Prehistory of the concept of mathematical structure:
Isomorphism between group theory, crystallography and philosophy

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In the historical note concluding his fascicle on structures published in 1957, the Bourbaki group asserts that "every structure carries within itself a notion of isomorphism"[1]. The same note adds that the general notion of isomorphism was perceived for the first time by Leibniz through the idea of *similarity*:

> "Making precise the 'accord' [between different branches of mathematics] of which Descartes spoke, [Leibniz] glimpses, in fact, for the first time, the general notion of isomorphism (which he calls 'similitude'), and the possibility of 'identifying' relations or operations that are isomorphic; he gives as examples addition and multiplication. But these audacious views remained without echo amongst his contemporaries, and one must await the expansion of Algebra which takes place around the middle of the 19th century to see the beginnings of the realisation of the Leibnizian dreams. [...] But it is only with the modern notion of structure that it was finally recognised that every structure carries within itself a notion of isomorphism, and that it is not necessary to give a special definition of it for each type of structure."[2]

Bourbaki was not the only one to link structure and isomorphism. In 1927 Hermann Weyl, another major figure of the 20th century mathematics, declared that "isomorphic domains may be said to possess the same *structure*."[3] In his book *Symmetry* (1952) which, by the way, is quoted by Bourbaki[4], Weyl also seems to attribute to Leibniz the origin of the connection between structure and isomorphism:

> "A transformation which preserves the *structure* of space [...] is called an *automorphism* by the mathematicians. Leibniz recognised that this is the idea underlying the geometric concept of similarity. An automorphism carries a figure into one that in Leibniz'words is 'indiscernable from it if each of the two figures is considered by itself'."[5]

But the definition of the Leibnizian concept of similarity given by Bourbaki (as identity of relations or operations) does not coincide with the one given by Weyl (as indistinctness of things perceived separately). In addition, the historical development of "Leibnizian dreams" "around the middle of the 19th century", as reported by Bourbaki, remains rather vague.

What I propose to do is to trace back a path starting from the bourbakist view on structure, to gradually get back to earlier authors and determine how the connection between the concepts of structure and of isomorphism was formed. Such a general quest could lead us

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[1] Bourbaki (1994 [1957]), p. 23.
[2] Bourbaki (1994 [1957]), p. 23.
[3] Weyl (1949 [1927]), p. 25.
[4] Weyl (1949 [1927]), p. 22.
[5] Weyl (1980 [1952]), p. 18.

very far. The history of mathematics is full of examples of discoveries of similarities between "structures". Just think to the equivalence classes in Gauss's arithmetic theory, or to the duality theorems in projective geometry. Yet the stake is not only a historical one; it also concerns to this question: why isomorphism? Why, while he was trying to define and historically situate the concept of mathematical structure, did Bourbaki insist to such an extent on isomorphism instead of just morphism, or homomorphism in general? First of all, let me make clear what we mean today by the concepts of "structure", "homomorphism" and "isomorphism" in *mathematics*[6].

A *structure* is a unified description of mathematical objects, characterised by one or more *sets A, A',… of elements* interrelated by one or more *relations ®, ®'…* These relations can be unary (as, for example, the fact to possess some property), binary (for example, the order relation), ternary (for example, a law of composition from AxA to A), etc. They can link one or more sets (for example, an operation from AxA' to A'), and so form a more or less complex system. For example, a *topological structure* implies several kinds of relations such as neighbourhood, union, intersection, inclusion, membership, which can operate on one or more sets[7].

A *homomorphism Φ*: $A{\rightarrow}A'$ is a mapping from A to A' which preserve the "structure relations" in A. In other words, A' is called "homomorphic" to A if, for all $x, y \in A$, $Φ(x{®}y) = Φ(x) ® Φ(y)$. For example, in the case of two groups $(G,*)$ and $(G',-)$, G' is homomorphic to G if, for all $x, y \in G$, $Φ(x*y) = Φ(x) - Φ(y)$. If a homomorphism $Φ$ is one-to-one, onto, and if its inverse mapping $Φ^{-1}$: $A'{\rightarrow}A$ is also a homomorphism, then $Φ$ is called an *isomorphism*. A homomorphism from A to itself is called an *endomorphism*. If this endomorphism is also an isomorphism, it is called an *automorphism*[8].

We see that, even if isomorphism is certainly a *sufficient* condition for having the same structure, it is not a *necessary* condition. There are "equivalent" structures which are equivalent in a lesser sense than isomorphism. One can perfectly conceive mappings from one set to another preserving the properties of structure, without preserving the individuality of elements. Such mappings that preserve structure are called homomorphisms. I will then explore how the connection between the concepts of structure and isomorphism was historically established, trying to keep in mind the following two questions:
1) Wouldn't homomorphism, as the only partial equivalence of two sets, be the true ancestor of the structural approach?
2) Since equivalences in the cases of homomorphism are always partial or relative to certain criteria, what would these criteria be? What defines the specificities of the structural approach?

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[6] I deal only with the question of the *prehistory* of the *mathematical* notion of structure. For a discussion on the interest of the Bourbakist undertaking to give a formal meaning to the notion of structure, see Corry (1992). For a discussion of the role of Bourbaki as "cultural connector" of philosophy, linguistics, literature, etc., see Aubin (1997). For a discussion of the meanings and issues of "structure" in social sciences, see Bastide (1962) and Boudon (1971 [1968]).

[7] See Bourbaki (1957). The notion of *algebraic structure* (that is, of sets with laws of composition and operations satisfying certain axiom systems) had already been approached by Noether (1927), Birkhoff (1935) and Ore (1935).

[8] Some authors reserve the term *morphism* (belonging to the category theory) to describe a mapping preserving *structure*, and the term *homomorphism* to describe a mapping preserving *algebraic structure*. I take here the term homomorphism in the broader sense given by the model theory, where *structures* involve not only binary or ternary relations, but relations in general.

Structure constants in Elie Cartan and Sophus Lie

Why does Bourbaki insist so much on the close connection between structure and isomorphism, while the mappings preserving "structure properties" are referred to as homomorphisms? A simple answer is that Bourbaki, in his effort to centralise all mathematics around structure, must obviously take into account the definition *already* given to structure in some departments of mathematics. Now, in the *group theory,* this definition was precisely expressed in terms of isomorphism. It can be found for instance in Elie Cartan (1937)[9] or Elie Cartan (1904)[10].

I summarise these definitions in simple terms. Two groups have the same *structure* if, and only if, there is *isomorphism*, i.e. bijective correspondence between the elements of both groups and between the products of these elements. Reading each of the definitions given by Cartan, we should not get misled by the shift in meaning of "isomorphism". Until around the beginning of Thirties, "isomorphism" was not always bijective. Indeed, according to a meaning dating back to Camille Jordan[11], isomorphism from *A* to *B* was simply defined as a *surjective* mapping from *A* to *B*. When this mapping became also injective, in short when the correspondence became bijective, isomorphism was called "holoedric". In the opposite case, it was called "meriedric". I will investigate the origin of this distinction further. In the definitions of group structure referred to, there is no ambiguity: one only speaks about a common structure of two groups when the isomorphism between both of them is *holoedric* – that is bijective – and not meriedric – that is surjective. This definition of group structure lies within the framework of the research undertaken by Sophus Lie and Wilhelm Killing around 1870, and by Elie Cartan around 1890, to identify and classify the "continuous" groups into different structures.

