One Person, Many Votes: Divided Majority and Information Aggregation*

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Abstract

This paper shows that information imperfections and common values can solve coordination problems in multicandidate elections. We analyze an election in which (i) the majority is divided between two alternatives and (ii) the minority backs a third alternative, which the majority views as strictly inferior. Standard analyses assume voters have a fixed preference ordering over candidates. Coordination problems cannot be overcome in such a case, and it is possible that inferior candidates win. In our setup the majority is also divided as a result of information imperfections. The majority thus faces two problems: aggregating information and coordinating to defeat the minority candidate. We show that when the common value component is strong enough, approval voting produces *full information and coordination equivalence*: the equilibrium is unique and solves both problems. Thus, the need for information aggregation helps resolve the majority's coordination problem under approval voting. This is not the case under standard electoral systems.

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1 Introduction

Coordination problems can undo even the best electoral systems. Consider the classical case in which the majority is divided between two alternatives, whereas the minority is unified behind a single one.[1] Such a majority faces at least three important problems. First, failing to coordinate ballots on a single alternative can lead to the victory of the minority. Just to mention one example, think of the US presidential election of 2000, in which A. Gore lost to G.W. Bush by 537 votes in Florida (a state which proved pivotal). In that election, R. Nader got 1.63% of the Floridians' votes (2.74% at the national level). Thus despite being a majority, left-wing voters lost because they had failed to coordinate their ballots on a single candidate. A second problem is that, to avoid such splits, some candidates may be compelled to withdraw. The best alternative may thus not even be in th running. Third, since the majority can coordinate on *any* of the running alternatives (there are multiple equilibria) the majority may fail to select the best among the ones running. No electoral system has been found to be immune to this coordination problem so far.[2]

In this paper, we take a step back and reconsider the *nature* of divisions in the electorate: majority voters are divided both by their individual preferences and by their information. The presence of information imperfections implies that, on top of overcoming coordination problems, majority voters must also aggregate information to identify which alternative is the full information Condorcet winner. We focus on the cases in which a common element of information affects voter valuations in a common direction.[3] The voters' valuation of alternatives thus includes both a *common value component* and a *private value component*. The literature on electoral systems has typically overlooked the information aggregation problem posed by the common value component[4] as it has instead focused on opposing preferences, *i.e.* the private value component.

Our main finding is that approval voting[5] resolves both problems at once, producing

[1] This so-called *problem of the divided majority* has been central to the analysis of electoral systems since at least the 18$^{\text{th}}$ Century (Borda 1781).

[2] The literature on electoral systems and coordination problems is vast. See e.g. Arrow (1951), Cox (1997), Dewan and Myatt (2007), Myatt (2007), Myerson (2002), and Myerson and Weber (1993).

[3] We are thinking for instance of a politician's moral qualities or of policies with *ex ante* uncertain costs and benefits. This is typical of social and tax policy, and of policy choices such as financial sector (de)regulation or of the "war on terror": as it unfolds, information about the actual costs and benefits of these policies can substantially affect electoral support. See also Osborne and Turner (2010, Section 5.2).

[4] Information aggregation in two-candidate elections is studied by the Condorcet Jury Theorem literature (see e.g. Austen-Smith and Banks 1996, Feddersen and Pesendorfer 1996, 1997, Myerson 1998a). For multicandidate elections, see e.g. Castanheira (2003) and Piketty (2000).

[5] Under approval voting, voters can "approve of" as many candidates as they want, each approval counts

full information and coordination equivalence. A necessary and sufficient condition for this result is that sufficiently many voters may need additional information to identify their best alternative (i.e. they have a "doubt" that information can dispel). This implies, quite surprisingly, that the presence of an information aggregation problem facilitates the resolution of the coordination problem. In contrast, a large class of electoral systems, including plurality and runoff, remain prone to coordination failures.

Conveying the intuition for this result is simpler when the private value component is assumed away (see sections 2 and 3). Divisions are then only due to information: consider three candidates (A, B and C), two states of nature (a and b) and three types of voters (t_A, t_B and t_C), whose actual number is random. Types t_C are the (expected) minority block; they always vote for C. Types t_A and t_B form the (expected) majority block: they prefer A to B to C in state a, and B to A to C in state b. Yet, they are divided because they hold opposite information about which state, a or b, is the true state of nature.

The distinctive characteristic of approval voting is to allow voters to kill two birds with one ballot: they can vote for their most preferred alternative *and* lend support to their second choice. For instance, A-supporters would vote for both A and B if A has little chance of defeating C. This is the *coordination motive* for double voting. We also identify a *common-value motive* for double voting: neither types t_A nor t_B want A to win in state b nor B to win in state a. Thus, all majority voters would (also) vote B if they deem that A has too high a chance of winning in state b. Finally, the *information motive* implies that majority voters avoid double voting "excessively": they value giving exclusive support to their preferred alternative in order to aggregate information. Our results show that the equilibrium strategy is unique and balances these three motives: some majority voters double vote to ensure that C's expected vote share is the lowest one, and others single-vote to ensure that the full information Condorcet winner has the largest share. Thus, full information and coordination equivalence *always* hold under approval voting.

In contrast, such a desirable property fails to hold with the two most used electoral systems around the world, plurality and runoff, as well as with a broad class of scoring rules. The reason is that the coordination motive tends to swamp the other motives. Voters are then trapped into coordinating on either A or B. Deviating is too costly because one would primarily reduce the vote count of the leading alternative when lending support to the trailing one: one cannot vote both for A and for B as in approval voting. This prevents information aggregation and the election of the full information Condorcet winner in these systems.

as one vote and the candidate that obtains the largest number of votes wins (Weber 1977, 1995, Brams and Fishburn 1978, 1983, Laslier 2009).

2 A common value model

This section lays out a simplified model in which voters have purely common valued preferences. Divisions are thus only due to information. This simplified case is sufficient to convey the main intuitions. It also clarifies the comparison of electoral systems in a setup polar to the classical one, in which divisions are only rationalized by the private value component. The general case in which voters are divided by both the private and the common value components is introduced in Section 4.

We consider three alternatives, $P \in \{A, B, C\}$, two states of nature, $\omega \in \{a, b\}$, and three types of voters, $t \in T \equiv \{t_A, t_B, t_C\}$. Types t_A and t_B have purely common valued preferences: conditional on the state of nature, they all want to elect the same alternative, which is A in state a and B in state b:

$$\begin{aligned} U(P, t_A|\omega) = U(P, t_B|\omega) &= 1 \text{ if } (P,\omega) = (A,a) \text{ or } (B,b) \\ &= 0 \text{ if } (P,\omega) = (A,b) \text{ or } (B,a) \\ &= -1 \text{ if } P = C, \end{aligned} \tag{1}$$

where $U(P, t|\omega)$ denotes the utility of a voter with type t when alternative P is elected and the true state is ω. The values 1, 0 and -1 are only meant to simplify exposition.

Given their private signal (see below), types t_A and t_B have opposite convictions regarding which state is most likely, and therefore which alternative is best. A voter with type t believes that the true state is ω with a probability $q(\omega|t)$. We impose that:

$$\infty > \frac{q(a|t_A)}{q(b|t_A)} > 1 > \frac{q(a|t_B)}{q(b|t_B)} > 0. \tag{2}$$

Importantly, (2), implies that the voters' information is imperfect: types t_A do not put a probability 1 on the true state being a (otherwise the probability ratio would be infinite), and types t_B do not put a probability 1 on the true state being b (otherwise the probability ratio would be zero). Yet, these priors may be arbitrarily close to 1. The relevant difference between priors being close or equal to 1 is that the voters' beliefs can change through Bayesian updating if other voters reveal some of their information.

For the sake of simplicity, we assume that types t_C are partisans: they always vote for alternative C.[6]

[6] This strategy can be supported by t_C-voters having dichotomous preferences. Alternatively, one may assume that t_C-voters also have a common value component and preferences $C \succ A \succ B$ in state a and $C \succ B \succ A$ in state b. Yet, in this case, some of these voters could lend support to types t_A and t_B by voting for A or B in equilibrium. This may only reduce tensions between types t_C and types t_A and t_B, and therefore the problems posed by C.

Timing. At the beginning of the game (**time 0**), nature chooses the state ω with probability $q(\omega)$. The state remains unobserved until after the election. The probabilities of states a and b are common knowledge.

At **time 1**, nature selects a random number of voters from a Poisson distribution of mean n and, conditional on the state, assigns them a type t by iid draws. The conditional probability of being assigned type t is $r(t|\omega)$, with $\sum_t r(t|\omega) = 1$, $\forall \omega$. These probabilities are also common knowledge.

The distribution of voters determines which type is expected to be in the majority. We focus on the case:

$$r(t_C|\omega) < 1/2, \tag{3}$$

which implies that, in expected terms, types t_C are a strict minority.[7] Hence, types t_A and t_B compose the *majority block*, whereas types t_C form the *minority block*. We assume for simplicity that $r(t_C|a) = r(t_C|b)$.

The election is held at **time 2**. Neither the actual state of nature nor the actual number of voters of each type is observed: voters only know their own type, t.[8] Through Bayesian updating, a voter with type t infers that the probability of state ω is $q(\omega|t)$:

$$q(\omega|t) = \frac{q(\omega)\, r(t|\omega)}{q(a)\, r(t|a) + q(b)\, r(t|b)}, \tag{4}$$

and, clearly, condition (2) imposes that:

$$\begin{aligned} r(t_A|a) &> r(t_A|b), \text{ and} \\ r(t_B|a) &< r(t_B|b). \end{aligned}$$

Payoffs are realized at **time 3**: the winning alternative $W \in \{A, B, C\}$ is selected and each voter receives utility $U(W, t, \omega)$.

Action set under approval voting. Under *approval voting*, each voter can cast a ballot on as many (or as few) alternatives as she wishes. Each approval counts as one vote: when a voter only approves of A, then only alternative A is credited with one vote. If the voter approves of both A and B, then both A and B are credited with one vote, and so on. Hence, the voters' action set is:

$$\Psi = \{A, B, C, AB, AC, BC, ABC, \varnothing\},$$

[7] For $r(t_C|\omega) > 1/2$, an expected majority of the electorate prefers that C wins, independently of ω. This case is trivial to investigate: by the law of large numbers, the realized fraction of types t_C will be larger than $1/2$ with a probability that converges to 1 as $n \to \infty$ and C wins.

[8] With common initial priors $q(\omega)$, a voter's type is uniquely defined by the signal she receives (t_A or t_B). Our results however directly extend to any other value of $q(\omega|t)$ that satisfies (2).

where, by an abuse of notation, action A denotes a ballot in favor of A only, action BC denotes a joint approval of B and C, etc., and $\varnothing$ denotes abstention. Thus, the difference between approval voting and more common electoral rules is that a voter can cast a single, a double or a triple approval.

A ballot affects the winning probabilities of each alternative. The value of a ballot thus depends on its probability of being pivotal across alternatives.[9] Single approvals ($\psi = A$, B and C) act as positive votes: for instance, an A-vote can only be pivotal in favor of A, either against B or against C. Double approvals ($\psi = AB$, BC and AC) act as negative votes. For instance, if the voter plays AC, her ballot can only be pivotal against B, either in favor of A or of C. Finally, a triple approval (ABC) can never be pivotal: it is strategically equivalent to abstention.

Let $x(\psi)$ denote the number of voters who played action $\psi \in \Psi$ at time 2. The *total number of approvals* received by alternatives A, B, and C are respectively:

$$\begin{aligned} X(A) &= x(A) + x(AB) + x(AC) + x(ABC), \\ X(B) &= x(B) + x(AB) + x(BC) + x(ABC), \\ X(C) &= x(C) + x(AC) + x(BC) + x(ABC). \end{aligned} \tag{5}$$

The alternative with the largest total number of approvals wins the election. Ties are resolved by the toss of a fair coin.

Strategy space and equilibrium concept. A type t's strategy is a mapping $\sigma : T \to [0,1]^8$ where $\sigma_t(\psi)$ denotes the probability that a randomly sampled voter of type t plays action ψ, and the usual constraint applies: $\sum_\psi \sigma_t(\psi) = 1$, $\forall t$. Note that a voter can only condition her strategy on her type t.[10] Given σ, an *expected share of voters*:

$$\tau(\psi|\omega, \sigma) = \sum\nolimits_t r(t|\omega) \;\; \sigma_t(\psi) \tag{6}$$

is expected to play action ψ in state ω. The expected number of ballots ψ is:

$$\mathsf{E}[x(\psi)|\omega, \sigma] = \tau(\psi|\omega, \sigma) \cdot n.$$

The realized number of ballots, $x(\psi)$ follows a Poisson distribution of mean $\tau(\psi|\omega, \sigma) \cdot n$, which thus depends on the strategy, σ, and on the state of nature. Let an action profile $x \in \mathbb{N}^8$ be the vector that lists, for each action ψ, the realized number of ballots ψ. The set of possible action profiles for the players is denoted $Z(\Psi)$.

[9] Appendix A1 summarizes the properties of Poisson games, which we apply to approval voting.

[10] Note that the equilibrium mapping $\sigma_t(\psi)$ *must* be identical for all voters of a same type t, by the very nature of population uncertainty (see Myerson 1998b, p377, for more detail). Section 4 extends the model to a continuum of types, in which case the equilibrium is in cutoff strategies.

For this voting game, we analyze the limiting properties of symmetric Bayesian Nash equilibria when the expected population size n becomes infinitely large. As shown by Myerson (2000, Theorem 0), there must be at least one equilibrium in this game. In such an equilibrium, each voter plays an undominated strategy given the expected vote share of each alternative, and the expected vote share τ in turn results from this strategy.

As shown in Lemma 2 (in Appendix A2), the set of undominated actions is actually $\{A, B, AB\}$. We will thus omit actions AC, BC, and ABC from now on.

3 Common value: approval voting vs. other voting systems

This section lays out our main two results in the pure common value setup of Section 2: under approval voting, the equilibrium is unique and such that the full information Condorcet winner is the only likely winner of the election. Under other systems, such as plurality voting or runoff (two-round) elections, voters may instead be trapped in equilibria in which a poor candidate is the only likely winner.

Given the information available, the best outcome would be obtained if, before the election, voters could freely aggregate all the elements of information available in the electorate, update their beliefs about the actual state of nature, and coordinate their votes on the best alternative (*i.e.* the full information Condorcet winner). In reality, which is the best alternative is unclear at the time of election, and coordination problems may arise. We introduce the concept of *full information and coordination equivalence*[11] to identify when an election produces (almost surely when population size is large) the best outcome:

Definition 1 *A strategy σ satisfies* ***full information and coordination equivalence*** *if its associated expected vote shares satisfy:*

$$\begin{aligned} &\tau(A|a) + \tau(AB|a) > \max\{\tau(B|a) + \tau(AB|a), \tau(C)\} \ \text{ in state } a, \text{ and} \\ &\tau(B|b) + \tau(AB|b) > \max\{\tau(A|b) + \tau(AB|b), \tau(C)\} \ \text{ in state } b. \end{aligned} \tag{7}$$

That is, alternative A's (B's) expected vote share is the largest one in state a (b).

