Localized patterns in nonlinear optical cavities

G. Kozyreff

Optique Nonlinéaire Théorique, Université Libre de Bruxelles (U.L.B.), CP 231, Belgium

Abstract

The subcritical Turing instability is studied in two classes of models for laser-driven nonlinear optical cavities. In the first class of models, the nonlinearity is purely absorptive, with arbitrary intensity-dependent losses. In the second class, the refractive index is real and is an arbitrary function of the intra-cavity intensity. Through a weakly nonlinear analysis, a Ginzburg-Landau equation with quintic nonlinearity is derived. Thus, the Maxwell curve, which marks the existence of localized patterns in parameter space, is determined. In the particular case of the Lugiato-Lefever model, the analysis is continued to seventh order, yielding a refined formula for the Maxwell curve and the theoretical curve is compared with recent numerical simulation by Gomilà, Firth, and Scroggie [Physica D 227, 70 (2007)].

Keywords: Localized patterns, Maxwell curve, Lugiato-Lefever, nonlinear cavity, Dissipative solitons

1. Introduction

A large number of homogenous, spatially extended dynamical systems can support dissipative localized structures. Often, these can be defined as local back and forth switches between two stable states. Hence, they can be of several kinds, depending on the nature of the states involved: both spatially homogenous [1–5], homogeneous and periodic [6, 7], both periodic [8], or even periodic and biperiodic in space [9]. The present paper is concerned with the “homogeneous and periodic” case, so that localized states are localized patterns. The resulting localized structures thus typically consist of a finite set of regularly spaced peaks over an homogenous background. A large number of numerical studies of optical [7, 10–14], chemical [15–17] or fluid mechanical models [18–22] together with an increasing body of experiments [23–28] have demonstrated the widespread relevance of these localized patterns. Mathematically, in one spatial dimension, localized patterns are homoclinic solutions passing in the vicinity of a limit cycle, where the independent variable is the space coordinate. By examining the topology of their phase space trajectory, a lot of useful informations have been deduced [4, 29–31], notably the fact that they follow a specific bifurcation sequence, called the snaking bifurcation diagram [29]. Further structures in the bifurcation diagram were later revealed through a combination of numerical and asymptotic studies [32–43].

Only one limit is analytically tractable in general. It is when the pattern is of small-amplitude. In that case, spatial oscillations are essentially sinusoidal and higher harmonics are one order of magnitude smaller. Unfortunately, whereas this limit is the most amenable to analytical predictions, it is at the same time the most challenging one for numerical investigations. Indeed, two
widely separated spatial scales naturally arise in this situation, and the system is subjected to critical slowing down. As a result, a large number of spatial discretization points are necessary and the equations must be integrated for a long time before reaching a stable state. Moreover, the domain of existence of localized patterns in the parameter space is exponentially thin [37, 40, 44] and can therefore easily be missed. The aim of this paper, therefore, is to locate the region in parameter space where localized patterns can be found for two important classes of optical models. More specifically, we will determine the Maxwell curve, where localized patterns are restricted to exist in first approximation [45]. This curve was so named in reference to Maxwell’s construction of the coexistence curve between two phases in thermodynamics [46–48].

Most of previous analytical results are based on the Swift-Hohenberg equation because it is the simplest one that describes a subcritical Turing bifurcation, from which localized patterns emerge. Moreover, it can be derived in a specific limiting in many contexts [49–51] and it is well-known that close to bifurcation points, the dynamics is universal, i.e. that its qualitative features are model-independent [52]. However, this equation is sometimes criticized as physically inapplicable. Firstly, it is variational and, as such, only admits steady states in the long run. It thus misses a wide range of unsteady dynamical behaviors that can be observed in practice. The influence of non-variational effects was recently listed in a series of open questions regarding localized patterns [53] and the Swift-Hohenberg equation can definitely not answer it. Secondly, although this equation and non-variational variants of it can be derived in specific limits [51], its range of validity as an asymptotic reduction of more general models is not known.

For these reasons, models of nonlinear optical cavities with an injected laser field are an interesting alternative to the Swift-Hohenberg equation. One example is

\[
\frac{\partial E}{\partial t} = E_I - [1 + L(I) + i\theta] E + i\frac{\partial^2 E}{\partial x^2}, \quad I = |E|^2, \tag{1}
\]

where \(E\) and \(I\) are respectively the intracavity field and intensity, \(E_I\) is the injected field, and \(\theta\) is the detuning between the injected field frequency and the closest cavity resonance, normalized with the cavity decay rate. The term \(L(I)\) represents saturable losses and is often given by

\[
L(I) = \frac{2C}{1 + I}, \tag{2}
\]

where \(C\) is the coupling strength between the cavity field and a saturable absorber inside the cavity [54]. This particular form for the losses corresponds to an assembly of two-level atoms in resonance with the intracavity field.