I briefly recall in what consists the general principle of Lie's *Theorie der Transformationsgruppen*. Lie starts from a set of "continuous"[12] functions *f* depending on a finite number r of parameters. This set must include the identity function. Lie proposes to differentiate the functions in the directions of the parameters in question, i.e. to consider their "slopes" (vector fields) with respect to parameter variations when these are very close to identity. He associates with these vector fields $X_1, X_2...$, which he calls "infinitesimal transformations", a new composition law, known as "bracket operation":

$$[X_1, X_2] = X_1(X_2(f)) - X_2(X_1(f))$$

The bracket operation then consists in grasping the difference between the successive differentiations whether they take place in the one order X_1X_2, or in the other X_2X_1. The interest of this operation is that it allows, by linking between them any tangent vector at the identity function, to produce a *result always belonging to the same system* (i.e. to the same tangent space at the identity function): according to the bracket law, the composition of two

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[9] "Deux groupes ont la même structure ou, comme on dit, sont *isomorphes*, quand on peut établir entre leurs opérations une correspondance biunivoque, telle que si S',T' correspondent à S,T, S'T' corresponde à ST." (Cartan, 1937, p. 2)

[10] "Nous dirons enfin que deux groupes isomorphes holoédriques ont la même *structure*." (Cartan, 1904, p. 184)

[11] Jordan (1870), p. 56.

[12] "Continuous" must be understood as "analytic", that is to say: expressible in series of powers in the neighbourhood of each point of their domain of definition. Functions must be *invertible*, and their parameters must be "essential", in other words the functions cannot be expressed using a lower number of parameters than that given initially.

"infinitesimal transformations" still gives an "infinitesimal transformation" of the parameter values fixing the identity function. This causes a closed composition system to emerge, which Lie calls "r-parameter group of infinitesimal transformations" (*infinitesimale Transformationen einer r-gliedrigen Gruppe*). This type of system, which can indeed be put in correspondence with the set of "continuous" functions given at the start, and which reveals the group structure of this latter, will later be called "Lie algebra"[13].

At the beginning of chapter 17 of his *Theorie der Transformationsgruppen* (volume 1)[14], Lie shows that bracket operation can also be defined as a linear combination of the basis vectors $X_l, ... X_r$ belonging to the tangent space at the identity function:

$$[X_i, X_k] = \sum_{s=1}^{r} c_{iks} X_s f$$

Lie calls "composition of the group" (*Zusammensetzung der Gruppe*) the set of constants c_{iks} multiplying the basis vectors. If, in each of two *r*-parameter "groups" (actually: Lie algebras), there exists a basis $X_l, ... X_r$ such that the *coefficients c_{iks} are the same,* these "infinitesimal groups" (Lie algebras) are said to have "identical composition" (*gleichzusammengesetzt sein*), or are said to be "holoedrically isomorphic" (*holoedrisch isomorph*). No doubt the word "composition" (*Zusammentsetzung*) was not perfectly suitable to designate the multiplying coefficients of the basis transformations of an "infinitesimal group". Indeed, these coefficients can also be found in groups known as *simple* (*i.e.* without invariant subgroups except the group itself and the identity element), as well as in groups known as *composite*. This objection was raised by Wilhelm Killing, who proposed to replace the word "composition" by "shape" (*Gestaltung*)[15], or by "construction" (*Bau*)[16]. But it is the term "structure" that has emerged, following the translation suggested by Ernst Vessiot and Wladimir de Tannenberg in their reporting of the first volume of Lie's *Theorie der Transformationsgruppen* :

> "Mr. Lie makes the notion of isomorphism dependent on another important notion, namely that of *structure (Zusammensetzung)* of a group. [...] The system of the constants c_{iks} defines the structure of the group. [...] Two r-parameter groups which have the same structure are said to be *isomorphic*, and the *isomorphism* is holoedric."[17]

The young Elie Cartan has popularised the term by carrying out the vast classification project of the various continuous groups of transformations, as a function of to their structure. As to the coefficients c_{iks} common to two holoedrically isomorphic groups, they will consequently be called "structure constants" (*constantes de structure*).

So from the 1890s onwards, the concept of structure acquires a precise mathematical meaning. It is the set of constant values c_{iks} multiplying the basis vectors of a "group of infinitesimal transformations", later known as "Lie algebra". If, for two distinct Lie algebras with the same number of parameters, these values are all identical, then there is a bijective correspondence between the infinitesimal transformations of each of the algebras, and between the results of their composition according to the bracket law. The *structure* of these

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[13] Weyl (1934-1935), p. 5.
[14] Lie (1970 [1888]), p. 189.
[15] Killing (1890), p. 163. See Hawkins (1982), pp. 161-162.
[16] Killing (1904).
[17] Vessiot and de Tannenberg (1889), pp. 137-138.

two algebras is then said to be identical because the number of their parameters and the law of combination of these parameters are the same[18]. Even if the structure concept has not yet taken the widened or "abstract" meaning that it will take later, it is already clearly related to the isomorphism notion. But Lie does not only note that two *r*-parameters "groups" with the same composition (*Zusammensetzung*) are *holoedrically isomorphic*. He adds that one can also *compare* the composition of "groups" *not* having the same number of parameters. Let a "group" which, instead of *r*, has (*r-q*) parameters:

> "One can also compare the composition of such groups, which do not have both the same number of parameters. This is made possible by the introduction of the general concept: isomorphism. [...] One distinguishes between *holoedric* and *meriedric* isomorphism. The *holoedric* occurs when the number *q*, which takes place in the definition of isomorphism, has the zero value; the *meriedric* occurs when *q* is greater than zero."[19]

So isomorphism is not necessarily holoedric. Identifying *one and the same structure* on two Lie algebras having the same number of parameters allows to work, so to speak, on a single object; but discovering identical coefficients of structure on two Lie algebras having a *different* number of parameters is also possible. In this case, isomorphism is said *meriedric*. This second approach, which focuses not on the *identification* of identical Lie algebras, but on the *transfer* of certain properties from one to another, has not been supported by Cartan, who pursued a general objective of classification of all the continuous groups. But the concern of Lie was not only classificatory. Several of his contemporaries noted his interest in interconnectedness and pathways[20]. Lie was probably more prone than Cartan, if not to follow the track of meriedric isomorphisms, at least to point out their possibility.

Meriedry and conjugation in Auguste Bravais and Camille Jordan

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[18] It should be remarked that the (Lie) *groups* corresponding respectively to each of these Lie *algebras* are not, for all that, isomorphic between them, because isomorphism occurs only at the local level, i.e. for transformations infinitely close to the identity function. Elie Cartan and Hermann Weyl will determine in the 1920s the conditions of passage from the "local" to the "global" point of view. See Hawkins (2000) and Chorlay (2009).

[19] Lie (1970 [1888]), pp. 292-293.

[20] See E. Study and F. Engel quoted by Stubhaug (2002), p. 395.