Satisfying full information and coordination equivalence is not a trivial matter in three-alternative elections: first, C may win the election if the majority split their votes. Second, this equivalence cannot hold if all majority-block voters approve of either A or B: if all majority voters approve of A, for instance, then A necessarily leads the election. Third, as

[11] This concept is a natural extension to multicandidate elections of Feddersen and Pesendorfer's (1997) concept of *full information equivalence.*

explained in the introduction, coordination issues arise when there are multiple equilibria: for instance, all majority block voters may want to approve of a same alternative but cannot agree whether to coordinate on A or on B.

3.1 Approval voting

Because of these problems, most electoral systems fail to ensure that the full information Condorcet winner is elected. In contrast, when population size is large enough, approval voting addresses all these issues at once:

Theorem 1 *Under approval voting, there exists an expected population size $\bar{n}$, such that for any $n \geq \bar{n}$, the equilibrium is unique and satisfies full information and coordination equivalence.*

A step-by-step proof can be found in Appendix A2. Here, we focus on the main intuition for this result. It builds around the interaction among the three motives that shape voting behaviour – the *coordination*, *common-value*, and *information* motives.

The *coordination motive* is such that, if C may either be expected to win or to be the main challenger against A or B, majority voters develop an incentive to support the strongest of the majority alternatives. In classical electoral systems, this implies that majority voters must coordinate their votes only on that single alternative (we return to this point below). Yet, a distinguishing feature of approval voting is that voters are given the option to double vote: voters need not abandon their preferred alternative to support the strongest one. We find that majority voters actually prefer to exert that option whenever coordination against C is required: types t_A mix between A and AB but never single-vote B, and types t_B mix between B and AB but never single-vote A.

The *common-value motive* is the second rationale for double voting: common value implies that both types t_A and t_B want A to win in state a and B to win in state b. Since majority voters have an imperfect signal, they must compare pivot probabilities across states of nature to decide whether they should support their *a priori* second choice. For instance, imagine that B's vote share is expected to be higher than in equilibrium. The probability of being pivotal between A and B becomes much larger in state a than in state b, in which case even types t_B dislike being pivotal against A. Majority voters have two options to restore the balance of pivot probabilities across states: types t_A may double vote less (to reduce B's vote share) *or* types t_B may double vote more (to increase A's vote share). But double voting necessarily dominates both absention and single voting for one's second choice.

Finally, the *information motive* guarantees that majority voters do not double vote "excessively". Consider the case in which all majority voters double vote. Then, A and B's number of votes is necessarily equal and (almost surely) larger than C's. Thus, C ceases to be a threat. The focus is on the choice between A and B. To influence this choice, voters must single vote. Pivot probabilities being equal across states of nature, voters can only rely on their private signal to decide which alternative to vote for: types t_A single-vote A and types t_B single-vote B. More generally, this motive dominates whenever C's threat is perceived to be small and pivot probabilities are balanced across states of nature.

The unique equilibrium under approval voting is the one that balances these three motives (Section 3.3 illustrates this interaction with the help of numerical examples): first, because of the coordination motive, C must rank third in both states of nature. Second, by the common-value motive, A's vote share in state a cannot be smaller than that of B, and conversely in state b. Finally, by the information motive, the vote shares of A and B cannot be equal. That is, A must rank first in state a and B must rank first in state b. Approval voting produces full information and coordination equivalence.

3.2 Other voting systems

At this stage, a natural question is whether the results of Theorem 1 extend to other electoral systems: does it not immediately follow from the assumption of purely common values? To address this question, we introduce two benchmarks that cover more than 95% of the presidential elections held around the world over the 1990s (Golder 2005). Benchmark 1 studies plurality elections, a stylized version of the system used in the US and in the UK, and related scoring rules. Benchmark 2 covers runoff elections, used in France as well as many other countries. We find that these systems fail to produce full information and coordination equivalence.

3.2.1 Benchmark 1: Plurality voting and related scoring rules

Myerson (2002) introduces $\{\beta, \delta\}$ scoring rules in the following way: voters award one vote to one alternative, zero vote to another, and a score between β and δ to the third one. In that setup, approval voting is the scoring rule such that $\{\beta, \delta\} = \{0, 1\}$: the voter can assign either 0 or 1 point to her second-ranked alternative. Instead, under plurality voters must assign one vote to one alternative only. This constrains them to award a score of zero to all the others: $\{\beta, \delta\} = \{0, 0\}$.

Here, we study the class of scoring rules such that $0 = \beta \leq \delta < 1$, *i.e.* the rules comprised between plurality and approval voting. Our second theorem shows that the coordination motive may swamp the other two motives in all such scoring rules. That is:

Theorem 2 *For any scoring rule with $\delta < 1$, there exists an $\bar{n}$ such that, for any population $n \geq \bar{n}$, there is an equilibrium in which full information and coordination equivalence fails.*

Proof. See Appendix A2. ■

Let us first focus on plurality voting to provide an intuition for this result. In that system, voters abstain or cast a ballot on exactly one alternative. Minority voters cast their ballots on C, and majority voters must decide whether to vote for A *or* for B. Consider the following example: $q(b) = 0.03$, $r(t_A|a) = 0.6$, $r(t_B|a) = 0.01$ and $r(t_C|\cdot) = 0.39$. That is, alternative B is the best with a probability of 3%. In state a, 98.4% (60/61) of the majority voters receive the signal that A is the best candidate. Only 1.6% (1/61) believe that B may be best. In state b, $r(t_A|b) = 0.2$ and $r(t_B|b) = 0.41$.

Now, consider a voter who expects almost all other majority voters to coordinate on B (for instance, B may have a history of strong voting support). What is the best response of this voter? Since the expected vote share of A is below that of C, this voter also prefers to cast her ballot on B: A could only win if, in a highly unlikely draw, the realized scores of B and C were well below their expected values. There is thus an equilibrium in which all majority voters cast their ballot on B.

This is already true for relatively small expected population sizes: with $n = 100$, a ballot for A is 10^{-37} times less likely to be pivotal than a vote for B in this equilibrium, and the ratio would be yet smaller with larger population sizes.[12] Thus, the value of a ballot for A is much lower than that of a ballot for B: no majority voter wants to deviate from this strategy and the bad alternative, B, emerges as the only likely winner.

The reasoning is similar for all the other scoring rules considered in Theorem 2: if all majority voters play BA, the number of votes for B is the realized number of majority voters: $X(B) = x(t_A) + x(t_B)$, whereas the number of votes for A is $X(A) = \delta \cdot (x(t_A) + x(t_B))$. Thus, $X(B) > X(A)$ for any $x(t_A) + x(t_B) > 0$. A single A-ballot gives one vote to A and zero vote to B. This vote can be pivotal if and only if: $(1-\delta) \cdot (x(t_A) + x(t_B)) \leq 1$, which is much less likely than being pivotal between B and C if population size is not too small: shifting any *fraction* of a vote away from B primarily weakens that alternative against C. Its impact on the probability that A wins the election is only of second order importance.

[12] We use Properties 1 and 2 in Appendix A1 to compute pivot probabilities and probability ratios in numerical examples.

This identifies a specific property of approval voting that is central to Theorem 1: when the voter can give A an additional vote *without withdrawing any point from* B, deviating towards action AB becomes costless.[13]

3.2.2 Benchmark 2: Runoff elections

In runoff elections, a candidate wins outright in the first round if she satisfies two conditions: (i) she obtains the largest number of votes, and (ii) she receives more than a pre-defined threshold, here 50% of the votes.[14] If no candidate passes this threshold, then a second round opposes the top two candidates. This runoff system is often argued to be better than plurality with respect to information or preference aggregation (Duverger 1954, Cox 1997, Piketty 2000 and Martinelli 2002). Piketty (2000) for instance notes that runoff elections should be able to separate the "communication stage", in which voters learn whether A or B is best, from the "election stage". This intuition finds support in Martinelli (2002). However, as shown by Bouton (2010), uncertainty about the second-round outcome causes runoff elections to produce equilibria in which a bad candidate is the only likely winner. We illustrate this result here with the help of a second numerical example.

Assume that, in each round, the distribution of preferences follows the same Poisson distribution as in Benchmark 1. If a given voter expects the other majority voters to coordinate on B, then A's expected vote share in the first round is 0%, and that of B is 61%. This voter's ballot can be pivotal in favour of A if and only if both B and C receive at most one vote: in all the other cases, A is third and gets eliminated from the race. From (9) in the appendix, for an expected population size of $n = 100$, the probability of this event is approximately 10^{-40}.

Compare this with the probability that a ballot for B is pivotal against C. In a two-round system, the latter pivotability combines two events: that B and C nearly tie in the first round (probability: ≈ 0.009), and that C wins the second round (probability: ≈ 0.0135). Altogether, a first-round ballot for B is thus pivotal against C with probability

[13] In a setup with three potentially good alternatives and dichotomous preferences (which implies that there is no coordination problem), Goertz and Maniquet (2011) find that scoring rules with $\delta < 1$ may not even feature one equilibrium with information aggregation. With purely common valued voters, Ahn and Oliveiros (2011) show that, if any scoring rule admits a strategy that guarantees the election of the full information Condorcet winner, then approval voting also admits one.

[14] There exist other thresholds. For instance, in Costa Rica and New York City, the threshold for first-round victory is below 50%. In Sri Lanka in 1996, the threshold was at 55%. Finally, there are also countries which use a threshold that depends on the victory margin. For instance, in Nicaragua the threshold is 40% if the margin of victory is below 5% but 35% if it is above.

of approximately 10^{-4} (*i.e.* 0.009×0.0135), which is 10^{36} times higher than the probability of being pivotal in favour of A. Action B is thus 10^{36} times more valuable than action A.

Again, this ratio would increase with population size. In other words, and as formally proven by Bouton (2010), even if small, the risk of an upset victory of C in the second round dwarfs the incentive of majority voters to vote for A.

3.3 Numerical examples

To provide a more concrete interpretation of Theorem 1, this subsection proposes numerical examples that focus on symmetric priors: $q(a) = \frac{1}{2} = q(b)$ and a symmetric distribution of types: $r(t_A|a) = r(t_B|b)$. Symmetry is only meant to simplify exposition: from (32) and Lemma 8 in Appendix A2, it imposes that $\sigma^*_{t_A}(A) = \sigma^*_{t_B}(B)$. We illustrate the effect of the three motives that shape voting behaviour. We also use these examples to illustrate that population sizes of 10 or 100,000 voters are largely sufficient for our limit results to have bite.

Let $r(t_C) = 0.4$, $r(t_A|a) = 0.36$ and $r(t_A|b) = 0.24$. With these parameter values the Condorcet loser, C, would asymptotically win if the majority single voted for their *a priori* preferred candidate. Vote shares would be: $\tau(C) = 0.4 > \tau(A|a) = \tau(B|b) = 0.36 > \tau(A|b) = \tau(B|a) = 0.24$. This implies that we are in case (*ii*) of Lemma 8, and that there must be some double voting in equilibrium.[15] The equilibrium strategy profile is $\sigma_{t_A}(AB) = 0.57 = \sigma_{t_B}(AB)$, which leads to the expected vote shares and magnitudes illustrated in Table I.

Table I: equilibrium vote shares (left) and magnitudes (right).

Candidate	Vote shares in state a	Vote shares in state b
A	0.497 (first)	0.445 (second)
B	0.445 (second)	0.497 (first)
C	0.4 (third)	0.4 (third)
Total	1.342	1.342

and

Magnitudes	state a	state b
$mag(piv_{AC}\|\omega)$	$\mathbf{-0.0052}$	-0.0081
$mag(piv_{BC}\|\omega)$	-0.0081	$\mathbf{-0.0052}$
$mag(piv_{AB}\|\omega)$	$\mathbf{-0.0052}$	$\mathbf{-0.0052}$

As this example illustrates, double voting allows the majority to "inflate" the expected vote shares of both A and B above the share of C. This is why the sum of the three vote

[15] In a symmetric setup, $r(t_C) \leq r(t_A|b)$ is a necessary and sufficient condition for all voters to single-vote in equilibrium. For $r(t_C) > r(t_A|b)$, the larger $r(t_C)$, the more double voting in equilibrium.

shares exceeds 100% of the population. It also illustrates that, with an internal solution, the magnitudes of the pivot probabilities between A and B must be equal to the largest magnitudes against C.

Deviations. To illustrate the effect of the common value motive, consider for a moment a deviation by types t_A who increase their probability of playing AB by 3 percentage points, such that $\sigma_{t_A}(AB) = 0.6$. The largest magnitude is then the one between A and B in state a. This implies that, with 10,000 voters, the probability of being pivotal between A and B in state a becomes 4×10^{15} higher than in state b, and 10^8 higher than between A and C in state a (the figures increase respectively to 2×10^{156} and 2×10^{80} with 100,000 voters). In other words, the value of a single vote A becomes much larger than the value of an AB vote, since the dominant value of a ballot is precisely to be pivotal against B. This induces types t_A to decrease their probability of playing AB, moving back towards the equilibrium of Table I.

Next, we illustrate how the aggregate level of double voting affects the relative importance of the information and coordination motives. Consider a strategy function for which majority voters do not double vote enough: $\sigma_{t_A}(AB) = 0.54 = \sigma_{t_B}(AB)$ (recall that $\sigma_{t_A}(AB) = 0.57 = \sigma_{t_B}(AB)$ in equilibrium). Table II displays the magnitudes associated with this strategy:

Table II: The coordination motive: magnitudes after deviation

Magnitudes	state a	state b
$mag(piv_{AC}\|\omega)$	$\mathbf{-0.0045}$	-0.0078
$mag(piv_{BC}\|\omega)$	-0.0078	$\mathbf{-0.0045}$
$mag(piv_{AB}\|\omega)$	-0.0056	-0.0056

Thus, with n =10,000, a ballot is 4×10^4 times more likely to be pivotal against C than between A and B (with 100,000 voters, these figures increase to 5×10^{45}). All majority voters thus develop the incentive to double vote AB, and fight C, again moving back towards the equilibrium of Table I. Clearly, the effect goes in the other direction if there is more double voting than in equilibrium: the information motive dominates, and voters prefer to double vote less.

4 Approval voting: common vs. private values

This section presents the main results of the paper. We introduce an extended model of multicandidate elections in which the two sources of majority divisions coexist: majority voters have opposite information *and* heterogeneous preferences regarding the two majority alternatives. The former division is the same as in Sections 2 and 3. We call it the *common value component* of the voters' valuation of alternatives: additional information in favor of either A or B increases the voters' valuation of that alternative. The second division is the one considered in the rest of the literature. We call it the *private value component*: even with perfect information, majority voters may keep disagreeing about which alternative is best.