Another frequent situation is that of a cavity containing an optical element with a nonlinear refractive index \(N(I)\). This leads to the following model:

\[
\frac{\partial E}{\partial t} = E_I - [1 + i\theta - iN(I)] E + i\frac{\partial^2 E}{\partial x^2}. \tag{3}
\]

The simplest case \(N(I) = I\) corresponds to a Kerr nonlinearity and leads to the well-known Lugiato-Lefever model [55, 56]. Equations (1) and (3) are relatively simple-looking, non-variational, and realistic enough to account for experimental observations [24, 25, 28, 57, 58]. For these reasons, after having been the subject of numerical and experimental investigation, these systems are receiving renewed mathematical attention [31, 59]. In this frame, the control parameters are the injection amplitude \(E_I\) and the detuning parameter \(\theta\). However, in lieu of \(E_I\) it is customary to use the intracavity intensity \(I\) around which oscillations take place.
In this paper, we will construct spatially oscillating solutions of Eqs. (1) and (3) with a slowly varying envelope. More specifically, in the vicinity of the codimension-2 point

$$(\theta, I) = (\theta_c, I_c) \quad (4)$$

where the Turing bifurcation becomes subcritical and the oscillations vanish, we will derive the fifth-order Ginzburg-Landau equation

$$\frac{\partial a}{\partial T} = \lambda a + \mu |a|^2 a - \nu |a|^4 a + i \sigma |a|^2 \frac{\partial a}{\partial X} - i \omega |a|^2 \frac{\partial a}{\partial X} + d \frac{\partial^2 a}{\partial X^2} \quad (5)$$

for the complex oscillation envelope $a$. Above, $\lambda \propto I - I_c$, $\mu \propto \theta - \theta_c$, and $X$ and $T$ are rescaled space and time variables. The Maxwell curve is then given by

$$-\frac{\lambda}{\mu^2} = \frac{3d}{16\nu + (\sigma - \omega)(5\sigma + 3\omega)} \quad (6)$$

On this curve, (5) admits the front solution

$$a(X) = \frac{\sqrt{-4\lambda/\mu} e^{cX/2}}{(1 + e^{cX})^{1/2+\sqrt{2}/2}}, \quad c = \sqrt{-4\lambda/d}, \quad \beta = (\sigma - \omega) \sqrt{-\lambda/d}/\mu \quad (7)$$

that separates regions of homogeneous and spatially periodic intensity. The Maxwell curve and the ‘pinning range’ of parameters around it where localized patterns exist are depicted schematically in Fig. 1. Several methods can be used to derive (5). Miyaji, Ohnishi, and Tsutsumi [59] used the normal form reduction for the Lugiato-Lefever model. We will use the method of multiple-scale, whereby (5) arises as a solvability condition at fifth order of the analysis. The two methods are known to yield consistent results and indeed, we will see that it is so when we particularize our calculations to the Lugiato-Lefever model. The former method has the advantage that it can be the basis of rigorous arguments but the latter is generally more efficient.
from a computational point of view and is probably more physically appealing. Recently, the two methods were applied to the same problem with the Swift-Hohenberg equation [60]. For more complicated equations such as (1) and (3), the calculations quickly becomes inextricable as one progresses from one order to the next, unless the linearized problem is carefully studied beforehand.

Motivated by the special importance of the Lugiato-Lefever model [58], we will pursue the multiple-scale analysis two orders further for that equation. This will allow us to compute the first correction to formula (6). Such a refinement may appear superfluous at first sight, but close to the codimension-2 point, the width of the pinning range becomes so thin that the leading order calculation becomes insufficiently accurate. This was noted before in the two instances where a beyond-all-order analysis was carried out [37, 40, 44] and will be illustrated by comparing our results with the pinning range numerically computed by Gomilà, Scroggie, Firth in [31].

Note that equation (5) is general for a subcritical Turing bifurcation [61–63], although textbooks and reviews on pattern formation usually omit the terms proportional to $a^2 \partial \bar{a}/\partial X$ and $|a|^2 \partial a/\partial X$ in (5). While numerous fifth-order amplitude equations are derived for extended patterns, only a limited number of actual derivations of (5) can be found in the literature. Besides the derivation by Miyaji, Ohnishi, and Tsutsumi for the Lugiato-Lefever equation [59], most of them apply to the Swift-Hohenberg equation [34, 40, 45, 64, 65]. Another notable example was provided by Weinstein for Plane Poiseuille flows [66]. On the other hand, a normal form was derived in all generality for the 1:1 Hopf resonance [67], which is another name for Turing bifurcation in steady state. In the appendix, we show how (5) can be deduced from the normal form in steady state. Finally, rigorous argument for the derivation of (5) are given in [68].

The rest of the paper is organized as follows. In Sec. 2, we perform de detailed multiple-scale analysis of a cavity with general saturable losses. The special case (2) is then treated in more details. In Sec. 3, a general nonlinear refractive index is considered, and in Sec. 4 the Lugiato-Lefever model is analyzed further. Finally, we conclude.

2. Cavity with saturable losses

We first analyze Eq. (1). The homogeneous state is most conveniently parameterized by the intracavity intensity $I = |E|^2$:

$$E_I = \left[ I \left[ (1 + \mathcal{L}(I))^2 + \theta^2 \right] \right]^{1/2}, \quad E_I = \frac{E_I(I)}{1 + \mathcal{L}(I) + i \theta}.$$  (8)

It is customary, using the relations above, to substitute $I$ for $E_I$ as a control parameter. Our goal, therefore, is to derive the coefficients $\lambda, \mu, \nu, \sigma, \omega$ and $d$ of (5) in terms of the two control parameters $I$ and $\theta$.

