Capital and technical development in long-term projection models

par

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1. The production function

Assuming substitutability between factors, and more specifically a production function as suggested by Cobb and Douglas:

(1) \[ y = \beta a^\lambda k^\pi. \]

where \( y \) = national product, \( a \) = labour and \( k \) = capital, then \( \lambda \) and \( \pi \) represent the elasticities of national product with regard to labour and capital, respectively.

2. Representing technical development

As to technical development, there are two main possibilities of representing the influence of the latter:

i) By not forcing Eq. (1) into homogeneity, but allowing for the case:

(2) \[ \lambda + \mu > 1, \]

it is possible to take into account systematic relationships between the efficiency of production and the level of output. Examples are returns to scale in their narrower sense and such dynamic factors
as are related to volume of output, as e.g. induced inventions and manufacturing progress curves, the latter being defined as:

\[
\frac{a}{y} = \eta \log \left( \int_0^t y \, dt \right) + \zeta, \quad (\eta < 0)
\]

\[\text{ii) Nonsystematic or autonomous influences on factor efficiency are to be expected in the case of:}\]

\[a) \quad \text{Creative} \text{ } \text{entrepreneurial reactions as distinct from the} \text{ } \text{adaptive} \text{ } \text{reactions mentioned sub (i).}\]
\[b) \quad \text{General factors such as improved education, health care, etc.}\]

These result in shifts of the original production function (1) and the long-term time-path of national income can therefore be described as:

\[y = \beta a^\lambda \, k^\mu \, e^{\nu t},\]

where \(e\) is the base of the natural logs and \(\nu\) is a constant.

Differentiating (4) with respect to \(t\) we find:

\[\frac{\Delta y}{y} = \lambda \frac{\Delta a}{a} + \mu \frac{\Delta k}{k} + \nu\]

where normally \(\lambda + \mu \geq 1\).

3. A three equation model

This Eq. (5) might serve well as a forecasting equation were it not that the systematic component of capital formation depending upon the level of national product:

\[\Delta k = \Delta k (y)\]

is according to Eq. (4) not independent of \(a\) and \(\nu\).

For the purpose of clarifying this particular point, a simple three equation system will suffice:

\[y_t = y_o \left( \frac{a_t}{a_o} \right) \left( \frac{k_t}{k_o} \right) \mu \, e^{\nu t}\]
\( a_t = a_0 e^{\pi t} \)  

\( \triangle k_t = \alpha y_t. \)

Presupposing maintenance or full employment at \( t = 0 \) as well during the forecasting period, \( \pi \) represents the rate of increase of working population. Ignoring differences between depreciation and replacement, \( \alpha \) is the propensity to save. It will be assumed constant:

\( \alpha_0 = \alpha_t = \alpha. \)

Complications due to divergences between replacement and depreciation will be dealt with in sec. 7.

Solving the system for \( y_t \), we find at any point of time for the annual rate of growth, \( \gamma_t = \triangle y_t/y_t \):

\[
\gamma_t = \frac{(\pi \lambda + \nu) + ae^{(\pi \lambda + \nu)t} - \alpha(1 - \mu)}{\kappa_0 (\pi \lambda + \nu) + \alpha (1 - \mu) e^{(\pi \lambda + \nu)t} - \alpha (1 - \mu)}.
\]

where \( \kappa_0 \) is the capital-output ratio \( k_0/y_0 \) in the base year (1).

The obvious advantage of Eq. (11) over Eq. (5) as estimator equation is the possibility to project values \( \triangle y/y \) that are consistent with the corresponding, expected rate of capital formation.

The asymptotic value of \( \gamma_t \) in Eq. (11) can be defined as the «structural» or «dynamic equilibrium» rate of growth:

\[
\lim_{t \to \infty} \frac{\triangle y}{y} = \gamma_{lim} = \frac{\pi \lambda + \nu}{1 - \mu}.
\]

4. Numerical results

Starting from the 4 possible combinations of two extreme hypotheses with regard to both \( \lambda \) and \( \mu \), Table 1 shows the expected structural rate of growth of national income for different values of \( \pi \) and \( \nu \):

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(1) See Appendix.
5. The role of capital formation

Perhaps the most striking feature of Eq. (12) is the fact that — at least in the long run — capital formation as such does not enter the picture at all. Assuming a constant and positive rate of saving, the structural rate of growth of national income is apparently invariant as against the level of \( \alpha \).