Two results emerge: first, "a doubt is enough". That is, if majority voters have even the slightest doubt about which alternative is best, approval voting ensures the victory of the full information Condorcet winner. Second, some majority voters may have no doubt: they are certain about which alternative they prefer. In that case, a necessary and sufficient condition for full information and coordination equivalence is that these voters do not represent too large a fraction of the electorate. In other words, the classical finding that no electoral system can resolve the problem of the divided majority is due to the omission of the common value component from the analysis of electoral systems. Whenever this component is relevant for sufficiently many voters, there is at least one electoral system, approval voting, that does resolve this problem. By contrast, the other systems studied in Section 3 fail to ensure the election of the full information Condorcet winner (see Lemma 9 in Appendix A4).

4.1 An extended model

This extended model is related to Feddersen and Pesendorfer (1997): let us consider a voter-specific utility function $U(P, i|\omega)$, where $U(\cdot)$ is finite, $i \in T \equiv [-1, 1] \cup \{t_C\}$ summarizes voter preferences, $P \in \{A, B, C\}$ is the winning alternative and $\omega \in \Omega = [0, 1]$ is the state of nature. Each voter knows her preference i but is uncertain about the realized state of nature ω.

Population size is determined by the draw of a Poisson distribution of mean n.[16] Given n, each voter is attributed a preference i by iid draws. With probability $r(t_C) < 0.5$, the voter is part of the minority and has preferences $U(C, t_C|\omega) - U(P, t_C|\omega) = k > 0$, $\forall P \in \{A, B\}$, $\omega \in \Omega$. With probability $(1 - r(t_C))$, the voter is part of the majority. Her type i then belongs to $[-1, 1]$ with $F(i)$ denoting the probability that she has a preference $x \leq i$. $f(i) \equiv F'(i)$ denotes the probability distribution function, with $f(i) > 0, \forall i \in [-1, 1]$.

Private and common value components. Majority voters prefer both A and B to C: $U(A, i|\omega) - U(C, i|\omega) > 0$ and $U(B, i|\omega) - U(C, i|\omega) > 0$, for any i, ω. Yet, voters with

[16] In the next section, we show how the results extend to multinomial distributions.

a higher type i values A more as compared to B: for any $i > i'$, we have $U(A, i|\omega) - U(B, i|\omega) > U(A, i'|\omega) - U(B, i'|\omega)$.

This *private value component* implies that the majority is divided as to whether A or B is the best alternative for any interior state $\omega \in (0, 1)$. The (expected) *fraction of voters* who prefer A to B in state ω is defined by: $\phi(A|\omega) \equiv \int_{-1}^{1} \mathbf{1}[U(A, i|\omega) - U(B, i|\omega) > 0]\, dF(i)$, where $\mathbf{1}[c]$ is an indicator function that takes value 1 when condition c holds. $\phi(B|\omega)$, the fraction of voters who prefer B to A, is defined similarly. The majority being divided for any interior state $\omega \in (0, 1)$ implies:

$$\phi(A|\omega), \phi(B|\omega) > 0, \ \forall \omega \in (0, 1).$$

The two corner states $\omega = 0$ and $\omega = 1$ are discussed below.

Voter preferences also feature a *common value component.* That is, for any majority voter, the utility differential between A and B is strictly increasing in the state of nature ω:

$$\forall i \in [0, 1], \ \omega > \omega' \Rightarrow U(A, i|\omega) - U(B, i|\omega) > U\left(A, i|\omega'\right) - U\left(B, i|\omega'\right).$$

We impose that, for any $\delta > 0$ and $\omega \in (0, 1)$, the fraction of voters with $|U(A, i|\omega) - U(B, i|\omega)| < \delta$ is strictly positive. Hence:

$$\omega > \omega' \Rightarrow \phi(A|\omega) > \phi\left(A|\omega'\right) \text{ and } \phi(B|\omega) < \phi\left(B|\omega'\right).$$

Thus, $\phi(A|\omega)$ is lowest in $\omega = 0$ and highest in $\omega = 1$.[17]

Concerning the two corners states, we impose that:

$$\phi(A|0) = 0 \text{ and } \phi(A|1) = 1.$$

Definition 2 *We call* doubt *the fact that all majority voters* may *prefer either alternative. This happens when the two corner states materialize with strictly positive probability. In contrast, there are* partisan voters *if the probability of $\omega = 0$ and/or $\omega = 1$ is zero.*

States of nature. The distribution of states of nature is denoted by the CDF $H(\omega)$ with H differentiable. The subset $[\underline{\omega}, \bar{\omega}] \subseteq \Omega$ defines which states have strictly positive density: $H(\omega) = 0$, $\forall \omega < \underline{\omega}$, $H(\omega) = 1$, $\forall \omega > \bar{\omega}$, and $h(\omega) \equiv H'(\omega) > 0$, $\forall \omega \in [\underline{\omega}, \bar{\omega}]$.

Note that our pure common value setup assumed $H(\omega) = q(b)$, $\forall 0 \leq \omega < 1$ and $H(1) = 1$: all the probability mass was on the two corner states. At the other extreme,

[17] Our setup thus excludes the adversarial preferences and preference reversals studied by Kim and Fey (2002) and Bhattacharya (2007).

a pure private value setup would amount to setting $H(\omega) = 0$, $\forall \omega < z$ and $H(\omega) = 1$, $\forall \omega \geq z$: all the probability mass is on the interior state z. The present setup allows us to span between these two extremes: *doubt* requires that the two corner states materialize with strictly positive probability, *i.e.* $\lim_{\omega \searrow 0} H(\omega) > 0$ and $\lim_{\omega \nearrow 1} H(\omega) < 1$. Conversely, there are partisan voters if either $\underline{\omega} > 0$ or $\bar{\omega} < 1$.

Signals. Prior to the election, each voter receives an independent and identically drawn signal $s \in \{0, 1\}$. Each signal is received with probability $r(s|\omega) \in (0, 1)$, with $r(1|\omega) > r(1|\omega')$ for any $\omega > \omega'$. That is, signal 1 is associated with A being better valued. Like in the pure common value setup, the private signal is informative in the sense that some voters are "sensitive" to the signal:

$$\begin{aligned} \exists \boldsymbol{\iota} \subset T \text{ s.t.} \qquad & \int q(\omega|1) \left[U(A, i|\omega) - U(B, i|\omega)\right] d\omega > 0, \text{ and} \\ & \int q(\omega|0) \left[U(A, i|\omega) - U(B, i|\omega)\right] d\omega < 0, \ \forall i \in \boldsymbol{\iota} \end{aligned}$$

where $q(\omega|s) = r(s|\omega)\ h(\omega) / r(s)$ is the belief about the distribution of states conditional on receiving signal s. Thus, after private signals are received, the higher ω is, the larger is the expected fraction of voters who perceive that A is better than B.

Cutoff strategies. An equilibrium of this voting game is characterized by an ordered cutpoint strategy such that any majority voter with $i < \theta_B(s)$ votes B and any voter with $i > \theta_A(s)$ votes A (in a related setup, see Feddersen and Pesendorfer 1997, Proposition 1).

Definition 3 *A strategy σ satisfies* full information and coordination equivalence *if the full information Condorcet winner has the largest expected vote share. That is, for $n \to \infty$,*
(i) $\forall \omega$ such that $\phi(A|\omega) > \phi(B|\omega)$, A wins with a probability that converges to 1 and
(ii) $\forall \omega'$ such that $\phi(A|\omega') < \phi(B|\omega')$, B wins with a probability that converges to 1.

4.2 A doubt is enough

We can now demonstrate that, with a large population and when the common value component is sufficiently important to generate doubt, approval voting almost surely selects the full information Condorcet winner. Note that, if the common value component were an additive shock to utility differences, then unbounded support for this shock (*e.g.* if it followed a normal distribution) would also be a sufficient condition for full information and coordination equivalence to hold.

Theorem 3 *When the population size is large,* doubt *is a sufficient condition for approval voting to have a unique equilibrium strategy, which satisfies full information and coordination equivalence.*

The intuition for Theorem 3 is similar to that of Theorem 1: majority voters have three motivations for single or double voting. Their *coordination motive* induces them to double vote if C is the main threat against either A or B. This implies that C cannot win in any state of nature. Their *common value motive* induces them to double vote if they feel that one of the majority alternatives has too high an expected vote share: if, say, A was among the likely winners in state 0 (in which all voters prefer B to A) then even the staunchest A supporters would prefer to vote AB. Thus, neither A nor B can be a likely winner in all states of nature. This implies that there exists an *interior* state ω^* in which the probability of being pivotal is much higher than in other states. Here, the *information motive* comes to dominate: if i prefers A to B in state ω^*, and provided that C is not a threat, she strictly prefers to play A than AB. Conversely, any i who prefers B to A would vote B. Theorem 3 establishes that, for n sufficiently large, only one strategy can balance these three motives, and it necessarily leads to full information and coordination equivalence.

Note that the presence of a common value motive is necessary to obtain that result. To make that clear, we now introduce the more classical model of purely private values. This case has been treated in a different setup by Myerson and Weber (1993), who show that multiple equilibria coexist. The purpose of our third benchmark below is to show that inferior equilibria also exist in a Poisson game environment: approval voting fails to always select the Condorcet winner when preferences are in purely private values. After that, Theorem 4 shows that this result hinges on having a sufficiently large share of partisan voters.

4.3 Benchmark 3: approval voting in a purely private value setup

The case opposite to our setup is when the common value component is absent. All voters are thus partisan: they know with probability 1 which candidate they prefer. To suppress the common value component in our setup, it is sufficient to assume that the state of nature is known *ex ante* to be some interior state: $0 < \underline{\omega} = \bar{\omega} < 1$. Two additional equilibria then emerge: in one, B-partisans only approve of B, whereas A-partisans double vote. In that case, B is the only likely winner, even if $\phi(A|\bar{\omega}) > \phi(B|\bar{\omega})$. Conversely, in the other equilibrium A-partisans only approve of A, and B-partisans double vote, producing the opposite outcome.

A numerical example is sufficient to prove that the Condorcet winner may lose almost surely in equilibrium:[18] let $n = 100$, $\bar{\omega}$ be known with probability 1, $\phi(A|\bar{\omega}) = 0.51$, $\phi(B|\bar{\omega}) = 0.19$ and $r(t_C) = 0.30$. Clearly, A is the Condorcet winner. Yet, there exists an equilibrium in which B is the only likely winner: if B-partisans vote B and A-partisans vote AB, a ballot for A is about 42,000 times less likely to be pivotal against either B or C than a vote for B to be pivotal against C.[19] It is thus a best response for A-voters to include B in their ballot. Thus, A-partisans prefer to double vote, for the (unlikely) case in which A may be among the top contenders. For the same reason, B-partisans prefer *not* to approve of A: they always want to be pivotal against A.

As our next result shows, however, this inferior equilibrium requires that the group of B-partisans is sufficiently large.

4.4 Approval voting in the presence of partisan voters

Together, Theorem 3 and Benchmark 3 show that the equilibrium properties of approval voting feature a discontinuity at the point in which the doubt is arbitrarily small as opposed to being absent. This would be a major issue if the results of Theorem 3 ceased to hold as soon as some arbitrarily small fraction of the electorate was partisan. We now study the case in which partisan and non-partisan voters coexist in the population.

This coexistence is obtained by setting $0 < \underline{\omega} < \bar{\omega} < 1$. In this case, a fraction $\phi(B|\bar{\omega}) > 0$ of the electorate prefers B to A with probability 1 and a fraction $\phi(A|\underline{\omega}) > 0$ of the electorate prefers A to B with probability 1. These are the partisan voters. The *largest fraction of partisans* is $\max[\phi(A|\underline{\omega}), \phi(B|\bar{\omega})]$. The other majority voters, who represent a fraction $1 - r(t_C) - (\phi(A|\underline{\omega}) + \phi(B|\bar{\omega}))$ of the population, have a doubt. Recall that $r(t_C)$ is the expected share of the minority.

The next theorem demonstrates when full information and coordination equivalence remains valid despite the presence of partisan voters:

Theorem 4 *Approval voting produces full information and coordination equivalence if and only if the largest fraction of partisans is no larger than* $1 - 2\sqrt{(1 - r(t_C))\, r(t_C)}$.

To understand the rationale for this result, it is useful to go back to the voters' incentives. By definition, partisan voters have no common value motive. They double vote only when their coordination motive dominates their information motive. That is, they double vote

[18] Nuñez (2010) develops a similar argument in greater detail.

[19] The probability ratio increases to 10^{46} with $n = 1000$ (Property 2 in Appendix A1).

when C is a serious enough threat and their most preferred candidate has too small a chance of winning the election. Imagine that, as in the example of Benchmark 3, candidate B is expected to win in all states of nature $\omega \in [\underline{\omega}, \bar{\omega}]$. Then, B-partisans have no reason to double vote: lending support to A would increase the odds in favour of A at the expense of B. Next, when do A-partisans and voters with a doubt prefer to double vote? Only when their probability of being pivotal in favor of A is too small compared to their probability of being pivotal between B and C. This requires that the B group is sufficiently large (which reduces the probability of being pivotal between A and B) *and* that the minority group is sufficiently large (which increases the probability of being pivotal between B and C).

The flipside of this condition is that the fraction of voters with a "doubt" must not be too small. The empirical question is thus whether it is large in reality. One strategy would be to examine swings in opinion polls. Anecdotically, such swing are often substantial: good economic conditions or major achievements by a head of state increase her support significantly. Bad economic conditions or her misbehavior reduce popular support. Arguably, such swings only represent a fraction of the voters with a "doubt" as defined here. Indeed, "doubt" only requires that a preference reversal *may* happen, even if with very low probability.

Finally, Theorem 4 does not imply that the full information Condorcet winner cannot win in equilibrium when the fraction of partisan voters is large. It shows when there *also* exists an equilibrium in which he is not sure to win. But under approval voting, there always exists an equilibrium that ensures the victory of the full information Condorcet winner. This contrasts with other electoral systems like plurality, for which such a good outcome cannot be guaranteed, and for which C may be the only likely winner in some equilibria (Bouton and Castanheira 2009, section 4.2).

