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JEL Classifications: G32, G34, C71, D72, D74.

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Abstract

Cross-ownership breaks the traditional rule of one-sided corporate control. Using a novel approach based on stochastic voting processes, this paper suggests a general method to determine control stakes in the presence of cross-ownership. It offers a generalization of the Banzhaf index, which applies to cross-ownership-inclusive graphs. The original feature of this approach is its absolute sequentiality. Accordingly, we build a new algorithm that evaluates the shareholders' control powers in any corporate structure. From a governance viewpoint, we emphasize that cross-ownership may act as a powerful device for shareholder expropriation. To illustrate this point, we revisit the leading example of the German Allianz Group.

1. Introduction

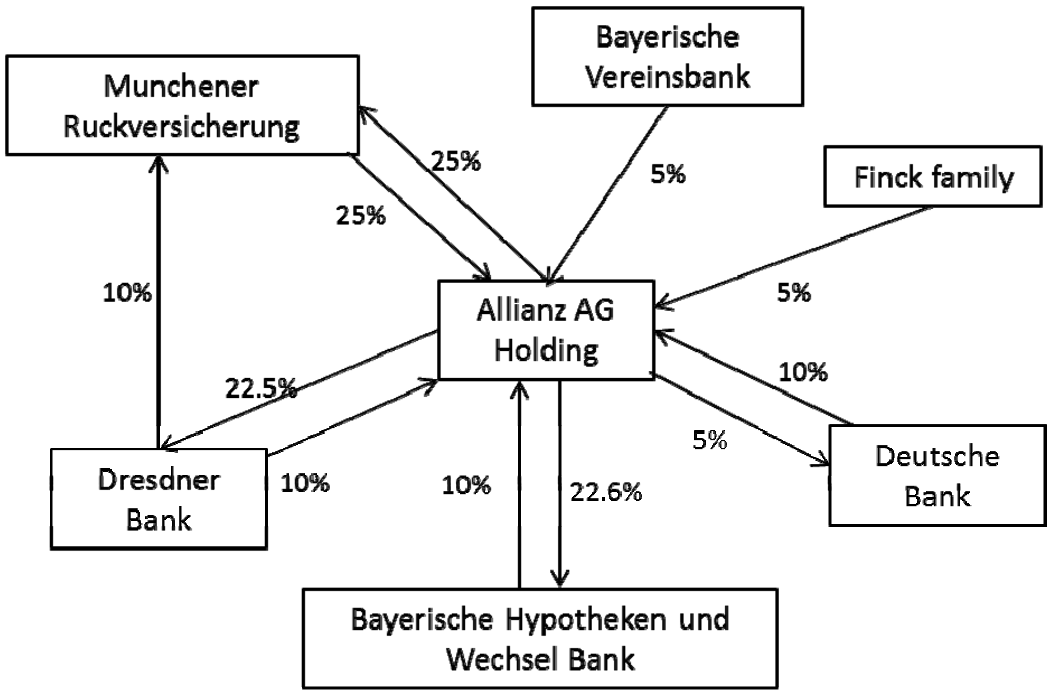
Cross-ownership is a device used by either dominant shareholders or managers to insulate firms from outside control.[1] Examples of cross-ownership abound in civil law countries, where control tunneling is more frequent (Johnson *et al.*, 2000). For instance, La Porta *et al.* (1999) cite the German Allianz Group as a leading case (see Fig. 1). Other examples are found in the Czech banking sector (Turnovec, 1999), in the Colombian brewery industry (Gutiérrez *et al.*, 2008), and in the Japanese banking sector (Temurshoev and Stakhovych, 2009). Conceptually though, cross-ownership is tricky since it breaks the traditional rule of one-sided control. For this reason, the literature on corporate governance is likely to disregard the control issues raised by cross-ownership. This paper intends to fill this gap. Using a novel approach based on stochastic voting processes, it proposes a general method to determine control stakes in the presence of cross-ownership.

Currently, two broad approaches coexist to work out the shareholders' control stakes in a corporate structure, which is represented by an ownership graph. Under the first, control stakes are consolidated through input-output matrix algebra (Brioschi *et al.*, 1989; Claessens *et al.*, 2000; Chapelle and Szafarz, 2005 and 2007). Secondly, the Banzhaf (1965) power index from game theory gives the probability that an ultimate shareholder decisively influences the decisions made in a subordinated firm (Crama and Leruth, 2007; Levy, 2009 and 2011). These two approaches work well as long as the ownership structure is a "directed

[1] Cross-ownership (*i.e.*, circular ownership links) is but one mechanism for exerting control over firms with low cash-flow rights. Bebchuk *et al.* (2000) identify three such mechanisms: pyramidal structures (Thesmar, 2001; Chapelle, 2001; Biebuyck *et al.*, 2005), non-voting stocks (Zingales, 1994; Nicodano, 1998), and cross-ownership (Weidenbaum, 1996). Additionally, the interplay of within-industry cross-ownership and collusion is analyzed by, e.g., Malueg (1992), Gilo *et al.* (2006), and Temurshoev and Stakhovych (2009).

acyclic graph", which *de facto* excludes cross-ownership.[2] In contrast, this paper offers a generalization of the Banzhaf index, which applies to cross-ownership-inclusive graphs.

Figure 1: Allianz' ownership structure in 1998



source: La Porta *et al.* (1999)

Technically speaking, cross-ownership brings unusual, and somewhat disturbing, features to corporate voting games. First, the outcome of a vote taking place in a subsidiary firm may depend on the order in which its parent firms organize their shareholder meetings ("path dependency"). Second, multiple voting equilibria may co-exist. As a consequence, the notion of "winning coalition" becomes ambiguous (Gambarelli and Owen, 1994). This problem drives a major methodological issue. The Banzhaf index of an ultimate shareholder in a firm is defined as the proportion of winning coalitions in which the shareholder is pivotal. Therefore, the impossibility to identify winning coalitions makes the Banzhaf index ill-

[2] Crama and Leruth (2007) partially address the problem of cross-ownership. For general definitions in graph theory, see Bondy and Murty (1976).

defined. This renders the existing game-theoretical approach incapable of dealing with cross-ownership structures.

To address these theoretical issues, we propose a new approach based on a stochastic voting process. The original feature of this process is its absolute sequentiality. To circumvent path-dependency, we impose that only one firm votes in each step. Moreover, the multiple-equilibrium problem is solved by proving the convergence of the stochastic voting process. This convergence makes it possible to consistently define a generalized Banzhaf index, which applies to corporate universes with cross-ownership. Accordingly, we build a new algorithm that determines control powers in any corporate structure.

Our new approach also helps clarify the interpretation of cross-ownership from a governance viewpoint. According to Bebchuk *et al.* (2000), cross-ownership is basically a way to control firms with an arbitrarily low fraction of shares. A cross-ownership ring represents an attractive prey for external shareholders because the firms in the ring artificially control each other. Controlling all of them from the outside of the ring is made easier. Thus, Bebchuk *et al.* (2000) view cross-ownership as a device for easing external control.

We challenge this view with two arguments. First, holding a very low fraction of shares may reveal a fragile instrument for controlling a cross-ownership ring. Other outside shareholders might then seize the same prey by simply buying slightly more shares than the existing shareholder. As a result, the control of cross-ownership rings with few shares is threatened by the entry of new shareholders, which considerably reduces its attractiveness. Second, cross-ownership is better understood as an entrenchment device for the insiders, namely the management of the firms belonging to the ring. Cross-ownership artificially creates own shares by leveraging the firm's shareholdings. According to this interpretation, cross-ownership leads to the expropriation of external shareholders through tunneling. To further illustrate our point, we revisit the leading example of the Allianz group with the help of our new methodology.