2.1. Linear preliminaries

Let us first analyze small steady perturbations of (8) in the form

$$E = \frac{E_I(I)}{1 + \mathcal{L}(I) + i \theta} (1 + \epsilon f(x)), \quad \epsilon \ll 1.$$  (9)

Substituting into (1) and keeping terms only up to $O(\epsilon)$, we get

$$0 = -\left[ 1 + \mathcal{L}(I) + \mathcal{L}'(I) + i \left( \theta - \frac{\partial^2}{\partial x^2} \right) f - \mathcal{L}'(I) \bar{f} \right].$$  (10)
and if \( f \propto \cos(kx) \), then
\[
1 + \mathcal{L}(I) + 2i \mathcal{L}'(I) f = -i \mathcal{L}'(I) f. \tag{11}
\]
A necessary condition to have a non trivial solution is that both sides of the above equality have the same modulus, i.e. that
\[
(1 + \mathcal{L}(I) + 2i \mathcal{L}'(I))^2 + (\theta + k^2)^2 = (\mathcal{L}'(I))^2 \tag{12}
\]
Had we decomposed \( f \) into its real and imaginary parts in (11), this is the characteristic equation we would have got for the resulting real linear algebraic system. This equation is quadratic in \( k^2 \) and must have two real positive roots for the linearized problem to have periodic solutions. The smallest intensity with which this happens is the instability threshold. At that intensity, \( k^2 \) is a double root of (12)\(^1\). This yields the threshold condition
\[
1 + \mathcal{L}(I_c) + 2I_c \mathcal{L}'(I_c) = 0, \tag{13}
\]
and the wave number is then, simply,
\[
k_c = \sqrt{-\theta}. \tag{14}
\]
Moreover, at the threshold, where (13) holds, (11) reduces to \( f = \bar{f} \), so that marginally unstable perturbations are of the form
\[
f = a \cos(k_c x), \tag{15}
\]
where \( a \) is real. In the following multiple-scale analysis, the following inhomogenous differential equation repeatedly arises:
\[
0 = \left[ I_c \mathcal{L}'(I_c) + i \left( k_c^2 + \frac{\partial^2}{\partial x^2} \right) \right] f - I_c \mathcal{L}'(I_c) \bar{f} + B \cos(k_c x). \tag{16}
\]
With such a resonant inhomogeneous term, it is easy to check that the solution grows proportionally to \((B + \bar{B}) \sin k_c x\). Hence, the solution will remain bounded and proportional to \(\cos(k_c x)\) only if the solvability condition
\[
B + \bar{B} = 0 \tag{17}
\]
is satisfied. This imposes that \( B = i b \), with real \( b \), in which case a particular solution is
\[
f = \frac{-i}{2I_c \mathcal{L}'(I_c)} b \cos(k_c x). \tag{18}
\]
On the other hand, if a non-resonant forcing term appears, like
\[
0 = \left[ I_c \mathcal{L}'(I_c) + i \left( k_c^2 + \frac{\partial^2}{\partial x^2} \right) \right] f - I_c \mathcal{L}'(I_c) \bar{f} + B_m \cos(mk_c x), \tag{19}
\]
then one can easily find the particular solution
\[
f = -\left(m^2 - 1\right)^{-2} k_c^{-4} \left[ (I_c \mathcal{L}' + i \left( m^2 - 1 \right) k_c^2) B_m + I_c \mathcal{L}' \bar{B}_m \right] \cos(mk_c x). \tag{20}
\]
Expressions (15), (17), (18) and (20) can be used to set up an automatic resolution of a perturbative multiple-scale analysis to any order.

\(^1\)This is sometimes referred to as 1:1 resonance.
2.2. Multiple-scale analysis

Let the reference intensity, around which oscillations take place, be

\[ I = I_c + \epsilon^4 i_4, \]  

(21)

where \( \epsilon \) is a small parameter, so that (13) becomes

\[ 1 + \mathcal{L}(I) + 2I \mathcal{L}'(I) = (3\mathcal{L}'(I_c) + 2I_c \mathcal{L}''(I_c)) \epsilon^4 i_4. \]  

(22)

We expand the solution as

\[ E = \frac{E_f(I)}{1 + \mathcal{L}(I) + i\theta} \left[ 1 + \frac{\epsilon}{2} \left( \alpha(X, T) e^{ikx} + \text{c.c.} \right) + \sum_{n>2} \epsilon^n f_n(X, T) \right], \]  

(23)

where \( X \) and \( T \) are slow space and time variable, defined as \( X = \epsilon^2 k_c x \) and \( T = \epsilon^4 t \). Substituting this ansatz into (1), the first nonzero terms come up at order \( \epsilon^2 \), where we get

\[ 0 = \left[ I_c \mathcal{L}' + i \left( k_c^2 + \frac{\partial^2}{\partial x^2} \right) \right] f_2 - I_c \mathcal{L}' f_2 - \left( \frac{3}{2} I_c \mathcal{L}' + I_c^2 \mathcal{L}'' \right) |a|^2 - \left( \frac{3}{4} I_c \mathcal{L}' + \frac{1}{2} I_c^2 \mathcal{L}'' \right) (a^2 e^{2ikx} + \text{c.c.}) \]  

(24)