5 Multinomial distribution

Until now, we assumed that the size of the population followed a Poisson distribution. To show that our results do not hinge upon this assumption, we analyze here the polar case in which the size of the population is known and fixed. A traditional way to analyze large voting games with fixed population size is to consider a multinomial distribution: the size of the population is fixed at n, and each voter is assigned a type $t \in \{t_A, t_B, t_C\}$ with probabilities $r(t_A|\omega)$, $r(t_B|\omega)$ and $r(t_C|\omega)$ respectively. For a strategy function σ, the expected fraction of voters playing ψ in state ω is still defined by (6). The probability of an action profile $x = \{x(A), x(B), x(C), x(AB), x(AC), x(BC)\}$ in state ω is given by:

$$\Pr(x|\omega) = n! \times \Pi_{\psi} \frac{\tau(\psi|\omega)^{x(\psi)}}{x(\psi)!}.$$

Myerson (2000, p24) shows that, for sufficiently large n, pivot probabilities under such Multinomial distributions are simply a monotone transformation of their Poisson equivalent. In particular, the magnitude of a pivot probability in a multinomial game is given by:

$$mag\left(piv_{PQ}|\omega, \text{multinomial}\right) = \log\left(1 + mag\left(piv_{PQ}|\omega, \text{Poisson}\right)\right). \tag{8}$$

It follows that the magnitude ratios behave exactly as in Poisson games. The only relevant difference between the multinomial and Poisson distributions is that, when the expected vote share of an alternative P is exactly zero, the magnitude of the pivot probabilities involving P are $-\infty$ with the multinomial distribution (instead of -1 in the Poisson distribution). That is, a vote in favour of P could *never* be pivotal in the Multinomial game. Instead, pivot probabilities are always strictly positive in Poisson games: the assumption of a Poisson distribution acts as a tremble when the expected vote share of an alternative is zero. It operates "as if" the probability that types t_A (t_B) vote for A (B) were bounded above zero. Theorem 3 extends directly to the case of a multinomial distribution if we introduce such a tremble.[20]

Formally, consider the setup of Section 4.2 in which there is doubt except for two differences:

Assumption 1 *An arbitrarily small fraction $\varepsilon \to 0$ of the electorate votes sincerely. That is, if $i \in [-1, 1]$, then with probability ε:*

$$\begin{aligned} i \text{ votes } A \text{ if } \mathrm{E}_\omega U_i\left(A|s\right) &> \mathrm{E}_\omega U_i\left(B|s\right), \text{ and} \\ i \text{ votes } B \text{ if } \mathrm{E}_\omega U_i\left(A|s\right) &< \mathrm{E}_\omega U_i\left(B|s\right). \end{aligned}$$

Assumption 2 *The distribution of voters follows a multinomial distribution instead of a Poisson distribution.*

In that case:

Theorem 5 *Under Assumptions 1 and 2, approval voting produces full information and coordination equivalence in equilibrium.*

Proof. First, note that, for $\varepsilon = 0$, there are strategies for which some pivot probabilities are exactly zero. For instance, if $\sigma_{i,s}(A) = 1$, $\forall i, s \in [0, 1] \times \{0, 1\}$, a vote can never be pivotal in favour of B:

[20] Interestingly, the Poisson assumption also refines away bizarre equilibria in which *all* voters vote for the same candidate (A or B or C). With multinomial distributions, all pivot probabilities are exactly zero for such strategy profiles.

$\Pr(piv_{BA}|\omega) = \Pr(piv_{BC}|\omega) = 0$, $\forall\omega$, which implies $G_i(B|s) = 0$ and $G_i(A|s) = G_i(AB|s), \forall i, s$. Therefore, all majority types are indifferent between actions $\psi = A$ and $\psi = AB$, and $\sigma_{i,s}(A) = 1$ is a (self-fulfilling) equilibrium. By symmetry, $\sigma_{i,s}(B) = 1$ $\forall s, i$ is also an equilibrium. Full information and coordination equivalence does not hold in that case.

By contrast, for any $\varepsilon \to 0$, all pivot probabilities are strictly positive for any σ:

$$\Pr(piv_{AB}|\omega), \Pr(piv_{BA}|\omega), \Pr(piv_{AC}|\omega), \Pr(piv_{BC}|\omega) > 0, \forall\omega.$$

Since, by (8), the ranking of magnitudes is the same under the Poisson and multinomial distributions, the proof of Theorem 3 then applies as such to the multinomial distribution. ■

Further, even in the absence of sincere voters ($\varepsilon = 0$), corner solutions would not be an equilibrium either if voters had a lexicographic preference for approving of their preferred candidate: if a vote can never be pivotal, say in favour of B, voters who *a priori* prefer B would vote AB instead of A. Theorem 3 thus extends to that case as well: since A and B have a strictly positive vote share, no pivot probability can be zero.

6 Conclusion

We analyzed a three-alternative election in which a majority of the electorate is divided between two alternatives, A and B, and a minority supports a single alternative, C. In contrast to standard analyses of electoral systems, we introduced slightly interdependent preferences in the electorate in the sense that additional information about the relative merits of different alternatives may affect the preferences of voters in a common direction. We showed that, when interdependencies are strong enough and the size of the electorate sufficiently large, approval voting ensures that the full information Condorcet winner is elected with a probability that approaches one. By contrast, classical electoral systems (plurality, runoff and a broad class of scoring rules) do not produce such a desirable result. This suggests that our understanding of coordination problems in elections is an artifact of the assumption that voters have fully independent preferences. Indeed, interdependencies in voters' preferences have typically been overlooked by voting theory. We therefore argue that one must consider such interdependencies when studying the properties of electoral systems in general.

A question that then springs to mind is why approval voting did not emerge in reality, through natural selection. Our results offer two rationales for this. First, multiple equilibria and coordination problems are still present if divisions are so deep that voters preferences cannot be altered by information (Benchmark 3). Second, as we just explained, the risk of coordination failures implies that opinion leaders, be they pundits, party leaders or lobbies, obtain the power to influence the outcome of the election. This may allow many actors to extract informational rents, that approval voting would dissipate.

Arguably, the results rely on elaborate calculations from voters but the trade-offs and strategies that emerge are quite natural. First, the equilibrium strategy proves extremely intuitive: voters only need to understand that a multiple ballot is valuable whenever a potentially good candidate is too weak or when a disliked candidate gets too strong. In the extended setup with a continuum of types and states of nature, the pattern of specialization that emerges is even more intuitive: voters who are closer to being indifferent between the two majority candidates double vote, and those most in favor of either candidate single-vote.

Second, these trade-offs should also be robust to several extensions not considered in the paper. Consider a world with more alternatives. If there are k alternatives in the majority (and k associated states of nature), and l alternatives in the minority, the trade-off remains identical: as long as their primary objective is to fight one another, both majority-block and minority-block voters "multiple-vote" for their own alternatives. Within the majority, voters can multiple-vote to maintain the right balance between their alternatives and make sure that each wins in its associated state of nature. Indeed, our results show that, whenever an alternative trails behind, *all* majority-block voters want to support her with a multiple ballot. Hence, although the analysis would become much more cumbersome given the number of deviations to consider, the main insights remain unchanged.[21] We can also think of a world in which C is not the worst alternative for majority-block voters: alternative A would still be the best in state a but would be the worst in state b, and vice versa for B. For that case, it is easy to prove that approval voting produces an equilibrium that satisfies full information and coordination equivalence: the strategy profile is exactly the same as in the initial setup.[22] Thus, despite different preferences, the full information Condorcet winner still ranks first and the alternative C still ranks last. Unfortunately, in such a case, the proof of uniqueness becomes intractable because ballots including alternative C are no longer strictly-dominated actions.

Finally, we considered a model in which alternatives/candidates are passive. A natural question for future research is to see how *candidates* behave when voters have interdependent preferences. This analysis would be worth pursuing not only for approval voting but also for other electoral systems such as plurality, runoff and the Borda count.

[21] Goertz and Maniquet (2011) may appear to contradict this statement, since they find a numerical example in which approval voting fails to aggregate information. Yet, that example crucially relies on a large fraction of common-valued voters having no doubt (they assign a probability zero to one state). This thus seems to be a manifestation of our Theorem 4.

[22] The proof is available upon request.

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Appendices

Appendix A1 provide a reminder of some fundamental properties of Poisson games (Myerson 2000 and 2002), and derive some specific properties for the case of Approval Voting. Appendices A2 and A3 contain the proofs of Theorems 1 and 2. Appendix A4 demonstrates the claims made in Section 4.

7 Appendix A1: Poisson games in approval voting

In a Poisson game, population size follows a Poisson distribution, of mean n. As shown by Myerson (2000), since types are attributed by iid draws, the number of voters of each type follows a Poisson distribution, of mean $r(t|\omega)\ n$, and the number of ψ-votes follows a Poisson distribution of mean $\tau(\psi|\omega,\sigma)\ n$. It follows that the probability of observing $x(\psi)$ voters playing action ψ is:

$$\Pr(x(\psi)|\omega,\sigma) = \exp(-\tau(\psi|\omega,\sigma)\ n)\ \tfrac{(\tau(\psi|\omega,\sigma)\ n)^{x(\psi)}}{x(\psi)!}. \tag{9}$$

For the sake of readability, we henceforth omit σ from the notation and simply write $\tau(\psi|\omega)$. From (9), the probability of the action profile $x \equiv \{x(\psi)\}_{\psi\in\Psi} \in \mathbb{N}^8$ is:

$$\Pr(x|\omega) = \prod_{\psi\in\Psi}\left(\exp(-\tau(\psi|\omega)\ n)\ \frac{(\tau(\psi|\omega)\ n)^k}{k!}\right). \tag{10}$$

An event $E \subset Z(\Psi)$ is a subset of the set of action profiles that satisfy given constraints. Myerson's (2000) *Magnitude Theorem* shows that:

Property 1 *For a large population of size n, the* magnitude *an event E in state ω is*

$$mag(E|\omega) \equiv \lim_{n\to\infty}\frac{\log[\Pr(E|\omega)]}{n} = \max_{x\in E}\sum_{\psi}\frac{x(\psi)}{n}\left(1-\log(\frac{x(\psi)}{n\tau(\psi|\omega)})\right)-1.$$

Thus, the probability that event E occurs in state ω is exponentially decreasing in n: from Myerson (2000, equation 3.1), the probability of event E can be approached by

$$\Pr(x|\omega) \simeq \frac{\exp(mag(E|\omega)\ n)}{\prod_{\psi\in\Psi}\sqrt{2\pi x(\psi)+\pi/3}}. \tag{11}$$

The absolute value of $mag(E|\omega) \in [-1,0]$ represents the "speed" at which the probability decreases towards 0: the more negative the magnitude, the faster the probability converges to 0 as n increases. It follows that (Myerson 2000, Corollary 1):

Property 2 *If two events E and E' have different magnitudes: $mag(E|\omega) < mag(E'|\omega')$, $\omega,\omega' \in \{a,b\}$, the probability ratio of the two events is approximately* $\exp[(mag(E|\omega) - mag(E'|\omega'))\cdot n]$, *which converges to zero as n increases:*

$$mag(E|\omega) < mag(E'|\omega') \Longrightarrow \frac{\Pr(E|\omega)}{\Pr(E'|\omega')} \underset{n\to\infty}{\longrightarrow} 0.$$

That is, unless two events have the same magnitude, their likelihood ratio necessarily converges either to zero or to infinity when electorate size increases.

Our focus will be on *pivotability events*, *i.e.* when one additional ballot ψ can change the outcome of the election. Under Approval Voting, this happens if increasing $x(\psi)$ to $x(\psi)+1$ modifies which alternative has the largest number of votes. For instance, if $X(A) = X(B) > X(C)$, both A and B win the election with probability $\frac{1}{2}$. An additional ballot A (or AC) changes the outcome to $X(A) > \max[X(B), X(C)]$, in which case A wins with probability 1. By contrast, if $X(C) - 1 > X(A) = X(B)$ an additional ballot A (or AC) cannot affect the outcome of the election: it is not pivotal. Let piv_{PQ} denote the event of being *pivotal between alternatives* P *and* Q: a single ballot ψ induces the victory of P instead of Q.

The magnitude of the pivot event piv_{PQ} is determined by the probability that two events realize jointly: (i) P either has the same number of votes as Q or one vote less, and (ii) the third alternative, R, does not have more votes than Q. The magnitude of event (i) is identified by Property 3, which is proven in Myerson (2002, pp231-2), exploiting the *Dual Magnitude Theorem*:

Property 3 *The magnitude of the event that alternatives* P *and* Q *have approximately the same number of votes is:*

$$mag(P, Q|\omega) \equiv -\left(\sqrt{\tau(P|\omega) + \tau(PR|\omega)} - \sqrt{\tau(Q|\omega) + \tau(QR|\omega)}\right)^2,$$

with $P \neq Q \neq R \in \{A, B, C\}$.

Lemma 1 identifies the ranking of pivot probabilities (the joint event (i) and (ii)) in three-candidate elections under Approval Voting:

Lemma 1 *For any triplet of alternatives* P, Q, $R \in \{A, B, C\}$, $P \neq Q \neq R$, *let:*

$$mag(piv_{PQ}|\omega) \equiv \lim_{n\to\infty} \frac{1}{n} \log\left[\Pr\left(X(Q) - X(P) \in \{0,1\} \ \&\ X(Q) \geq X(R)|\omega\right)\right].$$

Then:

$$\begin{aligned} mag(piv_{PQ}|\omega) &= mag(P, Q|\omega) \text{ if } P \text{ and } Q \text{ have the largest two expected vote shares,} \\ &< mag(P, Q|\omega) \text{ if } P \text{ and } Q \text{ have the lowest two expected vote shares.} \end{aligned}$$

It follows that, if $\mathsf{E}[X(P)|\omega] > \mathsf{E}[X(Q)|\omega] > \mathsf{E}[X(R)|\omega]$, *then:*

$$\lim_{n\to\infty} \Pr(piv_{QR}|\omega) / \Pr(piv_{PQ}|\omega) = 0.$$

Proof. We use the Magnitude Theorem of Myerson (2000) and focus first on the case in which $\mathsf{E}[X(A)|\omega] > \mathsf{E}[X(C)|\omega] > \mathsf{E}[X(B)|\omega]$ (the case $\mathsf{E}[X(A)|\omega] < \mathsf{E}[X(C)|\omega] < \mathsf{E}[X(B)|\omega]$ is symmetric).

The magnitude of the pivot probability between A and C (the other pivot events are computed in the same way) is defined as:

$$\begin{aligned} mag(piv_{AC}|\omega) &= \max_x \sum_\psi \frac{x(\psi)}{n}\left(1-\log\frac{x(\psi)}{n\tau(\psi|\omega)}\right)-1 \qquad (12)\\ & \quad s.t.\ x(A)+x(AB)=x(C)+x(BC) \text{ and } x(C)+x(BC)\geq x(B)+x(AB). \end{aligned}$$

If we abstract from the second constraint $x(C)+x(BC)\geq x(B)+x(AB)$, or if this constraint is not binding, (12) is actually defined by $mag(A,C|\omega)$ in Property 3 above. We refer to this as the *unrestricted* magnitude (denoted by *).