I continue to go back in time. To support his distinction between holoedric and meriedric isomorphisms, Lie refers in a footnote[21] to the *Traité des substitutions* of the mathematician Camille Jordan, published in 1870. It is indeed in the "Mémoire" of Camille Jordan on the groups of movements, published in 1868 and 1869[22], that the notion of meriedry appears for the first time in group theory. But right from the start of his memoir, Jordan openly admits that he has borrowed this notion from the *crystallographic* studies of Auguste Bravais (1848-1851)[23]. So here are some explanations on the meaning of meriedry in Jordan and in Bravais.

In his memoir, Jordan aims at classifying "all possible groups of movements in the Euclidean three-dimensional space" and discovers that certain "larger" groups contain other "smaller" groups. Jordan calls the larger groups "principal", and the smaller ones "meriedric". To him, a group is meriedric if it "contains only a given fraction of the movements which constitute one of the principal groups"[24]. It should be said in passing that the largest of all principal groups is the group of all helicoidal motions, i.e. the group composed by any rotation around an axis, and any translation for the same axis.

As to Bravais, he essentially makes a distinction between two great types of "structure": *molecular structure* and *crystalline structure* rather than two great types of groups. *Molecular structure* is the polyhedric arrangement of "components of the molecule"[25] around the centre of gravity of the molecule. In other words, it is the pattern which is periodically repeated inside the crystal. Bravais indexed all possible symmetries of these "molecular polyhedrons" in which at least one point remains fixed, and counted 32 crystallographic classes or groups[26].

In addition to this "concrete" structure of molecular polyhedrons, Bravais imagines a more "abstract" structure, the *lattice*[27] (which he calls either *réseau* if it is bi-dimensional, or *assemblage* if it is three-dimensional[28]) in which the centres of gravity of the various molecular polyhedrons are distributed. This time, Bravais does not take into account the shape of the polyhedrons. He calls *crystalline structure* this "relative arrangement of the centres of gravity of the molecules of the crystallised bodies"[29], and discovers fourteen possible lattice structures[30].

Clearly there is a difference of meaning of the term *structure* used respectively by Bravais and Lie, and a still bigger one as compared with the one used by Bourbaki. But what is striking here is that Bravais explicitly wonders how a *correspondence*, or a "*transmission*" *of symmetry properties* can occur between both structures, molecular and crystalline:

"Since to any given molecular structure there is a corresponding relative crystalline structure, most probably the molecular symmetry will determine that of the crystalline

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[21] Lie (1970 [1888]), p. 293.
[22] Jordan (1868-1869).
[23] These studies were later collected in Bravais (1866).
[24] Jordan (1964), p. 302.
[25] Bravais (1866 [1851]), p. 194.
[26] These 32 crystallographic groups had already been discovered by M.L. Frankenheim in 1825 and by J.F.C. Hessel in 1830. See below page 11.
[27] The inventor of this notion of lattice is a pupil of the French crystallographer René-Just Haüy: Gabriel Delafosse (1843).
[28] Bravais (1866 [1850]), p. 2.
[29] Bravais (1866 [1851]), p. 194.
[30] Frankenheim (1835) had counted fifteen of them, but Bravais notes that two of these "forment double emploi" (Bravais, 1866 [1850], p. 97).

assemblage. [...] we will tentatively show that the symmetry of the molecule tends to be *transmitted* to the *assemblage* which will be formed."[31]

This correspondence, as he notes, is sometimes "exact", or "complete", and sometimes "inaccurate", or "incomplete". If it is "complete", there is "holoedric isomorphism"; if not, "meriedric isomorphism" in the sense that an abstract lattice has more symmetries than the concrete pattern (molecular polyhedron) that it contains. To put it more precisely, there is hemiedric isomorphism when the lattice has twice the number of the pattern symmetries, of tetraoedric isomorphism when it has it four times, etc.

In his authoritative *Traité des substitutions* (1870) Jordan adopts Bravais's idea of a *partial isomorphism* transmitting certain properties from a group of transformations to another. In a move that lastingly imprinted the vocabulary of mathematics, he imports from crystallography the concepts of total and partial isomorphism:

"A group Γ is said to be *isomorphic* to another group G, if it is possible to establish between them a correspondence such that : 1° to each substitution of G there corresponds a unique substitution of Γ, and to each substitution of Γ one or more substitutions of G; 2° to the product of any two substitutions of G there corresponds the product of their respective corresponding substitutions. An isomorphism is said to be *meriedric* if many substitutions of G correspond to the same substitution of Γ, and *holoedric* in the opposite case."[32]

So, in the same way as a crystal's structure can completely or only partially reflect the properties of the lattice to which it is related, similarly any two groups of operations applying to any object (movements of solid objects, substitutions of the roots of an equation, etc.) can transmit, in whole or in part, some of their properties. "The notion of isomorphism", Jordan adds, "can often be helpful, because of the similarity of properties that isomorphic groups present between them. [...] One can therefore in many cases replace the direct consideration of a group by that of any of its isomorphs"[33].

Now, in this issue of "transmission" of certain symmetries (between principal and meriedric groups as stated by Jordan; or between holoedric lattice and meriedric crystalline classes in Bravais), I return to my second question: what are the criteria of such transmission? What allows us to state that certain symmetry properties of a set can be found in another one? I would like to particularly underline that two concepts, *conjugation* and *selfconjugation,* are ever-recurrent.

Let us start with Jordan. He aims at identifying groups of movements such as the combination of two movements always belongs to the same group. Intuitively, one could say that two movements, N and B, must always be connected by a third one, M, which, starting from one, brings it back to the other, and, from the other, carries out the reversed movement. Jordan formalises this by defining the concept of *transformed* (equivalent of our *conjugate*[34]) movement. A movement B is defined as the *transformed* (or *conjugate*) of another movement N- by a third M if there is the following relation[35]:

[31] Bravais (1866 [1851]), p. 202, my emphasis on the word "transmitted".
[32] Jordan (1870), p. 56.
[33] Jordan (1870), p. 60.
[34] Jordan uses the term "transformed (of S by T)" to designate $T^{-1}ST$ (Jordan 1870, p. 23). Burnside prefers the term "conjugate" (Burnside, 1897, p. 29).
[35] Jordan 1964 [1868-1869], p. 170-171.

NM = MB

that is:

$M^{-1}NM = B$

The characteristic of two transformed movements, Jordan notes, is that they preserve the relative situation of the points located around the axis attached to each of both movements. Hence we have a way to operate on the group of all possible movements a partition in *conjugacy classes* as we call them today.

Moreover, Jordan also uses what we nowadays call *self*conjugation. He "knows" that a specific group of movements – namely translations – is always transformed *into itself* by any other movement. This *invariance* of the group of translations is a very important characteristic, because it can be used as a classifying principle. Indeed, given an unspecified movement, one can ask which other movements can transform it in a translation. And every possible way that we find will determine a new sort of group each time. In his *History of algebra*[36], Van der Waerden summarizes Jordan's reasoning by noting that it amounts to find all possible extensions of the normal subgroup of translations, i.e. all the possible subgroups of helicoidal motions such as those helicoidal motions H combined with a translation T, then combined with inverse of H, still make a translation T :

$H^{-1}T\,H = T.$

Now what about the use of conjugation and selfconjugation in Bravais.