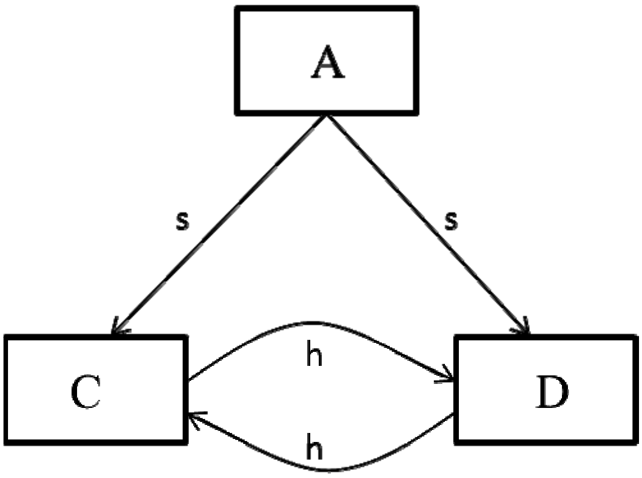
The rest of this paper is organized as follows. Section 2 presents the puzzling features brought by cross-ownership. Section 3 recalls the classical game-theoretic approach of corporate control. Section 4 introduces our new approach to cross-ownership structures. The new algorithm is described in Section 5, and applied to the Allianz Group in Section 6. Section 7 concludes.

2. The Puzzling Features of Cross-Ownership

Corporate cross-ownership obscures control mechanisms and allows ultimate shareholders to capture control with low cash-flow rights. Bebchuk *et al.* (2000) illustrate this feature with the symmetrical two-firm case shown in Fig. 2. In this graph, firm A is the sole ultimate shareholder of both firms C and D. Firm A owns fraction s of the shares of each of its subordinated firms, while firms C and D are linked by a symmetrical cross-ownership link of intensity h. Firm C owns fraction h of firm D's shares, and firm D owns fraction h of firm C's shares. For this case, Bebchuk *et al.* (2000) demonstrate that firm A's control over both firms C and D is entrenched. More precisely, they prove the following proposition:

Proposition (Bebchuk *et al.*, 2000, p. 300)
For any level of the cash-flow rights of A in C (and, symmetrically, in D), there exist values for s and h such that A controls C and D, meaning that A benefits from more than 50% of the voting rights in C and D.

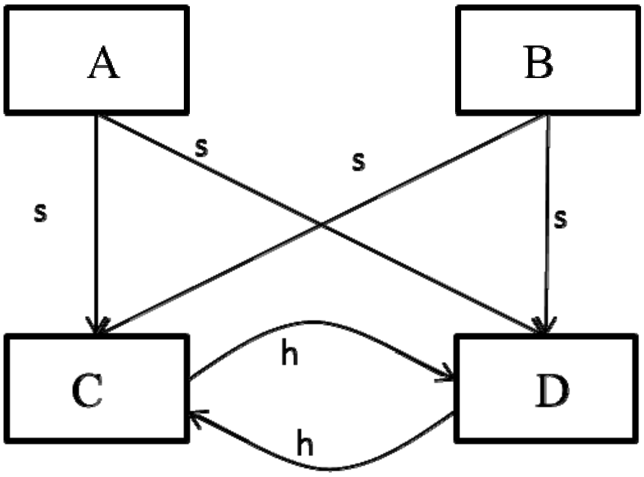
Figure 2: The symmetrical two-firm case with a single controller (Bebchuk *et al*., 2000)



We contend that this proposition is restrictive. It holds only when firm A is assumed to be the sole ultimate shareholder in the corporate universe under consideration. To make our point, let us add a second ultimate shareholder, say firm B, to the picture, as featured in Fig. 3, assuming that s is low enough to have $2s + h \leq 100\%$. In this new graph, the situations of firms A and B are symmetrical, and neither of them enjoys full control over firms C and D.[3] In fact, majority control over at least one firm in the cross-ownership ring formed by firms C and D is needed to secure command over this ring. On the other hand, (partial) control remains threatened by the entry of new shareholders. This results in entrenched control over a cross-ownership ring requiring more than an arbitrary small fraction of shares. We will show in Section 4 that cross-ownership can also result in actual control exerted by non-ultimate shareholders. A special case concerns control exerted on itself by a subsidiary firm. Therefore, determining actual shares of control in corporate structures involving cross-ownership requires examining the situation of all firms, and not only that of the ultimate shareholders.

[3] Interestingly, Dorofeenko *et al.* (2008) study the same situation but restrict themselves to pure Nash equilibria (Ritzenberger, 2005).

Figure 3: The symmetrical two-firm case with two controllers



The circularity of cross-ownership translates into multiple equilibria in voting games. To illustrate this point, let us consider again the situation in Fig. 3, with s = 30% and h = 40%. Let us further assume that the shareholders of firm D have to make a binary decision (0 or 1) through majority voting. If firms A and B vote alike, they dictate their common will to firm D, and firm C cannot influence the decision of firm D. But, what happens when firms A and B disagree?

To answer this question, let us suppose that firm A votes 0 and firm B votes 1. In this case, neither firm A nor firm B reaches majority in firm D alone. The outcome of the vote thus depends on firm C's vote, which, in turn, is determined by firm C's shareholders. This sends us back to the original problem (but for firm C instead of firm D). The decision of firm C is dictated by firm D, precisely because firms A and B disagree. Thus, cross-ownership makes the decision process circular, and consequently paradoxical. From a game-theoretical viewpoint, when firm A votes 0 and firm B votes 1, the game has two equilibria. Indeed, the only equilibrium condition will be achieved if firms C and D vote alike, which leaves us with two possibilities.

The voting game admits a single equilibrium when firms A and B agree, and two equilibria when they disagree. While there are four possible configurations for the votes of firms A and B, the voting game admits six equilibrium voting states. In particular, each case in which firms A and B disagree leads to dual equilibria. To address this issue, we will replace the usual deterministic framework by a probabilistic one and associate probabilities to multiple equilibrium voting states.

Setting multiple solutions aside, the previous discussion also points out the path-dependency of voting outcomes implied by cross-ownership. Let us suppose that firms A and B disagree. To actually reach an equilibrium voting state, firms C and D would therefore have to vote not only sequentially, but knowing the votes of all three other firms. For instance, when firm C votes according to the majority rule, it needs to be aware of the votes of firms A, B, and D. Otherwise, firm C would be unable to determine its vote. To circumvent this problem, we will adopt a recursive approach starting from initial voting states *for all the firms* in the corporate structure, and explicitly take into account the voting paths leading to equilibrium.

The general case of corporate structures with cross-ownership is considered in Section 4. However, before introducing our own contribution, we summarize, in the next section, the state of the art of voting games and corporate control.