Conversely, if the second constraint is binding at the optimum, the magnitude is maximized in $x(A)+x(AB)+x(AC) = x(C)+x(AC)+x(BC) = x(B)+x(AB)+x(BC)$, where the first equality is the event that $x(A)+x(AB)=x(C)+x(BC)$, and the second equality is the second constraint made binding. We refer to the magnitude of this binding event as the *restricted* magnitude (denoted by **):

$$mag\left(piv^{**}_{AC}|\omega\right)=mag\left(piv^{**}_{BC}|\omega\right)=mag\left(piv^{**}_{AB}|\omega\right),$$

which, by definition, are smaller than the lowest unrestricted magnitude ($mag\left(piv^*_{PQ}|\omega\right)$):

$$mag\left(piv^{**}_{AC}|\omega\right)\leq \min_{P,Q\in\{A,B,C\}} mag\left(piv^*_{PQ}|\omega\right).$$

Having observed this, we are now in a position to prove that, if the expected ranking is $\mathsf{E}[X(A)|\omega]>\mathsf{E}[X(C)|\omega]>\mathsf{E}[X(B)|\omega]$, then:

$$\begin{aligned} mag\left(piv_{AC}|\omega\right) &= mag\left(piv^*_{AC}|\omega\right) \text{ and}\\ mag(piv_{BC}|\omega) &= mag\left(piv^{**}_{BC}|\omega\right). \end{aligned}$$

Now, we prove that $mag\left(piv_{AC}|\omega\right)$ is unrestricted. This is true if $x(A)+x(AB)=x(C)+x(BC)$ implies $x(A)+x(AB)>x(B)+x(AB)$ at the optimum. By Lemma 2, we know that $\tau(AC|\omega)=\tau(BC|\omega)=0$. This implies $x(AC)=x(BC)=0$. If we denote $x(A)+x(AB)=x=x(C)$, $x(A)=\alpha x$, $x(AB)=(1-\alpha)x$, we find that (12) is maximized in:

$$\begin{aligned} \alpha^*_{AC} &= \frac{\tau(A|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}, \qquad (13)\\ x^*_{AC} &= n\sqrt{[\tau(C|\omega)][\tau(A|\omega)+\tau(AB|\omega)]},\\ x(B)^*_{AC} &= n\tau(B|\omega). \end{aligned}$$

It remain to check that these values imply $\alpha^*_{AC}x^*_{AC}>x(B)^*_{AC}$. Straightforward manipulations show that the latter inequality holds iff:

$$\sqrt{\frac{\tau(C|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}}>\frac{\tau(B|\omega)}{\tau(A|\omega)}, \qquad (14)$$

in which both sides are smaller than one. Hence, $\sqrt{\frac{\tau(C|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}}>\frac{\tau(C|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}$. By the assumed expected ranking $\mathsf{E}[X(A)|\omega]>\mathsf{E}[X(C)|\omega]>\mathsf{E}[X(B)|\omega]$, we have $\tau(A|\omega)+\tau(AB|\omega)>\tau(C|\omega)>$

$\tau(B|\omega)+\tau(AB|\omega)$, which in turn implies $\frac{\tau(C|\omega)}{\tau(A|\omega)+\tau(AB|\omega)} > \frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)} > \frac{\tau(B|\omega)}{\tau(A|\omega)}$, which proves that (14) holds and that $mag(piv_{AC}|\omega)$ is always unrestricted.

Second, we need to prove that $mag(piv_{BC}|\omega)$ is restricted. This is true if $x(B)+x(AB) = x(C)+x(AC)$ implies $x(A)+x(AB) > x(B)+x(AB)$ at the optimum, that is:

$$\alpha^*_{BC} x^*_{BC} < x(A)^*_{BC}\,, \tag{15}$$

where

$$\begin{aligned}
\alpha^*_{BC} &= \frac{\tau(B|\omega)}{\tau(B|\omega)+\tau(AB|\omega)},\\
x^*_{BC} &= n\sqrt{[\tau(C|\omega)+\tau(AC|\omega)][\tau(B|\omega)+\tau(AB|\omega)]},\\
x(A)^*_{BC} &= n\tau(A|\omega).
\end{aligned}$$

(the derivation of these critical values α^*_{BC}, x^*_{BC}, and $x(A)^*_{BC}$ is similar to that of α^*_{AC}, x^*_{AC}, and $x(B)^*_{AC}$ in (13)). To show that (15) holds, we proceed as with (14) and must show that:

$$\sqrt{\frac{\tau(C|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}} < \frac{\tau(A|\omega)}{\tau(B|\omega)},$$

in which both fractions are larger than one. This implies: $\sqrt{\frac{\tau(C|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}} < \frac{\tau(C|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}$. By the expected ranking, the latter is strictly smaller than $\frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}$, which is itself smaller than $\frac{\tau(A|\omega)}{\tau(B|\omega)}$, which proves that $mag(piv_{BC}|\omega)$ is always restricted and completes the proof.

The proof of the other possible expected rankings, namely $\mathsf{E}[X(A)|\omega] > \mathsf{E}[X(B)|\omega] > \mathsf{E}[X(C)|\omega]$ (the case $\mathsf{E}[X(B)|\omega] > \mathsf{E}[X(A)|\omega] > \mathsf{E}[X(C)|\omega]$ is symmetric) and $\mathsf{E}[X(C)|\omega] > \mathsf{E}[X(A)|\omega] > \mathsf{E}[X(B)|\omega]$ (the case $\mathsf{E}[X(C)|\omega] > \mathsf{E}[X(B)|\omega] > \mathsf{E}[X(A)|\omega]$ is symmetric), proceeds in the same manner. ■

Remark 1 *The correlation introduced by double voting implies that the largest magnitude need not be between the top two alternatives: for a given difference in expected vote shares, if two alternatives are more correlated through double voting, the probability of being pivotal between them is reduced. Thus, the largest magnitude can be the one between the alternatives that rank first and third in terms of expected vote shares.*

Finally, the following Property proves useful to compare two pivot probabilities with the same magnitude.

Property 4 *(Myerson 2000, Theorem 2) The probability that two actions, ψ and ψ' receive a number of votes that differs by a constant c ($c << n$) in state of the nature $\omega \in \{a, b\}$, is:*

$$\lim_{n\to\infty} \frac{\Pr\left(x(\psi) = x(\psi') + c|\omega\right)}{\Pr\left(x(\psi) = x(\psi')|\omega\right)} = \left(\frac{\tau(\psi|\omega)}{\tau(\psi'|\omega)}\right)^{c/2}.$$

8 Appendix A2: Proof of Theorem 1

This Appendix proves Theorem 1. Each step of the proof is presented in a different subsection. In the first, we identify the voters' expected payoffs and show that only three actions in Ψ are undominated. Then, Lemmas 3 and 4 show that in equilibrium the outcome of the election must reveal information about the actual state of nature. Third, Lemmas 5 and 6 show that voters specialize: types t_A mix between A and AB, whereas types t_B mix between B and AB. Fourth, Lemma 7 shows that A must have the largest expected vote share in state a, and similarly for B in state b. Finally, Lemma 8 shows that, for n large, the equilibrium is unique.

8.1 Payoffs and dominated strategies

Lemma 2 *For a majority-block voter $t \in \{t_A, t_B\}$, in equilibrium:*

$$\sigma_t(A) + \sigma_t(B) + \sigma_t(AB) = 1. \tag{16}$$

The other actions $\psi \in \{C, AC, BC, ABC, \varnothing\}$ are strictly dominated.

Proof. The expected utility of a majority voter $t \in \{t_A, t_B\}$ is:

$$\mathsf{E}U(t) = q(a|t)\ \left[\Pr(A \text{ wins}|a) - \Pr(C \text{ wins}|a)\right] + q(b|t)\ \left[\Pr(B \text{ wins}|b) - \Pr(C \text{ wins}|b)\right].$$

Compare actions AB and ABC: while the latter can never be pivotal, an AB-ballot can be pivotal against C, either in favor of A or in favor of B. Both events increase expected utility. Hence, AB strictly dominates ABC. All other strict dominance relationships are obtained by performing similar two-by-two comparisons: AB strictly dominates ABC, $\varnothing$ and C; A strictly dominates AC; and B strictly dominates BC. ■

The value of each undominated action is given by $G(\psi|t)$, the *expected gain* of action $\psi \in \{A, B, AB\}$ over abstention, $\varnothing$:

$$\begin{aligned} G(A|t) &= q(a|t)\ \left[\Pr(piv_{AB}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) - \Pr(piv_{AB}|b)\right], \end{aligned} \tag{17}$$

$$\begin{aligned} G(B|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) - \Pr(piv_{BA}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{BA}|b) + 2\Pr(piv_{BC}|b)\right], \end{aligned} \tag{18}$$

$$\begin{aligned} \text{and } G(AB|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) + 2\Pr(piv_{BC}|b)\right]. \end{aligned} \tag{19}$$

These gains depend on the voter's type only through $q(\omega|t)$, and on the state of nature through the pivot probabilities $\Pr(piv_{PQ}|\omega)$. These pivot probabilities depend on the strategy, σ, but we omit σ from the notation for the sake of readability.

8.2 No informational trap under approval voting

Now we show that the election result under approval voting must be expected to produce information about the state of nature. The only case in which the outcome of the election cannot generate information is when:

Definition 4 *A strategy σ^{IT} produces an **Informational Trap** if the expected result of the election is independent of the state of nature:*

$$\mathsf{E}\left(X\left(P\right)|a,\sigma^{IT}\right)=\mathsf{E}\left(X\left(P\right)|b,\sigma^{IT}\right),\ \forall P\in\left\{A,B,C\right\}.$$

Lemma 3 shows that only one undominated strategy could produce an informational trap. Lemma 4 then shows that this candidate strategy cannot be an equilibrium in a large population game.

Lemma 3 *If a strategy σ^{IT} produces an informational trap, then:*
(i) All majority voters must adopt the same strategy;
(ii) Given σ^{IT}, it is never a best response for a type t_A (resp. t_B) to play B (resp. A);
(iii) The only undominated strategy satisfying (i) *and* (ii) *is thus: $\sigma_{t_A}\left(AB\right)=1=\sigma_{t_B}\left(AB\right)$.*

Proof. $\sigma^{IT}_{t_A}=\sigma^{IT}_{t_B}$ follows immediately from (6). For such a strategy, (10) in Appendix A1 implies that $\Pr\left(piv_{PQ}|a\right)=\Pr\left(piv_{PQ}|b\right)$ for all $P,Q=A,B,C$. Introducing this in (17)-(19) directly shows that $G\left(AB|t_A\right)>G\left(B|t_A\right)$ and $G\left(AB|t_B\right)>G\left(A|t_B\right)$ for any strategy $\sigma^{IT}_{t_A}=\sigma^{IT}_{t_B}$. This in turn implies that $\sigma_{t_B}\left(A\right)=0=\sigma_{t_A}\left(B\right)$, which by (6) implies that $\sigma_{t_A}\left(AB\right)=1=\sigma_{t_B}\left(AB\right)$ is the only strategy profile in undominated strategies that produces an informational trap. ■

The intuition is that, when voters do not expect the election to elicit additional information, they want to play a "sincere strategy" of approving of their *a priori* preferred alternative (part *ii*). Thus, the only action that can be used both by types t_A and t_B is AB (part *iii*). In that case, A and B tie necessarily.

However, such a strategy cannot be an equilibrium when the electorate size, n, is sufficiently large:

Lemma 4 *There exists $\bar{n}$ such that, for any $n\geq\bar{n}$, there is* no *equilibrium with an informational trap.*

Proof. By Lemma 3, the only strategy function that may produce an informational trap is $\sigma_{t_A}\left(AB\right)=1=\sigma_{t_B}\left(AB\right)$, which would imply that $X\left(A\right)=X\left(B\right)$ for any realized number of voters. Hence, $\Pr\left(piv_{AB}|X\left(A\right)>X\left(C\right)\right)=1/2$. Since $r\left(t_C|\omega\right)<1/2$, we have: $\lim_{n\to\infty}\Pr\left(X\left(A\right)>X\left(C\right)\right)=1$. Hence, by (10) and (11) in Appendix A1, we have:

$$\begin{array}{ll}(i) & \Pr\left(piv_{AB}|\omega,n\right)<1/2 \text{ and } \lim\limits_{n\to\infty}\Pr\left(piv_{AB}|\omega,n\right)=1/2,\\ (ii) & \Pr\left(piv_{PC}|\omega,n\right)>0 \text{ and } \lim\limits_{n\to\infty}\Pr\left(piv_{PC}|\omega\right)=0 \text{ for } P\in\left\{A,B\right\}.\end{array}$$

Since these pivot probabilities are monotonic in n, there must exist some $\bar{n}$ such that

$$\max\left[G\left(A|t,n\geq\bar{n}\right),G\left(B|t,n\geq\bar{n}\right)\right]>G\left(AB|t,n\geq\bar{n}\right),\ \forall t\in\{t_A,t_B\}$$

■

The reason for this "no informational trap" result is that, when all majority voters double vote, $X(A)$ and $X(B)$ are necessarily equal. As population size increases, the probability that A and B tie for first place thus converges to 1, whereas the probability of being pivotal against C decreases to 0. Hence, all majority voters develop an incentive to deviate from σ^{IT} and single-vote for their *a priori* preferred alternative.

8.2.1 Voters specialize

The next lemma shows that no majority voter would ever mix between actions A and B. Then, we show that types t_A and t_B respectively mix between actions A and AB, and B and AB.

Lemma 5 *In equilibrium,* $\sigma_t(A)\times\sigma_t(B)=0$, $\forall t$.

Proof. A necessary condition for a type $t\in\{t_A,t_B\}$ to play both A and B with strictly positive probability is that:

$$G(A|t)=G(B|t)\geq G\left(AB|t\right). \tag{20}$$

From (17), (18) and (19), it is easy to check that:

$$G(A|t)\geq G\left(AB|t\right)\Longleftrightarrow\ \tfrac{q(b|t)}{q(a|t)}\leq\frac{1}{M_1}\equiv\tfrac{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)} \tag{21}$$

$$G(B|t)\geq G\left(AB|t\right)\Longleftrightarrow\ \tfrac{q(a|t)}{q(b|t)}\leq M_2\equiv\tfrac{\Pr(piv_{BA}|b)-\Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a)+2\Pr(piv_{AC}|a)}. \tag{22}$$

Hence, $G(A|t),G(B|t)\geq G\left(AB|t\right)$ require $\Pr\left(piv_{AB}|a\right)>\Pr\left(piv_{BC}|a\right)$ and $\Pr\left(piv_{BA}|b\right)>\Pr\left(piv_{AC}|b\right)$.

Using (17) and (18), a necessary condition for $G(A|t)=G(B|t)$ is:

$$\frac{q\left(a|t\right)}{q\left(b|t\right)}=\frac{\Pr\left(piv_{BA}|b\right)-\Pr\left(piv_{AC}|b\right)+\Pr\left(piv_{AB}|b\right)+2\Pr\left(piv_{BC}|b\right)}{\Pr\left(piv_{AB}|a\right)-\Pr\left(piv_{BC}|a\right)+\Pr\left(piv_{BA}|a\right)+2\Pr\left(piv_{AC}|a\right)}. \tag{23}$$

Now, we prove that (20) can never hold. Indeed, condition (20) requires:

$$M_1\leq M_2, \tag{24}$$

but we can identify a lower bound for M_1 and an upper bound for M_2 and show that the former is strictly larger than the latter, hence a contradiction.