Using the integration rule given in (20), we directly find

\[ f_2 = k_c^{-4} \left( 2I_c \mathcal{L}' - ik_c^2 \right) \left( \frac{3}{2} I_c \mathcal{L}' + I_c^2 \mathcal{L}'' \right) |a|^2 + \frac{k_c^2}{6} \left( 2I_c \mathcal{L}' + 3ik_c^2 \right) \left( \frac{3}{4} I_c \mathcal{L}' + \frac{1}{2} I_c^2 \mathcal{L}'' \right) (a^2 e^{2ikx} + \text{c.c.}). \]  

(25)

Next, at order \( \epsilon^3 \), a similar equation for \( f_3 \) arises. We omit the details for brevity. This time, a resonant driving term appears, and according to (17), its real part must vanish. This yields the solvability condition

\[ 19k_c^{-4} I_c \mathcal{L}' \left( I_c \mathcal{L}' + \frac{2}{3} I_c^2 \mathcal{L}'' \right)^2 + \left( \frac{3}{4} I_c \mathcal{L}' + 3I_c^2 \mathcal{L}'' + I_c^3 \mathcal{L}''' \right) = 0. \]  

(26)

Bearing in mind that \( k_c^2 = -\theta \), this is an equation for \( \theta \). This invites us to rewrite the perturbation expansion with

\[ \theta = \theta_c + \epsilon^2 \theta_2, \quad k_c^2 = -\theta, \]  

(27)

where \( \theta_c \) is given by

\[ \theta_c = -\left[ 19I_c \mathcal{L}' \left( I_c \mathcal{L}' + \frac{2}{3} I_c^2 \mathcal{L}'' \right)^2 / \left( \frac{3}{4} I_c \mathcal{L}' + 3I_c^2 \mathcal{L}'' + I_c^3 \mathcal{L}''' \right) \right]^{1/2}. \]  

(28)

Once \( \theta_c \) is so determined, the problem at this and the following order can automatically be solved using the integration rules (18) and (20). Eventually, at order \( \epsilon^5 \), a new solvability condition

---

2See print out of a mathematica file in the supplementary material.
arises, which is just equation (5), with the following coefficients.

\[
\lambda = -(3L' + 2L''L'')_i, \quad \mu = 38I_cL'(1 + \frac{2}{3}L''L')^2 \frac{\theta_c}{\theta_c'}, \quad d = \frac{2\theta_c^2}{L\theta_c'}, \quad (29)
\]

\[
\sigma = -I_cL' - \frac{2}{3}L'^2L'', \quad \omega = -\frac{80}{19}I_cL' - \frac{200}{57}I_c^3L''L' - \frac{16}{57}I_c^3L''', \quad (30)
\]

\[
\nu = -\frac{15680}{3L_c^6} \frac{(L')^6}{L''} - \frac{1960}{9L_c^4} (L')^2 - \frac{13}{9L_c^2} (L')^3 + \frac{1}{6} (L''L^{(4)})^2 + \frac{d}{4} \frac{1}{L_c^2} \frac{(L''')^2}{L''} \times \frac{15680}{3L_c^6} (L')^2 + \frac{1889}{2L_c^4} (L')^4,
\]

\[
+ (L_cL''')^2 \left( \frac{31360}{81L_c^6} (L''')^3 - \frac{3778}{3L_c^4} (L')^2 - \frac{613}{80L_c^2} (L')^3 \right)
\]

\[
+ (L_cL''')^2 \left( \frac{62720}{27L_c^4} (L''')^3 - \frac{1889}{3L_c^4} (L'')^2 \right)
\]

\[
+ (L_cL''')^2 \left( \frac{7556}{27L_c^4} (L''')^3 + \frac{29}{3L_c^2} (L')^4 - \frac{190L_c^2}{\theta_c'} (L'')^2 \right),
\]

(31)

where (26) was used to eliminate \( L''(I_c) \) in the expression for \( v \). Note that, since \( L(I) \) represent saturable losses, we must have \( L'(I_c) < 0 \) above. Hence, the diffusion coefficient \( d \) is positive, as it should.

### 2.3. Example

To illustrate the preceding formulas, we assume that the losses are given by (2), where we choose \( C = 5.4 \) as in the numerical simulations reported by McSloy et al.[13]. The threshold intensity, determined by (13) is found to be \( I_c = 1.650 \). Next, (28) yields \( \theta_c = -2.311 \) and the coefficients of the Ginzburg-Landau equations are

\[
\lambda = 0.783i, \quad \mu = 1.448\theta_c, \quad \nu = 2.569, \quad \sigma = 0.431, \quad \omega = 1.253, \quad d = 4.211. \quad (32)
\]

Using (6), the Maxwell curve is thus given by \( i = -0.201\theta_c \), i.e. by

\[
I - I_c = -0.201 (\theta - \theta_c)^2.
\]

(33)

Note that the curve starts at \( \theta = -2.311 \) and exists only for \(-2.311 < \theta \), that is where the bifurcation is subcritical.

In the same way as with \( C = 5.4 \), we can draw the Maxwell curve for any value of \( C \). To this end, we note that the solution of (13) is generally given by

\[
I_c = C - 1 - \sqrt{C(C - 4)}.
\]

(34)

This allows one to compute \( \theta_c \) at once from (28) and the coefficients of the Ginzburg-Landau equation. The resulting curves are shown in Fig. 2 for several values of \( C \).
Figure 2: Theoretical Maxwell curves for a cavity with nonlinear losses given by (2) drawn for different values of the coupling parameter $C$. The vertical axis is the intracavity intensity $I$.