3. Voting Games and Corporate Control: State of the Art

Consider set $V = \{1, \ldots, n\}$ representing n firms connected by ownership links. The corresponding *ownership structure* is the directed graph of (direct) shareholdings: $H = (V, R, A)$, where V is the vertex set, $R \subseteq V \times V$ is the arc set, and $A = (a_{ij})$ is the $n \times n$ matrix of direct ownership, with $a_{ij} \in [0,1]$ being the share that firm i owns in firm j. The set of *ultimate shareholders* – or *sources* – of V, denoted by $S = \{1, 2, 3, ..., s\} \subset V$, includes all firms/shareholders having no direct shareholders in V. We denote by $F = \{s+1, s+2, ..., n\}$ the set of *non-source firms*. Some firms in F may have unidentified shareholders, which are grouped together and form the so-called *float*. Hence, float $\tilde{a}_j$ of firm $j \in F$ is defined by:

$$\tilde{a}_j = 1 - \sum_{i=1}^{n} a_{ij} \tag{1}$$

The ownership structure is said to be *complete* when all firms in F have no float:

$$\forall j \in F : \tilde{a}_j = 0 \tag{2}$$

Cross-ownership refers to the presence of cycles in the ownership structure. More precisely, a *cycle* is set of vertices $\{i_1, i_2, ..., i_k\} \subset F$ such that:

$$\forall j \in \{1, 2, ..., k-1\},\ a_{i_j i_{j+1}} > 0 \text{ and } a_{i_k i_1} > 0 \tag{3}$$

In particular, own shares are special cycles (for $k=1$). On the other hand, *acyclic graphs* have no cycles and correspond to ownership structures referred to as *pyramids*. The next subsection addresses binary voting games in pyramids.[4]

Consider a voting game taking place in pyramid $H=(V,R,A)$. Each firm in *V* is asked to vote 0 or 1 while being consistent with the majority voting rule. As the sources have no shareholders in the structure, their votes are unconstrained. Moreover, for the sake of simplicity, we assume that the vote of any non-source firm's float is split, with in equal shares voting 0 and 1, respectively.[5]

A *voting state* is any $X=\left(x_1,\ldots,x_n\right)\in\left\{0,1\right\}^n$, where $x_j\in\left\{0,1\right\}$ represents the vote of firm *j*. An *equilibrium voting state* is a voting state consistent with the majority voting rule for all firms in *V*. Simply put, in equilibrium each firm in *V* votes in the same way as the majority of its direct shareholders. An equilibrium voting state may, therefore, be represented as a fixed point of the following mapping (Crama and Leruth, 2007; Levy, 2009):

$$G:\left\{0,1\right\}^n\rightarrow\left\{0,1\right\}^n:X\rightarrow\left(g_1(X),g_2(X),\ldots,g_n(X)\right) \tag{4}$$

where:

$$g_j:\left\{0,1\right\}^n\rightarrow\left\{0,1\right\}:X\rightarrow\begin{cases} x_j & \text{if } j\in S \\ 1 & \text{if } j\in F \text{ and } \sum_{i=1}^{n}a_{ij}\,x_i+\dfrac{\tilde{a}_j}{2}>0.5 \\ 0 & \text{if } j\in F \text{ and } \sum_{i=1}^{n}a_{ij}\,x_i+\dfrac{\tilde{a}_j}{2}\leq 0.5 \end{cases} \tag{5}$$

[4] Actually, the term "pyramids" is often loosely used to refer to ownership structures including chains of participations. We introduce here a stricter definition to make a clear distinction between pyramids and cross-ownership inclusive structures.
[5] See Levy (2011) for further developments on the modelization of the float.

Consequently, deriving all the equilibria of the game, $X^* = \left(x_1^*, \ldots, x_n^*\right)$, boils down to solving a fixed-point problem for any given votes of the sources, $X_S^0 \in \{0,1\}^s$, where X_S denotes the sub-vector of vector X made up of its first s components. Hence, the equilibrium condition in structure $H = (V, R, A)$ is the following:

$$X^* = G\left(X^*\right) \tag{6}$$

Gambarelli and Owen (1994) show that in pyramids, equilibrium X^* exists and is unique for any given voting states of the sources. Namely, we have:

$$\forall X_S^0 \in \{0,1\}^s, \exists! X^* \in \{0,1\}^n : X_S^* = X_S^0 \text{ and } X^* = g\left(X^*\right) \tag{7}$$

Each firm's vote is thus uniquely determined by the votes of the sources. In other words, in pyramids, only the ultimate shareholders (the sources) can impact a firm's decision. Therefore, non-source firms have absolutely no impact on the decisions of firms, including themselves.

The Banzhaf index measures the intensity of control of a source in a non-source firm. More precisely, the *Banzhaf* (1965) *index*[6] is the frequency with which source *i*'s vote is pivotal in non-source firm *j*. This definition may be formulated by using winning coalitions. A set of sources, $T \subset S$, is a *winning coalition* in $j \in F$ if the firms in *T* together make the vote of *j* in equilibrium:

[6] Originally, the Banzhaf index was created for measuring the influence of the US states in the American Electoral College. In that framework, the Banzhaf index associated to a state represents the probability for this state to be the decisive voter under the assumption that each state votes 0 or 1 with equal probability. In corporations though, the decision rule is often more complex since it depends on the entire ownership structure.

$$\left[\forall i \in T : x_i^* = x\right] \Rightarrow x_j^* = x \tag{8}$$

Accordingly, Banzhaf index z_{ij} of firm $i \in S$ in firm $j \in F$ is given by:

$$z_{ij} = \frac{1}{2^{s-1}} \sum_{T \subset \left(S \setminus \{i\}\right)} \left[v_j\left(T \cup \{i\}\right) - v_j\left(T\right)\right] \tag{9}$$

where $v_j(.)$ identifies the winning coalitions in firm *j*:

$$v_j(T) = \begin{cases} 1 & \text{if } T \text{ is a winning coalition in } j \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Additionally, $v_j(T)$ represents the voting state of firm *j* when all firms in *T* vote 1 and all other sources vote 0. Equation (9) may then be rewritten as:

$$z_{ij} = \frac{1}{2^{s-1}} \left(\sum_{X_s^* \in \{0,1\}^s : x_i^* = 1} x_j^* - \sum_{X_s^* \in \{0,1\}^s : x_i^* = 0} x_j^* \right), \quad i \in S,\ j \in F \tag{11}$$

The equilibrium uniqueness for given votes of the sources is instrumental to the definition of the Banzhaf index. In pyramids, once the votes of the sources are given, the equilibrium voting states of all non-source firms are determined, and so are the winning coalitions. Therefore, the Banzhaf index of each source in each non-source firm is defined without ambiguity.

In practice, however, computing Banzhaf indices may prove to be tedious because the combinatorial complexity of (11) is $O(2^n)$. To address this issue, Crama and Leruth (2007) use Monte-Carlo simulations. Alternatively, Levy's (2011) algorithm computes all Banzhaf indices in a single step. This algorithm is based on the coincidence matrix, $C = \left(c_{ij}\right)$, defined by:

$$c_{ij} = \frac{1}{2^s} \sum_{X_s^* \in \{0,1\}^s} \left[x_i^* \; x_j^* + \left(1 - x_i^*\right)\left(1 - x_j^*\right) \right] = \frac{1}{2^s} \sum_{X_s^* \in \{0,1\}^s} \delta_{ij}^* \qquad (12)$$

where $\delta_{ij}^* = \begin{cases} 1 & \text{if } x_i^* = x_j^* \\ 0 & \text{otherwise} \end{cases}$

Entry c_{ij} measures the occurrence of firms *i* and *j* voting alike. The sum in (12) is taken over all possible voting states of the sources, and then divided by the number of such possibilities. Levy (2011) shows that the Banzhaf index is linked to the coincidence matrix in the following way:

$$z_{ij} = 2.c_{ij} - 1, \quad i \in S,\; j \in F \qquad (13)$$

The intuition behind this result goes as follows. First, if firms *i* and *j* vote independently, their vote should be identical in 50% of the cases and $c_{ij} = 0.5$. According to Eq. (13), this produces a zero Banzhaf index. In the second polar case, where firm *i* dictates its vote to firm *j*, both firms should always vote alike and $c_{ij} = 1$. In this case, Eq. (13) produces a Banzhaf index equal to 1. Third, in the middle scenario, where firm *i* has partial influence on firm *j*, we have $c_{ij} \in (0.5,1)$. Let us consider the example where $c_{ij} = 0.7$. This value means that, on top of the 50% cases where firms *i* and *j* vote alike by chance, there is an additional 20% of active influence of firm *i* on firm *j*. However, this 20% level of influence is to be understood with respect to a maximal value of 50% (and not 100%). Therefore, on a 0-100% scale the influence of firm *i* on firm *j* is 2*20% = 40%, which is indeed the value of the Banzhaf index resulting from the application of Eq. (13) for $c_{ij} = 0.7$.