In his search for a classification of crystal's symmetries, Bravais is brought to compare different structures amongst themselves. A symmetry can be established around points, axes, or plans. For Bravais, two symmetries will be "similar", or "of the same kind", or "*conjugate*" (the vocabulary of Bravais varies in this regard) if these points, axes or plans are disposed in such manner that "the configuration of the tops around one is the same around the other"[37]. It is to be noted that Jordan had underlined the very same characteristics about transformed movements. But more than that, we will remember how Jordan used the property of a certain group of movements - the translations – to always be transformed in themselves by any other movement. Bravais notes that there are axes which always preserve the same configuration of tops around them, whatever other symmetries occur for other axes elsewhere. He calls "principal" this particular type of axes, whose symmetries remain unchanged whatever other symmetries may additionally occur. Indeed Bravais uses this invariance, or "selfconjugation", to ensure the "transmission" of properties from one structure to another:

> "The axes of symmetry of the molecular polyhedron tend to be transmitted, as axis of the same symmetry order, and similarly placed, to the assemblage (the lattice) where crystallisation occurs."[38]

So much for a brief presentation of the use by Bravais and Jordan of the concepts of conjugation and selfconjugation. Today, the importance of the concept of *selfconjugation*, and

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[36] Van der Waerden (1985), p. 120.
[37] Bravais (1866 [1849b]), p. xxiv.
[38] Bravais (1866 [1851]), p. 203.

especially of *normal subgroup*[39], is recognised to characterise *group homomorphisms*[40] and even *homomorphisms in general*[41]. The question now is whether both authors did not borrowed their concepts from e.g. the man known as the inventor of the conjugation concept, namely Augustin Cauchy, who developed it in his 1845 memoir on arrangements and substitutions[42]. After all, doesn't Jordan himself note "the remarkable analogy"[43] between his theorem on transformed movements and Cauchy's theorem on similar substitutions? And doesn't Bravais wholeheartedly thank Cauchy for his eulogistic reports on his own work[44]?

Actually, there is an enormous difference between the approaches of Bravais and Jordan on the one hand, and of Cauchy's one on the other. This difference is very simple: whereas Cauchy always specified that the variables on which his substitutions operate are *independent* from each other[45], Bravais and Jordan state that the elements on which symmetries or movements operate, are *dependent*. They are linked by such forces that "all mutual distances remain the same"[46], as Bravais states. This difference is substantial, as it allows to understand why Cauchy never expressed any interest for *self*conjugate subgroups. Obviously there is no interest to look for those substitutions *preserving* an unspecified connection between variables, at least if there is nothing to link those variables between themselves.

In fact, it seems to me that the comparison with Cauchy reveals the specificity of a certain approach, foreign to Cauchy though common to a number of other authors. This approach consists in wondering how a whole of given connections can be *preserved despite certain transformations*. How can a "structure" attached to a certain field be "transmitted" to other fields, other forms, other beings? Auguste Bravais did not only ask this question about the growth of crystals. By the end of the 1830s, he himself, his brother Louis Bravais and the geographer and botanist Charles Martins wondered about the conditions of the growth of *plants*, and more particularly the transmission of symmetries in their inflorescence. For example, knowing that the leaves of the wild plum are not arranged on both sides of the stem but that they form a spiral around it, can we determine how many times it is necessary to spiral up around the stem to find a leaf in the same location as the one we left? Moreover :

> "If we *imagine*, around a cylinder, all possible systems of intertwined helixes, and if we place a leaf at each of their intersections, [can we not fix] the arrangement of all known systems, and that of a prodigious number of unknown systems, but which are symmetrical and similar to those lying in the real domain of science?"[47]

The reasoning of the Bravais brothers and of Charles Martins about the growth of plants appears to me as very similar to Auguste Bravais's later reasoning about crystals. Each time,

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[39] *H* is a *self*conjugate or *normal* subgroup of *G* if every element of *G* is permutable with *H*, that is, if, for each element *h* in *H* and each *g* in *G*, the element $g^{-1}hg$ is still in *H* (Jordan, 1870, p. 41; Burnside, 1897, p. 29; Weber, 1898, p. 553).

[40] The image of any group homomorphism Φ: $G \rightarrow G'$ can be represented as a quotient group *G/H*, where *H* is a normal subgroup of *G* (Hölder, 1889 ; Noether, 1927).

[41] Every normal subgroup *H* of *G* is the *kernel* of a group homomorphism Φ: $G \rightarrow G'$. That is, *H* defines the subset of the elements of *G* that are mapped by Φ to the identity element of *G'*. Now, it is this notion of *kernel* that generally allows to measure the degree to which an homomorphism fails to be injective (Noether, 1927).

[42] Cauchy calls *similar* two substitutions *S'* and *S* satisfying the relation $T^{-1}ST = S'$ (Cauchy, 1974 [1845-1846], p. 190).

[43] Jordan (1868-1869), p. 235.

[44] Bravais (1866 [1850]), p. 128.

[45] Cauchy (1974 [1845-1846]), p. 171.

[46] Bravais (1866 [1850]), p. 3.

[47] L. and A. Bravais (1837), p. 50.

what is sought after is the preservation of a certain configuration around oneself, despite the transformations undergone. In this reasoning, mathematics only intervene to identify the *outstanding* features of such or such vegetal or crystalline form. The point is not to focus on the numbers or figures themselves but to determine how certain *properties* are *transmitted* from some beings to others, or from one field to another. The "consideration of the external forms of living beings", as Martins writes, gives precedence to the study of their "inner organisation"[48]. To Louis Bravais, it is necessary to pay attention "rather to the varieties of organisation, than to those of form"[49]. This approach is characteristic of the 19th century: the classification, "science of order", yields ground to the genetic science of *organisation*. The objective of science is no longer to reproduce, as accurately as possible, the diversity of appearances, but to tentatively penetrate their logic, to grasp their dynamics and, therefore, to return to the more abstract world of *operations* by which beings, living or not, organise or structure themselves.

Equivalence and selfconjugation
in Johann F.C. Hessel and Heinrich A. Rothe

Pursuing my enquiry, I realise that this approach of phenomena is not specific to the Bravais brothers and to Martins. It has been notably practised for some time by a number of German scientists. Right from his earliest works, Auguste Bravais reads and positively quotes German authors, both in the fields of botany (Goethe, Schimper, Braun[50]) and crystallography[51]. In regard to crystallography, Erhard Scholz[52] has clearly shown the connection of Bravais's work with German authors such as Moritz Ludwig Frankenheim, the discoverer of the 32 crystallographic groups in 1825, and Christian Samuel Weiss, the inventor of hemiedrism.