3. Cavity with nonlinear refractive index

3.1. Linear preliminaries

We now consider a cavity containing a material with nonlinear (real) refractive index. We follow the same steps as in the previous section. With the model given in (3), the homogeneous state is given by

$$E_I = \left[ I \left[ 1 + (\theta - N(I))^2 \right] \right]^{1/2}, \quad E = \frac{E_I(I)}{1 + i(\theta - N(I))},$$

where, as before, $I$ denotes the intracavity intensity $|E|^2$. The linear stability analysis now yields the threshold condition

$$I_c N'(I_c) = 1,$$

for the spatial instability, while the critical wave number is determined by

$$k_c^2 = N(I_c) + 1 - \theta.$$

Moreover, at threshold, the marginally stable mode has the following form

$$f = (1 + i) a \cos(k_c x),$$

where $a$ is real. At each order of the multiple-scale analysis, linear problems of the type

$$0 = \left[ -1 + i \left( k_c^2 + \frac{\partial^2}{\partial x^2} \right) \right] f + i \tilde{B} m \cos(mk_c x)$$

arise. If $m \neq 1$, one has the particular solution

$$f = (m^2 - 1)^{-2} k_c^{-4} \left[ (1 - i \left( m^2 - 1 \right) k_c^2) B_m + i \tilde{B}_m \right] \cos(mk_c x).$$

On the other hand, if $m = 1$, the following solvability condition

$$B_1 + i \tilde{B}_1 = 0$$
must be satisfied for \( f \) to remain bounded. This imposes that \( B_1 \) be of the form \((1 - i) b\), where \( b \) is real, and a corresponding particular solution is then

\[
f = \frac{1 - i}{2} b \cos(k_c x).
\]  

(42)

Expressions (38), (40), (41), and (40) can be used to set up an automatic resolution of a perturbative multiple-scale analysis to any order.

3.2. Multiple-scale analysis

Working near the instability threshold, we write

\[
I = I_c + \epsilon^4 i \delta
\]

so that

\[
IN'(I) = 1 + (N'(I_c) + I_c N''(I_c)) \epsilon^4 i \delta.
\]

(44)

We then look for a solution of the form

\[
I = \epsilon I_0 + \epsilon^2 I_2 + \epsilon^3 I_3 + \epsilon^4 I_4 + \cdots
\]

\[
I' = \epsilon I_1 + \epsilon^2 I_3 + \epsilon^3 I_5 + \epsilon^4 I_6 + \cdots
\]

\[
I'' = \epsilon I_2 + \epsilon^2 I_4 + \epsilon^3 I_6 + \epsilon^4 I_8 + \cdots
\]

\[
I''' = \epsilon I_3 + \epsilon^2 I_5 + \epsilon^3 I_7 + \epsilon^4 I_9 + \cdots
\]

(45)

The numerical factors in the definition of \( X \) and \( T \) are chosen in anticipation of the final results but are not otherwise necessary for the analysis. With the ansatz above, Eq. (3) is satisfied up to order \( \epsilon \). At order \( \epsilon^2 \), we find

\[
0 = \left[ -1 + i \left( k_c^2 + \frac{\partial^2}{\partial x^2} \right) \right] f_2 + i \bar{f}_2 + \frac{i}{4} \left( 2 + \bar{I}_c N'' + i \right) \left( \alpha \bar{e}^{2i k_c x} + 2|\alpha|^2 + \bar{\alpha}^2 e^{-2i k_c x} \right),
\]

(47)

which can be solved immediately by application of (40). This yields

\[
f_2 = \left[ 1 + i - (2 + 3i) k_c^2 + (1 - k_c^2 + i) \bar{I}_c^2 N'' \right] \left[ \frac{\alpha \bar{e}^{2i k_c x} + \bar{\alpha}^2 e^{-2i k_c x}}{36k_c^2} \right]
\]

(48)

At order \( \epsilon^3 \), resonant terms are present and application of (41) yields the condition

\[
38 - 60 k_c^2 + \bar{I}_c^2 N'' \left( 76 - 90 k_c^2 + 27 k_c^4 \right) + 2 I_c^4 N'' \left( 19 - 15 k_c^2 \right) + 9 k_c^4 I_c^2 N'''' = 0.
\]

(49)

This is a condition to be imposed on \( k_c^2 \), i.e. on \( \theta \). This equality marks the transition between a supercritical and a subcritical bifurcation. As in the study of the saturable absorber, we revise our perturbation scheme and choose \( \theta \) in such a way that

\[
38 - 60 k_c^2 + \bar{I}_c^2 N'' \left( 76 - 90 k_c^2 + 27 k_c^4 \right) + 2 I_c^4 N'' \left( 19 - 15 k_c^2 \right) + 9 k_c^4 I_c^2 N'''' = 60 \epsilon^2 \theta_2
\]