In Section 6, Eq. (13) is used to generalize the definition of the Banzhaf index to ownership structures involving cross-ownership. However, to make this possible, we first need to address the existence and (non-)uniqueness of equilibrium voting states in such structures.

4. Voting Games in Corporate Structures with Cross-Ownership

Cross-ownership challenges the game-theoretical approach to voting games in corporate structures because it creates cycles in ownership graphs. Existing theories – explicitly or not – concentrate on top-down corporate relationships, which *de facto* excludes cross-ownership. Only a few authors (e.g. La Porta *et al*., 1999; Dorofeenko *et al*., 2008; Crama and Leruth, 2011; Aminadav *et al.*, 2011) mention cross-ownership as a problem to be addressed, but no general theory properly accounts for it. In particular, the current definition of the Banzhaf (1965) index does not accommodate multiple equilibria. Moreover, multiple equilibria are detrimental to the convergence of existing algorithms, which compute the ultimate shareholders' control stakes in a set of interlocked firms (Crama and Leruth, 2007; Levy, 2011).

The goal of this Section is to propose a general definition of the Banzhaf index that holds in cross-ownership structures. As shown in Section 2, cross-ownership brings three new features. First, the outcome of a voting game is no longer necessarily unique for given voting states of the sources. Second, non-source firms can influence this outcome – and the equilibrium voting state of a given firm can even be influenced by itself. Third, the outcome of a voting game may also depend on the voting order of the firms in the structure.

The basic tool used here is a stochastic voting process, represented by a Markov chain. This process starts from initial voting states, not only of the sources, but also of the non-source firms. An additional crucial feature of our definition of the stochastic voting process refers to the voting sequence. To avoid inconsistencies due to path-dependency, we make only one firm vote at a time. This is a key characteristic of our approach because any voting scheme involving simultaneous voting may bring indeterminacy in the final outcome.

In order to proceed step-by-step, the next two subsections successively introduce first deterministic sequential voting, and then the stochastic processes. The last subsection defines the generalized Banzhaf index.

4.1. Sequential Voting States

Consider again graph $H = (V, R, A)$ of direct shareholdings, but now cross-ownership is allowed. In corporate structures with cross-ownership, although multiple equilibria are possible, equilibrium condition (6) still holds. This subsection addresses the existence of the fixed point problem in (6) when the assumption of acyclic graph is relaxed.

When graph *H* includes cross-ownership, the votes of *all* the firms may influence the voting outcome in any non-source firm. Let us therefore consider an initial voting state, $X^0 \in \{0,1\}^n$, which concerns all firms in *V*, and is no longer restricted to the sources. This initial voting state may be split in two sub-vectors: $X^0 = \left(X_S^0, X_F^0\right)$, where $X_S^0 = \left(x_1^0, ..., x_s^0\right)$ concerns the sources (in *S*), and $X_F^0 = \left(x_{s+1}^0, ..., x_n^0\right)$ concerns the non-source firms (in *F*). Starting from X^0, we consider successive voting states obtained by imposing the majority voting rule to the vote of one non-source firm in each step. The order in which this rule is applied is dictated by a voting path.

Definition 1:

Voting path $W = (w_k) \in F^{\mathbb{N}_0}$ is a sequence of non-source firms.

The sources do not belong to voting paths because they have no shareholder and their votes are unconstrained. Any non-source firm may appear several times (or none) in a given voting path. The voting paths are used to introduce voting sequences.

Definition 2:

The *voting sequence* associated to initial voting state $X^0 \in \{0,1\}^n$ and voting path $W = (w_k) \in F^{\mathbb{N}_0}$ is the sequence of voting states $(X^k) \in \{0,1\}^{n \times \mathbb{N}}$ starting from X^0 and such that:

$\forall k \in \mathbb{N}: X^k = (x_1^k, \ldots, x_n^k)$

and:

$$\forall k \in \mathbb{N}_0, \forall j = 1, \ldots, n: \quad x_j^k = \begin{cases} g_j\left(X^{k-1}\right) & \text{if } w_k = j \\ x_j^{k-1} & \text{if } w_k \neq j \end{cases} \tag{14}$$

where $g_j(.)$ is defined by (5).

According to Eq. (14), the transition from voting state X^{k-1} to voting state X^k is obtained by making the direct shareholders of firm w_k vote under the majority rule. The votes of these shareholders are dictated by their voting states in X^{k-1}. They define the new voting state of firm w_k. Meanwhile, the voting states of the other firms remain unchanged.

Definition 3:

a) Voting state $X \in \{0,1\}^n$ is *accessible from* $X^0 \in \{0,1\}^n$ *in K steps* if there exists a voting sequence $\left(X^k\right) \in \{0,1\}^{n\times\mathbb{N}}$ starting from X^0 such that $X^K = X$.

b) The set of all voting states accessible from X^0 in K steps is denoted by:

$$B_K\left(X^0\right) = \left\{X \in \{0,1\}^n : X \text{ is accessible from } X^0 \text{ in } K \text{ steps}\right\} \tag{15}$$

c) The set of all voting states accessible from X^0 regardless of the number of steps is denoted by:

$$B\left(X^0\right) = \bigcup_{k\in\mathbb{N}_0} B_k\left(X^0\right) \tag{16}$$

Definition (16) is particularly useful when X is an equilibrium voting state, *i.e.*, a fixed point of mapping $G(.)$ defined by Eqs. (4) and (5). To represent all equilibrium voting states accessible from X^0, we adopt the following notation:

$$B^*\left(X^0\right) = \left\{X^* \in \{0,1\}^n : X^* \text{ is an equilibrium voting state accessible from } X^0\right\} \tag{17}$$

Importantly, any voting sequence reaching an equilibrium voting state remains in this state forever. This is because equilibrium voting states are fixed points of mapping $G(.)$. Reaching an equilibrium voting state thus represents a sufficient condition for a voting sequence to converge. The next theorem addresses the existence of a converging voting sequence starting from an arbitrary initial voting state.

Theorem 1

$\exists K \in \mathbb{N}, \forall X^0 \in \{0,1\}^n, \exists$ voting path $W \in F^{\mathbb{N}_0}$ such that X^K is an equilibrium voting state, where $\left(X^k\right)$ is the voting sequence associated to X^0 and *W*.

Proof:

See Appendix A.[7]

Theorem 1 shows that there exists an upper limit to the number of steps needed to reach equilibrium. Remarkably, this upper limit uniformly holds for all initial voting states. Theorem 1 also implies that any (initial)[8] voting state of the sources (in $\{0,1\}^S$) is compatible with at least one equilibrium voting state. We thus have the next result.

Theorem 2

$\forall X^0 \in \{0,1\}^n : B^*\left(X^0\right) \neq \phi.$

Theorem 2 is a direct corollary of Theorem 1. It establishes the existence of an equilibrium voting state irrespective of the presence of cross-ownership in the structure.[9]

Both Theorems 1 and 2 hold for any initial voting state, but not for any voting path. Different voting sequences starting from the same initial voting state may indeed lead to different

[7] Appendix A proves Theorem 1 with $K = 2(n - s)$. However, for many corporate structures, smaller values of *K* likely exist.

[8] Actually, the voting states of the sources remain unchanged all through the voting sequences. Therefore, there is no need to specify whether a source's voting state is initial or not.

[9] Crama and Leruth (2007) adopt a similar methodology to address cross-ownership. However, in each step of their voting sequences, all non-source firms vote simultaneously. In comparison to ours, their definition does not require voting paths and yields a faster evolution of the voting sequences. Unfortunately, their approach is not recursive enough to ensure convergence. For instance, in the case drawn in Fig. 3, starting from $X^0 = (0,1,0,1)$, the voting sequence of Crama and Leruth (2007) would perpetually alternate between two voting states: $(0,1,0,1)$ and $(0,1,1,0)$. To get convergent voting sequences, we are bound to make one firm vote at a time.

equilibrium voting states or even to no equilibrium at all. For instance, a voting path corresponding to a cycle in the graph may create a periodic voting sequence, with no limit. This issue is addressed in the next subsection.