In contrast to the *mechanistic* interpretation of the crystals that prevails in France, the mineralogist Christian Samuel Weiss defends a "*dynamic perspective on crystallisation*" (*Dynamische Ansicht der Krystallisation*[53]) in the early years of the 19th century. At that time, any scientific or philosophical approach conceiving *movement* or *change* of either animate or inanimate beings as caused by *internal* or *immanent* and not *external* or *mechanical* forces could be called "dynamist". For Weiss, the diversity of the crystals should not be explained by the stacking or the addition of identical molecules, but by the "product", the *combination* of forces which are internal to the crystal. This "product" is first expressed by the combination of the various orientations and axes of different lengths. But it is also translated, in a more complex way, by the possible permutation of certain coordinates of the different faces of each crystal. In pursuing this path, namely in permuting or swapping these coordinates in any way authorised, another German naturalist, Moritz Ludwig Frankenheim arrives, in 1825, to index thirty-two "surface groups" (*Flächengruppen*), that is thirty-two ways to orient the faces of the crystals, depending on how their axes are organised[54]. Ten years later,

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[48] Martins (1871), p. 764.
[49] Bravais (1839), p. 2.
[50] L. and A. Bravais (1837), p. 42.
[51] Bravais (1866), pp. 105-117, 222, 234-235, 237, 243, 245, 244.
[52] Scholz (1989).
[53] Weiss (1804).
[54] Frankenheim (1826).

Frankenheim considers this time the "density" of the crystal, i.e. the arrangement of its "particles" (*Teilchen*) relatively to the axes which structure it. He distinguishes fifteen crystallographic "orders" (*Ordnungen*), or fifteen ways to distribute the molecules in the system of reference constituted by its axes[55].

Eventually Johann Friedrich Christian Hessel[56], in a rather confidential publication of 1830, will actually confer a mathematically interesting format to the dynamist speculation of the German crystallographers. Unlike Weiss and Frankenheim, Hessel does not proceed by permutation of the *coordinates* of reference axes and their relative magnitudes, but by examining how the *symmetries* or "equivalences" (*Gleichwertigkeiten*) around these axes, combined between them, modify each other:

> "It is no longer simply a case of equivalence (*Gleichwertigkeit*) between some parts of a body, but an equivalence related to a given system of axes; this latter equivalence requires equivalence between the parts of the body in question, but not the reverse; it is also necessary that the body parts, which are equivalent to each other, are presented as equivalent with respect to the given axis system."[57]

Hessel thus calls equivalence (*Gleichwertigkeit*) such property of certain axes of leaving unchanged the other equivalences (symmetries) of the system to which they are attached. Building on this property, he manages to find the 32 surface groups brought to light by Frankenheim. We can see that, from 1830 onwards, the requirement for invariance of certain transformations compared to others is clearly stated as a guiding principle for the classification of crystals, and of the development of their forms (*Gestalten Entwicklung*[58]).

While Hessel was developing his classification of crystals based on the invariance property of some transformations compared to others, another isolated researcher, in a quite different field, resorts to a similar argument. This researcher is Evariste Galois who notably submitted in 1830 a Memoir on the conditions for solvability of equations by radicals considered as incomprehensible by the members of the Academy of Sciences in Paris. In this Memoir, Galois uses the same invariance property as that which will be formalised later by mathematicians under the name of invariant, or normal, subgroup[59]. But Galois approaches this notion of invariance or selfconjugation in too laconic or too cryptic a manner to promote understanding of it[60].

Looking at the side of Hessel, and even of Galois, I have tried to understand what are the *mechanisms* behind the "structural approach", what are the *criteria* for a "transmission" of symmetries from an organised set to another one. But I must not lose sight of the use of the *words* "structure" and "isomorphism". They too can provide information on the time when, and the conditions under which, the need was felt to think the problems in terms of "structural correspondences".

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[55] Frankenheim (1835).
[56] Hessel (1830).
[57] Hessel (1897 [1830]), p. 68-69.
[58] Hessel, 1897 [1830], p. 70.
[59] See above, note 39.
[60] It is necessary, Galois writes, that the "knowledge of the adjoint quantity" "does not alter the group of the equation" (Galois, 1976, p. 59).

The term "structure", of Latin origin, is very old. At first it had the concrete meaning of masonry, construction[61], or the more abstract sense of order, arrangement, in particular of words and arguments in rhetoric[62]. Only in the 18th century the terms of *vegetal*[63] and *crystalline*[64] structures appeared to refer to organised wholes of vegetal or mineral elements.

The word "isomorphism" is much more recent. It was used for the first time by the German chemist Eilhard Mitscherlich around 1819[65]. He observed that substances with different properties can crystallise in an almost identical way -what he called "isomorphism"-, but also that the same chemical substance can sometimes crystallise in different forms -what he called "dimorphism". Thus, the *crystalline form* of a substance and its *chemical composition* do not coincide. Students of Haüy, like the mineralogist Beudant, casted doubt on the conclusions of this "geometrical" German science which proceeds by analogies and similarities:

> "The word *isomorphic* can not be taken in a rigorous sense, frequently it indicates only a great analogy."[66]

But the theses of Mitscherlich were supported in particular by the famous Swedish chemist Berzelius, who saw them as a way to strengthen his own electrochemical theory of combinations. According to this theory, different substances can possess similar groups of atoms, provided that these chemical combinations are due to connections between atoms with opposite electric charges. In 1836 the chemist Auguste Laurent returned to this hypothesis, by proposing that the substitutions within chemicals compounds take place independently of the electric charge of atoms. This time, it was Berzelius who was hostile, convinced that the substitutions cannot go against the electrical forces. But Laurent even imagined a geometric representation of these possibilities of substitutions: a same prism, or primitive form (kernel), could be found in different types of compounds and could allow, as in crystallography, various substitutions[67].

It is worth noting that the notion of chemical isomorphism established by Mitscherlich was not defined in precise mathematical or logical terms: it mixed several aspects like having the same crystal form as another substance (*homeomorphism*); the capacity for two crystals to form together a single one (*sincrystallisation*); and the fact of having the same chemical composition, except for a few substitutions of chemically close elements. It is not until much later that the rules for establishing the identity of the relative distribution of sites occupied by atoms (*structural isomorphism*) have been discovered[68]. Nevertheless we see that, as early as the 1820s, there was a trend of thought closely related to the German dynamist science which, well before the "official" mathematical definition of structure and isomorphism concepts, reflected on how certain principles of organisation preserve themselves *through transformations*.

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[61] Vitruvius. *De architectura libri decem*, 5, 12.
[62] Quintilianus. *Institutio oratoria*, 1, 10, 23.
[63] Rousseau (1789 [1771]), p. 10.
[64] "The manner in which the shapes of the constituent molecules of the crystals are arranged in each crystal" (Haüy, 1784, p. 9).
[65] Mitscherlich (1818-1819).
[66] Beudant (1830), p. 401, quoted by Schütt (1984), p. 191.
[67] Elie Cartan pointed out the similarity of this approach with that of Galois (Cartan, 1974 [1931], p. 17).
[68] Goldschmidt (1924).

This said, could it be that the technical concept of selfconjugation, allowing "transmission" of properties from one "structure" to another, was already approached in the context of dynamist's science and philosophy *even before* Hessel's works in crystallography, and debates on isomorphism in chemistry?

University of Erlangen, 1819. Here teaches Heinrich August Rothe, a mathematician who trained Hessel at the combinatorial technique. That year, Rothe publishes in collaboration with two of his colleagues, the botanist and *Naturphilosoph* Christian Gottfried Nees von Esenbeck and the chemist Carl Gustav Bischof, a rather strange book entitled *Development of vegetal substance physiologically, chemically and mathematically explained, with the combinative tables of possible vegetal substances and the laws of their stoichiometric composition*. In this book, written in the purest style of "dynamist" philosophy, the authors propose to examine how the general forces in nature act on plant generation and transformation. In particular, they wonder how natural forces link between them the three fundamental components of organic matter, namely: oxygen, hydrogen and carbon.