(50)
In this way, the problems at orders $\epsilon^3$ and $\epsilon^4$ can easily be solved using the rules of integration derived in the previous section\textsuperscript{3}. Eventually, at order $\epsilon^5$, the solvability condition yields (5), with

$$\lambda = (N'' + iN'')i_4, \quad \mu = \frac{100\theta_2}{3k^2_c}, \quad d = 4, \quad \sigma = \frac{-9k^4_c + 66k^2_c - 2I^2_cN''(9k^4_c - 18k^2_c + 19) - 38}{9k^4_c},$$

$$\omega = \frac{-2\sqrt{7}}{27k^6_c}[-54k^6_c + 144k^4_c - 90k^2_c + 4(172N''(3k^2_c + 2))^2 + I^2_cN''(-27k^6_c + 54k^4_c - 78k^2_c + 16) + 8],$$

$$\nu = [-162I^2_cN^5(\epsilon^2(12 + 2187k^4_c - 21060k^2_c - 1081k^4_cN''(15k^4_c - 35k^2_c + 29))k^6_c + 140400k^8_c - 293544k^6_c + 311304k^4_c - 184576k^2_c - 80(172N''(3k^2_c + 2))^2 + 1011k^4_c + 1336k^2_c - 588) + 4(172N'')^3(10935k^6_c - 60624k^6_c + 130182k^4_c - 126304k^2_c + 47040) - 3(172N'')^2(729k^12_c + 10908k^10_c - 81180k^8_c + 239112k^6_c - 370040k^4_c + 291456k^2_c - 94080) + 2172N''(3645k^12_c - 33372k^10_c + 54I^2_cN''(21k^2_c - 29))k^8_c + 170100k^6_c - 394272k^6_c + 490788k^4_c - 330304k^2_c + 94080) + 47040]/(1944k^12_c),$$

where (49) was used to eliminate $N'''(i_4)$ in the expressions for $\sigma, \omega,$ and $\nu$.

4. Lugiato-Lefever model

4.1. Leading-order of the Maxwell curve

For the Lugiato-Lefever model, $N(I) = I$. Hence, (37), (50), and (44) yield, respectively,

$$I = 1 + \epsilon^1i_4, \quad k^2_c = 19/30 - \epsilon^2\theta_2, \quad \theta = 41/30 + \epsilon^2\theta_2.$$

On the other hand, the coefficients of the Ginzburg-Landau equation greatly simplify, giving

$$\lambda = i_4, \quad \mu = \frac{30006\theta_2}{361}, \quad \nu = \frac{3067411529}{376367048}, \quad \sigma = \frac{\sqrt{7}}{19}, \quad \omega = \frac{9916\sqrt{7}}{6859}, \quad d = 4$$

For extended patterns, $\partial a/\partial X = 0$ and it is easy to see that the above coefficients are consistent with the results of Miyaji et al. [59, pp. 2070-2071] for extended patterns. Next, applying (6), we obtain the equation for the Maxwell curve

$$i_4 = \frac{-19494000000}{11968157201}\theta_2^2, \quad \text{i.e.} \quad I = 1 \approx -1.629\left(\theta - \frac{41}{30}\right)^2,$$

which is correct to second order in the deviation ($\theta - 41/30$).

\textsuperscript{3}See print out of a mathematica file in the supplementary material.
4.2. Higher order analysis

We now proceed to a higher order calculation. But first, we wish to define $\epsilon$ is such a way that $\lambda = i_4 = -1$. In other words, we set

$$\theta = \frac{41}{30} + \epsilon^2 \theta_2, \quad \theta_2 = \sqrt{\frac{11968157201}{194940000000}}. \quad (58)$$

The advantage of this choice is to have $c = 1$ in (7), giving the front solution

$$a(X) = \frac{\sqrt{4/\mu} e^{X/2}}{(1 + e^X)^{1/2 + \sigma/2}}, \quad \beta = \frac{\sigma - \omega}{2\mu}.$$ \quad (59)

In the following, we will need to express that solution in the polar form

$$a(X) = \Re e^{i\phi}, \quad \Re = \frac{\sqrt{4/\mu} e^{X/2}}{(1 + e^X)^{1/2}}, \quad \phi = \frac{\beta}{2} \ln (1 + e^X). \quad (60)$$

On the other hand, once (58) is fixed, we must now include higher order correction to $I$ and $k_c$:

$$I = 1 - \epsilon^4 + \epsilon^6 i_6, \quad k_c^2 = 19/30 - \epsilon^2 \theta_2 - \epsilon^4 q_4. \quad (61)$$

In addition, the solution expansion should also be revised to include a resonant term at order $\epsilon^3$:

$$E = \frac{E_I(I)}{1 + i(\theta - N(I))} \left[ 1 + \epsilon + \frac{i}{2} \left( a(X) + \epsilon^2 \beta(X) e^{iX} + c.c. \right) + \sum_{n \geq 2} \epsilon^n f_n(x, X) \right].$$ \quad (62)

