \sum_{i=1}^N \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial \partial w_k} \quad \forall k \leq m. \tag{2}$$

Then Lemma 1 states that for any $\mathbf{z}$, there exists a function P such that $p(\mathbf{z},\mathbf{w}) = P(\sum_{i=1}^N q(\mathbf{z},\mathbf{w}), \mathbf{z})$. Given that $p(\mathbf{z},\mathbf{w})$ and $q_i(\mathbf{z},\mathbf{w})$ are sufficiently smooth, the function $P(Q, \mathbf{z})$ is also sufficiently smooth. Finally, by condition nec2-CC.1, this function is decreasing in its first argument.

Next, assume that condition nec-CC.2 holds, and consider the following function $\gamma_i(\mathbf{z},\mathbf{w})$:

$$\gamma_i(\mathbf{z},\mathbf{w}) = p(\mathbf{z},\mathbf{w}) + \tau(\mathbf{z},\mathbf{w}) q_i(\mathbf{z},\mathbf{w}).$$

One can easily verify that condition nec-CC.2 implies that, for all $k, \ell \leq n$,

$$\frac{\partial \gamma_i(\mathbf{z},\mathbf{w})}{\partial z_k} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} = \frac{\partial \gamma_i(\mathbf{z},\mathbf{w})}{\partial z_\ell} \frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}$$

Now take any $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ and assume that z_k ($k \leq n$) satisfies the inequality condition in Assumption 1. Then, we can define,

$$\delta_i(\mathbf{z},\mathbf{w}) = \frac{\dfrac{\partial \gamma_i(\mathbf{z},\mathbf{w})}{\partial z_k}}{\dfrac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}}.$$

As above, this yields that, for all $k \leq n$,

$$\frac{\partial \gamma_i(\mathbf{z},\mathbf{w})}{\partial z_k} = \sigma_i(\mathbf{z},\mathbf{w})\frac{\partial q_i(\mathbf{z},\mathbf{w})}{\partial z_k}$$

Similar to before, Lemma 1 implies that there exists a sufficiently smooth function MC_i such that $\gamma_i(\mathbf{z},\mathbf{w}) = MC_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})$ for all $(\mathbf{z},\mathbf{w})$. Integrating out this function gives us the desired cost function $C_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})$.

Given the marginal cost function $MC_i(q_i(\mathbf{z},\mathbf{w}),\mathbf{w})$ and the slope of the inverse demand function $\tau(\mathbf{z},\mathbf{w})$, it is easy to see that the second order condition (CC.3) is satisfied whenever (nec-CC.3) is satisfied.

To finish the proof, we still need to consider the case with n and/or m equal to one. If $m = 1$, then condition nec1-CC.1 is of course redundant and condition nec2-CC.1 is equivalent to condition (2). An argument that is readily similar to the one above shows that conditions (nec-CC.2) and (nec-CC.3) are both necessary and sufficient for the Cournot model to hold. A similar argument holds for the case $n = 1$.

A.2 Proof of Corollary 1

Consider the vectors $(\mathbf{z},\mathbf{w}) \in \mathcal{O}$ and let $Q = \sum_j^N q_j(\mathbf{z},\mathbf{w})$. Then, let w_k satisfy the condition of Assumption 1. Keeping the vectors w_ℓ ($\ell \neq k$) fixed, we can locally invert the function $\sum_j q_j(\mathbf{z},\mathbf{w})$ with respect to w_k in a neighborhood of $(Q,\mathbf{z})$. we denote this inverse function, by $\theta_{\mathbf{w}}(Q',\mathbf{z}')$. As such, for all $(Q',\mathbf{z}')$ in a neighborhood of $(Q,\mathbf{z})$, we have the identity $Q' \equiv \sum_j^N q_j(\mathbf{z}',\tilde{\mathbf{w}})$ where $\tilde{w}_k = \theta_{\mathbf{w}}(Q',\mathbf{z}')$ and $\tilde{w}_\ell = w_\ell$ for all $\ell \neq k$. In order to show that $P(Q,\mathbf{z})$ is locally identified at $(Q',\mathbf{z}')$ in a neighborhood of $(Q,\mathbf{z})$, we only have to consider the vector $\tilde{\mathbf{w}}$ such that $\tilde{w}_k = \theta(Q',\mathbf{z}')$ and $\tilde{w}_\ell = w_\ell$ for all $\ell \neq k$, it follows from condition (CC.1) that

$$\begin{aligned} P(Q',\mathbf{z}') &= P\left(\sum_j q_j(\mathbf{z}',\tilde{\mathbf{w}}),\mathbf{z}'\right) \\ &= p(\mathbf{z}',\tilde{\mathbf{w}}). \end{aligned}$$