4.2 Stochastic Voting Process

We introduce uncertainty to the picture by randomizing the voting order of non-source firms.

Definition 4:

The *stochastic voting order*, $\tilde{W} = \left(\tilde{w}_k\right)_{k\in\mathbb{N}_0}$, is the independent and identically distributed (i.i.d.) stochastic process such that the probability distribution of random variable $\tilde{w}_k = \tilde{w}$ is:

$$\forall k \in \mathbb{N}_0, \forall j \in F : \Pr\left(\tilde{w}_k = j \middle| \tilde{w}_{k-1}, \tilde{w}_{k-2}, ..., \tilde{w}_1\right) = \Pr\left(\tilde{w}_k = j\right) = \frac{1}{n-s}$$

While there are infinitely many deterministic voting paths (see Definition 1), there is only one stochastic voting order process. Each realization of this process is a voting path. As a consequence, to each (deterministic) initial voting state $X^0 \in \{0,1\}^n$ is attached a single stochastic voting process defined as follows.

Definition 5:

The *stochastic voting process starting from* $X^0 \in \{0,1\}^n$ is $\tilde{X} = \left(\tilde{X}^k\right)_{k\in\mathbb{N}}$ such that:

$\forall k \in \mathbb{N}_0, \forall X, Y \in \{0,1\}^n :$

$$\Pr\left[\tilde{X}^k = X \,\middle|\, \tilde{X}^{k-1} = Y\right] = \begin{cases} 0 & \text{if } X \notin B_1(Y) \\ \dfrac{1}{n-s} & \text{if } X \in B_1(Y) \text{ and } X \neq Y \\ \dfrac{\#\left\{ j \in F : y_j = g_j\left(Y\right)\right\}}{n-s} & \text{if } X \in B_1(Y) \text{ and } X = Y \end{cases} \tag{18}$$

with $\Pr\left[\tilde{X}^0 = X^0\right] = 1$.

This definition is the stochastic counterpart of Definition 3. It gives the transition probabilities driven by the stochastic voting-order process. Three cases are possible depending on the one-step accessibility of voting state *X* from voting state *Y*. In the first case, *X* is not accessible in one step from *Y*. The transition from *Y* to *X* is then impossible. In the second case, *X* is accessible in one step from *Y*, and *X* is different from *Y*. Since there is only one firm *i* in *F* such that $y_i \neq x_i = g_i(Y)$, the transition probability is equal to $\frac{1}{n-s}$. In the last case, *X* is accessible in one step from *Y*, and *X* is equal to *Y*. The equalities $x_j = y_j = g_j\left(Y\right)$ may then be met for several firms in *F*. This explains the numerator of the transition probability, while the denominator, $\frac{1}{n-s}$, is the probability of any firm being the *k*-th voting firm.

It turns out that the stochastic voting process is a Markov chain with 2^n states (each state being an *n*-dimensional vector with binary components). Its transition matrix is given by Eq. (18). The next theorem states the convergence of this Markov chain.

Theorem 3:

$$\forall X^{0} \in \{0,1\}^{n}, \Pr\left[\exists K \in \mathbb{N}, \forall k > K : \tilde{X}^{k} \in B^{*}\left(X^{0}\right)\right] = 1 \tag{19}$$

Proof

We know from Theorem 2 that $B^{*}\left(X^{0}\right)$ is not empty. For any $X^{*} \in B^{*}\left(X^{0}\right)$, Eq. (18) yields:

$$\Pr\left[\tilde{X}^{k} = X^{*} \middle| \tilde{X}^{k-1} = X^{*}\right] = 1$$

This is because once the Markov chain has reached an equilibrium voting state, it stays there forever. Put formally, $B_1\left(X^{*}\right) \setminus \left\{X^{*}\right\} = \phi$ and $\#\left\{j \in F : x_j^{*} = g_j\left(X^{*}\right)\right\} = (n-s)$. This implies that all equilibrium voting states of the Markov chain in Eq. (18) are absorbing (Fu and Lou, 2003). Moreover, Theorem 1 ensures that, for any initial voting state, there exists an accessible, and hence absorbing, equilibrium state. The Markov chain defined by Eq. (18) is therefore, absorbing. Finally, in an absorbing Markov chain, the probability that the process is absorbed in a finite number of steps is equal to one (Grinstead and Snell, 1997, p. 417).

QED.

Theorem 3 ensures both the existence and uniqueness of random variable $\tilde{X}^{*}\left(X^{0}\right)$ representing the limit voting state of the process starting from any X^{0}. Although $B\left(X^{0}\right)$ may include multiple (deterministic) equilibrium voting states, each of them has a specific probability of being reached in the limit. In particular, equilibrium voting states reached *via* fewer and/or longer voting paths have lower probabilities.

4.3 Generalized Banzhaf Index

In pyramids, the Banzhaf index is consistently defined thanks to the existence of a unique deterministic equilibrium voting state associated with any initial voting state. When cross-ownership enters the picture, this is no longer true. However, the stochastic voting process defined by Eq. (18) makes it possible to define a generalized Banzhaf index. The new definition is built by replacing the previously deterministic equilibrium voting state by its stochastic counterpart, $\tilde{X}^*\left(X^0\right)$, and taking mathematical expectations to get a scalar. Namely we adopt the following definition inspired from Eq. (11).

Definition 6:

The *generalized Banzhaf index* of firm *i* in non-source firm *j* is:

$$Z_{ij} = \frac{1}{2^{n-1}}\left(\sum_{X^0\in\{0,1\}^n:\, x_i^0=1} E\left(\tilde{x}_j^*\left(X^0\right)\right) - \sum_{X^0\in\{0,1\}^n:\, x_i^0=0} E\left(\tilde{x}_j^*\left(X^0\right)\right)\right), i\in V,\, j\in F \qquad (20)$$

where $\tilde{x}_j^*\left(X^0\right)$ is the j^{th} component of $\tilde{X}^*\left(X^0\right)$.

In Eq. (20), firm *i* may denote any firm in *V*, and not necessarily a source as in Eq. (11). Moreover, when the corporate structure is a pyramid, the generalized Banzhaf index boils down to its standard counterpart ($\forall i\in S, \forall j\in F: Z_{ij} = z_{ij}$). Eq. (11) and (20) differ however in two respects. First, due to the stochastic setting, Eq. (20) is expressed with mathematical expectations. Second, the sum in Eq. (20) is taken over the initial voting states of *all* firms.

Contrasting with the approaches of Bennedsen and Wolfenzon (2000) and Gambarelli and Owen (1994) for pyramidal structures, our probabilistic approach does not require identifying winning coalitions. This is a notable advantage. Given the circularity of cross-ownership, the same coalition may be a winning one in some situations, and a losing one in others.

To make Definition 6 operational, we must compute the expectation of variable $\tilde{x}_j^*\left(X^0\right)$. This variable is binary. Therefore, its expectation is the probability that it takes value 1, *i.e.*, that firm *j* votes 1 in an equilibrium voting state of the stochastic voting process starting from X^0. Because the limit equilibrium is a unique random variable, this probability only depends on *j* and X^0. We may thus simplify its notation:

$$f_j\left(X^0\right)=E\left(\tilde{x}_j^*\left(X^0\right)\right)=\Pr\left[\tilde{x}_j^*\left(X^0\right)=1\right]. \tag{21}$$

Eq. (20) then becomes:

$$Z_{ij}=\frac{1}{2^{n-1}}\left(\sum_{X^0\in\{0,1\}^n:\,x_i^0=1} f_j\left(X^0\right)-\sum_{X^0\in\{0,1\}^n:\,x_i^0=0} f_j\left(X^0\right)\right) \tag{22}$$

Eq. (22) shows that the generalized Banzhaf index of firm *i* in firm *j* is the difference between the average probability (over all initial voting states) for firm *j* to vote 1 in equilibrium if the initial vote of firm *i* is 1 and the average probability for firm *j* to vote 1 in equilibrium if the initial vote of firm *i* is 0.