In the first part, entitled *Speculation*, Nees von Esenbeck recalls the great principles of *Naturphilosophie*: the diversity of natural forces can be reduced to the opposition of two large poles, cancelling one another when they meet in equal quantities. In the second part, entitled *Empirie*, the chemist Bischof explains that in chemistry this "indifferenciation of the opposites" involves the possibility of interchanging oxygen and hydrogen while carbon remains unchanged. According to Bischof, from this statement derive the five fundamental binary connections which are the basis of any decomposition of organic matter. These five fundamental connections are as follows:

Table 1 — Chemical compounds included in the combinatorics of Rothe

a	"water"	OH	(instead of H_2O)
b	"carbonic oxide gas" (carbon monoxide)	CO	
c	"carbonic acid gas" (carbon dioxide)	CO_2	
d	"olefiant gas" (our ethylene)	CH	(instead of C_2H_4)
e	"carbureted hydrogen gas" (our methane)	CH_2	(instead of CH_4)

Incidentally, Bischof makes a series of errors which are normal for his time. Back then, scientists were unaware of how many atoms (or how many "elementary quantities") were necessary to combine, for example, hydrogen with oxygen or carbon.

Of interest to us is the mathematical reasoning of the third and last part entitled *Mathematics*. Rothe now enters the scene. He first builds a gigantic table of all possible sums of components *a, b, c, d, e*, when each one is multiplied by zero, one, two or three. This produces 961 possible combinations. Rothe wonders how to classify or regroup all these combinations. The true innovation will take place here.

I wish to insist on the fact that for Rothe, as for Bischof and von Esenbeck, such various combinations are more than just simple arrangements. As stated at the beginning of the book,

the basic assumption of the authors is that "vegetal life [comes from] the combinatorial composition (*combinatorische Zusammenstellung*) of all chemical connections legally possible"[69]. The various chemical combinations are thus the "causes", or at least the "explicative factors", of the development of the vegetal substance (*die Entwickelung der Pflanzensubstanz*). Rothe wonders which amongst these combinations are those which leave a certain relation invariant, namely the proportion between quantities of oxygen, hydrogen and carbon. Hence he puts the combinations in various subsets, which he calls "groups of equivalent complexes" (*Gruppen von gleichgeltenden Complexionen*). For example, the group of "equivalent complexes" which keeps the proportion: 2 oxygens (O) – 2 carbons (C) – 1 hydrogen (H) unchanged, includes the following operations[70]:

Table 2 — Example of a "group" of "equivalent complexes"

d+3*b*+*a*	4O-4C-2H
d+*c*	2O-2C-1H
2*d*+*c*+3*b*+*a*	6O-6C-3H
3*d*+2*c*+3*b*+*a*	8O-8C-4H
e+*c*+2*b*	4O-4C-2H
e+*d*+2*c*+2*b*	6O-6C-3H
e+2*d*+3*c*+2*b*	8O-8C-4H

So far, there is nothing really new. Rothe is obviously not the first one to carry out that kind of partition. But he also remembers[71] – and this is the second stage of his reasoning – that the most general operation of "Nature" consists in exchanging oxygen and hydrogen, while carbon remains unchanged. Now, by applying this general operation, a group T of "equivalent complexes" can be changed or transformed into another group T':

"These complexes, which arise one from the other, because *b* and *d* , and *c* and *e*, are exchanged while *a* remains unchanged, I call them *conjugate complexes* (*verwandte Complexionen*) […]. So we note, for example, that the two groups of seven equivalent complexes, represented in the table below, are conjugate."[72]

Table 3 — Example of two "groups" of "conjugates complexes"

T	⇔	**T'**
	Permutation *b-d* and *c-e* (tantamount to the permutation between O and H)	

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[69] Nees von Esenbeck et al. (1819), p. 2.
[70] Nees von Esenbeck et al. (1819), p. 226.
[71] As the book was published as a compilation of epistolary correspondence between the three authors, this particular point was brought to Rothe's attention by Bischof (Nees von Esenbeck et al., 1819, p. 220).
[72] Nees von Esenbeck et al. (1819), p. 221, 226.

d+3b+a	*3d+b+a*
d+c	*e+b*
2d+c+3b+a	*e+3d+2b+a*
3d+2c+3b+a	*2e+3d+3b+a*
e+c+2b	*e+2d+c*
e+d+2c+2b	*2e+2d+c+b*
e+2d+3c+2b	*3e+2d+c+2b*
proportion :	proportion :
2O-2C-1H	1O-2C-2H

If Rothe is so quick on proposing the term *verwandte Complexionen,* it is because he has already defined the notion of *verwandte Permutationen* in a previous work on combinatorics:

"Two permutations of the numbers 1,2,3,...n, are called *conjugate* (*verwandte*) when each number and the number of the place which it occupies in the one permutation are interchanged in the case of another permutation [e.g. the permutations (3, 8, 5, 10, 9, 4, 6, 1, 7, 2) and (8, 10, 1, 6, 3, 7, 9, 2, 5, 4) are conjugate]."[73]

The expression *verwandte Permutationen* could be literally translated as "related permutations". Yet Thomas Muir (1890) has retained the term "conjugate permutations"[74]. Today, this notion appears to us as a particular case of the relation between a permutation and its *inverse*. If a permutation σ is an ordering of the elements of a set, its inverse σ^{-1} sends each element back to where it came from. Rothe's pattern is still sometimes used to give an example of inverse permutation: "An inverse permutation is a permutation in which each number and the number of the place it occupies are exchanged."[75] Permutations that are their own inverse are called "involutions"[76] or sometimes "involutary"[77]. This notion of *sich selbst verwandte Permutationen* also appears in Rothe's work on combinatorics (e.g. the permutation (3, 4, 1, 2) is "selfconjugate")[78].

What is the relationship between this and the "structural approach" by isomorphisms ? Obviously, the "groups" discussed by Rothe are not "groups" in the later (algebraic) sense, and his definition of "conjugation" is very different from the one given by Jordan in terms of NM=MB, or M^{-1}NM=B. This said, the relatively "abstract" character of Jordan's definition does not mean that the *idea* and the *use* of elements "conjugate" between themselves is recent in mathematics. There's always been a tendency to identify objects between which there is a *correspondence*. These objects are sometimes referred to as *conjugate*. For example, Cauchy calls "conjugate" two complex numbers *a+bi* and *a-bi* symmetric with respect to the axis of reals[79]. Clearly there is an operation (M = +*bi*) which, when applied to *a-bi* in one "sense" and to *a+bi* in the "reverse sense" (M^{-1} = -*bi*), produces the same result *a*. Well before that, Apollonius in his Treatise on the *Conic sections* made extensive use of the notion of *conjugation* (*zugon* or *suzugia*) to designate points, straight lines or curves related to each

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[73] Rothe (1800), p. 278.
[74] Muir (1890), p. 59. See also Müller (1900), p. 88 and Eisenreich and Sube (1996), p. 176.
[75] Sterling (2009), p. 181.
[76] Gowers et al. (2008), p. 554.
[77] Salomon (2003), p. 44.
[78] Rothe (1800), pp. 281-282.
[79] Cauchy (1821), p. 180.

other in such a way that the "movement" of the one *corresponds* to the other's with respect to an axis, a point, or a pole[80]. As to Rothe, he of course does not split into two inverse operations the permutation between a number and the number of its place (in 1800) or between oxygen and hydrogen (in 1819), as this permutation is in both cases a mere transposition.