As shown in the appendix, the equilibrium growth rates of capital and output are identical. Consequently, a constant propensity to save implies in the long run constancy of the capital-output ratio, \( \kappa \), if at the same time \( \pi \) and \( \nu \) remain unchanged:

\[
\kappa = a \frac{1 - \mu}{\pi \lambda + \nu} = \frac{\alpha}{\gamma_{lim}}.
\]

(13)

Comparing two different cases of dynamic equilibrium, the case with the higher value of \( \alpha \) will also show the higher value of \( k \). Working population being the same, it will necessarily show the higher income of both, although the economy's rate of growth will be the same in either case.

Finally, as easily seen, Eq. (13) corresponds with the Harrod-Domar theorem:

\[
\frac{\Delta y}{\alpha} = \frac{\Pi}{\gamma_{lim}}.
\]

(14)
It will be noted, however, that constancy of the capital-output ratio occurs in the present case as a consequence of equilibrium development only (i.e. by assuming \( t = \infty \)), whereas in the Harrod-Domar case it has been introduced as a basic assumption.

6. Changes in investment decisions

The whole picture changes drastically, if starting from a state of dynamic equilibrium, with a rate of saving \( \alpha_0 \), this rate is permanently raised to a level \( \alpha_1 \). In this case the time path of \( \gamma = \Delta y/y \) is characterized by (2):

\[
\gamma_t = \frac{\pi \lambda + \nu}{1 - \mu} \left( 1 + \mu \frac{\alpha_1 - \alpha_0}{\alpha_1 (e^{\pi \lambda + \nu} t - 1) + \alpha_0} \right).
\]

The formula shows the take off rate for \( t = 0 \) to exceed the previous structural rate, \( \gamma_{lim} \), by the ratio:

\[
\frac{\gamma_0 - \gamma_{lim}}{\gamma_{lim}} = \mu \frac{\alpha_1 - \alpha_0}{\alpha_0}.
\]

Since generally \( \mu < 1 \), the percentage increase as shown by the take off rate will be smaller than the percentage increase of the rate of saving.

Although according to Eq. (15), \( \gamma_t \) tapers off gradually to reach asymptotically \( \gamma_{lim} \) for \( t \to \infty \), the absolute level of \( y \) thus reached is of course higher than the level of income \( (y') \) that would have occurred without any intervening change of the saving ratio (3):

\[
\lim_{t \to \infty} \frac{y_t}{y'} = \left( \frac{\alpha_1}{\alpha_0} \right) \frac{1 - \mu}{\mu}.
\]

Since the exponent of Eq. (17) reaches unity only for \( \mu = .50 \), the relative shift of \( y \) will, as a rule, be smaller than the corres-

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(2) See Appendix.

(3) See Appendix.
ponding increase of the saving ratio. This is brought out by Table 2:

<table>
<thead>
<tr>
<th>( \alpha_1 )</th>
<th>( \lim_{t \to \infty} \frac{y}{y'} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_0 )</td>
<td>( \mu = .25 )</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>1.25</td>
<td>1.08</td>
</tr>
<tr>
<td>1.50</td>
<td>1.14</td>
</tr>
<tr>
<td>2.00</td>
<td>1.26</td>
</tr>
<tr>
<td>5.00</td>
<td>1.71</td>
</tr>
<tr>
<td>10.00</td>
<td>2.15</td>
</tr>
</tbody>
</table>

7. Depreciation and replacement

In equation (9) the rate of saving \( \alpha \), when applied to the national income \( y_t \), immediately produces net additions to capital stock \( \Delta k \), i.e. gross investment minus replacement. Usually, however, savings are defined so as to equal net investment, i.e. gross investment minus depreciation. Domar has shown (4) that in an economy with an increasing stock of capital, replacement \( r_t \) is always lower than depreciation \( d_t \). If the average life-time of capital is \( m \), and the growth rate \( \gamma_{lim} \), the ratio between both, assuming linear depreciation, is

\[
\frac{r_t}{d_t} = \frac{m \gamma_{lim}}{e m \gamma_{lim}}.
\]

When \( m \gamma_{lim} \) is large, the difference can become appreciable. The coefficient \( \alpha \) as used in Eq. (9) should therefore be looked upon as the ratio between net additions to capital stock and income, rather than the rate of saving as usually defined. A model which assumes such a ratio constant over time is, of course, no less realistic than one that presupposes the rate of saving to be constant. A third model might just as well assume the ratio of gross investment to income to remain unchanged.

After the results of the preceding sections it will be no surprise that in all three models the economy tends towards a dynamic equilibrium with a growth rate as defined in Eq. (12).

The transition from one state of dynamic equilibrium to another, however, takes a different form for each model.

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In this Appendix, the suffix $t$ will be omitted. Equations numbered below 41 are taken from the main text.