The amplitude $b$ was previously omitted for simplicity but it now becomes necessary. Note that, in principle, one should also introduce a super-slow scale $\xi = \epsilon^4 x$ on which $a$ and $b$ could vary. However, based on previous experience [40], we anticipate that the only thing that happens on this scale is a wave-number shift, and this is already taken into account with $q_4$. With the aid of a symbolic software, we may solve the $\epsilon^5$ and $\epsilon^6$ problems and eventually, we find, at order $\epsilon^7$, the solvability condition

$$0 = b \left( -1 + 2\mu |a|^2 - 3\nu |a|^4 + 2i \sigma a\bar{a}_x - i \omega a\bar{a}_x \right) + b \left( \mu \bar{a}^2 - 2\nu |a|^2 a^2 - i \omega a a_x \right)$$

$$+ i \sigma a^2 b_x - i \omega |a|^2 b_x + 4b_{XX} + \mathcal{W},$$ \quad (63)

where $b_X, a_X, \ldots$ denotes $\partial b/\partial X, \partial a/\partial X, \ldots$ and

$$\mathcal{W} = -\frac{4140193862671745}{196194120249632} a^3 \bar{a}^3 + i \frac{11492908635}{1787743478 \sqrt{2}} a^3 a\bar{a}_X - \frac{22135545750b}{893871739} a^2 \bar{a}^2$$

$$- \frac{10277}{13718} a^2 \bar{a}^2 - i \frac{12600 \sqrt{2} b_2}{6859} a^2 a\bar{a}_X + i \frac{149444787675}{3575486956 \sqrt{2}} a^4 a_x + \frac{180000\sigma}{6859} a^2 \bar{a}$$

$$+ \frac{350833}{130321} a^2 \bar{a}^2 - i \frac{2846400 \sqrt{2} b_2}{130321} a^2 a_x - \frac{320180}{130321} a\bar{a}_X a_x = \frac{1896150}{130321} a\bar{a}_X, \quad (64)$$

$$+ \frac{23}{19} - q_4 \right) a_X.$$
Apart from the inhomogeneous term \( W \), the equation for \( b \) is a linearized version of (5), of which the two solutions \( b = a_1 \), \( b = ia_1 \) can easily be spotted. Two other solutions of the homogenous problem can be found as in [40], by letting

\[
 b = (R_1 + iR_1) e^{ib},
\]

(65)

where \( R \) and \( \phi \) are given in (60). One solution is given by

\[
 R_1 = R_0 \int_{X}^{X'} \frac{dX'}{R_0^2}, \quad \phi_1 = \frac{X - \sigma}{2d} \int_{X}^{X'} \sigma R_0 dX'.
\]

(66)

A fourth solution is

\[
 R_1 = \frac{3\sigma + \omega}{4d} R_0 \int_{X}^{X'} R_0^2 dX', \quad \phi_1 = \int_{X}^{X'} \left( \frac{\omega - \sigma}{2d} R_0 - \frac{1}{2} R_0^2 \right) dX'.
\]

(67)

We now look for a particular solution of the homogeneous problem. Using the decomposition (65), we find that the equation for \( \phi_1 \) can easily be integrated and it yields

\[
 \phi_1 = \int_{X}^{X'} \left( \frac{X - \sigma}{2d} R_0 + \frac{a_4 - 23/19}{2 \sqrt{2}} + \frac{Q_4}{100} \right) dX'.
\]

(68)

Using this, the equation for \( R_1 \) is

\[
 4R_1, \alpha = R_1 \left( 1 - \frac{9000\theta}{361} R_0^2 + \frac{59840786005}{1505468192} R_0^4 \right) = \frac{\partial g}{\partial R}.
\]

(69)

where

\[
 g(R) = \frac{i}{2} R_0^2 - \frac{10135977701}{2971318800} R_0^4 + \frac{739442825}{3575486956} R_0^6 + \frac{21605132780200195}{4708658885991168} R_0^8.
\]

(70)

This has the solution

\[
 R_1 = \frac{1}{4} R_0 \int_{X}^{X'} R_0^2 \cdot g(R) dX'.
\]

(71)

Finally, the general solution is given by

\[
 b = K_1 a + iK_2 a + K_3 b_3 + K_4 b_4 + b_{(part)},
\]

(72)

where \( b_3 \) and \( b_4 \) are given by (66), and (67), respectively, \( b_{(part)} \) is the particular solution given by (68) and (71), and the \( K_i \) are real constants. By evaluating the solution as \( X \to -\infty \), we see that

\[
 b \sim \sqrt{\frac{3000\theta}{361}} e^{-X/2} \left( -K_3 + \frac{1}{2} K_4 \right).
\]

(73)

Hence, we must set \( K_3, K_4 = 0 \) to avoid divergence in that direction. Next, we look at the large, positive-\( X \) limit. There, we find that

\[
 b \sim \left[ \left( \frac{141137643}{1531924121728} \right)^{1/4} i_6 + \frac{1873147963175802007 \times 3^{3/4}}{100\sqrt{22} \times 11968157201^{7/4}} (1 + i\beta) e^{X - \varphi X/2} \right. \\
+ i \left[ \left( \frac{141137643}{23936314402} \right)^{1/4} q_4 - \frac{19953^{3/4} \sqrt{19\sqrt{19}}}{263036422^{3/4}} i_6 \\
- \frac{27910272207792205233}{7870153120211437055 \sqrt{23936314402}} \right] X e^{-\varphi X/2}.
\]

(74)
Figure 3: Theoretical Maxwell curves for the Lugiato-Lefever model. The dashed blue (upper) curve, is the lowest-order approximation (57); the red (lower) curve is the third-order approximation (76). Dots: limits of pinning range, from [31]

In order to avoid that $b$ diverge at infinity on the slow scale, the two terms above must vanish. This yields the new solvability conditions

$$i_6 = -1.941 \ldots, \quad q_4 = -0.021 \ldots \quad (75)$$

Eventually, combining these values with (58) and (61), we obtain a new approximation for the Maxwell curve

$$I = 1 - 1.629 \left( \theta - \frac{41}{30} \right)^2 - 4.036 \left( \theta - \frac{41}{30} \right)^3, \quad (76)$$

correct to third order in $(\theta - 41/30)$. One can conclude that the higher-order correction of the expression for Maxwell curve arises as a solvability condition on the solvability condition (63). Indeed, the appropriate choice of values of $i_6$ and $q_4$ precludes that $b$ diverges as $X \to \infty$ and therefore that the asymptotic expansion (62) breaks down.