Yet it seems to me that something interesting happened in 1819. So far, the "conjugate" (i.e. exchanged, or simply "set in correspondence") elements were *points*, *straight lines*, *curves* or *numbers*, though never chemical combinations which would be the "explicative factors" of the development of plants ! In 1819 the very purpose of the work of Nees von Esenbeck, Bischof and Rothe was to show that the transposition of some chemical elements had to be *transmitted* to more complex sets. Rothe calls *groups of complexes* (*Complexionen Gruppen)* these more complex sets. Now – and this is the important point – Rothe draws attention to the case where the replacement of each complex in a group by its "conjugate" still produces the same group :

> "If, in a group of equivalent complexes, we replace each complex by its conjugate and obtain the same group [...], this gives groups which are selfconjugate. For example, there is a group of 21 equivalent complexes [where the proportion between O, C and H is 1:1:1] which, as shown in the table below, is selfconjugate. These selfconjugate groups can also be called *isolate.*"[81]

Clearly, complexes of type $\alpha O - \beta C - \gamma H$ with $\alpha = \gamma$ lead to "selfconjugated groups" in Rothe's sense. This case, which could have previously seemed insignificant, here takes on a particular importance : it implies that a same chemical signature can be *transmitted* to more complex vegetal forms. In particular, the group of 21 equivalent complexes quoted above is identified by Rothe and Bischof as corresponding to a "fundamental proportion of *sugar*" (*Grundverhältnis des Zuckers*[82]), which Bischof rightly describes as one of the most widespread substances in the organic world. The book concludes on this enthusiastic note :

> "[...] the vegetal substances, with which the whole vegetal process begins, and which seem to play the most important part in the activity of vegetation (*die Akte der Vegetation*), correspond to the isolate complexes (*den isolierten Complexionen*)."[83]

So the idea of "selfconjugation" in Rothe's sense, i.e. interpreted by Rothe as a kind of invariance of (vegetal) transformations of "Nature" under transposition of chemical elements, here emerges not in a purely mathematical context, but in a context centred on philosophy of "Nature". This might contribute to the recognition of certain "dynamistic" origins of the "structural thought". For a "dynamist philosophy", the problem is not to identify fixed and universal laws which would pass from one field to another in totality, or in a holoedric way. The problem is: how combinations or relations can be transmitted from one field to another in an only *partial*, or *relative*, way ? Rothe is probably the first one to express this type of transmission in such a technical way. But I see no reason to consider him at the basis of this type of approach. Rothe was not just the teacher of Hessel. He himself was formed in Leipzig

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[80] Apollonius Pergaeus. *Conicorum libri octo,* 1, 15ff.
[81] Nees von Esenbeck et al. (1819), p. 226.
[82] Nees von Esenbeck et al. (1819), p. 178, 182. Let us bear in mind that scientists of that time were unaware of how many "elementary quantities" were necessary to combine for example hydrogen with oxygen and carbon. Today, the term *sugar* refers in chemistry to a type of compounds that have the formula $C_nH_{2n}O_n$, where $n = 3$ to 9.
[83] Nees von Esenbeck et al. (1819), p. 231.

at the combinatorial school founded in 1778 by Carl Friedrich Hindenburg[84]. Now, that school is a part of the legacy left by Leibniz. Thus, as Bischof writes in 1819, "this idea of applying the combinatorial theory to Nature, however, is not new: the great Leibniz had already expressed it with as much truth as enchanting eloquence"[85]. Here I am sent back to Leibniz, whom Bourbaki and Weyl precisely identify as being at the origin of the isomorphism concept.

Similarity in Gottfried W. Leibniz

As a reminder, Weyl and Bourbaki give a different definition of the Leibnizian concept of similarity: identity of relations or operations according to Bourbaki, indistinctness of things perceived separately according to Weyl. What exactly does the similarity consist of in Leibniz's view? And what could be the connection with what has been seen so far?

To support his definition of similarity in terms of identity of relations or operations, Bourbaki refers to the pages 301-303 of Louis Couturat's great book *La logique de Leibniz*[86]. Couturat there analyses two fragments. In the first one, entitled *Matheseos universalis. Pars prior*, Leibniz proposes to represent the "similarity", or the formal equality between two relations, by a new sign:

"In addition to the signs of proportion and ratio, I sometimes add a sign for relations in general. It only indicates, in the simplest way, the proportion between classes of relations (*species relationis*) [...]. From there I get, in addition to equality, the sign of *similarity* ∽ [...] For example, if you have:

$a^2 - b^2 = c^2$ and $l^2 - m^2 = n^2$,

Then I say we have:

$$\overset{\backsim}{a} ; \overset{\backsim}{b} ; c \; \backsim \; l ; m ; n$$

that is to say : the relations between *a*, *b*, *c* are the same as the respective relations (taken in the same order) between *l*, *m*, *n*."[87]

In the second fragment commented by Couturat, Leibniz again conceives "similarity" as a formal identity, this time between different *operations*, allowing us to interchange them:

"Nothing prevents, when two different but similar operations are brought together, to swap them, if we see that thereby they are not distinguished, but just turned in different ways. Thus, $ab + cd$ can correspond to $(a + b).(c + d)$ if we represent the multiplication by '+', and the addition by ' .'"[88]

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[84] On Hindenburg and his school, see Jahnke (1990), pp. 161-232 ; Panza (1992), vol. 2, pp. 651-689 ; Séguin (2005).
[85] Nees von Esenbeck (1819), p. 65.
[86] Couturat (1901), pp. 301-303.
[87] Leibniz (1694-1695), VII, p. 57.
[88] Leibniz (1680-1685), p. 31.

It seems to me that these two passages show, if not that Leibniz got a "glimpse", to quote Bourbaki, of the general notion of isomorphism, at least that his work toward a formal symbolism in mathematics could be a breeding ground for the development of the notion of isomorphism, even if this development actually took a long time. Different mathematical objects are indeed connected, considering the formal identity between the relations or the operations which are in each one of them. However, if Leibniz indeed draws attention to the importance of the "similarity" between relations and operations, he does not explain, in these two passages, what it does exactly consist of. How does Leibniz define similarity?