A. The rate of growth

Substitution of

$$a = a_0 e^{\sigma t}$$

into (7)

$$y = y_0 \left( \frac{a}{a_0} \right)^\lambda \left( \frac{k}{k_0} \right)^\mu e^{\nu t}$$

gives (41)

$$y = y_0 \left( \frac{k}{k_0} \right)^\mu e^{(\sigma \lambda + \nu) t}.$$

This is substituted in

$$\frac{dk}{dt} = \alpha y,$$

rendering

$$\frac{dk}{dt} = \alpha y_0 \left( \frac{k}{k_0} \right)^\mu e^{(\sigma \lambda + \nu) t},$$

or

$$k^{-\mu} dk = \alpha y_0 k_0^{-\mu} e^{(\sigma \lambda + \nu) t} dt.$$

Upon integration we find

$$k^{1-\mu} = k_0^{1-\mu} \left\{ 1 + \frac{\alpha (1 - \mu)}{\pi \lambda + \nu} \frac{y_0}{k_0} \left( e^{(\sigma \lambda + \nu) t} - 1 \right) \right\}.$$
wich for \( \mu \neq 1 \) can be transformed into:

\[
(45) \quad k_{\mu} = k_{0\mu} \left\{ 1 + \frac{\alpha (1 - \mu)}{\pi \lambda + \nu} \cdot \frac{y_0}{k_0} \left( e^{(\pi \lambda + \nu) t} - 1 \right) \right\}^{\frac{\mu}{1 - \mu}}.
\]

This can be substituted in (41) to give:

\[
(46) \quad y = y_0 \left\{ 1 + \frac{\alpha (1 - \mu)}{\pi \lambda + \nu} \cdot \frac{1}{k_0} \left( e^{(\pi \lambda + \nu) t} - 1 \right) \right\}^{\frac{\mu}{1 - \mu}} e^{(\pi \lambda + \nu) t}.
\]

By logarithmic differentiation and re-arrangement:

\[
(47) \quad \frac{\Delta y}{y} = \left( \frac{\pi \lambda + \nu}{\kappa_0 (\pi \lambda + \nu) + \alpha e^{(\pi \lambda + \nu) t} - \alpha (1 - \mu)} \right) \cdot \frac{\kappa_0 (\pi \lambda + \nu) + \alpha e^{(\pi \lambda + \nu) t} - \alpha (1 - \mu)}{\kappa_0 (\pi \lambda + \nu) + \alpha (1 - \mu) e^{(\pi \lambda + \nu) t} - \alpha (1 - \mu)}.
\]

For \( t \to \infty \) this is simplified to:

\[
(48) \quad \gamma_{\ell m} = \frac{\pi \lambda + \nu}{1 - \mu}.
\]

In cases where \( \alpha = \alpha_1 \) and where (13) applies to \( \kappa_0 \), i.e. where:

\[
(49) \quad \kappa_0 = \alpha_0 \frac{1 - \mu}{\pi \lambda + \nu}
\]

equation (47) can be transformed into

\[
(50) \quad \frac{\Delta y}{y} = \frac{\pi \lambda + \nu}{1 - \mu} \left\{ 1 + \mu \frac{\alpha_1 - \alpha_0}{\alpha_1 \left( e^{(\pi \lambda + \nu) t} - 1 \right) + \alpha_0} \right\}.
\]
B. The final level of output

For \( t \to \infty \) equation (46) can be simplified to:

\[
(51) \quad \lim_{t \to \infty} y = \lim_{t \to \infty} y_0 \left( \frac{\alpha (1 - \mu)}{1 - \mu} \right) \frac{\mu}{(\pi \lambda + \nu) \left( \frac{\mu}{1 - \mu} \right)} t \quad e \quad (\pi \lambda + \nu) \left( \frac{\mu}{1 - \mu} \right)
\]

In case where \( \alpha = \alpha_1 \) and where (13) applies to \( \kappa_0 \) this reduces to:

\[
(52) \quad \lim_{t \to \infty} y = \lim_{t \to \infty} y_0 \left( \frac{\alpha_1}{\alpha_0} \right) \frac{1 - \mu}{e^{\gamma_{11} t}}.
\]

The level before the transition to a higher rate of saving \( y' \) can be extrapolated to

\[
(53) \quad \lim_{t \to \infty} y' = \lim_{t \to \infty} y_0 e^{\gamma_{11} t}.
\]

Obviously,

\[
(54) \quad \lim_{t \to \infty} \frac{y}{y'} = \left( \frac{\alpha_1}{\alpha_0} \right) \frac{1 - \mu}{}.
\]