The generalized Banzhaf indices are summarized in matrix $\mathrm{Z}=\left(Z_{ij}\right)$. We similarly generalize the definition of the coincidence matrix in Eq. (12).

Definition 7:

The g*eneralized coincidence matrix* $\Gamma=\left(C_{ij}\right)$ is defined by:

$$C_{ij}=\frac{1}{2^n}\sum_{X^0\in\{0,1\}^n}\left(x_i^0.f_j\left(X^0\right)+\left(1-x_i^0\right).\left(1-f_j\left(X^0\right)\right)\right),\qquad i\in V,\, j\in F \tag{23}$$

The last theorem states that the link between the Banzhaf index and the coincidence matrix remains true for their generalized counterparts.

Theorem 4:

$\forall i\in V,\forall j\in F: Z_{ij}=2.C_{ij}-1$

Proof:

Simple algebra yields:

$$\begin{aligned}
C_{ij}&=\frac{1}{2^n}\left(\sum_{X^0\in\{0,1\}^n:x_i^0=1}\left(x_i^0.f_j\left(X^0\right)+\left(1-x_i^0\right).\left(1-f_j\left(X^0\right)\right)\right)+\sum_{X^0\in\{0,1\}^n:x_i^0=0}\left(x_i^0.f_j\left(X^0\right)+\left(1-x_i^0\right).\left(1-f_j\left(X^0\right)\right)\right)\right)\\
&=\frac{1}{2^n}\left(\sum_{X^0\in\{0,1\}^n:x_i^0=1}f_j\left(X^0\right)+\sum_{X^0\in\{0,1\}^n:x_i^0=0}\left(1-f_j\left(X^0\right)\right)\right)\\
&=\frac{1}{2^n}\left(\sum_{X^0\in\{0,1\}^n:x_i^0=1}f_j\left(X^0\right)-\sum_{X^0\in\{0,1\}^n:x_i^0=0}f_j\left(X^0\right)+2^{n-1}\right)\\
&=\frac{1}{2^n}\left(\sum_{X^0\in\{0,1\}^n:x_i^0=1}f_j\left(X^0\right)-\sum_{X^0\in\{0,1\}^n:x_i^0=0}f_j\left(X^0\right)\right)+\frac{1}{2}
\end{aligned}$$

Finally, we obtain:

$$2.C_{ij}-1=\frac{1}{2^{n-1}}\left(\sum_{X^0\in\{0,1\}^n:x_i^0=1}f_j\left(X^0\right)-\sum_{X^0\in\{0,1\}^n:x_i^0=0}f_j\left(X^0\right)\right)=Z_{ij}\,.$$

QED

For example, the generalized coincidence matrix, Γ, and the generalized Banzhaf matrix, Z, of the ownership structure featured in Fig. 3 with s = 30% and h = 40%, are given by:

$$\Gamma = \begin{pmatrix} 1 & 0.5 & 0.75 & 0.75 \\ 0.5 & 1 & 0.75 & 0.75 \\ 0.5 & 0.5 & 0.625 & 0.625 \\ 0.5 & 0.5 & 0.625 & 0.625 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.25 & 0.25 \end{pmatrix}$$

In matrix Z, 50% of control power in both firms C and D is attributed to both firms A and B. The generalized Banzhaf index is also designed to address the control stakes of non-source firms. The voting power of firm C in firm D (or, equivalently here, of firm D in itself) is 25%. When firms A and B disagree (which happens with probability 0.5), the decision must be taken by the firms in the cycle. For obvious reasons of symmetry, the resulting control power is then equally split between firms C and D, each of them having a 25% chance to be decisive.

Two major tasks have been accomplished in this section. First, the game-theoretical approach to binary voting games has been extended to cycle-inclusive ownership graphs. As a consequence, the generalized Banzhaf index closes the only remaining gap in this stream of literature. Now, control in any ownership structure may be understood with the voting-game methodology.

Second, the generalized Banzhaf index makes it possible to evaluate the control stake of any shareholder, ultimate or not, in any non-source firm. This is especially meaningful in cross-ownership structures because cross-ownership may confer actual voting power to subsidiaries. Indeed, in cross-ownership structures, control is no more restricted to top-down relationships. While the generalized Banzhaf index of a non-source firm in a non-source firm is always zero in pyramids, this index may take strictly positive values in cross-ownership structures. In particular, firms may control themselves to some extent through circular shareholdings. Section 6 will further elaborate on this.

5. The Algorithm

This section extends Levy's (2011) algorithm to corporate structures with cross-ownership. Levy's (2011) algorithm computes the classical Banzhaf index thanks to the coincidence matrix. The use of coincidence matrices makes it possible to derive all indices in a single Monte-Carlo simulation. Moreover, Levy (2011) considers several modelizations of the float. Here, we restrict ourselves to a single modelization and assume that the float is split, with equal parts voting 0 and 1 respectively. This split is assumed for the sake of simplicity, but our algorithm could easily be adapted to other situations for the float.

The new algorithm simulates the Markov stochastic voting process and stops as soon as an equilibrium voting state is reached. The two steps of the algorithm are driven by the theoretical framework presented in Section 4. First, for a given initial voting state, we determine the limit equilibria through simulations. Second, we proxy the coincidence matrix by replacing probability $f_j\left(X^0\right)$ in Eq. (23) with the corresponding simulated frequency.

Step 1: Simulations

For each initial voting state, $X^0 = \left(x_1^0, x_2^0, ..., x_n^0\right) \in \{0,1\}^n$, we generate *M* random voting paths. In each simulation, the voting path is driven by integers randomly taken in set $\{s+1, s+2, ..., n\}$.

In the first step, we randomly pick $i_1 \in \{s+1, s+2, ..., n\}$, which designates the first voting firm. The first simulated voting state is $X^1 = (x_1^1, x_2^1, ..., x_n^1)$, where:

$$x_i^1 = \begin{cases} g_i\left(X^0\right) & \text{if } i = i_1 \\ x_i^0 & \text{otherwise} \end{cases}$$

In the $\left(k+1\right)^{\text{th}}$ step, $i_{k+1} \in \left\{s+1, s+2, ..., n\right\}$ defines voting state $X^{k+1} = (x_1^{k+1}, x_2^{k+1}, ..., x_n^{k+1})$ where:

$$x_i^{k+1} = \begin{cases} g_i\left(X^k\right) & \text{if } i = i_k \\ x_i^k & \text{otherwise} \end{cases}$$

The m^{th} simulated voting path stops after k steps if $\forall j \in \left\{s+1, s+2, ..., n\right\}$ we have $x_j^k = g_j\left(X^k\right)$. Then, we write:

$X^k = X_m^*$.

Step 2: Computation of the Banzhaf indices

For each $X^0 \in \left\{0,1\right\}^n$ and each corresponding equilibrium voting state, X_m^*, $m \in \left\{1, \ldots, M\right\}$, we compute matrix $S\left(X^0, m\right) = \left(s_{ij}\left(X^0, m\right)\right)_{i \in V, j \in F}$ such that:

$$s_{ij}\left(X^0, m\right) = \begin{cases} 1 & \text{if } x_j^* = x_i^0 \\ 0 & \text{otherwise} \end{cases}$$

Entry $s_{ij}\left(X^0, m\right)$ represents the number of times that, in the m^{th} simulated path starting from X^0, firm *j* votes in equilibrium in the same way as firm *i* voted initially.