M_1 is strictly increasing in $\Pr\left(piv_{BC}|a\right)$ and $\Pr\left(piv_{BC}|b\right)$. A lower bound to M_1 is thus found by setting these two pivot probabilities equal to 0. Similarly, an upper bound to M_2 is found by setting $\Pr\left(piv_{AC}|a\right)$ and $\Pr\left(piv_{AC}|b\right)$ equal to zero. This establishes that:

$$\frac{\Pr\left(piv_{AB}|b\right)}{\Pr\left(piv_{AB}|a\right)}<M_1\text{ and }M_2<\frac{\Pr\left(piv_{BA}|b\right)}{\Pr\left(piv_{BA}|a\right)}, \tag{25}$$

and hence that a necessary condition for (24) is that:

$$\frac{\Pr\left(piv_{AB}|b\right)}{\Pr\left(piv_{BA}|b\right)}\frac{\Pr\left(piv_{BA}|a\right)}{\Pr\left(piv_{AB}|a\right)} < 1.$$

Using Property 4 (in Appendix A1), the left-hand side of this expression is equal to:

$$\sqrt{\frac{\tau\left(A|a\right)}{\tau\left(A|b\right)}\frac{\tau\left(B|b\right)}{\tau\left(B|a\right)}},$$

which cannot be smaller than 1. Indeed, by (23), types t_A must vote for A with a higher probability than types t_B, since $\frac{q(a|t_A)}{q(b|t_A)} > \frac{q(a|t_B)}{q(b|t_B)}$. Hence, in any equilibrium:

$$\frac{\tau\left(A|a\right)}{\tau\left(A|b\right)} \geq 1 \text{ and } \frac{\tau\left(B|b\right)}{\tau\left(B|a\right)} \geq 1. \tag{26}$$

It follows that $G(A|t) = G(B|t)$ implies $G\left(AB|t\right) > G(A|t)$, and therefore that $\sigma_t\left(A\right) > 0$ implies $\sigma_t\left(B\right) = 0$ and conversely. ■

The intuition relates to the swing voter's curse (Feddersen and Pesendorfer 1996; Myerson 1998a): in a setup with two candidates and common valued voters (*i.e.* our setup but without candidate C), voters avoid mixing between actions A and B because they fear being "mistakenly pivotal," for instance in favor of B against A when the actual state is a. In our three-candidate setup, voters avoid mixing between actions A and B for the same reason but, because of C, do not either want to abstain. Approval voting allows them to play action AB. This allows them to abstain between A and B, while maximizing their probability of being pivotal against C.

Lemma 6 *There exists $\bar{n}$, such that for any $n \geq \bar{n}$, in equilibrium, $\sigma_{t_A}\left(A\right) > 0$ and $\sigma_{t_B}\left(B\right) > 0$. Thus: $\sigma_{t_A}\left(A\right) + \sigma_{t_A}\left(AB\right) = 1$ and $\sigma_{t_B}\left(B\right) + \sigma_{t_B}\left(AB\right) = 1$.*

Proof. We need to show that $\sigma_{t_A}\left(A\right)$ and $\sigma_{t_B}\left(B\right)$ are strictly positive in equilibrium. To this end, we show that:

$$\sigma_{t_B}\left(B\right) > 0 \text{ and } \sigma_{t_A}\left(A\right) = 0 \tag{27}$$

leads to a contradiction. Indeed, (27) implies $\tau\left(A|\omega\right) = 0$ in both states. Hence, by Property 3 (in Appendix A1):

$$mag(A, B) = -\tau\left(B|\omega\right).$$

By (26), we have: $\tau\left(B|a\right) < \tau\left(B|b\right)$. One can check that whether magnitudes are restricted or not, this implies that $\lim_{n\to\infty} \Pr\left(piv_{BA}|b\right) / \Pr\left(piv_{BA}|a\right) = 0$ and therefore that $\lim_{n\to\infty} M_2 = 0$ in the proof of Lemma 5. Instead, $\sigma_{t_B}\left(B\right) > 0$ imposes that M_2 be strictly positive, which shows that $\sigma_{t_A}\left(A\right) = 0$ contradicts the possibility that $\sigma_{t_B}\left(B\right) > 0$. By symmetry, we cannot either have: $\sigma_{t_A}\left(A\right) > 0$ and $\sigma_{t_B}\left(B\right) = 0$.

Together with Lemmas 4 and 5, this proves that, in equilibrium, we must have $\sigma_{t_A}\left(A\right) > 0$ and $\sigma_{t_B}\left(B\right) > 0$. ■

Lemma 6 implies that majority-type voters always include their *a priori* preferred alternative in their ballot.[23] Hence, and perhaps surprisingly, since types t_A always include A in their ballot, their strategy only influences the vote share of B: the more types t_A double vote, the higher the expected vote share of B, whereas the expected vote share of A is left unchanged. Similarly, it is types t_B's strategy that determines the expected vote share of A.

8.2.2 Electing the Full Information Condorcet winner

Here, we show that the full information Condorcet winner must be the only likely winner:

Lemma 7 *Any equilibrium strategy must produce the following ranking of expected vote shares:*

$$\begin{aligned}\tau(A|a)+\tau(AB|a) &> \tau(B|a)+\tau(AB|a) \geq \tau(C), \text{ and} \\ \tau(B|b)+\tau(AB|b) &> \tau(A|b)+\tau(AB|b) \geq \tau(C).\end{aligned}$$

Proof. We show that any strategy satisfying Lemma 6 must produce a specific ranking of pivot probabilities which can only be satisfied if the full information Condorcet winner is the only likely winner.

From Lemma 6, types t_A and t_B must single-vote with positive probability in equilibrium. This implies that the differences $G(A|t_A)-G(AB|t_A)$ and $G(B|t_B)-G(AB|t_B)$ must be non-negative. Formally, using $(17)-(19)$, the voters' relevant comparison of payoffs is the following:

$$\begin{aligned}G(A|t_A)-G(AB|t_A) &= q(a|t_A)\ \left[\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)\right] \\ &\quad -q(b|t_A)\ \left[2\Pr(piv_{BC}|b)+\Pr(piv_{AB}|b)\right] \gtrless 0,\end{aligned} \tag{28}$$

$$\begin{aligned}G(B|t_B)-G(AB|t_B) &= q(b|t_B)\ \left[\Pr(piv_{BA}|b)-\Pr(piv_{AC}|b)\right] \\ &\quad -q(a|t_B)\ \left[2\Pr(piv_{AC}|a)+\Pr(piv_{BA}|a)\right] \gtrless 0.\end{aligned} \tag{29}$$

By (28), a necessary condition to have $G(A|t_A)-G(AB|t_A)\geq 0$ is that $\Pr(piv_{AB}|a)$ be sufficiently large compared to the other pivot probabilities in (28). Similarly, $\Pr(piv_{BA}|b)$ must be sufficiently large compared to the other pivot probabilities in (29). That is, from Property 2 (in Appendix A1):

$$\begin{aligned}mag(piv_{AB}|a) &\geq \max\{mag(piv_{AB}|b), mag(piv_{BC}|a), mag(piv_{BC}|b)\}, \\ mag(piv_{BA}|b) &\geq \max\{mag(piv_{BA}|a), mag(piv_{AC}|a), mag(piv_{AC}|b)\}.\end{aligned} \tag{30}$$

For (30) to be satisfied, $mag(piv_{AB}|a)$ and $mag(piv_{BA}|b)$ must be equal. By Property 3 and Lemma 1 (in Appendix A1), a necessary condition is that:

$$\begin{aligned}&\left(\sqrt{r(t_A|a)\cdot\sigma_{t_A}(A)}-\sqrt{r(t_B|a)\cdot\sigma_{t_B}(B)}\right)^2 = \\ &\qquad\left(\sqrt{r(t_A|b)\cdot\sigma_{t_A}(A)}-\sqrt{r(t_B|b)\cdot\sigma_{t_B}(B)}\right)^2.\end{aligned} \tag{31}$$

[23] In a private value setup, Brams and Fishburn (2007, Theorem 2.1) show that a voter always includes her most preferred alternative in her ballot. Lemma 6 thus extends their Theorem to voters with state-dependent preferences.

This condition depends on $\sigma_{t_A}(A)$ and $\sigma_{t_B}(B)$. Defining $\rho \equiv \sigma_{t_A}(A)/\sigma_{t_B}(B)$,one readily sees that condition (31) is satisfied iff:

$$\left|\sqrt{r(t_A|a)\cdot\rho} - \sqrt{r(t_B|a)}\right| = \left|\sqrt{r(t_B|b)} - \sqrt{r(t_A|b)\cdot\rho}\right|,$$

which has a unique solution in $\mathbb{R}^+$:

$$\rho^* = \left(\frac{\sqrt{r(t_B|a)} + \sqrt{r(t_B|b)}}{\sqrt{r(t_A|a)} + \sqrt{r(t_A|b)}}\right)^2. \tag{32}$$

This solution in turn implies: $\tau(A|a) > \tau(B|a)$ and $\tau(A|b) < \tau(B|b)$. By Lemma 1 and since there is no double voting involving alternative C, condition (30) cannot be satisfied if $\tau(C) > \tau(B|a) + \tau(AB|a)$ and/or $\tau(C) > \tau(A|b) + \tau(AB|b)$. Therefore, in equilibrium:

$$\begin{aligned} \tau(A|a) &> \tau(B|a) \geq \tau(C) - \tau(AB|a)\text{, and} \\ \tau(B|b) &> \tau(A|b) \geq \tau(C) - \tau(AB|b). \end{aligned}$$

■

The proof builds on the fact that types t_A and t_B must single-vote with positive probability in equilibrium (Lemma 6). Both types of majority voters are willing to single vote if neither the *coordination motive* nor the *common-value motive* dominates the *information motive*. Otherwise, majority voters would double vote, either to support a trailing majority candidate (common-value motive) or to fight C (coordination motive). Therefore, in equilibrium, no majority candidate can trail in this way, i.e. A (resp. B) must have the largest expected vote share in a (resp. b) and in both states of nature, C's vote share must be the lowest of all three.

In proving Lemma 7, we also found that a necessary condition for the information motive to be stronger than both the coordination and common-value motives is that $\sigma_{t_A}(A) = \sigma_{t_B}(B)\rho^*$ (see 32). This condition ensures that the pivot probabilities between A and B have the same magnitude in the two states of nature. It now remains to prove that the equilibrium strategy is unique.

8.2.3 Equilibrium uniqueness

To prove uniqueness, we focus on the case in which $\rho^* \leq 1$ (the complementary case amounts to switching labels A and B). We show that $\sigma_{t_A}(A) = \rho^*\sigma_{t_B}(B)$ is the unique best response of types t_A given a strategy $\sigma_{t_B}(B)$ and that there is a unique value of $\sigma_{t_B}(B)$ that can be chosen in equilibrium:

Lemma 8 *There exists an expected population size $\bar{n}$, such that for any $n \geq \bar{n}$, the equilibrium strategy is unique and such that, in the limit $n \to \infty$:*
i) $\sigma_{t_B}(B) = 1$, $\sigma_{t_A}(A) = \rho^*$ *iff, for this strategy,*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) \geq \max_{\omega}\left\{mag\left(piv_{AC}|\omega\right), mag\left(piv_{BC}|\omega\right)\right\}.$$

ii) *Otherwise,* $\sigma_{t_B}(B) = \bar{\sigma}$, $\sigma_{t_A}(A) = \rho\bar{\sigma}$ *with* $\bar{\sigma} \in (0,1)$ *such that:*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) = \max_{\omega}\left\{mag\left(piv_{AC}|\omega\right), mag\left(piv_{BC}|\omega\right)\right\}. \tag{33}$$

Proof. We proceed in two steps: first, we show that $\sigma_{t_A}(A) = \rho^*\sigma_{t_B}(B)$ is the unique best response of types t_A given the strategy of types t_B. Second, we prove that there is a unique equilibrium strategy $\sigma^*_{t_B}(B)$.

From (30) and (32), we must have in equilibrium:

$$\begin{aligned} mag\left(piv_{AB}|a\right) = mag\left(piv_{AB}|b\right) \geq \quad & \max\{mag\left(piv_{BC}|a\right), mag\left(piv_{BC}|b\right), \\ & mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right)\}. \end{aligned} \tag{34}$$

We can check that types t_A never want to deviate from $\sigma_{t_A}(A) = \rho^*\sigma_{t_B}(B)$: for any $\sigma_{t_A}(A) < \rho^*\sigma_{t_B}(B)$, the expected share of alternative B increases in both states. Hence, $mag\left(piv_{AB}|a\right)$ increases above $mag\left(piv_{AB}|b\right)$, whereas $mag\left(piv_{BC}|a\right)$ and $mag\left(piv_{BC}|b\right)$ decrease. This implies (see the proof of Lemma 5):

$$\frac{q\left(b|t_A\right)}{q\left(a|t_A\right)} < \lim_{n\to\infty}\frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)} = \infty,$$

and hence: $G(A|t_A) > G\left(AB|t_A\right)$. Therefore, $\sigma_{t_A}(A) < \rho^*\sigma_{t_B}(B)$ cannot be true in equilibrium.

For any $\rho^*\sigma_{t_B}(B) < 1$, we also have to check that $\sigma_{t_A}(A) > \rho^*\sigma_{t_B}(B)$ cannot be an equilibrium either. Following the same procedure as above, one can check that $\sigma_{t_A}(A) > \rho^*\sigma_{t_B}(B)$ implies:

$$\frac{q\left(b|t_A\right)}{q\left(a|t_A\right)} > \lim_{n\to\infty}\frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)} \leq 0,$$

which in turn implies $G(A|t) < G\left(AB|t\right)$. Hence, $\sigma_{t_A}(A) > \rho^*\sigma_{t_B}(B)$ cannot be true in equilibrium. Therefore, when (34) holds, $\sigma^*_{t_A}(A) = \rho^*\sigma_{t_B}(B)$ is the unique best response of types t_A to $\sigma_{t_B}(B)$.

It remains to prove that $\sigma^*_{t_B}(B)$ is unique. Two cases must be considered:

<u>Case 1: $G(B|t_B) - G(AB|t_B) \geq 0$ in $\sigma_{t_B}(B) = 1$, $\sigma_{t_A}(A) = \rho^*$.</u>

In that case, $\sigma_{t_B}(B) = 1$ is the only possible best response for types t_B. Indeed, $\sigma_{t_B}(B) < 1$ would imply $\sigma_{t_B}(AB) > 0$. This increases the expected vote share of A in both states of nature. Hence $mag\left(piv_{BA}|b\right)$ increases above $mag\left(piv_{BA}|a\right)$, whereas $mag\left(piv_{AC}|a\right)$ and $mag\left(piv_{AC}|b\right)$ decrease. Using M_2 (see proof of Lemma 5), this implies:

$$\frac{q\left(a|t_B\right)}{q\left(b|t_B\right)} < \lim_{n\to\infty} M_2 \equiv \frac{\Pr(piv_{BA}|b)-\Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a)+2\Pr(piv_{AC}|a)} = \infty,$$

and hence $G(B|t_B) > G\left(AB|t_B\right)$. Therefore, $\sigma_{t_B}(B) = 1$ is the unique best response to $\sigma_{t_A}(A) = \rho^*$.