In September 1677 Leibniz wrote to Jean Gallois, the director of the *Journal des Savants,* a letter in which he explains his new definition of similarity:

> "After having searched carefully, I have found that two things are perfectly similar, when they cannot be discerned other than by *compresence* […]. This proposition is also important in metaphysics and also in geometry and analysis, that the whole is greater than its part. And nonetheless, to the best of my knowledge, no one has enounced it."[89]

From that moment on, Leibniz keeps repeating that he owns a concept – that of similarity – whose use does not only concern mathematics, but also all qualitative aspects of reality[90]. Two things are said "similar" if "they cannot be distinguished when seen by themselves in isolation from each other"[91]. Consider, for example, two circles, one large and one smaller. Leibniz says they are "similar" because:

> "They could not be discerned other than by seeing them together, for in this way we can well see that the one is bigger than the other. You may object: I shall measure the one today, the other tomorrow, and thus I will discern them even without seeing the two of them together. But I say that this still is a way of discerning them *not by memory, but by compresence* : because the measure of the first one is not present to your memory […], but in a material measure engraved on a ruler or some other thing."[92]

What is the connection with what has been said above concerning the importance of the homomorphism in transmission of structures? Leibniz defines similarity, not in relation to the notions of *size* and *shape,* but from the more general notions of *perception* and *space*. One could say that two things are similar if, considered in the same "space", they are perceived as different, and if, considered in different "spaces", they are perceived as identical; or that similarity is a relation perceived as an *identity* "outside the space", and as a *difference* "inside the space". But we well know that, for Leibniz, there is no single "space", absolute, fundamental receptacle of all the objects of our perceptions. Space is not a *being*, but an *order* of coexistences[93] which may involve very different objects and may thereby take very different forms. This principle of "relativity of space(s)" explains why the Leibnizian definition of similarity does not only apply to the geometrical field of the *mathesis*, but also to the "qualitative", or not measurable, aspects of reality[94].

For Leibniz, the perception of a similarity always implies a form of *selection*. We select some points of comparison, and leave others aside. If we should compare things in all

[89] Leibniz (1923-1999), II, i, p. 380.
[90] Leibniz (1679?), pp. 179-180; Leibniz (1686), p. 380; Leibniz (1715), pp. 17-18.
[91] Leibniz (1679?), p. 180.
[92] *Letter to Gallois, September 1677* (Leibniz, 1923-1999, II, i, p. 380).
[93] *Letter to Remond, March 1714* (Leibniz, 1960-1961, III, p. 612).
[94] See Timmermans (2005).

respects, they would always appear as irreducibly different, according to the principle of indiscernibles: "there is no perfect similarity anywhere"[95]. Therefore, "*perfect* similarity occurs only", Leibniz says, "in incomplete and abstract notions, where matters are conceived, not in their totality, but according to a certain single viewpoint"[96]. What he calls "perfect *similarity*" has thus nothing to do with a "perfect *resemblance*", "which holds all the way through"[97], but is closer to what he means by *expression*:

> "That is said to express a thing in which there are relations, or ways of being [*habitudines*], which correspond to the relations of the thing expressed."[98]

Unlike what occurs in the entire resemblance, not *all* aspects, nor *all* the points of comparison are used to build a connection between things, but only *one* or *some* of them. The concept of similarity considered in its "perfection", as Leibniz says, serves not only to create *equalities* between beings or sizes, or even between operations or relations, but also to create much more general, and sometimes very tenuous, *connections* between situations or objects which can be very different from each other. Leibniz's combinatorial art is not just a technique of exhaustion of all possibilities. It rather consists in combining objects which, while they can be qualitatively different, can also be seen as equivalent or substitutable in respect to some of their properties, as these are perceived in the space in which objects are embedded.

Several commentators have stressed the importance of similarity in Leibniz. Oscar Esquisabel notes that "the general formulation [of the concept of similarity] entails the consideration of the possibility of establishing structural identities between objects, the term 'object' being understood in a very general sense"[99]. Vincenzo De Risi, referring in particular to Stiegler (1972) and Swoyer (1995), adds that he "would rather use here the concept of homomorphism, i.e. a partial representation of *one and the same* isomorphism that all monads have in common"[100]. But it is probably Hermann Weyl who first emphasised the originality of Leibniz's approach, by relating it to the current mathematical definition of similarity.

Weyl deals with this issue not only in a side remark of his *Symmetry* (1980 [1952]), but also in several places of his *Philosophy of mathematics and natural science* (1949 [1927])[101]. In particular, he notes that similarity conceived as an identity of shape must be compared to congruence conceived as an identity of shape and size. While congruence is a "point-by-point mapping"[102] allowing to identify objects taken individually between them, similarity provides a striking example of a mapping which preserves only *some* invariant aspects of an object:

> "Two figures have the same shape if and only if they are similar. […] the concept of shape results from that of figure by abstracting from position and magnitude. In scientific practice the introduction of a concept thus abstracted expresses the intention of exclusively considering *invariant* properties and relations among the originally given objects." [103]

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[95] Leibniz (1698), p. 514.
[96] Leibniz (1689), pp. 519-520.
[97] Leibniz (1704), II, viii, §13.
[98] Leibniz (1678), p. 263.
[99] Esquisabel (2008), p. 8.
[100] De Risi (2007), p. 321.
[101] Weyl (1949 [1927]), p. 7ff, 25f, 73ff.
[102] Weyl (1949 [1927]), p. 78.
[103] Weyl (1949 [1927]), p. 9.

For Weyl this ability to capture invariant properties of an object may be associated with the mathematical ability to generalize (concepts) or to "homogenize" (spaces) that Leibniz had in mind when he spoke of similarity:

> "Any property derived from the basic geometric relations without reference to individual points, lines, or planes that applies to any one point applies to every point. This conceptual homogeneity reflects the intuitive homogeneity of space. Leibniz has this in mind when he gives the following 'philosophical' definition of similar configurations in geometry, 'Things are similar if they are indistinguishable when each is observed by itself'."[104]

Similarity according to Leibniz then consists, as Weyl points out, in *selecting* or *abstracting* certain properties, to leave others aside. It allows to see things not only as individual objects, but also in their *generality*. By comparing these things partially, by selecting certain features of similarity, we can form, as Weyl says, spaces of "conceptual homogeneity"[105] or, as Leibniz says, "all sorts of similarities or agreements"[106] which are essential to thought :

> "As far as you conceive the similarity of things you conceive something more, and the universality consists only in that."[107]

To conclude, it seems to me that Bourbaki and Weyl are right, each in their own way, when they see in Leibniz a kind of "precursor" of structural isomorphisms in mathematics. Bourbaki is right in stating that Leibniz had for the first time some inkling of the concept of formal identity between operations or relations. Weyl is right too in stating that for Leibniz mathematical abstraction consists in "[seeking] an identity, a thing which would truly be the same, and [imagining] it to be in a manner outside of the subjects."[108] Leibniz has inaugurated a truly new and original approach, whose principle consisted in selecting certain features belonging to distinct spaces or to separate perceptions, to find them elsewhere. In my effort to trace back the clue, I realised that this approach was gradually linked with a combinatorial treatment of elements of vegetal, chemical and crystallographic "structures", to finally produce one of the most beautiful and powerful tools of mathematics.

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[104] Weyl (1949 [1927]), pp. 7-8.
[105] Weyl (1949 [1927]), p. 8.
[106] Leibniz (1704), III, i, §3.
[107] Leibniz (1704), IV, xvii, §8. See also III, i, §3.
[108] Leibniz, *Fifth letter to Clarke, June 1716*, § 47 (Leibniz, 1960-1961, VII, p. 401) quoted by Weyl (1949 [1927], p.11).

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