Matrix *S* is computed *M* times, even when some simulated paths are leading to the same equilibrium. This is how the algorithm acknowledges the possibility that multiple equilibria may have different probabilities. Indeed, the frequency of an equilibrium voting state indicates how probable this particular equilibrium is when the stochastic voting process starts from X^0.

The coincidence matrix C is then obtained by averaging all the $s_{ij}\left(X^0, m\right)'s$:

$$C_{ij} = \frac{1}{2^n.M} \sum_{X^0 \in \{0,1\}^n} \sum_{m=1}^{M} s_{ij}\left(X^0, m\right)$$

This generates the simulated counterpart of the generalized coincidence matrix. According to Theorem 4, the generalized Banzhaf indices are then proxied by:

$$Z_{ij} = 2.C_{ij} - 1$$

This algorithm requires $M.2^n$ simulated samples. Each sample is characterized by an initial voting state and a random voting path. Therefore, when the number of firms (especially non-source firms) is large, the algorithm can become highly time-consuming. In such a case, it may be suitable to work with a random sample of initial states X^0 along the lines proposed by Crama and Leruth (2007). This, in turn, would reduce the precision of the algorithm.

6. Who Controls Allianz?

The case of Allianz, a German insurance company, is taken from La Porta *et al*. (1999). The ownership situation of Allianz in 1998, featured in Fig. 1, has now become a typical example of cross-ownership. Allianz is mentioned in several papers (Franks and Mayer, 2001; Gugler and Yurtoglu, 2003; Dorofeenko *et al*., 2008) to illustrate the complexity of determining actual control stakes when cross-ownership is involved. However, only Dorofeenko *et al*.

(2008) really try to identify the actual controller of Allianz, starting from the assumption that control is held by, at most, one ultimate shareholder. Strikingly, these authors conclude that all ultimate shareholders could be *the* controller of Allianz, so that the control of Allianz is probably hidden among the firms with which Allianz has cross-participations.

The Allianz ownership structure (see Table 1) has two ultimate shareholders, Bayerische Vereinsbank and the Finck family. Both of them own 5% of the company's shares. Four other shareholders have cross-participations with Allianz. The largest direct shareholder of Allianz, Munchener Ruckversicherung, holds 25% of Allianz' shares while Allianz owns 25% of its shares. Moreover, Allianz also benefits from indirect control over Munchener Ruckversicherung *via* Dresdner Bank.

Table 1: Direct Ownership Shares in the Allianz Group (in %)

Shareholder / **Firm**	Mun. Ruck.	Dresdner Bank	Deutsche Bank	Bay. Hyp. und Wech. Bank	Allianz
Bayerische Vereinsbank	0	0	0	0	**5**
Finck family	0	0	0	0	**5**
Munchener Ruckversicherung	0	0	0	0	**25**
Dresdner Bank	10	0	0	0	**10**
Deutsche Bank	0	0	0	0	**10**
Bayrische Hypotheken und Wechsel Bank	0	0	0	0	**10**
Allianz	25	22.5	5	22.6	**0**

Source: La Porta *et al*. (1999)

We have applied our algorithm to two different scenarios, and reported the resulting generalized Banzhaf indices in Table 2. First, we have simulated the situation of Allianz without cross-ownership, meaning that we have put to zero all the numbers of Table 1 that are not in bold. In other words, scenario 1 ignores the shares owned by Allianz (25% in Munchener Ruckversicherung, 22.5% in Dresdner Bank, 22.6% in Bayrische Hypotheken und

Wechsel Bank, and 5% in Deutsche Bank) as well as the 10% shares Dresdner Bank holds in Munchener Ruckversicherung. In this first scenario, the graph includes six ultimate shareholders and only one non-source firm, Allianz. Second, we have considered the full cross-ownership-inclusive structure described in Table 1. Scenario 2 thus corresponds to the real situation with two ultimate shareholders, Bayerische Verenisbank and the Finck family, and five non-source firms. In both scenarios, we assume that the float of each non-source firm is split, with equal parts voting 0 and 1 respectively.

Our algorithm computes the generalized Banzhaf index in Allianz for any firm in the group (including Allianz itself), and is, therefore, not restricted to ultimate shareholders (see Table 2). The total of each column in Table 2 can exceed 100%, which is a feature of the Banzhaf index.

Table 2: Generalized Banzhaf Indices for the Allianz Group

Shareholder / **Scenario**	**Scenario 1:** Simulation excluding cross-ownership	**Scenario 2:** Real situation including cross-ownership
Bayerische Vereinsbank	6,3%	5,0%
Finck Family	6,3%	5,0%
Munchener Ruckversicherung	81,3%	35,9%
Dresdner Bank	18,8%	7,7%
Deutsche Bank	18,8%	7,8%
Bayrische Hypotheken und Wechsel Bank	18,8%	7,7%
Allianz	0,0%	59,4%

The second scenario, i.e. the real situation, delivers striking results. While cross-ownership does not significantly decrease the ultimate shareholders' voting power, Allianz has the highest control power (59.4%) over itself. The actual voting power of its ultimate shareholders, Bayerische Vereinsbank and the Finck family, equals their direct ownership share (5%). Moreover, the control power of the other shareholders of Allianz is substantially lower in scenario 2 than in scenario 1, and even lower than their direct ownership shares, except for Munchener Ruckversicherung.

Two effects concur to explain these results. First, in scenario 2, each shareholder of Allianz has to share control with Allianz itself. To better understand how this feature acts on control shares, let us consider the extreme and fictitious situation where two firms, say A and B, own 100% of each other's shares. At first sight, one could think that each firm fully controls the other one. In fact, this is not the case. The voting power in A is to be split between A and B, each of them having a generalized Banzhaf index of 0.5.[10] Thus, although unrealistic and counter-intuitive, this extreme case illustrates how cross-ownership insulates a firm from outside control, even when the latter is exerted by a dominant shareholder.

The second effect of cross-ownership stems from the impact of the voting order. If a shareholder of Allianz wishes to oppose Allianz's will (all firms have initial votes in our setting), then Allianz can use cross-ownership to force other firms to vote in its favor, provided that the votes are organized first in these other firms.[11]

This example reveals that the main impact of cross-ownership is not likely the capture of control by outsiders with low cash-flow rights, as Bebchuk *et al.* (2000) argue, but rather the empowerment of the management. Indeed, while the control of Allianz over itself is trivially zero in the absence of cross-ownership, it amounts to 59.4% when cross-ownership is taken

[10] In this extreme case, there are four possible initial voting states and two equilibrium voting states: (0,0) and (1,1). Voting states (0,1) (1,0) both lead to equilibrium (0,0) with probability 0.5, and to equilibrium (1,1) with probability 0.5. As a consequence, the generalized coincidence index of any firm with itself is 0.75, and the corresponding generalized Banzhaf index is 0.5. The same holds true for the indices of A in B, and of B in A.
[11] Interestingly, using our algorithm shows that in all equilibrium states, the votes of the firms with which Allianz has cross-participations, are the same as that of Allianz. This means that after a few iterations all the firms start acting as a coalition. This prevents external shareholders from taking control over Allianz.

into account. Evidence thus shows that cross-ownership turns Allianz into its own dominant controller.[12]

7. Conclusion

This paper addresses both the theoretical and the practical challenges brought by cross-ownership. It proposes a general theory for voting games in corporate structures with cross-ownership. This new theory is based on fully sequential voting. In line with this theory, we propose an algorithm to compute the control stakes of all shareholders (ultimate or not) in all subsidiary firms through Monte-Carlo simulations.