It remains to show that types t_B would deviate from any $\{\sigma_{t_A}(A), \sigma_{t_B}(B)\} = \{\rho^*\sigma, \sigma\}$ if $\sigma < 1$. To this end, we need to show that

$$\lim_{n\to\infty}\frac{G(B|t_B) - G(AB|t_B)}{\Pr(piv_{AB}|a)} = q(b|t_B)\frac{\Pr(piv_{BA}|b)}{\Pr(piv_{AB}|a)} - q(a|t_B)\frac{\Pr(piv_{BA}|a)}{\Pr(piv_{AB}|a)} > 0, \tag{35}$$

for any $\{\sigma_{t_A}(A), \sigma_{t_B}(B)\} = \{\rho^*\sigma, \sigma\}$, $\sigma < 1$.

The strategy of the types t_A implies:

$$\begin{aligned}\lim_{n\to\infty}\frac{G(A|t_A)-G(AB|t_A)}{\Pr(piv_{AB}|a)} &= q(a|t_A)-q(b|t_A)\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)}=0\\ &\implies \frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)}=\frac{q(a|t_A)}{q(b|t_A)}.\end{aligned}$$

By Myerson's offset theorem (Property 4 in Appendix A1): $\Pr(piv_{BA}|\omega) = \Pr(piv_{AB}|\omega)\sqrt{\frac{\tau(A|\omega)}{\tau(B|\omega)}}$. Hence, (35) can be rewritten as:

$$\frac{q(b|t_B)}{q(a|t_B)}\frac{q(a|t_A)}{q(b|t_A)} > \sqrt{\frac{\tau(A|a)\,\tau(B|b)}{\tau(B|a)\,\tau(A|b)}}.$$

By (4), the left-hand side of this inequality is equal to: $\frac{\tau(A|a)\tau(B|b)}{\tau(B|a)\tau(A|b)} > 1$, which proves that (35) holds.

<u>Case 2: $G(B|t_B) - G(AB|t_B) < 0$ in $\sigma_{t_B}(B) = 1$, $\sigma_{t_A}(A) = \rho^*$.</u>

In this case, there must exist a $\bar{\sigma} \in (0,1)$ such that, for $\{\sigma_{t_A}(A), \sigma_{t_B}(B)\} = \{\rho^*\bar{\sigma}, \bar{\sigma}\}$, we have: $G(B|t_B) - G(AB|t_B) = 0$. Indeed, by Lemma 4, $G(B|t_B) - G(AB|t_B) > 0$ for $\sigma_{t_A}(A) = 0 = \sigma_{t_B}(B)$. The existence of $\bar{\sigma}$ immediately follows from the continuity of the G function.

This value of $\bar{\sigma}$ is unique and such that:

$$\begin{aligned}mag\,(piv_{AB}|a) = mag\,(piv_{AB}|b) = \quad &\max\{mag\,(piv_{BC}|a)\,, mag\,(piv_{BC}|b)\,,\\ &mag\,(piv_{AC}|a)\,, mag\,(piv_{AC}|b)\}.\end{aligned} \tag{36}$$

Indeed, any $\sigma < \bar{\sigma}$ implies that the total expected vote shares of alternatives A and B increase. Since (36) implies that C is third in both states, the magnitudes $mag\,(piv_{PC}|\omega)$ must decrease, for any $P \in \{A, B\}$ and $\omega \in \{a, b\}$. In contrast, the magnitudes $mag\,(piv_{AB}|\omega)$ must increase, since:

$$mag\,(piv_{AB}|a) = mag\,(piv_{AB}|b) = -\left(\sqrt{r\,(t_A|a)\cdot\rho^*} - \sqrt{r\,(t_B|a)}\right)^2\sigma$$

is strictly decreasing in σ. Hence (34) holds with a strict inequality for any $\sigma < \bar{\sigma}$. This implies that (35) holds, and hence that $G(B|t_B) - G(AB|t_B) > 0$ for any $\{\sigma_{t_A}(A), \sigma_{t_B}(B)\} = \{\rho^*\sigma, \sigma\}$, $\sigma < \bar{\sigma}$.

Similarly, one can check that (34) is violated for any $\sigma > \bar{\sigma}$: it implies $G(B|t_B) - G(AB|t_B) < 0$. This proves that (36) must hold in $\sigma_{t_A}(A) = \rho^*\bar{\sigma}$ and $\sigma_{t_B}(B) = \bar{\sigma}$, and that the solution to $\bar{\sigma}$ is unique. ■

The logic of the proof is straightforward: if there is "too much" double voting, both A's and B's vote shares become large as compared to that of C, and the information motive dominates: both types t_A and t_B strictly prefer to single-vote for their initially preferred alternative. Single voting increases the vote gap between A and B and hence the precision of the voting signal. The only obstacle to furthering this gap is the threat posed by C: if (33) binds, then the coordination motive induces both types t_A and t_B to keep double voting, to ensure that A and B remain sufficiently ahead of C. The equilibrium is reached at the unique value of $\sigma_{t_A}(A)$ and $\sigma_{t_B}(B)$ for which the coordination motive balances the information motive – unless a corner solution is reached.

9 Appendix A3: Proof of Theorem 2

It is sufficient to show that there exists at least one equilibrium in which a given candidate, say B, is the only likely winner in both states of nature. Here, we show that there is an equilibrium in which all majority-group voters play action $\psi = BA$ that gives 1 vote to B and $\gamma < 1$ vote to A.

Denote by $x(t)$ the realized number of t-voters. For the above-mentioned strategy, the number of votes received by B is $X(B) = x(t_A) + x(t_B)$. The number of votes received by A is $X(A) = \gamma \cdot X(B)$, and the number of votes received by C is $x(t_C)$.

A deviation by one voter who plays action $\psi = A$ can be pivotal in favour of A iff, with her additional vote, A has at least as many ballots as either B or C. This requires:

$$\begin{aligned} X(B) - X(A) &\in [0,1] \Leftrightarrow x(t_A) + x(t_B) \leq \tfrac{1}{1-\gamma}, \text{ and} \\ x(t_C) &\leq \tfrac{1}{1-\gamma}. \end{aligned}$$

The probability of this joint event is:

$$\overbrace{\exp\left(-(1-r(t_C))\ n\right) \sum_{k=0}^{\lfloor(1-\gamma)^{-1}\rfloor} \tfrac{((1-r(t_C)\ n)^k}{k!}}^{\text{Probability that } x(t_A)+x(t_B)\ \leq\ (1-\gamma)^{-1}} \times \overbrace{\exp\left(-r(t_C)\ n\right) \sum_{k=0}^{\lfloor(1-\gamma)^{-1}\rfloor} \tfrac{(r(t_C)\ n)^k}{k!}}^{\text{Probability that } x(t_C)\ \leq\ (1-\gamma)^{-1}}$$

$$= \exp(-n) \times \sum_{k=0}^{\lfloor(1-\gamma)^{-1}\rfloor} \tfrac{((1-r(t_C)\ n)^k}{k!} \times \sum_{k=0}^{\lfloor(1-\gamma)^{-1}\rfloor} \tfrac{(r(t_C)\ n)^k}{k!},$$

which has magnitude -1. In other words, $mag\left(piv_{AC}|\omega,\psi\right) = -1 = mag(piv_{AB}|\omega,\psi)$ where $mag(piv_{PQ}|\omega,\psi)$ is the magnitude of the probability that voting ψ be pivotal between P and Q in state ω.

This voter must compare the value of playing this action A with that of playing BA. If she plays BA then B necessarily collects more votes than A, since A has a most $\gamma < 1$ vote and B has at least 1 vote. Thus a vote BA can only be pivotal in favour of B. Hence:

$$\begin{aligned} G(A|t) &= q(a|t)\left[2\Pr\left(piv_{AC}|a,A\right) + \Pr(piv_{AB}|a,A)\right] \\ &\quad + q(b|t)\left[\Pr\left(piv_{AC}|b,A\right) - \Pr(piv_{AB}|b,A)\right], \end{aligned}$$

$$\begin{aligned} G(BA|t) &= q(a|t)\left[\Pr\left(piv_{BC}|a,BA\right) - \Pr\left(piv_{BA}|a,BA\right)\right] \\ &\quad + q(b|t)\left[2\Pr\left(piv_{BC}|b,BA\right) + \Pr\left(piv_{BA}|b,BA\right)\right], \end{aligned}$$

Since types t_A and t_B play the same strategy, vote shares and pivot probabilities are identical across states of nature. We thus have that:

$$G(A|t) > G(BA|t) \text{ iff}$$
$$q\,(a|t)\,[2\Pr(piv_{AC}|\omega,A) + \Pr(piv_{AB}|\omega,A) + \Pr(piv_{BA}|\omega,BA) - \Pr(piv_{BC}|\omega,BA)] > ...$$
$$... > q\,(b|t)\,[2\Pr(piv_{BC}|\omega,BA) + \Pr(piv_{AB}|\omega,A) + \Pr(piv_{BA}|\omega,BA) - \Pr(piv_{AC}|\omega,A)].$$

From Property 2 in Appendix A1, for $n \to \infty$, the pivot probabilities with a magnitude of -1 become infinitesimal compared to the probability of being pivotal between B and C:

$$\lim_{n\to\infty} \frac{2\Pr(piv_{AC}|\omega,A)+\Pr(piv_{AB}|\omega,A)+\Pr(piv_{BA}|\omega,BA)}{\Pr(piv_{BC}|\omega,BA)} = \lim_{n\to\infty} \frac{\Pr(piv_{AB}|\omega,A)+\Pr(piv_{BA}|\omega,BA)-\Pr(piv_{AC}|\omega,A)}{2\Pr(piv_{BC}|\omega,BA)} = 0.$$

Hence, as $n \to \infty$, the factor multiplying $q\,(a|t)$ converges to a negative value, whereas the factor multiplying $q\,(b|t)$ converges to a positive value. This shows that $G(A|t) < G(BA|t)$.

A voter may also consider voting AB. That ballot can only be pivotal in favour of A if $X(B) - X(A) \in [0, 1-\gamma]$, which requires that $x\,(t_A) + x\,(t_B) \leq 1$. Repeating the same steps as above shows that the magnitudes of pivot probabilities involving A are also equal to -1 in this case, and hence that $G(BA|t) > G(AB|t)$.[24]

Appendix A4: Proofs for Section 4

Lemma 9 *The scoring rules $\{0, \delta\}$ with $\delta < 1$ and the runoff electoral system do not produce full information and coordination equivalence when $n \to \infty$.*

Proof. Consider a strategy $\bar{\sigma}$ in which all majority voters play the same action, e.g. A. This strategy produces an informational trap (see Definition 4 in Appendix A2), in which A is the only likely winner of the election. Given $\bar{\sigma}$, pivot probabilities are state-of-nature invariant (i.e. $\Pr(piv_{PQ}|\omega) = \Pr(piv_{PQ}|\omega')\ \forall \omega, \omega' \in \Omega$ and any $P, Q \in \{A, B, C\}$). Hence, for such a strategy, the payoff associated, for instance, with action A boils down to:

$$\begin{aligned} G_i\,(A|s) &\equiv \int_0^1 q\,(\omega|s)\,[\Pr(piv_{AB}|\omega)\ \ \Delta U\,(AB,i|\omega) + \Pr(piv_{AC}|\omega)\ \ \Delta U\,(AC,i|\omega)]\,d\omega \\ &= \Pr(piv_{AB}) \underbrace{\int_0^1 q\,(\omega|s)\ \ \Delta U\,(AB,i|\omega)\,d\omega}_{E_{\omega|s}\Delta U(AB,i)} + \Pr(piv_{AC}) \underbrace{\int_0^1 q\,(\omega|s)\ \ \Delta U\,(AC,i|\omega)\,d\omega}_{E_{\omega|s}\Delta U(AC,i)} \end{aligned}$$

Payoffs simplify in the same way for any other action permitted by the electoral system.

If all majority voters play A, we have:

$$\lim_{n\to\infty} \frac{\Pr(piv_{AB})}{\Pr(piv_{AC})} = \lim_{n\to\infty} \frac{\Pr(piv_{BC})}{\Pr(piv_{AC})} = 0.$$

[24] The action B produces exactly the same payoffs as BA, which implies that there exists another equilibrium, in which all voters play B.

Therefore, independently of their private preference, for all majority voters, the value of action A, $G_i(A|s) \simeq \Pr(piv_{AC}) E_{\omega|s} \Delta U(AC, i)$, becomes infinitely larger than $G_i(B|s)$, and than $G_i(BA|s)$ in the scoring rules considered in Theorem 2. The strategy $\bar{\sigma}$ is thus an equilibrium for all scoring rules $\{0, \delta\}$ with $\delta < 1$. The same logic applies to runoff elections: the Duverger's Law equilibria in Bouton (2010) also exist in this extended setup. ■

Proof of Theorem 3. Like in the simple setup, majority voters choose one out of three actions: vote A, vote B, or vote AB. Denote $\Delta U(PQ, i|\omega) \equiv U(P, i|\omega) - U(Q, i|\omega)$, $P, Q \in \{A, B, C\}$. The payoff associated with each action is:

$$\begin{aligned} G_i(A|s) &= \int_0^1 q(\omega|s) \left[\Pr(piv_{AB}|\omega) \;\; \Delta U(AB, i|\omega) + \Pr(piv_{AC}|\omega) \;\; \Delta U(AC, i|\omega)\right] d\omega \\ G_i(B|s) &= \int_0^1 q(\omega|s) \left[-\Pr(piv_{BA}|\omega) \;\; \Delta U(AB, i|\omega) + \Pr(piv_{BC}|\omega) \;\; \Delta U(BC, i|\omega)\right] d\omega \\ G_i(AB|s) &= \int_0^1 q(\omega|s) \left[\Pr(piv_{AC}|\omega) \;\; \Delta U(AC, i|\omega) + \Pr(piv_{BC}|\omega) \;\; \Delta U(BC, i|\omega)\right] d\omega. \end{aligned}$$

We know that pivot probabilities are determined by the action profile of the voters. Given an action profile σ, let mag_1 denote the largest magnitude:

$$mag_1 \equiv \max_{\omega} \left[\max\left\{mag(piv_{AB}|\omega), mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\right\}\right],$$

and let ω_1 be the arg max of that expression. Similarly, the second largest magnitude (*i.e.* excluding the event piv_{PQ} associated with the largest magnitude) is denoted mag_2 and its arg max is ω_2.

Now, we show by contradiction that the largest magnitude must be the one between A and B: imagine that $mag_1 = mag(piv_{AC}|\omega_1)$. Then, it is straightforward to check that $G_i(AB|s) > G_i(B|s)$, $\forall i, s$. Thus, all majority voters would either vote A or AB. Yet, since there cannot be an informational trap (see Definition 4. It is straightforward to show that Lemma 4 in Appendix A2 also holds in the present setup), the excess vote share of A over B must be increasing in ω, which implies: $mag_2 = mag(piv_{AB}|\omega_2 = 0)$. But in that case, no majority voter wants to be pivotal against B ($U(A, i|0) - U(B, i|0) < 0$, $\forall i$): all would vote AB, which leads to a contradiction. By symmetry, the same holds for $mag_1 = mag(piv_{BC}|\omega_1)$. Thus, mag_1 must be $mag(piv_{AB}|\omega_1)$.