This result is compared in Fig. 3 to the numerical results in [31], where the pinning range was computed numerically for the Lugiato-Lefever equation, with excellent agreement. On the other hand, it is seen that expression (76) is much more efficient than its second order counterpart to locate the pinning range for small values of $(\theta - 41/30)$. This may seem surprising at first, since the third-order term becomes negligible compared to the second-order term in (76). However, the pinning range becomes thinner at a much faster rate as one approaches the codimension-2 point. As a result, the curve given by (57) leaves the pinning range for very small values of $(\theta - 41/30)$.

**5. Conclusion**

Motivated by the importance of optical cavity models in the study of pattern formation on the one hand, and by the increasing attention brought to localized patterns on the other hand, we have derived the equation for the Maxwell curve for two wide classes of optical models. Such calculation was until recently only theoretically possible, because the linear problems encountered
at each order of the analysis, while in principle explicitly soluble, rapidly become very difficult to solve by hand. With the advent of symbolic softwares, the problem has become tractable for the Lugiato-Lefever model and its variants, provided one knows one’s way through the various orders of the multiple-scale analysis. The present paper may serve as a guide in this respect. With the Lugiato-Lefever model, it also shows how to improve the leading order expression of the Maxwell curve. Besides its purely technical interest, this calculation is useful if one wishes to compute localized patterns numerically in the small-amplitude limit. Indeed, that limit is notoriously difficult to investigate numerically both through direct numerical simulation and with continuation software. In the latter case, it is essential to have a good guess on the parameter values where homoclinic snaking happens, and the Maxwell curve is the best possible one.

Acknowledgments. I am grateful to Damià Gomila for promptly communicating the data of his paper. This work was supported by the Fonds de la Recherche Scientifique-FNRS (Belgium).

Appendix A. Derivation of Ginzburg-Landau equation (5) from the 1:1 resonance normal form

The normal form for a dynamical system near reversible 1:1 resonance is [67]

\[
\begin{align*}
\frac{dA}{dx} &= i(kA + B + iAP(\mu,|A|^2, i(AB - \bar{A}\bar{B})), \quad (A.1) \\
\frac{dB}{dx} &= i(kB + B + iBP(\mu,|A|^2, i(AB - \bar{A}\bar{B}))) + AQ(\mu,|A|^2, i(AB - \bar{A}\bar{B})), \quad (A.2)
\end{align*}
\]

where \(A\) and \(B\) are the amplitudes of the unstable modes and

\[
P(\mu, u, v) = p_1\mu + p_2u + p_3v, \quad Q(\mu, u, v) = -q_1\mu + q_2u + q_3v + q_4\mu^2 \quad (A.3)
\]

These equations are asymptotically valid in the limit of small \(A, B, \) and \(\mu\). Above, \(k\) is the wave number of the unstable mode. As noted in [30], in order to ignore higher-order terms in the normal form equations above, it is necessary to assume that \(q_2 = O(\epsilon), A = O(\sqrt{\epsilon}), B = O(\epsilon^{3/2}), \mu = O(\epsilon^2)\) and that the amplitude vary slowly as \(O(\sqrt{\epsilon})\). Hence, let us write

\[
A(x) = \sqrt{\epsilon}a(x)e^{ikx}, \quad B(x) = \epsilon^{3/2}b(x)e^{ikx}, \quad q_2 = \epsilon q_2', \quad \mu = \epsilon^2\mu', \quad X = \sqrt{\epsilon}x, \quad (A.4)
\]

where \(\epsilon \ll 1\). Equation (A.1) then becomes

\[
\frac{da}{dx} = b + ip_2a|a|^2 + \epsilon a \left[ip_1\mu' + \frac{p_3}{2}(ab - \bar{a}\bar{b})\right]. \quad (A.5)
\]

In the limit of small \(\epsilon\) it is easy to solve this equation for \(b\):

\[
b \sim \frac{da}{dx} - ip_2a|a|^2 - \epsilon a \left[ip_1\mu' + \frac{p_3}{2}\left(\bar{a}\frac{da}{dx} - a\frac{d\bar{a}}{dx} - 2ip_2|a|^2\right)\right] + O(\epsilon^2). \quad (A.6)
\]

Substituting that result into (A.2) then yields

\[
0 = q_1\mu' a - q_2' a|a|^2 - \left[p_2^2 - p_2q_3 + q_4\right]a|a|^4 - i\left(p_2 + q_3\right)a^2\frac{da}{dx} - i\left(3p_2 - \frac{q_3}{2}\right)|a|^2\frac{da}{dx} + \frac{d^2a}{dx^2} + O(\epsilon), \quad (A.7)
\]

which is of the expected form. Let us remark that, while Eqs. (A.1-A.2) contain eight parameters, only five combinations of them –the coefficients of (A.7)– actually matter.
References