As a matter of fact, cross-ownership was the only remaining feature of corporate graphs thus far unaddressed by game theory in a general way. Moreover, authors typically feel uncomfortable when it comes to drawing the practical consequences of cyclical ownership structures on governance (Crama *et al.*, 2011).[13] Our approach fills this gap by offering a generalization of the Banzhaf index. As a result, researchers are now equipped to compute control stakes in any corporate structure. Nevertheless, further empirical applications are still needed to test the relevance of our approach in such complex cross-ownership structures, as are observed in the banking sector[14] and in the US mutual fund industry.[15]

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[12] The figures in Table 2 should, nevertheless, to be taken with caution. Indeed, the generalized Banzhaf indices in scenario 2 are likely over-estimated because Allianz's shareholders have owners who are not reported in the data. Therefore, our algorithm puts all these owners in the float, and neutralizes their votes. In this way, the voting power of Allianz in its shareholders is inflated.

[13] These authors also mention an approach based on Markov chains, but the cited paper is reported as a working paper in preparation, and therefore is not available. We were thus unable to compare this approach to ours.

[14] The fact that Allianz and several other real-life examples of cross-ownership structures belong to the banking/insurance sector is puzzling.

[15] For instance, there is strong cross-participation between BlackRock, State Street, and PNC.

We also stress that firms embedded in cross-ownership rings should be viewed as potential controllers of themselves. This evidence contradicts the existing literature, which typically takes for granted that only ultimate shareholders can control a firm. Although considering only top-down control relationships makes perfect sense in pyramidal structures, cross-ownership changes the picture dramatically. The example of the Allianz Group perfectly illustrates this principle. We have shown that taking cross-ownership into account reveals how Allianz has managed to be its own majority controller.[16] Actually, the leading example of Allianz originally pointed out by La Porta *et al.* (1999) has attracted the attention of several researchers. For instance, Dorofeenko *et al*. (2008) state that many shareholders could be the controller of Allianz, and conclude that the actual control is probably held by the firms embedded in the cross-ownership ring. While this analysis captures part of the story, our approach goes further because it disentangles the roles of each firm in this ring.

Our findings may also be instrumental for regulators for at least two reasons. First, cross-ownership, like share buybacks, artificially inflates capital, and hence provides a distorted image of real financial situations. Second, cross-ownership can increase management entrenchment. Although most countries restrict companies from purchasing their own shares, cross-ownership is mostly disregarded, likely because of the impossibility to measure its actual consequences. By lifting away this impossibility, our method could pave the way to a better regulatory appraisal of control tunneling through cross-ownership.

Our paper opens additional avenues for research. Recent literature documents the impact of governance variables on different dimensions of corporate performance. However, cross-ownership is rarely taken into consideration as a relevant governance feature. This should be

[16] Boubaker and Labégorre (2008) mention this possibility of own control through cross-ownership, although in an informal way.

changed, especially regarding management entrenchment. Second, in terms of methodology, the fundaments of our model could be applied to other fields involving voting issues in cyclic graphs, such as political science. Lastly, the robustness of our results with respect to the modelization of the float could be checked. In this paper, we assume that the vote of the float is neutral (*i.e.*, equally split). This assumption may reveal too strong, especially in situations like mergers and acquisitions, where small shareholders come to the shareholder meetings with a concerted agenda (Levy, 2011).

In sum, this paper provides a general method for measuring the impact of cross-ownership on corporate control, and emphasizes that cross-ownership may act as a powerful device for shareholder expropriation. As such, cross-ownership certainly deserves more attention than it has received so far.

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Appendix A: Proof of Theorem 1

Let X^0 be an initial voting state. To prove the existence an equilibrium state accessible from X^0, we build a finite voting path made of two parts. Then we show that its length is smaller than or equal to $2(n-s)$.

1) In the first part of this voting path, we place all the firms that, according to the majority voting rule, will change their voting state from 0 to 1. Namely:

- If $\forall i \in F : x_i^0 = 0 \Rightarrow g_i\left(X^0\right) = 0$, then $L = 0$ and $X^L = X^0$.
- Otherwise there exists $w_1 \in F$ such that: $x_{w_1}^0 = 0$ and $g_{w_1}(X^0) = 1$. We start the voting path with w_1, and Eq. (14) defines voting state X^1.

We similarly define the second firm in the voting path, if any. Namely:

- If $\forall i \in F : x_i^1 = 0 \Rightarrow g_i\left(X^1\right) = 0$, then $L = 1$, $X^L = X^1$.
- Otherwise there exists w_2 such that $x_{w_2}^1 = 0$ and $g_{w_2}(X^1) = 1$. Then, firm w_2 is the second in the voting path, and Eq. (14) defines voting state X^2.

And so on, until the following condition is met:

$$\forall i \in F : x_i^k = 0 \Rightarrow g_i\left(X^k\right) = 0$$

This is fulfilled in a finite number of steps, say L, because the graph is finite (and $L \le n-s$). As a consequence, after L steps, we obtain a finite voting path $(w_1, w_2, ..., w_L)$ and its associated voting sequence $\left(X^1, X^2, ..., X^L\right)$ such that:

$$\forall j \in \{1, 2, ..., L\}, x_{w_j}^{j-1} = 0 \text{ and } x_{w_j}^j = g_{w_j}\left(X^{j-1}\right) = 1 \tag{A.1}$$

$$\text{And: } \forall i \in F : x_i^L = 0 \Rightarrow x_i^L = g_i\left(X^L\right) \tag{A.2}$$

2) In the second part of the voting path, we place all the firms that will change their voting state from 1 to 0.

- If $\forall i \in F : x_i^L = 1 \Rightarrow g_i\left(X^L\right) = 1$, and $X^{L+M} = X^L$.
- Otherwise, there exists w_{L+1} such that: $x_{w_{L+1}}^L = 1$ and $g_{w_{L+1}}(X^L) = 0$. Then, the voting path goes on with w_{L+1}, and Eq. (14) defines voting state X^{L+1}.

And so on, until the following condition is met:

$\forall i \in F : x_i^{L+k} = 1 \Rightarrow g_i(X^{L+k}) = 1$

Again, this is fulfilled in a finite number of steps, say *M*. Hence, $(w_{L+1}, w_{L+2}, ..., w_{L+M})$ and its associated voting sequence, $(X^{L+1}, X^{L+2}, ..., X^{L+M})$, are such that:

$$\forall j \in \left\{L+1, L+2, ..., L+M\right\} : x_{w_j}^{j-1} = 1 \text{ and } x_{w_j}^j = g_{w_j}\left(X^{j-1}\right) = 0 \tag{A.3}$$

and: $\forall i \in F : x_i^{L+M} = 1 \Rightarrow g_i\left(X^{K+L}\right) = 1$ (A.4)

3) Now consider the last voting state X^{K+L}. From Eq. (A.4), we have:

If $x_i^{K+L} = 1$, then $x_i^{K+L} = g_i\left(X^{K+L}\right)$.

Alternatively, if $x_i^{K+L} = 0$, then either $x_i^K = 0$ and Eq. (A.2) yields $x_i^K = g_i(X^K) = 0$, or $\exists l \in \{1, 2, ..., L\}$ such $x_i^{K+l-1} = 1$ and $x_i^{K+l} = g_i(X^{K+l-1}) = 0$.

In both cases, $\exists l \in \{0, 1, 2, ..., L\}$ such that $g_i(X^{K+l}) = 0$. Condition (A3) then implies that $g_i\left(X^{K+l}\right) = 0 \Rightarrow g_i\left(X^{K+L}\right) = 0 = x_i^{K+L}$

In sum, in all possible cases we obtain $x_i^{K+L} = g_i(X^{K+L})$, which implies that X^{K+L} is an equilibrium voting state. Moreover, the length of the complete voting path is $L + K \leq 2(n - s)$.

QED