Now, we need to show that (a) ω_1 is an interior state, and (b) that $\phi(A|\omega_1) = \phi(B|\omega_1)$: (a) if $\omega_1 \in \{0, 1\}$, then all majority voters would vote "en bloc" either A or B, which leads to an informational trap; a contradiction. If instead $\omega_1 \in (0, 1)$, then some voters i prefer A to B in state ω_1, and either vote A or AB (and conversely for those who prefer B to A).

(b) we show that $\phi(A|\omega_1) = \phi(B|\omega_1)$: if $\phi(A|\omega_1) > \phi(B|\omega_1)$, then the expected number of voters who include A in their ballot must be larger than the expected number of voters who include B in their ballot. But there must then also exist a state $\omega < \omega_1$ for which $\phi(A|\omega_1) > \phi(A|\omega) > \phi(B|\omega) > \phi(B|\omega_1)$, which implies $mag(piv_{AB}|\omega) > mag(piv_{AB}|\omega_1)$. This contradicts the definition of ω_1. Since the same reasoning can be applied to $\phi(A|\omega_1) < \phi(B|\omega_1)$, it must be that

$\phi(A|\omega_1) = \phi(B|\omega_1)$, which in turn implies that $mag(piv_{AB}|\omega_1) = 0$. Finally, this also implies that A has a larger vote share than B iff $\phi(A|\omega) > \phi(B|\omega)$, and conversely. ■

Proof of Theorem 4. Let $\Omega(A)$ denote the set of states ω for which $\phi(A|\omega) > \phi(B|\omega)$, and let $\Omega(B)$ denote the set of states ω for which $\phi(A|\omega) < \phi(B|\omega)$. There is full information and coordination equivalence (FICE) when, for any equilibrium strategy σ, alternative A (resp. B) has the largest expected vote share iff $\omega \in \Omega(A)$ (resp. $\Omega(B)$).

We are looking for the conditions on the distribution of voter preferences such that FICE does *not* hold under approval voting. To do this, we conjecture a strategy σ^F for which FICE is not satisfied and proceed in 4 steps: first, we show that C cannot be the only likely winner in any state, otherwise σ^F is not an equilibrium. Second, we show that if, say B wins in some states $\omega \in \Omega(A)$ under σ^F, then B must actually be winning in all states, otherwise σ^F is non generic. Third, we identify the necessary condition for B to win in all states. Fourth, we show that this necessary condition is also sufficient.

Step I: C cannot be the only likely winner in any state.

Let $\bar{\Omega}_C(\sigma^F) \subseteq \Omega$ denote the subset of states of nature in which C has the largest expected vote share, for a voting strategy σ^F. If $\bar{\Omega}_C(\sigma^F) \neq \varnothing$, then $\max_\omega mag(piv_{AB}|\omega) < \max\{mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\}$ by Lemma 1: the vote share of A being weakly increasing in ω and the vote share of B being weakly decreasing in ω (remember that the fraction of voters who prefer A to B is increasing in ω), either A or B must have the lowest expected vote share in every state $\omega \in \Omega$, while C either has the largest expected vote share if $\omega \in \bar{\Omega}_C(\sigma^F)$, or the second largest, if $\omega \in \Omega \backslash \bar{\Omega}_C(\sigma^F)$ .It is straightforward to see that if, say, $mag(piv_{BC}|\omega)$ is the largest of all magnitudes, then $G_i(AB|s) > G_i(A|s)$, for all i, s. This implies that all majority voters would include B in their ballot, which in turn implies that the expected vote share of B must be larger than that of C, a contradiction. This proves that $\bar{\Omega}_C(\sigma^F) = \varnothing$ if σ^F is an equilibrium strategy.

Step II: the same alternative must win in all states if FICE does not hold.

Let $\bar{\Omega}_P(\sigma^F)$ denote the set of states in which $P \in \{A, B\}$ has the largest expected vote share under σ^F. We focus on the case in which $\bar{\Omega}_B(\sigma^F) \supset \Omega(B)$, that is B has the largest expected vote share for some set of states in which A is actually the full information Condorcet winner. We now show that, generically, if FICE does not hold, then $\bar{\Omega}_B(\sigma^F) = \Omega$. That is, B must always have the largest expected vote share.

Assume $\bar{\Omega}_B(\sigma^F) \neq \Omega$. Since the vote shares of A and B are continuous in ω and C cannot lead in any state of nature (see Step I), there must exist a state ω^{AB} at the junction between $\bar{\Omega}_B(\sigma^F)$ and $\bar{\Omega}_A(\sigma^F)$, where the two majority alternatives tie for first place. Thus, the magnitude of the pivot probability between A and B is zero in ω^{AB}. Now, consider the following difference:

$$G_i(A|s) - G_i(AB|s) = \int_0^1 q(\omega|s)\left[\Pr(piv_{AB}|\omega)\ \Delta U(AB, i|\omega) - \Pr(piv_{BC}|\omega)\ \Delta U(BC, i|\omega)\right] d\omega. \tag{37}$$

If $mag\left(piv_{AB}|\omega^{AB}\right) = 0 > mag\left(piv_{BC}|\omega\right)$, $\forall\omega$, then this difference is strictly positive for a fraction $\phi\left(A|\omega^{AB}\right)$ of the voters, who would thus actually play A as a best response to the strategy σ^F. Likewise, comparing $G_i\left(B|s\right)$ and $G_i\left(AB|s\right)$ shows that a fraction $\phi\left(B|\omega^{AB}\right)$ would actually play B. Since $\omega^{AB} \in \Omega\left(A\right)$, we have $\phi\left(A|\omega^{AB}\right) > \phi\left(B|\omega^{AB}\right)$, and A must thus be the only likely winner, a contradiction.

Thus, for B to be among the likely winners in $\omega^{AB} \in \Omega\left(A\right)$ under an equilibrium strategy σ^F, it must be that $\max_\omega mag\left(piv_{BC}|\omega\right) = mag\left(piv_{AB}|\omega^{AB}\right) = 0$. This requires that B and C tie for first place in state ω^{AB} (remember that the vote share of B is weakly decreasing in ω). Yet, we just saw that, in state ω^{AB}, the vote shares of A and B must also be equal. That is, the fraction of voters who prefer to vote A or AB must be equal to the fraction of voters who prefer to vote B or AB *and* to the fraction of voters who prefer C. This distribution of preferences is clearly non-generic. This shows that if σ^F is an equilibrium strategy, then $\Omega \supset \bar{\Omega}_B\left(\sigma^F\right) \supset \Omega\left(B\right)$ cannot happen: either $\bar{\Omega}_B\left(\sigma^F\right) = \Omega\left(B\right)$ and there is FICE, or $\bar{\Omega}_B\left(\sigma^F\right) = \Omega$, and there is no information aggregation.

Step III: necessary condition.

Now we identify a necessary condition for $\bar{\Omega}_B\left(\sigma^F\right) = \Omega$: we prove that $\phi\left(B|\bar{\omega}\right)$ must be large enough ($\bar{\omega}$ is the state in which this share is lowest). The complementary case $\bar{\Omega}_B\left(\sigma^F\right) = \varnothing$ (such that A leads in all states) is similar.

Since $\phi\left(A|\omega\right) > \phi\left(B|\omega\right)$ when $\omega \in \Omega\left(A\right)$, a large enough fraction of the voters who prefer A to B must approve of B (i.e. play AB) to obtain $\bar{\Omega}_B\left(\sigma^F\right) = \Omega$. Likewise, a large enough fraction of the voters who prefer B to A must be playing action B. Since the vote share of A is weakly increasing in ω ($\phi\left(A|\omega\right)$ is increasing in ω) and that of B weakly decreasing in ω, we have: $\bar{\Omega}_B\left(\sigma^F\right) = \Omega \Rightarrow \arg\max_\omega mag\left(piv_{AB}|\omega\right) = \bar{\omega}$. By (37), a necessary condition for the voters who prefer alternative A to B in state $\bar{\omega}$ to play AB instead of A is:

$$\max_{\omega} mag\left(piv_{BC}|\omega\right) \geq mag\left(piv_{AB}|\bar{\omega}\right). \tag{38}$$

Now, compare the value of actions B and AB:

$$G_i\left(B|s\right) < G_i\left(AB|s\right)$$

$$\Leftrightarrow$$

$$\int_0^1 q\left(\omega|s\right)\left[-\Pr\left(piv_{BA}|\omega\right)\ \Delta U\left(AB,i|\omega\right) - \Pr\left(piv_{AC}|\omega\right)\ \Delta U\left(AC,i|\omega\right)\right] d\omega < 0. \tag{39}$$

Clearly, none of the voters who prefer alternative A to B in state $\bar{\omega}$ would ever play B: both terms are strictly negative.

Now, concerning the voters who prefer alternative B to A in state $\bar{\omega}$:

–1– $G_i\left(A|s\right) - G_i\left(AB|s\right)$ is strictly negative for all of them;

–2– $G_i\left(B|s\right) \geq G_i\left(AB|s\right)$ requires that $mag\left(piv_{BA}|\bar{\omega}\right) \geq \max_\omega mag\left(piv_{AC}|\omega\right)$.

Thus, a necessary condition for B to have the largest vote share in all states turns out to be:

$$\max_{\omega} mag\left(piv_{BC}|\omega\right) \geq mag\left(piv_{BA}|\bar{\omega}\right) \geq \max_{\omega} mag\left(piv_{AC}|\omega\right), \tag{40}$$

where the latter magnitude is always restricted since A and C have the lowest two expected vote shares (see Lemma 1 in Appendix A1). This condition allows for up to 4 cases:

Case (i) $\max_\omega mag\left(piv_{BC}|\omega\right) = mag\left(piv_{BA}|\bar{\omega}\right) = \max_\omega mag\left(piv_{AC}|\omega\right)$. This case is non generic (see Step II).

Case (ii) $\max_\omega mag\left(piv_{BC}|\omega\right) > mag\left(piv_{BA}|\bar{\omega}\right) > \max_\omega mag\left(piv_{AC}|\omega\right)$. In this case, all the voters who prefer alternative A to B in state $\bar{\omega}$ play AB, and all the voters who prefer alternative B to A play B. A vote can be pivotal between B and C if:

$$x\left(B\right) + x\left(AB\right) \in \left\{x\left(C\right), x\left(C\right) - 1\right\}, \text{ and } x\left(B\right) + x\left(AB\right) \geq x\left(AB\right),$$

where the latter condition is never binding. Thus, $mag\left(piv_{BC}|\omega\right) = -\left(\sqrt{1 - r\left(t_C\right)} - \sqrt{r\left(t_C\right)}\right)^2$ $\forall\omega$ and the first inequality becomes:

$$-\left(\sqrt{1 - r\left(t_C\right)} - \sqrt{r\left(t_C\right)}\right)^2 > -\phi\left(B|\bar{\omega}\right) = mag\left(piv_{BA}|\bar{\omega}\right).$$

Rearranging terms yields the necessary condition:

$$\phi\left(B|\bar{\omega}\right) > 1 - 2\sqrt{r\left(t_C\right)}\sqrt{1 - r\left(t_C\right)}. \tag{41}$$

To identify the condition behind the second inequality, $mag\left(piv_{BA}|\bar{\omega}\right) > \max_\omega mag\left(piv_{AC}|\omega\right)$. Knowing that $mag\left(piv_{AC}|\omega\right)$ is restricted, it is sufficient to show when $mag\left(piv_{BA}|\bar{\omega}\right)$ is unrestricted. Since no voter plays A, a vote can only be pivotal between A and B if:

$$\begin{aligned} x\left(B\right) &= 0, \text{ and} \\ x\left(AB\right) &\geq x\left(C\right), \end{aligned} \tag{42}$$

meaning that $mag\left(piv_{BA}|\bar{\omega}\right)$ is unrestricted iff $\phi\left(A|\bar{\omega}\right) > r\left(t_C\right)$.

Case (iii) $\max_\omega mag\left(piv_{BC}|\omega\right) = mag\left(piv_{BA}|\bar{\omega}\right) > \max_\omega mag\left(piv_{AC}|\omega\right)$. By the same arguments as for Step II, this case is non-generic.

Case (iv) $\max_\omega mag\left(piv_{BC}|\omega\right) > mag\left(piv_{BA}|\bar{\omega}\right) = \max_\omega mag\left(piv_{AC}|\omega\right)$. As explained in (ii) above, the equality requires that $\phi\left(A|\bar{\omega}\right) \leq r\left(t_C\right)$. Using the fact that $\phi\left(A|\bar{\omega}\right) + \phi\left(B|\bar{\omega}\right) + r\left(t_C\right) = 1$, one can check that this inequality actually amounts to:

$$\phi\left(B|\bar{\omega}\right) \geq 1 - 2r\left(t_C\right). \tag{43}$$

Since (43) implies that (41) holds, we have proven that, both in case (ii) and in case (iv), (41) must be satisfied for FICE to fail.

Step IV: sufficient condition.

This step proves that (41) is also a sufficient condition for a failure of FICE: from Step III, there are two cases to consider.

Case (a): $\phi(A|\bar{\omega}) > r(t_C)$. In this case, the following strategy is an equilibrium:

$$\begin{aligned} \sigma_i(AB) &= 1, \forall i \text{ s.t. } \Delta U(AB, i|\bar{\omega}) \geq 0 \\ \sigma_i(B) &= 1, \forall i \text{ s.t. } \Delta U(AB, i|\bar{\omega}) < 0. \end{aligned}$$

Indeed, for these strategies, (41) implies that (40) holds with strict inequalities, and therefore:

$$G_i(AB|s) > \max\{G_i(A|s), G_i(B|s)\}, \forall i \text{ such that } \Delta U(AB, i|\bar{\omega}) \geq 0 \text{ and} \tag{44}$$

$$G_i(B|s) > \max\{G_i(A|s), G_i(AB|s)\}, \forall i \text{ such that } \Delta U(AB, i|\bar{\omega}) < 0. \tag{45}$$

Case (b): $\phi(A|\bar{\omega}) \leq r(t_C)$. In this case, the voters who prefer alternative B to A in state $\bar{\omega}$ may take either of two actions: B or AB. Let $\gamma\phi(B|\bar{\omega})$ denote the fraction of the electorate who plays B in state $\bar{\omega}$. There exists $\bar{\gamma} \equiv \frac{1-2r(t_C)}{\phi(B|\bar{\omega})}$ (>0) such that if $\gamma < \bar{\gamma}$, then the expected vote share of A is strictly larger than that of C, and hence: $mag(piv_{BA}|\bar{\omega}) > \max_\omega mag(piv_{AC}|\omega)$. This implies that (45) holds for any $\gamma < \bar{\gamma}$. Therefore, the fraction of voters who play B is bounded below by $\bar{\gamma}\phi(B|\bar{\omega})$. For that lower bound, we have:

$$mag(piv_{BC}|\omega) = 2\sqrt{r(t_C)}\sqrt{1-r(t_C)} - 1 > -\bar{\gamma}\phi(B|\bar{\omega}) = mag(piv_{BA}|\bar{\omega}),$$

which implies (44), and hence that no voter wants to play A. B is then the only likely winner for any $\omega \in \Omega$. ∎